Lemma 6.8 For regular uncountable λ and regular θ with $(2^{\lambda})^{+} \leq \theta$

(1) $S \subseteq \lambda$ is stationary iff there is $M \prec H(\theta)$ s.t. $\lambda, S \in M$ and $\lambda \cap M \in S$

(2) $S \subseteq \lambda$ contains a club subset iff for all $M \prec H(\theta)$ s.t. $\lambda, S \in M$, $\lambda \cap M \in \lambda$ we have $\lambda \cap M \in S$

Lemma 8.1 For regular $\underbrace{\lambda}_{\text{uncountable}}$ $S \subseteq \lambda$ contains a club subset of λ iff there is $g : {}^{\omega>}\lambda \to \lambda$ s.t. $C_g \subseteq S$

where

$$C_g = \{\alpha < \lambda \mid \alpha \text{ is closed wrt } g \}$$

(i.e. $\forall \beta_0, \dots, \beta_{n-1} \in \alpha$ $g(\langle \beta_0, \dots, \beta_{n-1} \rangle) \in \alpha$)

proof "$\Leftarrow$": Clear since $C_g \subseteq_{\text{club}} \lambda$.

"$\Rightarrow$" Let θ be regular with $\theta \geq (2^{\lambda})^{+}$. Let $\mathfrak{A} = \langle H(\theta), \in, \lambda, S, \lhd \rangle$

well ordering of $H(\theta)$

Let $\{f_n : n \in \omega\}$ be the set of all definable functions in $\mathfrak{A}$.

Let $g : {}^{\omega>}\lambda \to \lambda$ be defined by

$$g(\langle \alpha_0 \cdots \alpha_\ell \rangle) = \begin{cases} n+1 & \text{if } \ell = 0,\ \alpha_0 = n \quad (*) \\ f_{\alpha_0}(\alpha_1, \dots, \alpha_\ell) & \text{if } \ell \geq 1,\ \alpha_0 \in \omega,\ f_{\alpha_0}(\alpha_1, \dots, \alpha_\ell) \in \lambda \quad (**) \\ 0 & \text{otherwise.} \end{cases}$$

Claim $C_g \subseteq S$

$\vdash$ Suppose $\alpha \in C_g$. Then $\alpha \neq 0$ by the def of C_g. $\alpha \geq \omega$ by $(*)$

$\underbrace{\mathrm{sk}_{\mathfrak{A}}(\alpha)}_{= M} \cap \lambda = \alpha$ $\quad \lambda, S \in M$ by the def of $\mathfrak{A}$

By Lemma 6.8, (2) it follows that $\alpha \in S$ $\dashv$

Thus this g is as desired! $\boxtimes$

For uncountable regular θ, ...

$M \prec \mathcal{H}(\theta)$ is said to be *internally cofinal* if $[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$ w.r.t. $\subseteq$. (i.e. $\forall a \in [M]^{\aleph_0}\ \exists b \in [M]^{\aleph_0} \cap M\ (a \subseteq b)$)

Lemma 8.2 For any regular $\theta > \omega_1$ and $X \in [\mathcal{H}(\theta)]^{\leq \aleph_1}$ there is an internally cofinal $M \prec \mathcal{H}(\theta)$ s.t. (1) $|M| = \aleph_1$ (2) $X \subseteq M$

Proof. Let $X = \{x_\alpha : \alpha < \omega_1\}$ (wlog $X \neq \emptyset$)

Let $\langle M_\alpha : \alpha < \omega_1 \rangle$ be a continuously increasing chain of countable elementary submodels of $\mathcal{H}(\theta)$ s.t.

(1) $x_\alpha \in M_{\alpha+1}$

(2) $M_\alpha \in M_{\alpha+1}$ (3) $\alpha \in M_{\alpha+1}$

Possible by Downward-Löwenheim-Skolem

(Take $\underbrace{M_\alpha \cup \{x_\alpha, M_\alpha, \alpha\}}_{\text{countable}} \subseteq \underbrace{M_{\alpha+1}}_{\text{countable}} \prec \mathcal{H}(\theta)$)

Let $M = \bigcup_{\alpha<\omega_1} M_\alpha$. $M \prec \mathcal{H}(\theta)$ by Union of Chain Lemma.

