Lemma 6.1 Suppose that $2^\kappa < \theta$ for a (κ: infinite cardinal) regular θ.

Then, for any $a \in [H(\theta)]^{\le 2^\kappa}$ there is

$M \prec H(\theta)$ s.t. $a \subseteq M$

$|M| \le 2^\kappa$ and (*) $[M]^\kappa \subseteq M$

Note $2^\kappa < \theta$ implies $[H(\theta)]^{\le 2^\kappa} \subseteq H(\theta)$

(1) In particular $[H(\theta)]^\kappa \subseteq H(\theta)$

Lemma 6.2 From (*) it follows that $\underbrace{\kappa \in M}_{(1)}$ and (2) $[M]^{\le\kappa} \subseteq M$.

$[X]^{\le\lambda} = \{Y \in \mathcal{P}(X) \mid |Y| \le \lambda\}$

$[X]^\lambda = \{ \;\;" \;\; \mid \;" = "\}$

$[X]^{>\lambda} = \{ \;\;" \;\; \mid \;" > "\}$ etc.

Proof of Lemma 6.2 (2): Let $a \in [M]^{\le\kappa}$. If $|a| = \kappa$ then $a \in M$ by (*). If $|a| < \kappa$ then let $u \in [M]^\kappa$ be s.t. $a \cap u = \emptyset$. By (*) $u \in M$. $a \cup u \in [M]^\kappa$. Hence again by (*) $a \cup u \in [M]^\kappa$. By elementarity, $M \models \exists x (x = \underset{\in M}{a \cup u} \setminus \underset{\in M}{u})$

Thus $a \in M$.

(1): By induction we prove $\alpha \in M$ for any $\alpha \le \kappa$.

$0 = \emptyset \in M$

If $\alpha \in M$ then $\alpha + 1 \in M$ (by $M \prec H(\theta)$)

If γ is a limit $\le \kappa$, $\gamma \subseteq M$. By (2), $\gamma \in M$.

Proof of Lemma 6.1

Let $\langle M_\alpha \mid \alpha < \kappa^+ \rangle$ be a continuously increasing sequence of elementary submodels of $H(\theta)$ s.t.

(1) $a \subseteq M_0$

(2) $|M_\alpha| \le 2^\kappa$, $M_\alpha \prec H(\theta)$ for all $\alpha < \kappa^+$

(3) $[M_\alpha]^\kappa \subseteq M_{\alpha+1}$ for all $\alpha < \kappa^+$

since $|M_\alpha| \le 2^\kappa$, $|[M_\alpha]^\kappa| \le \underbrace{2^\kappa}_{(2^\kappa)^\kappa}$

Let $M = \bigcup_{\alpha < \kappa^+} M_\alpha$. By (2) $|M| \le 2^\kappa$.

$M \prec H(\theta)$ by (2) and the Lemma on "union of chains".

$[M]^\kappa \subseteq M$: Suppose $a \in [M]^\kappa$. Then there is $\alpha < \kappa^+$ s.t. $a \subseteq M_\alpha$. It follows $a \in M_{\alpha+1}$ by (3). Hence $a \in M$. ☑

Theorem 6.3 (Arhangel'skii)
If $X = \langle X, \mathcal{O}\rangle$ is Hausdorff space with $w(X) = \kappa \geq \aleph_0$. Then $|X| \leq 2^{\kappa}$

$w(X) = \min\{|\mathcal{B}| \mid \mathcal{B} \subseteq \mathcal{O}$ $\mathcal{B}$ is a base of the topology $\mathcal{O}\}$
weight of X

i.e. $\forall x \in X\ \forall O \in \mathcal{O}$ if $x \in O$ then there's $O' \in \mathcal{B}$ s.t. $x \in O' \subseteq O$

$w(X) = \aleph_0 \iff X$ is separable

Cor 6.4 If $X = \langle X, \mathcal{O}\rangle$ is a separable Hausdorff space then $|X| \leq 2^{\kappa}$

Proof of Theorem 6.3 by the method of elementary substructures: Let $(X, \mathcal{O})$ be a Hausdorff space with $w(X) = \kappa \geq \aleph_0$. Let $\mathcal{B}$ be a base of $\underbrace{\mathcal{O}}_{\text{the topology}}$ s.t. $|\mathcal{B}| = \kappa$.

Let θ be sufficiently large (e.g. $\langle X, \mathcal{B}\rangle \in \mathcal{H}(\theta)$, $2^{\kappa} < \theta$)

Let $M \prec \mathcal{H}(\theta)$ be s.t. $\langle X, \mathcal{B}\rangle \in M$, $|M| \leq 2^{\kappa}$, $[M]^{\kappa} \subseteq M$ by Lemma 6.1.

