Lemma 10.1 (the missing detail in the proof of Thm 8.3)

For any top. space $(X, \mathcal{O})$ and $\kappa \geq \aleph_0$ if $w(X) \leq \kappa$, if $\mathcal{B}$ is an open base of top. $(\mathcal{O})$ of X then there is $\mathcal{B}_0 \in [\mathcal{B}]^{\leq\kappa}$ s.t. $\mathcal{B}_0$ is also an open base of X.

Proof Let $\mathcal{B}^*$ be an open base of X of cardinality $w(X) \leq \kappa$

For $B_0, B_1 \in \mathcal{B}^*$ with $B_0 \subseteq B_1$, Let $B_{B_0 B_1}$ be s.t.

(*) $$B_{B_0,B_1} = \begin{cases} \text{some } B \in \mathcal{B} \text{ s.t. } B_0 \subseteq B \subseteq B_1 & \text{if there is such } B \\ \emptyset & \text{otherwise} \end{cases}$$

Let $\mathcal{B}_0 = \{B_{B_0,B_1} : B_0, B_1 \in \mathcal{B}^*\} \setminus \{\emptyset\}$

$|\mathcal{B}_0| \leq \kappa$, $\mathcal{B}_0 \subseteq \mathcal{B}$

Claim $\mathcal{B}_0$ is an open base of X

$\vdash$ Suppose $x \in X$ $O \in \mathcal{O}$ s.t. $x \in O$

Let $B_1 \in \mathcal{B}^*$ be s.t. $x \in B_1 \subseteq O$

(possible since $\mathcal{B}^*$ is an open base)

Let $B \in \mathcal{B}$ be s.t. $x \in B \subseteq B_1$

(possible " $\mathcal{B}$ ")

Let $B_0 \in \mathcal{B}^*$ s.t. $x \in B_0 \subseteq B$

(possible " $\mathcal{B}^*$ " ")

Then, by (*), we have $B_{B_0,B_1} \in \mathcal{B}$ and $x \in B_0 \subseteq B_{B_0,B_1} \subseteq B_1 \subseteq O$

Thm 8.3 (Juhász)

For a topological space $(X, \mathcal{O})$ if all subspaces of X of card. $\leq \aleph_1$ have weight $\leq \aleph_0$ then $w(X) \leq \aleph_0$

Thm 9.3 A countably compact Hausdorff sp. $X = (X, \mathcal{O})$ is metrizable iff $w(X) \leq \aleph_0$

Proof "$\Leftarrow$"

"$\Leftarrow$" By Urysohn's Thm and Lemma 9.1

Hausdorff regular space X with $w(X) \leq \aleph_0$ is metrizable

X first countable countably compact $\Rightarrow$ X is regular
$\underbrace{}_{\text{Hausdorff}}$

"$\Rightarrow$" Suppose X countably compact Hausdorff and metrizable. Let θ be sufficiently large regular cardinal, Let d be a metric on X which induces the top of X

Let $M \prec \mathcal{H}(\theta)$ be countable with $\langle X, d\rangle \in M$.

Since $d(X) \leq w(X)$ and $d(X) = w(X)$ for metrizable spaces,

It is enough to show $d(X) \leq \aleph_0$. This follows from:

<u>Claim</u> $X \cap M$ is dense in X

$\vdash$ Suppose not. Then $X \setminus \overline{X \cap M} \neq \emptyset$.

Let $x^* \in \underbrace{X \setminus \overline{X \cap M}}_{O}$

Let $\mathcal{U} = \{O\} \cup \overbrace{\{U : U \in M \cap \mathcal{O},\ x^* \notin U\}}^{\mathcal{U}'}$. $|\mathcal{U}| \leq \aleph_0$

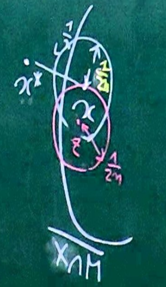
<u>Subclaim</u> $\mathcal{U}$ is an open covering of X

$\vdash$ Suppose $\boxed{x \in \overline{X \cap M}}$. We have to show that there is some $U \in M \cap \mathcal{O}$ s.t. $x \in U$ $x^* \notin U$ (this implies $\bigcup \mathcal{U}' \supseteq \overline{X \cap M}$)

Since $x \neq x^*$ there is some $n \in \omega \setminus 1$ s.t. $\frac{1}{n} < d(x, x^*)$. Let $z \in B(x, \frac{1}{2n}) \cap M$

Then $\underset{\overset{\parallel}{U}}{B(z, \frac{1}{2n})} \in M$. $U \in M \cap \mathcal{O}$

$x \in U$ But $x^* \notin U$ i.e. $U \in \mathcal{U}'$ $\dashv$



By countable compactness of X there are $U_0, \ldots, U_{k-1} \in \mathcal{U}'$ s.t.

$O \cup U_0 \cup \cdots \cup U_{k-1} \supseteq X$ In particular $U_0 \cup \cdots \cup U_{k-1} \supseteq \overline{X \cap M}$
$\supseteq X \cap M$

$M \models U_0, \ldots, U_{k-1}$ is an open covering of X.

