Lemma 6.8 Suppose that λ is a $\underbrace{\text{regular}}_{\text{uncountable}}$ cardinal and $\theta \geq (2^{\lambda})^{+}$ also regular

(1) $S \subseteq \lambda$ is stationary iff there is $M \prec H(\theta)$ s.t. $\lambda, S \in M$ and $\lambda \cap M \in S$

(2) $S \subseteq \lambda$ contains a club subset of λ iff for all $M \prec H(\theta)$ with $\lambda, S \in M$ and $\lambda \cap M \in \lambda$

we have $\lambda \cap M \in S$

Lemma 7.1 If $\mathfrak{A} = \underbrace{\langle A, \ldots \rangle}_{\text{with the signature of size } <\lambda}$ and $\lambda \subseteq A$ for a $\underbrace{\text{regular}}_{\text{uncountable}}$ cardinal λ then

$$S = \{ \alpha \in \lambda \mid \text{there is } B \subseteq A \text{ s.t. } \mathfrak{A} \restriction B \prec \mathfrak{A} \text{ and } \alpha = \lambda \cap B \}$$

contains a club subset of λ.

Proof Let $\sqsubset$ be a wellordering on A and $\tilde{\mathfrak{A}} = \langle \underbrace{A, \ldots}_{\mathfrak{A}}, \sqsubset \rangle$

Let $C = \{ \alpha \overset{\in\lambda}{\ } \mid \mathrm{sk}_{\tilde{\mathfrak{A}}}(\alpha) \cap \lambda = \alpha \}$ Note $|\mathrm{sk}_{\tilde{\mathfrak{A}}}(\alpha)| < \lambda$ for $\alpha < \lambda$

Claim 1 $C \underset{\text{club}}{\subseteq} \lambda$.

$\vdash$ For any $\alpha_0 < \lambda$ let $\alpha_i < \lambda$, $i \in \omega$ be defined by

(a) $\alpha_0 =$ the α_0 above.

(b) $\alpha_{i+1} = \sup(\lambda \cap \underset{\cup\!|\ \alpha_i}{\mathrm{sk}_{\tilde{\mathfrak{A}}}(\alpha_i)}) + 1$

Then $\alpha = \sup \alpha_i < \lambda$, $\alpha_0 < \alpha$

And $\lambda \cap \mathrm{sk}_{\tilde{\mathfrak{A}}}(\alpha) = \lambda \cap (\bigcup_{i \in \omega} \mathrm{sk}_{\tilde{\mathfrak{A}}}(\alpha_i)) \underset{\uparrow (b)}{=} \alpha < \lambda$

Thus $\alpha \in C$

if $\langle \alpha_\xi \mid \xi < \delta \rangle$ $\delta < \lambda$ is an increasing sequence in C then $\alpha = \sup_{\xi < \delta} \alpha_\xi < \lambda$

and

$$\lambda \cap \mathrm{sk}_{\tilde{\mathfrak{A}}}(\alpha) = \lambda \cap \bigcup_{\xi < \delta} \mathrm{sk}_{\tilde{\mathfrak{A}}}(\alpha_\xi)$$
$$= \bigcup_{\xi < \delta} \lambda \cap \mathrm{sk}_{\tilde{\mathfrak{A}}}(\alpha_\xi)$$
$$= \bigcup_{\xi < \delta} \alpha_\xi = \alpha.$$

Thus $\alpha \in C$. $\dashv$

Claim 2 $C \subseteq S$

$\vdash$ Clear by the def. of C $\dashv$ $\blacksquare$

Rem S in Lemma 7.1 need not to be closed!

For an increasing sequence α_ξ $\xi < \lambda$ in S

we can find B_ξ, $\xi < \lambda$ $\alpha_\xi = \lambda \cap B_\xi$

but $\langle B_\xi \mid \xi < \lambda \rangle$ need not to be an increasing sequence.

An application of Lemma 6.8

Theorem 7.2 (Fodor's Lemma)

Suppose that λ is an $\overbrace{\text{regular}}^{\text{uncountable}}$ cardinal. $S \underset{\text{stat}}{\subseteq} \lambda$.