We have $\mathcal{B} \subseteq M$: $\mathcal{H}(\theta) \models \exists f : \kappa \xrightarrow{\text{onto}} \mathcal{B}$ implies

$M \models$ " " (we know $\kappa \in M$ by Lemma 6.2)

So there is $f \in M$ s.t. $f : \kappa \xrightarrow{\text{onto}} \mathcal{B}$

Thus for each $\alpha \in \kappa$ ($\alpha < \kappa$) $M \models \exists x\,(x \equiv f(\alpha))$

Hence $f(\alpha) \in M$. Since f is a surjection it follows that $\mathcal{B} \subseteq M$.

The following claim finishes the proof:

Claim $X \subseteq M$.

$\vdash$ Let $x \in X$. We prove that $x \in M$

Let $\mathcal{B}_x = \{O \in \mathcal{B} \mid x \in O\}$

Since $\mathcal{B}_x \subseteq \mathcal{B} \subseteq M$ and $|\mathcal{B}_x| \leq \kappa$ and by Lemma 6.2 (2) $\mathcal{B}_x \in M$

$\mathcal{H}(\theta) \models \bigcap \mathcal{B}_x = \{x\}$. In particular

$\mathcal{H}(\theta) \models \exists y \bigcap \mathcal{B}_x = \{y\}$. By elementarity

$M \models$ " "

There is some $a \in M$ s.t. $M \models \bigcap \mathcal{B}_x = \{a\}$

But then $\mathcal{H}(\theta) \models a = x$. Thus $a = x$ and $x \in M$. $\dashv$

$\boxtimes$

Theorem 6.5 (Arhangel'skii's Theorem)
For any Hausdorff space X we have
$$|X| \leq 2^{\chi(X) \cdot L(X)}.$$

$\chi(X)$: character of X

For $x \in X$

$\chi(x, X) = \min\{|\mathcal{U}| \mid \mathcal{U}$ is a nbhd base of x in $X\}$

$\chi(X) = \max\{\chi(x, X) \mid x \in X\}$

$L(X)$ Lindelöf number of X

$L(X) = \min\{\kappa \mid \forall \mathcal{U}\,(\mathcal{U}$ is an open covering of X ($\mathcal{U} \subseteq \mathcal{O}$ and $\bigcup\mathcal{U} = X$) $\exists \mathcal{U}' \in [\mathcal{U}]^{<\kappa}\,(\bigcup\mathcal{U}' = X)\}$

X is Lindelöf $\Longleftrightarrow$ $L(X) = \aleph_1$

Exercise:

(1) Show that Theorem 6.3 follows from Theorem 6.5.

(2) Provide a proof of Theorem 6.5 similar to that of Theorem 6.3. ⊠

Club and stationary subsets of ordinals

Let λ be an ordinal of cofinality $> \omega$

$C \subseteq \lambda$ is said to be closed and unbounded (club) if

① C is closed (for any limit $\beta < \lambda$ if $C \cap \beta$ is cofinal in β then $\beta \in C$)

② C is unbounded $\forall \beta < \lambda\ \exists \alpha \in C\ (\beta \leq \alpha < \lambda)$.

Example 6.6 $\lambda \subseteq \lambda$ is club. $\emptyset$ is not club.

Lemma 6.7

If $\{C_\alpha \mid \alpha < \rho\}$ $\rho < cf(\lambda)$ is a family of club subsets of λ then $\bigcap_{\alpha<\rho} C_\alpha$ is also club in particular $\bigcap_{\alpha<\rho} C_\alpha \neq \emptyset$

Cor If $\mathcal{F} = \{X \subseteq \lambda \mid X$ contains a club subset of $\lambda\}$ Then $\mathcal{F}$ is $<cf(\lambda)$-complete filter.

$S \subseteq \lambda$ is said to be stationary if $S \cap C \neq \emptyset$ for all club subset of λ

Lemma 6.8

Suppose λ is a regular cardinal $\theta > (2^\lambda)^+$ regular

(1) $S \subseteq \lambda$ is stationary if and only if there is $M \prec \mathcal{H}(\theta)$ s.t. $S, \lambda \in M$ and $M \cap \lambda \in S$.

(2) $C \subseteq \lambda$ contains a club if and only if for any $M \prec \mathcal{H}(\theta)$ s.t. $C, \lambda \in M$ and $M \cap \lambda \in \lambda$ we have $M \cap \lambda \in C$