By elementarity $U_0, \ldots, U_{k-1}$ covers X

A contradiction $x^* \notin U_0 \cup \cdots \cup U_{k-1}$!

$\boxtimes$

Cor 10.2 (to Lemma 9.3) TFAE

(a) X is metrizable and countably compact (Hausdorff)

(b) X is metrizable countably compact and Lindelöf (Hausdorff)

(c) X is metrizable and compact (Hausdorff)

proof (b) $\Rightarrow$ (c) by (**).

(a) $\Rightarrow$ (b) by Lemma 9.3 and (***)

(c) $\Rightarrow$ (a) trivial

X is Lindelöf if any open covering of X has a countable subcovering

(**) If X is Lindelöf and countably compact then X is compact

(***) If $w(X) = \aleph_0$ then X is Lindelöf.

[Let $\mathcal{B}$ be a countable open base of the topology

For any open covering $\mathcal{U}$ of X Let

$$\mathcal{B}_0 = \{B \in \mathcal{B} \mid B \subseteq U \text{ for some } U \in \mathcal{U}\}$$

For each $B \in \mathcal{B}_0$ let $U_B \in \mathcal{U}$ be s.t. $B \subseteq B_B$

Let $\mathcal{U}_0 = \{U_B \mid B \in \mathcal{B}_0\} \subseteq \mathcal{U}$. Then $\mathcal{U}_0$ is countable and $\mathcal{U}_0$ is a covering of X.]

Lemma 10.3 If $X = \langle X, \tau \rangle$ has a point countable base $\mathcal{B}$ (i.e. for each $x \in X$ $\{B \in \mathcal{B} \mid x \in B\}$ is countable), then for any sufficiently large regular θ and $M \prec \mathcal{H}(\theta)$ with $\langle X, \tau \rangle \in M$, $\tau \cap M$ is an open base for $\overline{X \cap M}$.

(τ: open base)

proof Suppose $\langle X, \tau \rangle, \theta, M$ are as above. By elementarity there is a point countable base $\mathcal{B}$ of M s.t. $\mathcal{B} \in M$.

Remark all metrizable spaces have point countable base

[**Thm (Stone)** If X metrizable then X is σ-locally finite implies the statement.]

Suppose that $\boxed{x \in \overline{X \cap M}}$ and $O \in O_\tau$ with $x \in O$. We have to show that there is $O' \in \tau \cap M$ with $x \in O' \subseteq O$.

Let $B_0 \in \mathcal{B}$ be s.t. $x \in B_0 \subseteq O$.

Let $O_0 \in \tau$ be s.t. $x \in O_0 \subseteq B_0$

Let $C_0 \in \mathcal{B}$ be s.t. $x \in C_0 \subseteq O_0 \subseteq B_0$

By $\boxed{}$ there is some $y \in C_0 \cap (X \cap M)$

Since $\{B \in \mathcal{B} \mid y \in B\}$ is countable $C_0 \cap M$

$= \mathcal{B}_y \in M$

$\mathcal{B}_y \subseteq M$

C_0, B_0 Thus $C_0, B_0 \in M$ $\mathcal{H}(\theta) \models \exists O' (O' \in \tau \wedge C_0 \subseteq O' \subseteq B_0)$

By elementarity $O' \in \tau \cap M$ s.t.

$M \models C_0 \subseteq O' \subseteq B_0$ So this O' is as desired. $\boxtimes$

Lemma 10.4 Suppose that $X = \langle X, \tau \rangle$ is a countably compact Hausdorff space. If $M \prec \mathcal{H}(\theta)$ for sufficiently large regular θ with $\langle X, \tau \rangle \in M$, and if $\tau \cap M$ is not an open base of X. Then there is $z \in \overline{X \cap M}$ s.t. $\tau \cap M$ d.n. include any nbhd $\underset{\text{base}}{\text{of}}$ z.

proof If $X = \overline{X \cap M}$ then the claim of the Lemma is trivial.

Assume $x \in X \setminus \overline{X \cap M}$. Suppose, towards a contradiction, that each $z \in \overline{X \cap M}$ has a ... base $\subseteq \tau \cap M$.

For each $z \in \overline{X \cap M}$ we can choose $O_z \in \tau \cap M$ s.t. $x \notin O_z$.

$$\mathcal{U} = \{X \setminus \underbrace{\overline{X \cap M}}_{\shortparallel \atop O}\} \cup \{O_z \mid z \in \tau \cap M\}$$

is an open covering X. ($\boxed{\text{Exercise}}$)

Since $\mathcal{U}$ is countable then are $z_0, \dots, z_{k-1} \in \overline{X \cap M}$ s.t. $\{O_{z_0} \dots O_{z_{k-1}}\} \cup \{O\}$ is a covering of X.

$M \models \{O_{z_0} \dots O_{z_{k-1}}\}$ is a open covering of X.

A contradiction as in the proof of Thm 9.3! ⊠