Lemma 6 (Union of Chains)

(1) If $\langle \mathfrak{A}_\alpha \mid \alpha \leq \gamma \rangle$ is a continuously increasing sequence of $\mathcal{L}$-structures

If $\mathfrak{A}_\alpha \prec \mathfrak{A}_{\alpha+1}$ for all $\alpha < \gamma$

Then we have $\mathfrak{A}_\alpha \prec \mathfrak{A}_\beta$ for all $\alpha < \beta \leq \gamma$

(2) $\langle \mathfrak{A}_\alpha \mid \alpha < \gamma \rangle$ is increasing sequence of elementary substructures of $\mathfrak{A}$ then

$\mathfrak{A}_\gamma = \bigcup_{\alpha<\gamma} \mathfrak{A}_\alpha \prec \mathfrak{A}$

Lemma 5 If $\mathfrak{A}$ is an $\mathcal{L}$-structure $\sqsubseteq$ new relation symbol and $\hat{\mathfrak{A}} = \langle \underbrace{A, \dots}_{\mathfrak{A}}, \sqsubseteq^{\hat{\mathfrak{A}}} \rangle$ s.t.

$\sqsubseteq^{\mathfrak{A}}$ is a well-ordering on A. Then

For any $\emptyset \neq X \subseteq A$ $\mathfrak{A} \upharpoonright sk_{\hat{\mathfrak{A}}}(X) \prec \mathfrak{A}$

Theorem 8 (Downward Löwenheim Skolem Theorem)

If $\mathfrak{A}$ is an (infinite) $\mathcal{L}$-structure, $\emptyset \neq X \subseteq A$ ($\mathfrak{A} = \langle A, \dots \rangle$) then

there is $X \subseteq M \subseteq A$ s.t. $|M| \leq \max\{|X|, |\mathcal{L}|, \aleph_0\}$ and

If $\mathcal{L} = \langle \dots \rangle_{i \in I, j \in J, k \in K}$

we define $|\mathcal{L}| = |I| + |J| + |K|$

$\mathfrak{A} \upharpoonright M \prec \mathfrak{A}$

Proof Let $\hat{\mathfrak{A}}$ be as in Lemma 5

$M = sk_{\hat{\mathfrak{A}}}(X)$ will do. $\boxtimes$

Lemma 9 For any $\mathcal{L}$-structure $\mathfrak{A} = \langle A, \dots \rangle$ s.t. $\lambda \subseteq A$

λ: regular (uncountable) cardinal ($|\mathcal{L}| < \lambda$). Then for any $X \in [A]^{<\lambda}$

There is $M \in [A]^{<\lambda}$ s.t. $X \subseteq M$ $\mathfrak{A} \upharpoonright M \prec \mathfrak{A}$ and

$[A]^{<\lambda} = \{x \in \mathcal{P}(A) \mid |x| < \lambda\}$

$\lambda \cap M$ is a initial segment of M

and hence $\lambda \cap M \in \lambda$

Proof Let $\hat{\mathfrak{A}}$ be as in Lemma 5.

Let $\langle \mathfrak{A}_n \mid n \in \omega \rangle$ be an increasing chain of elementary submodels of $\mathfrak{A}$

with $\mathfrak{A}_n = \langle A_n, \dots \rangle$. defined by:

(0) $\mathfrak{A}_0 = \mathfrak{A} \upharpoonright sk_{\hat{\mathfrak{A}}}(X)$ ($|sk_{\hat{\mathfrak{A}}}(X)| < \lambda$) $|A_{n+1}| < \lambda$

(1) $\mathfrak{A}_{n+1} = \mathfrak{A} \upharpoonright sk_{\hat{\mathfrak{A}}}(A_n \cup \sup(\lambda \cap A_n)) \prec \mathfrak{A}^{by\ 5}_n$

Let $\mathfrak{b} = \bigcup_n \mathfrak{a}_n$ $\quad (M = \bigcup_n A_n)$

By Lemma 4 (2) $\quad \mathfrak{b} = \mathfrak{a} \restriction M \prec \mathfrak{a}$

For each $\alpha \in M$ there is $n \in \omega$ s.t. $\alpha \in A_n$ but then $\alpha \subseteq A_{n+1}$ by def. of A_{n+1}

Thus $\lambda \cap M$ is an initial segment of A.

$|M| < \lambda$ since $|A_n| < \lambda$.

M is as desired. ! ∎

Lemma 10 For an infinite cardinal κ and a regular $\lambda > \kappa$ s.t. $\oplus$ $|[\alpha]^{<\kappa}| < \lambda$ for all $\alpha < \lambda$.

Then for a sufficiently large regular θ and any $a \in [H(\theta)]^{<\lambda}$ there is $M \in [H(\theta)]^{<\lambda}$ s.t.

