Recap. and some corrections:

Thm 8.3 (Juhász) For any topological space $X = \langle X, \mathcal{O}\rangle$ family of all open sets in X

if all subspaces Y of X of cardinality $\leq \aleph_1$ have weight $\leq \aleph_0$

then $w(X) \leq \aleph_0$

proof Let $\theta > \omega_1$ be regular and sufficiently large. Let $M \prec \mathcal{H}(\theta)$ be s.t. $\langle X, \mathcal{O}\rangle \in M$

$|M| = \aleph_1$ M is internally cofinal.

Claim $\mathcal{O} \cap M$ is an open base of $X \cap M$ the subspace of X

(i.e. $\{\underbrace{O \cap M}_{O\cap(M\cap X)} \mid O \in \mathcal{O} \cap M\}$ is an open basis of the top space $\langle X \cap M, \{O \cap M \mid O \in \mathcal{O}\}\rangle$)

- **proof**. Suppose $x \in X \cap M$ $x \in O \cap M$ for some $O \in \mathcal{O}$. We have to show that there is $O' \in \mathcal{O} \cap M$ s.t. $x \in O' \cap M \subseteq O \cap M$.

$O' \cap (X \cap M)$ $\quad$ $O \cap (X \cap M)$

Since $|(X \cap M) \setminus O| \leq \aleph_1$; $(X \cap M) \setminus O$ as a subspace of X is of countable weight.

In particular, $(X \cap M) \setminus O$ is separable

There is $\underbrace{D_0 \underset{\text{dense}}{\subseteq} (X \cap M) \setminus O}_{(*)}$ with $|D_0| = \aleph_0$

By internally cofinality of M, and since $D_0 \subseteq M$

There is $D \in [M]^{\aleph_0} \cap M$ s.t. $D_0 \subseteq D$. By replacing D by $D \cap X$ $(\in M)$ we may assume $D \subseteq X \cap M$.

$D \cup \{x\} \in M$ Since $D \cup \{x\}$ is a countable subspace of X,

there is a countable open base $\mathcal{B}$ of $D \cup \{x\}$ (as subspace of X).

Since $D \cup \{x\} \in M$

$M \models$ "$D \cup \{x\} \underset{\text{countable}}{\subseteq} X$, $\boxed{}$"

$M \models$ "there is a countable open base of $D \cup \{x\}$"

Thus we may assume $\mathcal{B} \in M$.

Since $\mathcal{B}$ is countable, $\mathcal{B} \subseteq M$.

There is $O' \in \mathcal{B}$ s.t.

$x \in O' \cap (D \cap \{x\}) \subseteq O \cap (D \cap \{x\})$

We have

$$x \in \underset{\cup\!\shortmid}{O' \cap (D_0 \cup \{x\})} \subseteq \underset{\cup\!\shortmid}{O \cap (D_0 \cup \{x\})} \overset{(*)}{=} \{x\} \quad \text{or } (**)\ O' \cap D_0 = \emptyset$$

Now we have $x \in O' \cap M \subseteq O \cap M$:

Suppose otherwise $(O' \cap M) \setminus (O \cap M) \neq \emptyset$

$$\| $$

$$(O' \setminus O) \cap M = O' \cap ((X \cap M) \setminus O)$$

↑ open in the open $(X \cap M) \setminus O$

Since D_0 is dense in $(X \cap M) \setminus O$

There must be $d \in D_0 \cap O' \cap (\cdots)$

A contradiction to $(**)$ ⊣

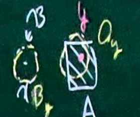
By Claim and by assumption  there is a countable open base $\mathcal{B}_0 \in [\mathcal{O} \cap M]^{\aleph_0}$ [redacted] by Lemma 9.0.

Since M is internally refined there is $\mathcal{B} \in [\mathcal{O}]^{\aleph_0} \cap M$ s.t. $\mathcal{B}_0 \subseteq \mathcal{B}$. $\mathcal{B}$ is also an open base of $X \cap M$.

i.e. for all $x \in X \cap M$ and for all $O \in \mathcal{O}$ there is $O' \in \mathcal{B}$ s.t. $x \in O' \subseteq O$.

It follows $M \models \mathcal{B}$ is an open base of X.

By elementarity it follows that $\mathcal{B}$ is really an open base of X. ∎



Lemma 9.0 [redacted] For a [redacted] top space $\langle X, \mathcal{O} \rangle$ and $\kappa \geq \aleph_0$, [redacted] if $w(X) \leq \kappa$ and $\mathcal{B} \subseteq \mathcal{O}$ open base then there is an open base $\mathcal{B}' \subseteq \mathcal{B}$ with $|\mathcal{B}'| \leq \kappa$.

Lemma 9.1 A first countable and countably compact Hausdorff space $\langle X, \mathcal{O} \rangle$ is regular.

Proof Let $x \in X$ and $A \underset{\text{closed}}{\subseteq} X$ with $x \notin A$.

Let $\mathcal{B}$ be a countable open nbhd base of x.

For each $y \in A$ let $B_y \in \mathcal{B}$ and $O_y \in \mathcal{O}$ be s.t. $B_y \cap O_y = \emptyset$ and $y \in O_y$. For each $B \in \mathcal{B}$ let $O_B = \bigcup\{O_y \mid y \in A,\ B_y = B\}$,

⑦

$\langle X, \mathcal{O} \rangle$ is first countable $\Leftrightarrow$ each point in X has countable (open) nbhd base.

$\langle X, \mathcal{O} \rangle$ is second countable iff $w(X) \leq \aleph_0$ (i.e. of countable weight)

$\langle X, \mathcal{O} \rangle$ countably compact $\Leftrightarrow$ (X is Hausdorff and) for any countable open covering $\mathcal{O}_0 \in [\mathcal{O}]^{\aleph_0}$ (with $\bigcup \mathcal{O}_0 = X$) has a finite subcovering $\mathcal{O}_1 \in [\mathcal{O}_0]^{<\aleph_0}$ (with $\bigcup \mathcal{O}_1 = X$)

$\langle X, \mathcal{O} \rangle$ is regular if

(7) $O_B \cap B = \emptyset$ by the def. of O_B.

$\mathcal{U} = \{O_B \mid B \in \mathcal{B}\}$ is a countable open covering of A. Then $\mathcal{U} \cup \{X \setminus A\}$ is an countable open covering of X. Hence there are $B_0, \dots, B_{k-1} \in \mathcal{B}$ for some $k < \omega$

s.t. $(X \setminus A) \cup U_{B_0} \cup \dots \cup U_{B_{k-1}} = X$

(or $\underbrace{U_{B_0} \cup \dots \cup U_{B_{k-1}}}_{= O_1} \supseteq A$)

Letting O_1 and $O_0 = \bigcap_{i<k} B_i$ (open, $\ni x$)

We have $x \in O_0$, $O_0 \cap O_1 = \emptyset$.

Thus O_0 and O_1 separate x and A as desired! ∎

Theorem 9.3

A countably compact Hausdorff space is metrizable iff X has countable weight.

Theorem 9.2 (Urysohn's Metrization Thm)

Every $\underbrace{\text{Hausdorff space}}_{\text{regular}}$ of countable weight is metrizable.

$(X, \mathcal{O})$ metrizable $\iff$ there is a metric d on X s.t. the topology induced from this metric coincides with $\mathcal{O}$.

Proof of Thm 9.3

"$\Leftarrow$" By Lemma 9.1 and Theorem 9.2

"$\Rightarrow$" Done next time!