今回の後半から
自然科学…棟 3号棟
4階 瀬野グループ(内)
プレゼンテーション室に
移動してそこでやります

The lecture "mathematical logic" is held in
自然科学研究棟(?) 3号館
4階 瀬野グループ
プレゼンテーション室

A language (signature) is a set $\mathcal{L}$ of the following form:

$$\mathcal{L} = \underbrace{\{c_i \mid i \in I\}}_{\text{constant symbols}} \cup \underbrace{\{f_j \mid j \in J\}}_{\text{function symbols}} \cup \underbrace{\{r_k \mid k \in K\}}_{\text{relation symbols}}$$

each f_j has its arity $m_j \in \mathbb{N} \setminus \{0\}$

each r_k has its arity $m_k \in \mathbb{N} \setminus \{0\}$

I, J, K may be also empty

$\mathcal{L}$-structures are objects of the following form

$\mathfrak{A} = \langle A, c_i^{\mathfrak{A}}, f_j^{\mathfrak{A}}, r_k^{\mathfrak{A}} \rangle_{i \in I, j \in J, k \in K}$

A: non empty set (the underlying set of $\mathfrak{A}$)

$c_i^{\mathfrak{A}} \in A$ for $i \in I$

$f_j^{\mathfrak{A}} : A^{m_j} \to A;\ \langle a_0, \dots, a_{m_j-1} \rangle \mapsto f_j^{\mathfrak{A}}(a_0, \dots, a_{m_j-1})$

for all $j \in J$

$r_k \subseteq A^{m_k}$ for all $k \in J$

Examples (1) For $\mathcal{L} = \{\underset{\sqcup}{0}, \underset{\sqcup}{+}\}$ (0: constant symbol; $+$: binary function symbol)

any $\underbrace{\text{group}}_{\text{abelian}}$ $\mathcal{G} = \langle G, 0, + \rangle$ can be seen as an $\mathcal{L}$-structure.

(2) $\mathcal{L} = \{\underset{\sqcup}{<}\}$ ($<$: binary relation symbol)

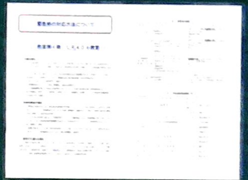
Then any partial ordering $\mathcal{P} = \langle P, < \rangle$ can be seen as an $\mathcal{L}$-structure.

(3) $\mathcal{L} = \{\underset{\sqcup}{A}\}$ (A: binary relation symbol.)

Any graph $\mathcal{G} = \langle V, E \rangle$, $E \subseteq \{\{x, y\} \mid x, y \in V, x \neq y\} = [V]^2$ (E: adjacent relation)

Let $A = \{\langle x, y \rangle \mid \{x, y\} \in E\} \subseteq V^2$

Then $\langle V, A \rangle$ is a $\mathcal{L}$-structure.



- For $\mathcal{L}$-structures $\mathfrak{A}$, $\mathfrak{B}$ with

$\mathfrak{A} = \langle A, c_i^{\mathfrak{A}}, f_j^{\mathfrak{A}}, r_k^{\mathfrak{A}} \rangle_{i\in I,\, j\in J,\, k\in K}$

$\mathfrak{B} = \langle B, c_i^{\mathfrak{B}}, f_j^{\mathfrak{B}}, r_k^{\mathfrak{B}} \rangle_{i\in I,\, j\in J,\, k\in K}$

$\mathfrak{A}$ is said to be a substructure of $\mathfrak{B}$

if $A \subseteq B$, $c_i^{\mathfrak{A}} = c_i^{\mathfrak{B}}$ for all $i \in I$,

$f_j^{\mathfrak{A}} = f_j^{\mathfrak{B}} \restriction A^{m_j}$ (i.e. $f_j^{\mathfrak{B}}$ is an extension of $f_j^{\mathfrak{A}}$)

for all $j \in J$, $r_k^{\mathfrak{A}} = r_k^{\mathfrak{B}} \cap A^{m_k}$ for all $k \in K$.

For $\mathcal{L}$-structure $\mathfrak{A} = \langle A, c_i^{\mathfrak{A}}, f_j^{\mathfrak{A}}, r_k^{\mathfrak{A}} \rangle_{i\in I,\, j\in J,\, k\in K}$

$X \subseteq A$ is closed in $\mathfrak{A}$ if

$c_i^{\mathfrak{A}} \in X$ for all $i \in I$

$f_j^{\mathfrak{A}}[X^{m_j}] \subseteq X$ for all $j \in J$

$(f_j^{\mathfrak{A}} \restriction X^{m_j} : X^{m_j} \to X)$

We write also

$f_j^{\mathfrak{A}} {}'' X^{m_j}$

If X is closed in $\mathfrak{A}$ then

$\subseteq X^{m_k}$

$\mathfrak{A} \restriction X = \langle X, c_i^{\mathfrak{A}}, f_j^{\mathfrak{A}} \restriction X^{m_j}, r_k^{\mathfrak{A}} \cap X^{m_k} \rangle_{i\in I,\, j\in J,\, k\in K}$

is an $\mathcal{L}$-structure and

$\mathfrak{A} \restriction X \subseteq \mathfrak{A}$.

