Lemma For an $\mathcal{L}$-structure $\mathfrak{A} = \langle A, \dots \rangle$ and any $B \subseteq A$ $\underbrace{\Xi^{\mathfrak{A}}}$ well-ordering of A

$$\tilde{\mathfrak{A}} = \langle A, \dots \Xi^{\mathfrak{A}} \rangle$$

$$\hat{\mathcal{L}} = \mathcal{L} \cup \{\Xi\}$$

$$\tilde{\mathfrak{A}} \restriction \mathrm{sk}_{\tilde{\mathfrak{A}}}(B) \prec \tilde{\mathfrak{A}}$$

(in particular $\mathfrak{A} \restriction \mathrm{sk}_{\tilde{\mathfrak{A}}}(B) \prec \mathfrak{A}$)

proof We have proved that $\tilde{\mathfrak{A}} \restriction \mathrm{sk}_{\tilde{\mathfrak{A}}}(B)$ is a substructure of $\tilde{\mathfrak{A}}$.

To prove the elementarity we check the condition in Tarski-Vaught test:

Suppose $\varphi = \varphi(x, x_0, \dots x_{k-1})$ is an $\mathcal{L}$-formula and $b_0, \dots b_{k-1} \in \mathrm{sk}_{\tilde{\mathfrak{A}}}(B)$. We have to show:

If $\tilde{\mathfrak{A}} \models \exists x \varphi(x, b_0, \dots, b_{k-1})$ then there is $b \in \mathrm{sk}_{\tilde{\mathfrak{A}}}(B)$ s.t. $\tilde{\mathfrak{A}} \models \varphi(b, b_0, \dots, b_{k-1})$.

Consider the function $f : A^k \to A$

defined by

$$f(a_0 \dots a_{k-1}) = \begin{cases} \text{the } \overbrace{\text{smallest}}^{\Xi^{\mathfrak{A}}} \ a \in A \text{ s.t.} \\ \mathfrak{A} \models \varphi(a, a_0, \dots, a_{k-1}) \\ \text{if } \mathfrak{A} \models \exists x \varphi(x, a_0, \dots, a_{k-1}) \\ \\ \text{the } \Xi^{\mathfrak{A}}\text{-smallest element of } A \\ \text{otherwise} \end{cases}$$

f is a definable function in $\tilde{\mathfrak{A}}$

Suppose $\mathfrak{A} \models \exists x \varphi(x, b_0, \dots b_{k-1})$

Then $f(b_0, \dots b_{k-1}) \in \mathrm{sk}_{\tilde{\mathfrak{A}}}(B)$. By def of f,

$$\mathfrak{A} \models \varphi(f(b_0, \dots b_{k-1}), b_0, \dots b_{k-1})$$

$\boxtimes$

Hereditarily $<\kappa$ sets

A set x is transitive (wrt $\in$)

if, for any $y \in x$, $y \subseteq x$ holds

$\Leftrightarrow$ for any $y \in x$ and $z \in y$ we have $z \in x$

If x is a set $\mathrm{trcl}(x)$ (the transitive closure of x) is the smallest transitive set y with $x \in y$ i.e.

$$\mathrm{trcl}(x) = \bigcap \{ y \mid x \in y \text{ } y \text{ is transitive} \}$$

If we define $t(x,n)$ for $n \in \omega$ inductively

by $t(x,0) = \{x\}$

$$t(x,n+1) = t(x,n) \cup \bigcup t(x,n)$$

then $\bigcup_{n \in \omega} t(x,n)$ is the transitive closure of x.

For any cardinal κ

$$\mathcal{H}(\kappa) = \{ x \mid |\mathrm{trcl}(x)| < \kappa \}$$

Lemma 6 (1) $\mathcal{H}(\kappa)$ is a set and $|\mathcal{H}(\kappa)| = 2^{<\kappa}$

(2) If κ is a regular cardinal then $\mathcal{H}(\kappa) \models ZFC^-$

$\aleph_0$

ZFC without Axiom of power set

(2') $\mathcal{H}(\omega) \models ZFC$ − Axiom of infinity

(4) If a property $\varphi(x_0, \dots, x_{k-1})$ is Δ_1 (in the language $\mathcal{L}_\in$ of set theory) then for any $a_0 \dots a_{k-1} \in \mathcal{H}(\kappa)$

$\mathcal{H}(\kappa) \models \varphi(a_0, \dots, a_{k-1}) \Leftrightarrow \varphi(a_0, \dots, a_{k-1})$ holds

