Report on Exercises (a list of them is will be posted very soon) Dead line 15. June

von Neumann Hierarchy V_α, $\alpha \in On$ ← the class of all ordinals

$V_0 = \emptyset$

$V_{\alpha+1} = \mathcal{P}(V_\alpha) \cup V_\alpha$ ← powerset of V_α (actually it is the same if this is replaced by "$V_{\alpha+1} = \mathcal{P}(V_\alpha)$")

$V_\gamma = \bigcup_{\alpha<\gamma} V_\alpha$ for a limit ordinal γ

The Axiom of Regularity (also often called Axiom of Foundation) implies that $\underline{\forall x \exists_{\alpha \in On} (x \in V_\alpha)}$

$\boxed{\text{we can prove } On \cap V_\alpha = \alpha}$

"$V = \bigcup_{\alpha \in On} V_\alpha$"

↑ the class of all sets

Axiom of Regularity:

For any $\underbrace{\text{set}}_{\text{non-empty}}$ A, there is a $\in$-minimal element a of A (i.e. such a that $\forall_{b \in A} (b \notin a)$)

Gödel's L-hierarchy ← constructible sets

L_α, $\alpha \in On$ are defined recursively by:

$L_0 = \emptyset$

$L_{\alpha+1} = \mathrm{def}(L_\alpha) \cup L_\alpha$ ← $\subseteq \mathcal{P}(L_\alpha)$

$L_\gamma = \bigcup_{\alpha<\gamma} L_\alpha$ if γ is a limit ordinal

$L_n = V_n$ for all $n \in \omega$ — Hence $L_\omega = V_\omega$

$L = \bigcup_{\alpha \in On} L_\alpha$

For a structure $M = \langle M, \in, \ldots \rangle$

$\mathrm{def}(M) = \{ A \in \mathcal{P}(M) \mid A \text{ is definable in } M \text{ (possibly with some parameters from } M) \}$

$|L_{\omega+1}| = \aleph_0$

$|V_{\omega+1}| = |V_\omega \cup \mathcal{P}(V_\omega)| = 2^{\aleph_0}$

Thus $L_{\omega+1} \neq V_{\omega+1}$

But $L = V$ is consistent relative to ZF.

L is a transitive class (i.e. $a \in L$, $b \in a \Rightarrow b \in L$) and $On \subseteq L$ ($On \cap L_\alpha = \alpha$)

Thm 14.1 (K. Gödel 194?) For any axiom φ of ZFC we have $ZFC \vdash \varphi^L$

(Note that this is a meta theorem!)

For a (transitive) class $K = \{x \mid \psi(x, \vec{z})\}$.
and for each formula φ in $\mathcal{L}_\varepsilon = \{\varepsilon\}$
φ^K is defined inductively by

$$(x \in y)^K = x \in y, \quad (x = y)^K = x = y$$

$$(\varphi_0 \wedge \varphi_1)^K = (\varphi_0^K \wedge \varphi_1^K) \quad (\neg \varphi_0)^K = \neg \varphi_0^K$$

$$(\exists x \varphi)^K = (\exists x \in K) \varphi^K$$
$$= \exists x (\psi(x, \vec{z}) \wedge \varphi^K)$$

A class K is transitive if $\forall a \in K \ \forall b \in a \ (b \in K)$

$$\forall x \left((\psi(x, \vec{z}) \wedge y \in x) \to \psi(y, \vec{z}) \right)$$

φ^L is also often written as $L \models \varphi$

But we do not have the relation $L \models (\cdot)$ for all $\varphi \in \underbrace{\mathrm{Fml}_\varepsilon}_{\text{a set in ZFC}}$

φ is absolute between L and V

Question: For which formula do we have $\varphi^L \leftrightarrow \varphi$ or

If M is a transitive class with $On \subseteq M$ and $ZFC \vdash \varphi^M$ for all axioms φ of ZF Then $ZFC \vdash L \subseteq M$.

φ is absolute between L and M

Question For which formula do we have $\varphi^L \leftrightarrow \varphi^M$ or $\varphi^M \leftrightarrow \varphi$

φ is absolute between M and V

Proposition 14.7 (ZF + DC) dependent choice

Suppose that $\mathfrak{A} = \langle A, E, \dots \rangle$ and

a set

a well founded relation

φ is a sentence in the signature of $\mathfrak{A}$. If $\mathfrak{A} \models \varphi$

Then there is countable structure $\mathfrak{B} = \langle B, E', \dots \rangle$ s.t. E' is well-founded $\mathfrak{B} \in L$ and $\mathfrak{B} \models \varphi$

$L \subsetneq L(\mathbb{R}) \subsetneq V$ is possible

"model" of ZF

A binary relation E on a set A is said to be well-founded if for any non-empty $A_0 \subseteq A$ there is an E-minimal element a of A_0.
(i.e. such $a \in A_0$ s.t. $\forall b \in A_0 (b \not\!E\, a)$)

DC If a tree does not have any maximal node then it has an infinite branch.