Suppose that $\mathcal{L}$ and $\mathcal{L}'$ are two languages with

$\mathcal{L} = \{c_i \mid i \in I\} \cup \{f_j \mid j \in J\} \cup \{r_k \mid k \in K\}$

$\mathcal{L}' = \{c'_i \mid i \in I'\} \cup \{f'_j \mid j \in J'\} \cup \{r'_k \mid k \in K'\}$.

If $I \subseteq I'$, $J \subseteq J'$, $K \subseteq K'$ and

$c_i = c'_i$ for all $i \in I$, $f_j = f'_j$ for all $j \in J$ and $r_k = r'_k$ for all $k \in K$. Then we say that $\mathcal{L}'$ extends $\mathcal{L}$ and simply write

$\mathcal{L} \subseteq \mathcal{L}'$

If $\mathcal{L} \subseteq \mathcal{L}'$

$\mathfrak{A}'$ is a $\mathcal{L}'$-structure with

$\mathfrak{A}' = \langle A, c_i'^{\mathfrak{A}'}, f_j'^{\mathfrak{A}'}, r_k'^{\mathfrak{A}'} \rangle_{i\in I',\, j\in J',\, k\in K'}$

Then

$\mathfrak{A}' \restriction \mathcal{L} = \langle A, c_i'^{\mathfrak{A}'}, f_j'^{\mathfrak{A}'}, r_k'^{\mathfrak{A}'} \rangle_{i\in I,\, j\in J,\, k\in K}$

is an $\mathcal{L}$-structure. The reduction of $\mathfrak{A}'$ to $\mathcal{L}$

$\mathfrak{A}'$ is an expansion of $\mathfrak{A}' \restriction \mathcal{L}$ to $\mathcal{L}'$

We introduce some new symbols:

Var a countable set of symbols called "variables"

we denote elements of Var by x, y, z, u, v, w etc.

We assume $Var \cap \mathcal{L} = \emptyset$. We also use the symbols '(', ')', ','

'$\equiv$', '$\wedge$', '$\vee$', '$\neg$', '$\exists$', '$\forall$'. We also assume that

these symbols are different from all symbols from Var and $\mathcal{L}$

For given language $\mathcal{L}$ $\mathcal{L}$-terms are defined recursively as follows:

(0) For any $x \in Var$ the sequence "x" of length 1 is a term

(1) for c_i, $i \in I$ "c_i" is an $\mathcal{L}$-term.

(2) If $t_0, \dots, t_{m_j-1}$ are $\mathcal{L}$-terms

$$f_j(t_0, \dots, t_{m_j-1})$$

(i.e. $f_{m_j} {}^\frown ({}^\frown t_0 {}^\frown , \dots\dots)$)

(3) nothing else.

Example if $\mathcal{L}$ contains constant symbols $0, 1$ and 2place function symbols f, g then

$f(x,y)$ $f(1,0)$ $f(g(f(1,0),x),1)$

are all $\mathcal{L}$-terms.

If $I = \emptyset$ $J = \emptyset$

then $\mathcal{L}$-terms are just "x", $x \in Var$

If $J = \emptyset$

then $\mathcal{L}$-terms are either of the form "x" for $x \in Var$ or "c_i" for $i \in I$

We write often simply x, c_i in place of "x", "c_i"

If an $\mathcal{L}$-term t contains only variables from the list $x_0, \dots, x_{l-1}$ we write this as $t = t(x_0, \dots, x_{l-1})$

If $\mathfrak{A}$ is an $\mathcal{L}$-structure and $t = t(x_0, \dots, x_{l-1})$ is an $\mathcal{L}$-term then

$= \langle A, \dots \rangle$

$t^{\mathfrak{A}}_{(x_0, \dots, x_{l-1})} : A^l \to A$ is defined recursively according to the recursive definition of t

(0) If $t = x_p$ $p < l$ then

$$t^{\mathfrak{A}}_{(x_0, \dots, x_{l-1})}(a_0, \dots, a_{l-1}) = a_p$$

(1) If $t = c_i$ then

$$t^{\mathfrak{A}}_{(x_0, \dots, x_{l-1})}(a_0, \dots, a_{l-1}) = c_i^{\mathfrak{A}}$$

(2) If $t = f_j(t_0, \dots, t_{m_j-1})$

Then $t_0 = t_0(x_0, \dots, x_{l-1}), \dots, t_{m_j-1} = t_{m_j-1}(x_0, \dots, x_{l-1})$

Then

$$t^{\alpha}_{(z_0,\dots,z_{k-1})}(a_0,\dots a_{k-1})$$

$$= f_j^{\alpha}\left(t^{\alpha}_{0(z_0,\dots,z_{k-1})}(a_0,\dots,a_{k-1}), \dots t^{\alpha}_{m_j-1(z_0,\dots,z_{k-1})}(a_0,\dots,a_{k-1})\right)$$