(4) actually holds for any transitive set (in place of $\mathcal{H}(\kappa)$)

(3) $\mathcal{H}(\kappa)$ is a transitive set

(3)' $\mathcal{H}(\kappa)$, $\kappa \in \mathrm{Card}$ is increasing and $\bigcup_{\kappa \in \mathrm{Card}} \mathcal{H}(\kappa) = V$

Remark If M is set then we identify M with $\langle V, \in \rangle \restriction M$

So $M = \langle M, \in \cap M^2 \rangle$ we often write $\langle M, \in \rangle$

we postpone the proofs of (1) (2)' (4)

(1): For $x \in \mathcal{H}(\kappa)$ let $|\mathrm{trcl}(x)| = \lambda$, $\lambda < \kappa$. Let $\langle \lambda, E \rangle$ be a copy of the structure $\langle \mathrm{trcl}(x), \in \cap (\mathrm{trcl}(x))^2 \rangle$. So $E \subseteq \lambda^2$ and

$$\langle \lambda, E \rangle \cong \langle \mathrm{trcl}(x), \in \cap (\mathrm{trcl}(x))^2 \rangle$$

Actually the structure $\langle \lambda, E\rangle$ decides $\mathrm{trcl}(x)$ and hence also x uniquely (if we take the Mostowski collapse of $\langle \lambda, E\rangle$ then we get $\langle \mathrm{trcl}(x), \in\rangle$)

Then there is a mapping

$$\Phi : \underbrace{\{\langle \lambda, E\rangle \mid \lambda < \kappa,\ E \subseteq \lambda^2\}}_{X} \xrightarrow{\text{onto}} H(\kappa)$$

$$\Phi(\langle \lambda, E\rangle) = \begin{cases} \text{the maximal element of the Mostowski collapse of } \langle \lambda, E\rangle & \text{if } E \text{ is well-founded extensional relation on } \lambda \text{ with the maximal element} \\ \emptyset & \text{otherwise} \end{cases}$$

By Axiom of Replacement of ZFC the image of Φ (i.e. $H(\kappa)$) is a set. Since

$|X| = 2^{<\kappa}$ $\quad |H(\kappa)| \leq 2^{<\kappa}$

On the other hand $[\kappa]^{<\kappa} = \{x \in P(\kappa) \mid |x| < \kappa\} \subseteq H(\kappa)$

$|[\kappa]^{<\kappa}| = 2^{<\kappa}$. Thus

$$|H(\kappa)| = 2^{<\kappa}.$$

(3) Suppose $x \in H(\kappa)$ $\quad y \in x$

(then $\mathrm{trcl}(y) \subseteq \mathrm{trcl}(x)$)

(3)' $y \in x$ $\quad y \subseteq \bigcup x \subseteq t(x, \lambda) \subseteq \mathrm{trcl}(x)$

$\kappa \leq \kappa'$ then if $|\mathrm{trcl}(x)| < \kappa$ then $|\mathrm{trcl}(x)| < \kappa'$

i.e. $H(\kappa) \subseteq H(\kappa')$. For any x if $\kappa = |\mathrm{trcl}(x)|^+$ then $x \in H(\kappa)$

Theorem 7

(Δ-System Lemma)

For any uncountable regular cardinal κ and any family $\{a_\alpha \mid \alpha < \kappa\}$ of finite sets, there is $I \in [\kappa]^\kappa = \{x \subseteq \kappa \mid |x| = \kappa\}$ and a set r s.t. for any $\alpha, \beta \in I$ $\alpha \neq \beta$

$a_\alpha \cap a_\beta = r$ ($\{a_\alpha \mid \alpha \in I\}$ is a Δ-system with root r) $\boxed{\text{Wlog. we may assume } a_\alpha \subseteq \kappa \text{ for all } \alpha \in \kappa}$

<u>Proof</u> Let θ be a regular cardinal s.t. $\langle a_\alpha \mid \alpha < \kappa\rangle \in H(\theta)$

Let $M \prec H(\theta)$ (i.e. $\langle M, \in\rangle \prec \langle H(\theta), \in\rangle$) s.t. $\langle a_\alpha \mid \alpha < \kappa\rangle \in M$ $\quad |M| < \kappa$

$\kappa \cap M$ is an initial segment of κ so there is $\alpha^* < \kappa$

$\kappa \cap M = \alpha^*$

(we can obtain this M by applying the theorem on union of chains and Lemma 5)