Thm 14.3 (Lévy–Shoenfield Absoluteness Lemma)

Let $W_0 \in W_1$ be transitive classes s.t. there are enough many axioms $\varphi_0, \dots, \varphi_{k-1}$ of ZFC s.t. $\varphi_0^{W_0} \dots \varphi_{k-1}^{W_0}$, $\varphi_0^{W_1} \dots \varphi_{k-1}^{W_1}$ hold.

Then for any Σ_1-formula $\varphi = \varphi(x_0, \dots x_{n-1})$ and parameters $\underbrace{\ }_{a_0, \dots, a_{n-1}}$ from $H(\aleph_1)^{W_0}$, $\varphi(a_0 \dots a_{n-1})$ is absolute between W_0 and W_1 [i.e.

$W_0 \models \varphi(a_0, \dots a_{n-1}) \Leftrightarrow W_1 \models \varphi(a_0 \dots a_{n-1})$

Lévy Hierarchy of formulas of ZFC.

A formula φ is said to be bounded (or Δ_0) if φ contains only quantifiers of the form $(\forall x \in y)$ $(\exists x \in y)$

where $(\forall x \in y)\varphi = \forall x (x \in y \to \varphi)$

$(\exists x \in y)\varphi = \exists x (x \in y \wedge \varphi)$

if we consider $\varphi \to \psi$ as an abbreviation of $(\neg\varphi \vee \psi)$

Then $\neg(\forall x \in y)\neg\varphi \Leftrightarrow \neg\forall x (\neg x \in y \vee \neg\varphi)$

$\Leftrightarrow \neg\forall x \neg(x \in y \wedge \varphi)$

$\Leftrightarrow (\exists x \in y)\varphi$

A formula φ is Σ_1 if φ is of the form $\exists x_0 \dots \exists x_{k-1} \psi$ where ψ is Δ_0.

The proof of Thm 14.3 without parameters from Prop 14.2

Suppose that φ is a formula of the form $\exists x \psi(x)$ where ψ is a Δ_0-formula

If $W_0 \models \varphi$ then we have $W_1 \models \varphi$

Suppose $W_1 \models \varphi$ (i.e. φ^{W_1}) Then by "Reflection Theorem" there is some transitive set M s.t. $(M, \in) \models \varphi$. By Prop 14.2 there is $\mathcal{B} = \langle B, E \rangle$ s.t. $\mathcal{B} \models \varphi \wedge$ Extensionality Axiom E is well founded $\mathcal{B}$ countable $\mathcal{B} \in L$ In L, Let $\mathcal{C} = \langle C, \in, \dots \rangle$ be a Mostowski-collapse $\mathcal{B} \cong \mathcal{C}$. $\mathcal{C} \models \varphi$, $\mathcal{C} \in L \subseteq W_0$ (by)

By Lemma 14.4 (2) it follows $W_0 \models \varphi$. ⊠

Lemma 14.4 (1) If φ is Δ_0 and $\varphi = \varphi(x_0, \dots x_{n-1})$ then for any transitive W_0, W_1 ($W_0 \subseteq W_1$) and $a_0 \dots a_{n-1} \in W_0$

$W_0 \models \varphi(a_0, \dots, a_{n-1}) \Leftrightarrow W_1 \models \varphi(a_0, \dots a_{n-1})$

(2) If φ is Σ_1 and $\varphi = \varphi(x_0, \dots, x_{n-1})$ then for any transitive W_0, W_1 with $W_0 \subseteq W_1$ and $a_0, \dots, a_{n-1} \in W_0$ we have

$W_0 \models \varphi(a_0, \dots, a_{n-1}) \Rightarrow W_1 \models \varphi(a_0, \dots, a_{n-1})$

Theorem 14.5 (Lévy–Montague Reflection Theorem)

If W is a transitive class which "satisfies" Axiom of Regularity + (enough amount of axioms of) ZFC, Then for any formula $\varphi = \varphi(x_0, \dots x_{n-1})$ there is a countable transitive set M s.t.

For all $a_0, \dots, a_{n-1} \in M$ $M \models \varphi(a_0, \dots a_{n-1}) \leftrightarrow W \models \varphi(a_0, \dots a_{n-1})$