$|M| = \aleph_1$ by (3).

$X \subseteq M$ by (1).

M is internally cofinal: For $a \in [M]^{\aleph_0}$ ($a \subseteq M$) there is some $\alpha < \omega_1$ s.t. $a \subseteq M_\alpha$. $M_\alpha \in M_{\alpha+1} \subseteq M$ ($\uparrow$ by (2)) Thus $a \subseteq M_\alpha \in [M]^{\aleph_0} \cap M$. ⊠

$\underset{\text{weight}}{w(X)} = \min\{|\mathcal{B}| : \mathcal{B}$ is an open base of the topology of $X\}$

$\mathcal{B} \subseteq \mathcal{O}$ is an open base $\iff \forall x \in X\ \forall O \in \mathcal{O}\ \exists O' \in \mathcal{B}\ (x \in O' \subseteq O)$

Thm 8.3 (Juhász) For any topological space $X = \langle X, \mathcal{O} \rangle$, if all subspaces of X of cardinality $\leq \aleph_1$ have countable weight, then [redacted] X is also of countable weight.

Remark. By Arhangelskii Theorem, if X as in the Thm above is Hausdorff then $|X| \leq 2^{\aleph_0}$.

__Cor__ If X is a Hausdorff space with $|X| > 2^{\aleph_0}$ then there is a subspace (Y) of X of size $\leq \aleph_1$ with $w(Y) > \aleph_0$.

__Proof__ Let $\mathcal{O}$ be the open sets of X.

Let θ be a sufficiently large regular card s.t. $\theta > \omega_1$, $\langle X, \mathcal{O} \rangle \in H(\theta)$

Let $M \prec H(\theta)$ be s.t. $\langle X, \mathcal{O} \rangle \in M$, $|M| = \aleph_1$ and M is internally cofinal (there is such M by Lemma 8.2 (X in Lemma 8.2 $= \{\langle X, \mathcal{O} \rangle\}$))

__Exercise__: If $\langle a, b \rangle \in M \prec H(\theta)$ then $a, b \in M$

- density of X $\leq$ weight of X
  $d(X) = \min\{|D| \mid D \subseteq_{\text{dense}} X\}$
- If $S \in M \prec H(\theta)$ and if S is countable $(\theta > \omega)$ then $S \subseteq M$.

__Claim__ $\mathcal{O} \cap M$ is an open base of $X \cap M$ (considered as a subspace of X).

$\vdash$ Suppose that $x \in X \cap M$ and $x \in O \in \mathcal{O}$.

We have to show that there is some $O' \in \mathcal{O} \cap M$ s.t.

$x \in O' \cap (X \cap M) \subseteq O \cap (X \cap M)$,

Since $|X \cap M \setminus O| \leq \aleph_1$, $(X \cap M) \setminus O$ is of countable weight $(X \cap M) \setminus O$ is separable.

Let $D_0 \subseteq_{\text{countable}} (X \cap M) \setminus O$ s.t. $D_0 \subseteq_{\text{dense}} (X \cap M) \setminus O$

Since M is internally cofinal there is some $D_0 \subseteq D \subseteq_{\text{countable}} X \cap M \setminus \{x\}$ s.t. $D \in M$ [redacted]

Now since $D \cup \{x\}$ is a countable subspace of X there is a countable open base of $D \cup \{x\}$.

Since $D \cup \{x\} \in M$ we can take a countable open base $\mathcal{B} \subseteq \mathcal{O}$ s.t. $\mathcal{B} \in M$. Since $\mathcal{B}$ is an open base of $D \cup \{x\}$ there is $O' \in \mathcal{B}$ ($\mathcal{B} \subseteq M$ since $\mathcal{B}$ is countable) with $x \in O'$, $O' \cap D = \emptyset$. Note $O' \in \mathcal{B} \subseteq M$.

This O' is as desired.