Also we can see this as follows: Let $\sqsubseteq$ be a well-ordering of $H(\theta)$. Let $A \subseteq H(\theta)$ s.t. $\langle a_\alpha \mid \alpha < \kappa\rangle \in A$ and $A \cap \kappa \leq \kappa$ is closed under all definable functions in $\langle H(\theta), \in, \sqsubseteq\rangle$ taking values in κ, and $|A| < \kappa$. Then

$M = \mathrm{dc}_{\langle H(\theta), \in, \sqsubseteq\rangle}(A)$ is as desired

$\kappa \cap M = \kappa \cap A < \kappa$ (by the closedness of A)

Let $r = a_{\gamma} \cap M$

Then $r \in \Pi$ $\quad \mathcal{H}(\theta) \models \forall x \forall y \forall z (z \in x \leftrightarrow z \in y) \to x = y)$

[$r = \{r_1, \dots, r_{n-1}\}$

$\mathcal{H}(\theta) \models \forall x_1 \dots \forall x_{n-1} \exists y \forall z (z \in y \leftrightarrow \bigvee_{i<n} z = x_i)$

$\Pi \models$ " "]

$M \models \forall \alpha < \kappa \exists \beta < \kappa (\alpha < \beta \wedge a_\beta \cap \beta = r)$

[For any $\alpha \in \kappa^M$ (i.e. $\alpha < \alpha^*$) we have

$\mathcal{H}(\theta) \models \alpha < \alpha^* \wedge a_{\gamma} \cap \alpha^* = r$

$\mathcal{H}(\theta) \models \exists \beta < \kappa (\alpha < \beta \wedge a_\beta \cap \beta = r)$

$M \models \exists \beta < \kappa (\dots\dots)$

Since α was arbitrary $\in \kappa^M$

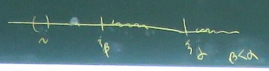
$M \models \forall \alpha < \kappa \exists \beta < \kappa (\alpha < \beta \wedge a_\beta \cap \beta = r)$

$\mathcal{H}(\theta) \models$ - - - - - -

In V (or in $\mathcal{H}(\theta)$) we can construct inductively

ξ_α $\alpha < \kappa$ s.t.

$a_{\xi_\alpha} \cap \xi_\alpha = r$ $\quad \xi_\alpha > \sup\{\bigcup a_{\xi_\beta} \mid \beta < \alpha\}$



Then $I = \{\xi_\alpha \mid \alpha < \kappa\}$ is as desired

i.e.

$\langle a_\xi \mid \xi \in I \rangle$ is a $\square$-sequence with the root r. $\boxtimes$

Let $r = a_{\alpha^*} \cap M$

Then $r \in M$ $\quad \mathcal{H}(\theta) \models \forall z \forall y \forall z ((z \in x \leftrightarrow z \in y) \to x = y)$

$[\, r = \{r_0, \dots, r_{m-1}\}$

$\mathcal{H}(\theta) \models \forall x_0 \dots \forall x_{m-1} \exists y \forall z (z \in y \leftrightarrow \bigvee_{i<m} z = x_i)$

$M \models$ " " $]$

$M \models \forall \alpha < \kappa \exists \beta < \kappa (\alpha < \beta \wedge a_\beta \cap \beta = r)$

[For any $\alpha \in \kappa^M$ (i.e. $\alpha < \alpha^*$) we have

$\mathcal{H}(\theta) \models \alpha < \alpha^* \wedge a_{\alpha^*} \cap \alpha^* = r$

$\mathcal{H}(\theta) \models \exists \beta < \kappa (\alpha < \beta \wedge a_\beta \cap \beta = r)$

$M \models \exists \beta < \kappa (\dots\dots)$

Since α was arbitrary $\in \kappa^M$

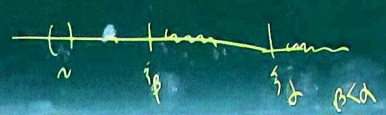
$M \models \forall \alpha < \kappa \exists \beta < \kappa (\alpha < \beta \wedge a_\beta \cap \beta = r)$

$\mathcal{H}(\theta) \models \dots\dots$

In V (or in $\mathcal{H}(\theta)$) we can construct inductively

ξ_α $\alpha < \kappa$ s.t.

$a_{\xi_\alpha} \cap \xi_\alpha = r \quad \xi_\alpha > \sup\{\bigcup a_{\xi_\beta} \mid \beta < \alpha\}$



Then

$I = \{\xi_\alpha \mid \alpha < \kappa\}$

is as desired

i.e.

$\langle a_\xi \mid \xi \in I \rangle$ is a Δ-system with the root r.

$\boxtimes$