For cardinal κ, and ordinals λ, μ

$$\kappa \to (\lambda, \mu)^2$$

$\iff$ For any mapping $f : [\kappa]^2 \longrightarrow 2 \overset{\{0,1\}}{=}$

either there is 1-homogeneous subset $O \subseteq \kappa$ of order type μ or there is 0-homogeneous subset $Z \subseteq \kappa$ of order type λ

$S \subseteq \kappa$ is i-homogeneous (wrt f) if $f''[S]^2 = \{i\}$

Graph theoretic reading of $\kappa \to (\lambda, \mu)^2$:
(in case λ and μ are regular cardinals.)

Thm 13.1 $\kappa \to (\lambda, \mu)^2$ implies:

(1) For any graph $G = (V, E)$ with $|V| = \kappa$, $E \subseteq [V]^2$
either G has a free (Complete) subgraph G' of size μ or G has a complete (free) subgraph of size λ

(2) . . .

13.2: **Thm** (Erdős–Dushnik–Miller Thm)

For any regular cardinal $\kappa > \omega$

$$\kappa \to (\kappa, \omega + 1)^2$$

proof (L. Soukup)

Let θ be sufficiently large and regular

Let $f : [\kappa]^2 \to 2$ and let

$M \prec H(\theta)$ s.t. $|M| < \kappa$, $f \in M$, $\kappa \cap M \in \kappa$.

Let $\xi \in \kappa \setminus M$ ($\kappa \cap M < \xi$)

Let A be maximal with the property

$A \subseteq \kappa \cap M$, $A \cup \{\xi\}$ is 1-homogeneous

If A is infinite we are done [Let $A' \subseteq A$ of order-type ω. Then $A' \cup \{\xi\}$ is 1-homogeneous and of order-type $\omega + 1$.] So assume that $|A| < \aleph_0$

Then $A \in M$. Let

$$B = \{\beta \in \kappa \setminus A \mid \forall \alpha \in A \; f(\{\beta, \alpha\}) = 1\}$$

$B \in M$ (Note $\kappa \in M$ since $f \in M$!)

Let $C \subseteq B$ be maximal (wrt $\subseteq$) 0-homogeneous set. $\gamma \in C$

Why $C \in M$.

Claim $|C| = \kappa$ (Since κ is regular it follows that there is $C' \subseteq C$ of order type κ which is 0-homogeneous)

⊢ Suppose otherwise. Then since κ is regular C is bounded. By elementarity $M \models$ "C is bounded". Then there is some $\beta \in \kappa \cap M$ s.t. $C \subseteq \beta$. If $\kappa_0 = |C|$, then $\kappa_0 \in M$ and hence $\kappa_0 \subseteq M$. It follows that $C \subseteq M$.

Subclaim $C \cup \{\gamma\}$ is 0-homogeneous.

[This leads to a contradiction since $\gamma \notin C$ by $C \subseteq M$ and C is maximal 0-homogeneous by def.] ($\beta \in B \subseteq \kappa \setminus A$ and)

⊢ Suppose $\beta \in C$. Then $\underbrace{A \cup \{\beta\}}_{1\text{-homo}} \cup \{\gamma\}$ is not 1-homogeneous by the maximality of A. Since $\beta \in B$ we have $\forall \alpha \in A\ f(\{\alpha, \beta\}) = 1$. Thus we should have $f(\{\beta, \gamma\}) = 0$. Since β was arbitrary $C \cup \{\gamma\}$ is 0-homogeneous.

⊣ Subclaim ⊣ Claim. ⊠

Thm 13.3 (Erdős–Rado)

$$(2^{\aleph_0})^+ \longrightarrow ((2^{\aleph_0})^+, \omega_1 + 1)^2$$

Proof Let θ be sufficiently large regular. Let $f : [(2^{\aleph_0})^+]^2 \to 2$ and let $M \prec H(\theta)$ be s.t. $|M| = 2^{\aleph_0}$, $f \in M$, $(2^{\omega})^+ \cap M < (2^{\omega})^+$ and ($\star$) $[M]^{\aleph_0} \subseteq M$. Let $\xi \in (2^{\omega})^+ \setminus M$. Let $A \subseteq (2^{\omega})^+ \cap M$ be s.t. $A \cup \{\xi\}$ is 1-homogeneous. If $|A| > \aleph_0$ then we are done

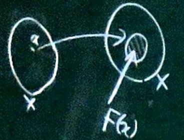
Exercise For any $\underbrace{\theta > (2^{\aleph_0})^+}_{\text{regular}}$ and any $a \in [H(\theta)]^{\le 2^{\aleph_0}}$ there is $M \prec H(\theta)$ s.t. $|M| = 2^{\aleph_0}$, $a \subseteq M$ and $[M]^{\aleph_0} \subseteq M$

More generally: For any regular κ, μ with $\kappa^{\mu} = \kappa$, for any regular $\theta > \kappa^+$ and for any $a \in [H(\theta)]^{\le \kappa}$ there is $M \prec H(\theta)$ s.t. $|M| = \kappa$, $a \subseteq M$, $[M]^{<\mu} \subseteq M$.

So assume $|A| = \aleph_0$ then (*) $A \in M$.

The rest is similar replacing κ with $(2^{\aleph_0})^+$. ⊠

Fact: $F : X \to \mathcal{P}(X)$. $Y \subseteq X$ is said to be F-free if



$F(x) \not\ni y$ for all distinct $x, y \in Y$

Thm 13.4 (Erdős-??)

For any uncountable regular κ and $F : \kappa \to [\kappa]^{<\aleph_0}$ $(\subseteq \mathcal{P}(\kappa))$ there's a F-free $C \in [\kappa]^\kappa$

Proof Let θ be sufficiently large and regular.
Let $M \prec H(\theta)$, $|M| < \kappa$, $\kappa \cap M \in \kappa$ and $F \in M$ (so $\kappa \in M$). Let $\xi \in \kappa \setminus M$ and let $A = F(\xi) \cap M$. A is finite and hence $A \in M$.
Let C be $\subseteq$-maximal F-free subset of $\kappa \setminus A$.
Since $A \in M$ we may choose $C \in M$.

Claim $|C| = \kappa$.

$\vdash$ Suppose otherwise. Then C is bounded in κ.
Hence $C \subseteq M$. But $C \cup \{\xi\}$ is F-free since $F(\xi) \cap M = A$ which is disjoint from C by def of C.
A contradiction to the maximality of C. $\dashv$ ⊠

Thm 13.5 For any regular κ, μ with $\kappa^\mu = \kappa$ and $F : \kappa \to [\kappa]^{<\mu}$, there is a F-free $C \in [\kappa]^\kappa$

Proof Just like in Thm 13.4 with Now generally .. switched in.. ⊠