The last missing gap (Lemma 20.1)

$(ZF + AC_\omega({}^\omega\omega))$

$A \subseteq ({}^\omega\omega)^l$ is $\Sigma^1_n(a)$ for some $n \in \omega$ and $a \in {}^\omega\omega$

iff there is some q. free $\hat\varphi$ in $\mathcal{L}_{\mathcal{A}^2}$

$\vec f \in A \iff \mathcal{A}^2 \models \exists^1 g_1 \forall^1 g_2 \dots Q^1 g_n Q^0 m\, \hat\varphi(m, \vec g \restriction m, \vec f \restriction m)$

where $Q^1 = \begin{cases} \forall^1 & \text{if } n \text{ is even} \\ \exists^1 & \text{if } n \text{ is odd} \end{cases}$ $Q^0 = \begin{cases} \exists^0 & \text{if } n \text{ even} \\ \forall^0 & \text{if } n \text{ odd} \end{cases}$

$A \subseteq ({}^\omega\omega)^l$ is $\Sigma^1_n(a)$ $(a \in {}^\omega\omega)$

$\overset{(*)}{\iff} \vec f \in A$

$\iff \mathcal{A}^2 \models \exists^1 \vec g \forall^1 \vec g \dots Q^1 \vec g\, \varphi(\vec f, \vec g, a)$

Q^1: $\forall^1$ if n is even, $\exists^1$ is odd

φ: only with first order q

for some φ with only first order quantifiers in $\mathcal{L}_{\mathcal{A}^2}$

$\mathcal{A}^2 = \langle {}^\omega\omega, \omega, ap(\cdot,\cdot), +, \times, \exp(\cdot,\cdot), p(\cdot), <, \underset{= c_0}{0}, \underset{= c_1}{1} \rangle$

ω: identified with $\{c_n : n \in \omega\}$, $c_n : \omega \to \omega;\ k \mapsto n$

$<$: ordering on ω

$p(c_n) = c_{p_n}$ ← nth prime

$ap(f, c_n) = c_{f(n)}$

"$f(n) = m$"

$\exp(n, m) = n^m$

$(\exp(c_n, c_m) = c_{n^m})$

All functions in $\mathcal{A}$ return 0 (i.e. c_0) if no standard value is provided.

All $\underbrace{\text{non atomic or constant}}_{\text{non trivial}}$ terms represent a number.

$\exists^1\ \forall^1$ usual quantifiers w/o any restriction

$\exists^0\ \forall^0$ are quantifiers restricted to ω

i.e. $\exists^0 x \dots \leftrightarrow \exists x(\omega(x) \wedge \dots)$

$\forall^0 x \dots \leftrightarrow \forall x(\omega(x) \to \dots)$

Bounded quantifiers are quantifiers of the form

$$(\exists x < t)(\cdots) \quad (\forall x < t)(\cdots)$$

non trivial $\mathcal{L}_{\mathcal{A}^2}$-term

Lemma 22.1

(a) $\mathcal{A}^2 \models \exists^1 f_1 \exists^1 f_2 \varphi(f_1, f_2, \ldots)$
$\qquad\quad \forall^1 \ \ \forall^1$

$\Leftrightarrow \mathcal{A}^2 \models \exists^1 f \ \varphi([f]_1, [f]_2, \ldots)$
$\qquad\quad \forall^1$

For $f \in {}^\omega\omega$ $\quad [f]_n : \omega \to \omega$

$$[f]_n(k) = f(\langle\langle n, k \rangle\rangle)$$

(b) $\mathcal{A}^2 \models \exists^0 m \exists^0 n \ \varphi(m, n, \ldots)$
$\qquad\quad \forall^0 \ \ \forall^0$

$\Leftrightarrow$
$\mathcal{A}^2 \models \exists^0 l \ \varphi((l)_1, (l)_2, \ldots)$
$\qquad\quad \forall^0$

Lemma 22.2 In (*) we may assume

(1) no equality between second order objects appear in φ

(i.e. equalities in φ are all of the form $t_1 \equiv t_2$ for some non trivial $\mathcal{L}_{\mathcal{A}^2}$-terms:

(2) the predicate ω does not appear in φ

Proof (1): We may replace $f \equiv g$ with $\forall^0 m(ap(f, m) \equiv ap(g, m))$