If $f : S \to \lambda$ is regressive

($\Leftrightarrow$ $\forall \alpha \in S$ $f(\alpha) < \alpha$) then there is $\beta_0 < \lambda$

s.t. $S_0 = \{\alpha \in S \mid f(\alpha) = \beta_0\}$ is stationary

proof Let $\theta \geq (2^\lambda)^+$ be regular

Let $M \prec \mathcal{H}(\theta)$ be s.t. $S, \lambda \in M$, [redacted] f
$\quad\ \ \overset{\shortparallel}{S}$

$\lambda \cap M < \lambda$ and $\alpha_0 \in S$
$\overset{\shortparallel}{\alpha_0}$ (there is such M by Lemma 7.1)

Let $\beta_0 = f(\alpha_0)$ Since $\beta_0 < \alpha_0$ $\beta_0 \in \lambda \cap M$

Claim $S_0 = \{\alpha \in S \mid f(\alpha) = \beta_0\}$ is stationary.

$\vdash$ Note $S_0 \in M$.

$\lambda \cap M = \alpha_0 \in S_0$

By Lemma 6.8 (1) it follows that $S_0 \underset{\text{stationary}}{\subseteq} \lambda$ $\boxtimes$

Proof of 6.8

(1): $\Rightarrow$: Suppose that $S \underset{\text{stat}}{\subseteq} \lambda$

Let $\sqsubset$ be a well-ordering of $\mathcal{H}(\lambda)$

Let $\hat{\mathfrak{A}} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$ and

consider

$C = \{\alpha < \lambda \mid \mathrm{sk}_{\hat{\mathfrak{A}}}(\alpha) \cap \lambda = \alpha\}$

as in the proof of Lemma 7.1 C is a club subset of λ.

Thus there is $\alpha^* \in S \cap C$.

Let $M = \mathrm{sk}_{\hat{\mathfrak{A}}}(\alpha^*)^{\prec \mathcal{H}(\theta)}$. Then $M \cap \lambda = \alpha^*$

This M is as desired!

$\Leftarrow$: Assume that $M \prec H(\theta)$

$\lambda, S \in M$ $\quad \alpha^* = \lambda \cap M \in S$.

$M \models \forall x\,(x \subseteq_{club} \lambda \to x \cap S \neq \emptyset)$

[Suppose that $M \models C \subseteq_{club} \lambda$ for $C \in \mathcal{P}(\lambda) \cap M$

then by elementarity $H(\theta) \models C \subseteq_{club} \lambda$. By the choice of θ $C \subseteq_{club} \lambda$.

$C \cap M = C \cap \alpha^*$

$C \cap \alpha^*$ is unbounded in α^*. Hence $\alpha^* \in C$.

Thus $C \cap S \neq \emptyset$ $\quad H(\theta) \models C \cap S \neq \emptyset$ By elementarity it follows $M \models C \cap S \neq \emptyset$]

By elementarity

$H(\theta) \models \forall x\,(x \subseteq_{club} \lambda \to x \cap S \neq \emptyset)$

Thus $\forall x\,(x \subseteq_{club} \lambda \to x \cap S \neq \emptyset)$ really holds!

This means that S is stationary.

(2): $\Rightarrow$ We prove the contrapositive of the statement: Suppose that there is $M \prec H(\theta)$ s.t. $\lambda, S \in M$ $\quad \underset{\alpha^*}{\underset{\parallel}{\lambda \cap M}} < \lambda$

$\alpha^* \notin S$. By (1), $\underbrace{\lambda \setminus S}_{\text{since } \alpha^* \in \lambda \setminus S}$ is stationary

Thus S does not contain a club

Remark $\quad \lambda \setminus S$ is stationary $\iff$ S does not contain a club

[For any club $C \subseteq \lambda$ $\quad C \cap (\lambda \setminus S) \neq \emptyset$

i.e. $C \not\subseteq S$. If S does not contain a club C then $C \cap (\lambda \setminus S) \neq \emptyset$]

$\Leftarrow$: Suppose that S does not contain a club. Then $\lambda \setminus S$ is stationary.

Hence by (1) there is $N \prec H(\theta)$ s.t.

$\lambda, S \in N$ and $\underset{\alpha^*}{\underset{\parallel}{\lambda \cap M}} \in \lambda \setminus S$

$\Updownarrow$

$\alpha^* \notin S$

Thus the right side of the equivalence of (2) does not hold. $\boxtimes$