Recap - For a Hausdorff sp X

Cor 10.2 TFAE

(a) X is metrizable and countably compact

(b) X is metrizable and compact

(c) (Thm 9.3) X is countably compact $w(X) \leq \aleph_0$

Cor 11.2 (d) X is countably compact with a point countable open base.

The following (d) is also equivalent to (a)~(c)

Thm 11.1 (Miščenko) A countably compact (Hausdorff) space $X = \langle X, \tau \rangle$ with a point countable open base has a countable open base (i.e. $w(X) \leq \aleph_0$) (This implies (d) $\Rightarrow$ (c))

Lemma 10.3 For top sp. $X = \langle X, \tau \rangle$ with a point countable base, if $\langle X, \tau \rangle \in M \prec \mathcal{H}(\theta)$ for sufficiently large regular θ then $\tau \cap M$ is an open base of $\overline{X \cap M} \subseteq X$.

Lemma 10.4 Suppose $X = \langle X, \tau \rangle$ is countably compact (Hausdorff).

If $\langle X, \tau \rangle \in M \prec \mathcal{H}(\theta)$ for sufficiently large regular θ and if $\tau \cap M$ is not an open base of X then there is $z \in \overline{X \cap M}$ s.t. there is no open nbhd base $\subseteq \tau \cap M$ for z (i.e. $\{O \in \tau \cap M \mid z \in O\}$ is not a nbhd base of z).

Lemma 10.4 bis Suppose X ...

... $\langle X, \tau \rangle \in M \prec \mathcal{H}(\theta)$ ... If $\tau \cap M$ is an open base for $\overline{X \cap M}$ then $\tau \cap M$ is an open base for X

proof of 11.1

Let $\langle M, \tau \rangle \in M \prec \mathcal{H}(\theta)$ be countable for sufficiently large regular θ.

By Lemma 10.3 $\tau \cap M$ is an open base for $\overline{X \cap M}$. By Lemma 10.4 bis it follows that $\underbrace{\tau \cap M}_{\text{countable}}$ is an open base for X

$\boxtimes$

Thm 11.3 (A variant of Dow's Metrization Theorem)

If X is a ~~regular and~~ countably compact (Hausdorff) space. If all $Y \in [X]^{<\aleph_2}$ has a point countable open base. Then X is metrizable.

Cor 11.4 (Dow's Metrization Theorem)

If X is a ~~regular and~~ countably compact (Hausdorff) space and if all subspace of X of cardinality $<\aleph_2$ are metrizable then X is metrizable.

Proof of Cor 11.4 from Thm 11.3

Suppose that X is regular and countably compact (Hausdorff) s.t. all $Y \in [X]^{<\aleph_2}$ are metrizable. Then all $Y \in [X]^{<\aleph_2}$ are with a point countable base. Hence by Thm 11.3 it follows that X is metrizable. $\boxtimes$

A metrizable space X has σ-locally finite base (i.e. such a base $\mathcal{B}$ that $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$, $\mathcal{B}_n$ is locally finite)

Thm 8.3 (Juhász)

For a top sp $\langle X, \mathcal{O}\rangle$ if all subspaces of X of cardinality $(\le \aleph_1)$ have weight $\le \aleph_0$ then $w(X) \le \aleph_0$

Proof Thm 11.3

Assume that $X = \langle X, \tau\rangle$ is a counter example to the Thm, i.e. we have
0. X is ~~regular and~~ countably compact (Hausdorff)
1. all $Y \in [X]^{<\aleph_2}$ have a point countable open base.
2. X is not metrizable.

Let θ be sufficiently large and regular, and let $\langle X, \tau\rangle \in M \prec \mathcal{H}(\theta)$ (s.t.) $|M| = \aleph_1$ and M is internally approachable.