(2): $\omega(f)$ can be replaced by

$$\exists^0 n \forall^0 m \ (ap(f, m) \equiv n + 0)$$

$\boxtimes$

By Lemma 22.1 and 22.2 we may assume

(**)

$A \subseteq ({}^\omega\omega)$ and

$$\vec{f} \in A \Leftrightarrow \exists^1 g_1 \forall^1 g_2 \ldots Q^1 g_n \ \varphi(\ ,\vec{g}, \vec{f})$$

$$Q^1 = \begin{cases} \forall^1 & n \text{ even} \\ \exists^1 & n \text{ odd} \end{cases}$$

contains only 1-st order quantifiers, only $\equiv$ for non trivial terms, no appearance $\omega(\cdot)$

Lemma 22.3 ($AC_\omega(^\omega\omega)$ is used here!)

(1) $\mathcal{A}^2 \models \forall^0_m \exists^1 f\, \psi(m, f, \vec{a})$

$\iff \mathcal{A}^2 \models \exists^1 f\, \forall^0_m\, \psi(m, [f]_m, \vec{a})$

(2) $\mathcal{A}^2 \models \exists^0_m \forall^1 f\, \psi(m, f, \vec{a})$

$\iff \mathcal{A}^2 \models \forall^1 f\, \exists^0_m\, \psi(m, [f]_m, \vec{a})$

<u>Proof</u> (2): Suppose $\mathcal{A}^2 \models \exists^0_m \forall^1 f\, \psi(m, f, \vec{a})$

Let $m^* \in \omega$ be a witness for this.

Then $\mathcal{A} \models \forall^1 f\, \psi(m^*, [f]_m, \vec{a})$

Thus $\mathcal{A} \models \forall^1 f\, \exists m\, \psi(m, [f]_m, \vec{a})$

Suppose now $\mathcal{A}^2 \not\models \exists^0_m \forall^1 f\, \psi(m, f, \vec{a})$

Then for each $m \in \omega$ let $f_m \in {}^\omega\omega$ s.t.

$\mathcal{A}^2 \not\models \psi(m, f_m, \vec{a})$

for this choice $AC_\omega(^\omega\omega)$ is needed!

Let $f^* \in {}^\omega\omega$ be s.t.

$[f^*]_m = f_m$ for all $m \in \omega$.

$\mathcal{A}^2 \not\models \psi(m, [f^*]_m, \vec{a})$ for all $m \in \omega$

Hence $\mathcal{A}^2 \not\models \exists^0_m \forall^1 f\, \psi(m, [f]_m, \vec{a})$

$\boxtimes$

(1): similar.

<u>Proof of Lemma 20.6</u>

We may assume $(**)$ holds

We may also assume that φ is in prenex form (with first order quantifiers)

We consider the case when n is even: so

We have:

$\vec{f} \in A \iff \mathcal{A}^2 \models \exists^1 g_1 \dots \forall^1 g_n \underbrace{}_{\text{first order}} \varphi_0(\dots)$

We change all $\forall^0$ to $\forall^1$ in $\underbrace{}$.

$\forall^0 m \dots m$

$\forall^1 m \dots m+0$

Then use Lemma 22.3 (2) to move all those $\forall^1$ in $\underbrace{}$ to left next to the $\forall^1 g_n$

$\dots \underbrace{\forall^1 g_n \forall \dots}_{\text{contract this to a single random universal quantifier by Lemma 22.1 (1)!}}$

Contract the remaining $\underbrace{\text{existential quantifier}}_{\text{1st order}}$ to a single $\underbrace{\text{existential quantifier}}_{\text{1st order}}$ to get

$\exists^1 g_1 \forall^1 g_2 \dots \forall^1 g_n \exists^0_m \tilde{\varphi}(m, \vec{g}, \vec{f})$

g free

$\exists^0_m \tilde{\varphi}(m, \vec{g}, \vec{f})$ is equivalent to

$\exists^0_m \tilde{\tilde{\varphi}}((m)_0, \vec{g}\restriction m, \vec{f}\restriction m)$

e.g. $ap(f, t_0) \equiv t_1$ is equivalent $f\restriction m(t_0) \equiv t_1$ for large enough m

$\hat{\varphi}(m, \vec{g}\restriction m, \vec{f}\restriction m) = \tilde{\tilde{\varphi}}((m)_0 \dots)$

will do! $\boxtimes$