Thm 1 (Tarski-Vaught Test)

$\mathfrak{A}$, $\mathfrak{B}$ $\mathcal{L}$-structures with $\mathfrak{B} \subseteq \mathfrak{A}$

$\mathfrak{B} \prec \mathfrak{A} \Leftrightarrow$ for any $\mathcal{L}$-formula $\varphi = \varphi(x, x_0, \ldots, x_{l-1})$ for any $b_0, \ldots, b_{l-1} \in B$ $(\mathfrak{B} = \langle B, \ldots \rangle)$

$\mathfrak{A} \models \exists x \varphi(x, b_0, \ldots, b_{l-1})$

then there is $b \in B$

$\mathfrak{A} \models \varphi(b, b_0, \ldots, b_{l-1})$

Lemma 3

(1) If $\mathfrak{A} \prec \mathfrak{B}$ and $\mathfrak{B} \prec \mathfrak{C}$ then $\mathfrak{A} \prec \mathfrak{C}$

(2) If $\mathfrak{A} \prec \mathfrak{C}$ $\mathfrak{A} \subseteq \mathfrak{B} \prec \mathfrak{C} \Rightarrow \mathfrak{A} \prec \mathfrak{B}$

proof (1): For $\mathcal{L}$-formula $\varphi = \varphi(x_0, \ldots, x_{l-1})$ and $a_0, \ldots, a_{l-1} \in A$ where $\mathfrak{A} = \langle A, \ldots \rangle$

$$\underline{\mathfrak{A} \models \varphi(a_0, \ldots, a_{l-1})} \underset{\mathfrak{A} \prec \mathfrak{B}}{\Leftrightarrow} \mathfrak{B} \models \varphi(a_0 \ldots a_{l-1}) \underset{\mathfrak{B} \prec \mathfrak{C}}{\Leftrightarrow} \underline{\mathfrak{C} \models \varphi(a_0, \ldots, a_{l-1})}$$

(2): For $\mathcal{L}$-formula $\varphi = \varphi(x_0, \ldots, x_{l-1})$ and $a_0, \ldots, a_{l-1} \in A$

$$\underline{\mathfrak{A} \models \varphi(a_0, \ldots, a_{l-1})} \underset{\mathfrak{A} \prec \mathfrak{C}}{\Leftrightarrow} \mathfrak{C} \models \varphi(a_0, \ldots, a_{l-1}) \underset{\substack{a_0, \ldots, a_{l-1} \in B \\ \mathfrak{B} \prec \mathfrak{C}}}{\Leftrightarrow} \mathfrak{B} \models \varphi(a_0, \ldots, a_{l-1})$$

□

Lemma 4 (Union of chains)

(1) If $\langle \mathfrak{A}_\alpha \mid \alpha \leq \gamma \rangle$ is a continuously increasing elementary chain of $\mathcal{L}$-structures (i.e. ⓪ $\mathfrak{A}_\alpha$, $\alpha \leq \gamma$ are $\mathcal{L}$-structures ① $\mathfrak{A}_\alpha \subseteq \mathfrak{A}_\beta$ for all $\alpha < \beta \leq \gamma$ (increasing) ② If $\xi \leq \gamma$ is a limit then [redacted] $\mathfrak{A}_\xi = \bigcup_{\alpha < \xi} \mathfrak{A}_\alpha$, and [redacted] ③ $\mathfrak{A}_\alpha \prec \mathfrak{A}_{\alpha+1}$ for all $\alpha < \gamma$)

Then $\mathfrak{A}_\alpha \prec \mathfrak{A}_\beta$ for all $\alpha < \beta \leq \gamma$ holds.

(2) If, $\langle \mathfrak{A}_\alpha \mid \alpha < \gamma \rangle$ is an increasing chain of elementary submodels of $\mathfrak{A}$ then

$$\mathfrak{A}_\gamma = \bigcup_{\alpha < \gamma} \mathfrak{A}_\alpha \prec \mathfrak{A}$$

If $\mathfrak{A}_\alpha = \langle A_\alpha, c_i^{\mathfrak{A}_\alpha}, f_j^{\mathfrak{A}_\alpha}, R_k^{\mathfrak{A}_\alpha} \rangle_{i \in I, j \in J, k \in K}$

and if $\langle \mathfrak{A}_\alpha \mid \alpha < \gamma \rangle$ is increasing

$$\bigcup_{\alpha < \gamma} \mathfrak{A}_\alpha = \left\langle \bigcup_{\alpha < \gamma} A_\alpha, c_i^{\mathfrak{A}_0}, \bigcup_{\alpha < \gamma} f_j^{\mathfrak{A}_\alpha}, \bigcup_{\alpha < \gamma} R_k^{\mathfrak{A}_\alpha} \right\rangle_{\substack{i \in I \\ j \in J \\ k \in K}}$$