$M \prec \mathcal{H}(\theta)$ is internally approachable if $|M| = \aleph_1$, $M = \bigcup_{\alpha<\omega_1} M_\alpha$ where

(1) $M_\alpha \prec \mathcal{H}(\theta)$, $|M_\alpha| = \aleph_0$ for all $\alpha < \omega_1$

(2) $\langle M_\alpha \mid \alpha < \omega_1\rangle$ is conti increasing (i.e. $\langle M_\alpha \mid \alpha<\omega_1\rangle$ is a filtration of M)

(3) $\langle M_\alpha \mid \alpha \le \beta\rangle \in M_{\beta+1}$ for all $\beta < \omega_1$ ($M_{\beta+1} \subseteq M$)

Lemma 11.5

For any regular $\theta > \aleph_1$ and $X \in [\mathcal{H}(\theta)]^{\le\aleph_1}$ there is $M \prec \mathcal{H}(\theta)$ s.t. $X \subseteq M$ and M is internally approachable.

proof The same proof as that for Lemma 8.6. (Exercise). $\boxtimes$

Let $\langle M_\alpha \mid \alpha < \omega_1 \rangle$ be as in the definition of internal approachability of M.

W.l.o.g. we may assume $\langle X, \tau \rangle \in M_0$

By ② and Thm 9.?, we have $w(X) > \aleph_0$

By Juhász's theorem, there is $Z \in [X]^{\aleph_1}$

s.t. $w(\overline{Z}) > \aleph_0$.

By elementarity we may assume that $Z \in M$

Now since $\overline{Z}$ is also countably compact (as closed subspace of X) $\overline{Z} \models$ ⓪, ①, ②

Thus w.l.o.g., we may assume (*) $X = \overline{Z}$

Furthermore Z is regular

by ⓪ ① and an argument similar to Lemma 9.1

For each $x \in X \cap M$ $|Z \cup \{x\}| = \aleph_1$

and hence by ① $Z \cup \{x\} \subseteq X \, (= \overline{Z})$ has a point countable base. In particular x has a countable nbhd base in $Z \cup \{x\}$.

By Lemma 10.3 $\tau \cap M$ is an open base for

$$\overline{Z \cap M} \underset{\text{by } (*)}{=} X \cap M$$

Thus we have

③ $\tau \cap M$ is an open base of $\langle X \cap M, \tau \rangle$

By ③ and ①

④ $\langle X \cap M, \tau \cap M \rangle$ has a point countable basis

Since $w(X) > \aleph_0$ and each M_α is countable, for all $\alpha < \omega_1$, $\tau \cap M_\alpha$ is not an open

(+) base of $\langle X, \tau \rangle$

By Lemma 10.4 there is $z \in \overline{X \cap M_\alpha}$ s.t. $\tau \cap M_\alpha$ does not provide an open base for z. Since $M_\alpha \in M_{\alpha+1}$ (this follows from (3)) We can find such z in $M_{\alpha+1}$ by elementarity

Let $N \prec H(\theta)$ be countable s.t. $X, \tau, M, \langle M_\alpha \mid \alpha < \omega_1 \rangle \in N$

Let $\alpha^* = \omega_1 \cap N \in \omega_1$

Exercise For any regular $\theta > \aleph_1$ $N \prec H(\theta)$

$\omega_1 \cap N$ is an initial segment of ω_1

[If $\alpha \in \omega_1 \cap N$ then there is a surjection $f : \omega \to \alpha$ with $f \in N$. Since $\omega \subseteq N$, $\underset{\alpha}{\overset{\shortparallel}{f''\omega}} \subseteq N$]

② By (+) there is $z^* \in M_{\alpha^*+1}$ s.t. $z^* \in \overline{X \cap M_{\alpha^*}}$ and $\tau \cap M_{\alpha^*}$ does not provide a nbhd base of z^*.

On the other hand, we have

$$(\tau \cap M) \cap N = \bigcup \{ \tau \cap M_\beta \mid \beta < \alpha^* \} = \tau \cap M_{\alpha^*}$$

$$\left(\begin{array}{l} M_\beta \in N \text{ for } \beta < \alpha^* \\ \Rightarrow M_\beta \subseteq N \end{array} \right)$$

Hence by (4) and Lemma 10.4

$\tau \cap M_{\alpha^*}$ is an open base of

$$\overline{(X \cap M) \cap M_{\alpha^*}} = \overline{X \cap M_{\alpha^*}}$$

A contradiction. $\boxtimes$