(0) $\langle M, \in \rangle \prec \langle H(\theta), \in \rangle$ (we also write simply: $M \prec H(\theta)$)

(1) $a \subseteq M$

(2) $\lambda \cap M \in \lambda$

(3) $[M]^{<\kappa} \subseteq M$

Proof Let $\delta_0 = \max\{|a|, \kappa\}$ and

$$\delta = \begin{cases} \delta_0 & \text{if } cf(\delta_0) \geq \kappa \\ \delta_0^+ & \text{otherwise} \end{cases}$$

Then $cf(\delta) \geq \kappa$ and $\delta < \lambda$

[If $\delta = \delta_0$ then $cf(\delta) \geq \kappa$ by definition and $\delta < \lambda$

otherwise $cf(\delta) = \delta = \delta_0^+ > \delta_0 \geq \kappa$

$cf(\delta_0) < \kappa$ Thus

$$\underbrace{\delta_0 < \delta_0^{cof(\delta)}}_{\text{a part of König's Theorem}} \underbrace{<}_{\text{by } \oplus} \lambda \quad \text{and} \quad \delta = \delta_0^+ < \lambda \ ]$$

Let $\langle M_\alpha \mid \alpha < \delta \rangle$ be a continuously increasing sequence in $[H(\theta)]^{<\lambda}$ s.t.

(0) $M_\alpha \prec H(\theta)$ $\quad$ (1) $a \subseteq M_0$

(2) $\sup(\lambda \cap M_\alpha) \subseteq M_{\alpha+1}$

(3) $[M_\alpha]^{<\kappa} \subseteq M_{\alpha+1}$

(1)(2)(3) are possible by Thm 8 (together with (0))

Limit step is o.k. by Lemma 4.

Let $M = \bigcup_{\alpha < \delta} M_\alpha$.

$|M| < \lambda$.

by Lemma 4 (1)

$\langle M, \in \rangle = \langle H(\theta), \in \rangle \restriction M \prec \langle H(\theta), \in \rangle$

As in the proof of Lemma 9

We have $\lambda \cap M \in \lambda$

We show $[M]^{<\kappa} \subseteq M$

Let $x \in [M]^{<\kappa}$ (i.e. $x \subseteq M$ and $|x| < \kappa$)
$M = \bigcup_{\alpha<\delta} M_\alpha$

Since $cf(\delta) \geq \kappa$

there is some $\alpha < \delta$ s.t. $x \subseteq M_\alpha$

Then $x \in M_{\alpha+1}$ by (3) in the construction of M_α's. $\subseteq M$ This shows $[M]^{<\kappa} \subseteq M$.

$\boxtimes$

Th 11 (generalized Δ-system-Lemma)

Let κ be an infinite cardinal and $\lambda > \kappa$ regular s.t. $\oplus$ $\forall \alpha < \lambda \left(|[\alpha]^{<\kappa}| < \lambda \right)$ (i.e. λ is $<\kappa$-inaccessible)

If $\langle a_\alpha \mid \alpha < \lambda \rangle$ is a family of sets of size $<\kappa$ then there is $I \in [\lambda]^\lambda$ and some r s.t. $\langle a_\alpha \mid \alpha \in I \rangle$ is a Δ-system with the root r

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If $\kappa = \omega$ then $|[\alpha]^{<\omega}| = \alpha + \aleph_0 < \lambda$ & the condition $\oplus$ is automatically satisfied!

Proof We may assume $a_\alpha \subseteq \lambda$ since $|\bigcup\{a_\alpha \mid \alpha<\lambda\}| \leq \lambda$.

Let θ be a sufficiently large regular cardinal e.g. $\langle a_\alpha \mid \alpha < \lambda \rangle \in H(\theta)$

Let $M \prec H(\theta)$ be s.t.

(1) $\langle a_\alpha \mid \alpha < \lambda \rangle \in M$
κ, λ

(2) $|M| < \lambda$

(3) $\alpha^* = \lambda \cap M \in \lambda$

(4) $[M]^{<\kappa} \subseteq M$

Such M exists by Lemma 19.

Let $r = a_{\alpha^*} \cap \alpha^*$ $r \subseteq M$ and $|r| < \kappa$

Thus by (4) $r \in M$

Then

$M \models \forall \alpha < \lambda\, \exists \beta < \lambda\, (\alpha < \beta \wedge a_\beta \cap \beta = r)$

By elementarity we also have

$H(\theta) \models$ ------ ---

Then we can take strictly increasing $\langle \xi_\alpha \mid \alpha < \lambda \rangle$ in λ s.t.

$\xi_\alpha > \sup\{\bigcup a_{\xi_\beta} \mid \beta < \alpha\}$ and

$a_{\xi_\alpha} \cap \xi_\alpha = r$.

Then $I = \{\xi_\alpha \mid \alpha < \lambda\}$ is as desired.

$\boxtimes$