If $\langle \mathfrak{A}_\alpha \mid \alpha < \gamma \rangle$ increasing and $\mathfrak{A}_\alpha \subseteq \mathfrak{A}$ for all $\alpha < \gamma$

Then $\bigcup_{\alpha < \gamma} \mathfrak{A} \subseteq \mathfrak{A}$

(1): By induction on γ

Case 1 If $\gamma = 1$ then the assertion is trivial

Case 2 Suppose that (1) holds for γ and let $\langle \mathfrak{A}_\alpha \mid \alpha \leq \gamma+1 \rangle$ be a conti. incr. elementary chain. By ind hyp we have

$\mathfrak{A}_\alpha \prec \mathfrak{A}_\beta$ for all $\alpha < \beta \leq \gamma$

Then by Lemma 3 (1) and since

$\mathfrak{A}_\alpha \prec \mathfrak{A}_\gamma \prec \mathfrak{A}_{\gamma+1}$ it follows $\mathfrak{A}_\alpha \prec \mathfrak{A}_{\gamma+1}$

Case 3 Suppose that γ is a limit and (1) holds for all $\gamma' < \gamma$

By induction hypothesis we have $\mathfrak{A}_\alpha \prec \mathfrak{A}_\beta$ for all $\alpha < \beta < \gamma$ (for any $\alpha < \beta < \gamma$ let $\gamma' < \gamma$ be s.t. $\alpha < \beta \leq \gamma' < \gamma$ and apply the ind hyp to $\langle \mathfrak{A}_\alpha \mid \alpha \leq \gamma' \rangle$) We have to show that $\mathfrak{A}_\alpha \prec \mathfrak{A}_\gamma$

By induction on $\mathcal{L}$-formulas: Critical step:

Suppose that $\varphi = \varphi(x, x_0, \ldots, x_{l-1})$ is an $\mathcal{L}$-formula and $a_0, \ldots, a_{l-1} \in A_\alpha$

If $\mathfrak{A}_\gamma \models \exists x \varphi(x, a_0, \ldots a_{l-1})$ then there is $a \in A_\gamma$ s.t. $\mathfrak{A}_\gamma \models \varphi(a, a_0, \ldots, a_{l-1})$

Since $a \in A_\gamma = \bigcup_{\alpha<\gamma} A_\alpha$ there is $\alpha < \beta < \gamma$ $a \in A_\beta$. By ind hyp. and $\mathfrak{A}_\gamma \models \varphi(a, a_0, \ldots a_{l-1})$ $\mathfrak{A}_\beta \models \varphi(a, a_0, \ldots a_{l-1})$ it follows that $\mathfrak{A}_\beta \models \exists x \varphi(x, a_0, \ldots a_{l-1})$ But since $\mathfrak{A}_\alpha \prec \mathfrak{A}_\beta$, it follows that $\mathfrak{A}_\alpha \models \exists x \varphi(x, a_0 \ldots a_{l-1})$

If $\mathfrak{A}_\alpha \models \exists x \varphi(x, a_0, \ldots a_{l-1})$ then there is $a \in A_\alpha$ s.t. $\mathfrak{A}_\alpha \models \varphi(a, a_0, \ldots, a_{l-1})$

By ind hyp. $\mathfrak{A}_\gamma \models \varphi(a, a_0, \ldots a_{l-1})$

Then $\mathfrak{A}_\gamma \models \exists x \varphi(x, a_0, \ldots, a_{l-1})$.

(2): If γ is successor then the assertion is trivial. (If $\xi+1 = \gamma$ then $\bigcup_{\alpha<\gamma} \mathfrak{A}_\alpha = \mathfrak{A}_\xi$) So suppose that γ is a limit

By Tarski-Vaught Test: Fix $\varphi = \varphi(x, x_0, \ldots x_{l-1})$

If $\mathfrak{A} \models \exists x \varphi(x, a_0, \ldots a_{l-1})$ for $a_0, \ldots, a_{l-1} \in A_\gamma$

Then there is $\xi < \gamma$ s.t. $a_0, \ldots, a_{l-1} \in A_\xi$ Since $\mathfrak{A}_\xi \prec \mathfrak{A}$ $\mathfrak{A}_\xi \models \exists x \varphi(x, a_0, \ldots a_{l-1})$. Let $a \in A_\xi$ s.t.

(b) $\mathfrak{A}_\gamma \models \varphi(a, a_0, \dots, a_{k-1})$

Again by $\mathfrak{A}_\gamma \prec \mathfrak{B}$ it follows that

$\mathfrak{A} \models \varphi(a, a_0, \dots, a_{k-1})$ But $a \in \mathfrak{A}_\gamma$

$\square$

For $\mathcal{L}$-structure $\mathfrak{A} = \langle A, \dots \rangle$ $\quad \hat{\mathcal{L}} = \mathcal{L} \cup \{\sqsubseteq\}$

Let $\sqsubseteq$ be a binary relation symbol not in $\mathcal{L}$)

An expansion $\tilde{\mathfrak{A}} = \langle \underbrace{A, \dots}_{=\mathfrak{A}}, \sqsubseteq^{\tilde{\mathfrak{A}}} \rangle$ is said to be a well-ordered expansion of $\mathfrak{A}$ if $\sqsubseteq^{\mathfrak{A}}$ well-orders the set A

(i.e. $\sqsubseteq^{\mathfrak{A}}$ is a linear ordering and for any $B \subseteq A$, $B \neq \emptyset$, B has the minimal element wrt $\sqsubseteq^{\mathfrak{A}}$)

Under Axiom of Choice, any set A has a well ordering on it.

For a well ordered expansion $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$ with $\sqsubseteq^{\tilde{\mathfrak{A}}}$ and $B \subseteq A$ we denote with

$\mathrm{sk}_{\tilde{\mathfrak{A}}}(B)$ = the closure of the set B in A wrt all definable functions in $\tilde{\mathfrak{A}}$

A function $f: A^n \to A$ is definable in $\tilde{\mathfrak{A}}$

$\Leftrightarrow$ there is an $\hat{\mathcal{L}}$ formula $\varphi = \varphi(x_0, \dots, x_{n-1}, y)$ s.t.

$$f = \{\langle\langle a_0, \dots, a_{n-1}\rangle, b\rangle \mid \tilde{\mathfrak{A}} \models \varphi(a_0, \dots, a_{n-1}, b)\}$$

$C \subseteq A$ is closed wrt a function $f: A^n \to A$ if $f(b_0, \dots, b_{n-1}) \in C$ for all $b_0, \dots, b_{n-1}$

$\mathrm{sk}_{\tilde{\mathfrak{A}}}(B) = \bigcap\{C \subseteq A \mid B \subseteq C,\ C$ is closed wrt. all definable functions in $\tilde{\mathfrak{A}}\}$

$\mathrm{sk}_{\tilde{\mathfrak{A}}}(B)$: the Skolem-hull of B in $\tilde{\mathfrak{A}}$

Lemma 5 For any $\emptyset \neq B \subseteq A$ $(\mathfrak{A} = \langle A, \dots \rangle)$ and for any well-founded expansion $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$

$\tilde{\mathfrak{A}} \restriction \mathrm{sk}_{\tilde{\mathfrak{A}}}(B) \prec \tilde{\mathfrak{A}}$ In particular

$\mathfrak{A} \restriction \mathrm{sk}_{\tilde{\mathfrak{A}}}(B) \prec \mathfrak{A}$

Proof

$\tilde{\mathfrak{A}} \restriction \mathrm{sk}_{\tilde{\mathfrak{A}}}(B)$ is well defined i.e.

$\mathrm{sk}_{\tilde{\mathfrak{A}}}(B)$ is closed wrt $\mathcal{L}$ in $\tilde{\mathfrak{A}}$

If c is a constant symbol then the constant function $A \to A$; $a \mapsto c^{\mathfrak{A}}$ is definable by $\varphi(x, y) = y \equiv c$.

Hence $c^{\tilde{\mathfrak{A}}} \in \mathrm{sk}_{\tilde{\mathfrak{A}}}(B)$

For a function symbol $f = f(x_0, \dots, x_{m-1})$ the function $A^m \to A$ $(a_0, \dots, a_{m-1}) \mapsto f^{\mathfrak{A}}(a_0, \dots, a_{m-1})$ is definable by $\varphi(x_0, \dots, x_{m-1}, y) = y \equiv f(x_0, \dots, x_{m-1})$

Thus $\mathrm{sk}_{\tilde{\mathfrak{A}}}(B)$ is closed wrt $f^{\mathfrak{A}}$. $\square$