An outline of independence proofs by forcing

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The first two sections of the following note grew out of a lecture in a course for graduate students I gave in the Spring semester 2015 at Kobe university. This text was then used a base for the first lecture of the intensive course on forcing for graduate students I gave at the University of Tokyo in the Autumn semester 2015. The later sections (based on the lectures of this course at the University of Tokyo) are still being added to provide all details of the proofs of the assertions I cited in the first two sections without proof. The last section contains some other simple applications (i.e. applicatins which can be treated without iteration of forcing).

This note in the present version is still a work in progress. Any comments or suggestions are appreciated. (1)

Contents

1 Metamathematical framework

metamathematics

Let T be a finite fragment of ZFC. (2)

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(1) **Caution:** This text is obsolete. Further updates of this note are posted as:
`http://fuchino.ddo.jp/notes/forcing-outline-katowice-2017.pdf`

(2) A finite fragment of ZFC is a (concretely given) finite set of axioms of ZFC containing finitely many instances of Axiom of Separation and Axiom of Replacement and all other axioms. Note that, in any of such a fragment of ZFC, we we have all the axioms needed to develop the first order logic (coded in

We show that we can find a sufficiently large finite fragment T^* of ZFC containing all axioms of T such that[3], for any countable transitive model M of T^* (cf. Corollary 3.12), we can construct, by means of forcing method, a countable transitive model

(1.1) $M^* \supseteq M$ of T such that $M^* \models \neg\mathrm{CH}$. meta-0

This shows the nonprovability of CH from ZFC: Suppose toward a contradiction that there is a proof P_0 with

(1.2) $\mathrm{ZFC} \vdash^{P_0} \mathrm{CH}$. meta-1

Let T be a finite fragment of ZFC which contains all the axioms used in P_0. Then we have $T \vdash^{P_0} \mathrm{CH}$. Let T^* be as above for this T.

From here on, we are working in ZFC: Let M be a countable transitive model of T^* and $M^* \supseteq M$ a countable transitive[4] model of $T + \neg\mathrm{CH}$. In particular, $M^* \models \neg\mathrm{CH}$, or: "$M^* \models \mathrm{CH}$ does not hold". Form this we obtain a contradiction since the formal proof (1.2) can be translated to a code of the proof and thus, in ZFC, from $M^* \models T$, it follows that $M^* \models \mathrm{CH}$.

Since this argument does not depend on the choice of T [5], we can distill an algorithm from the argument for constructing a proof P_0' with $\mathrm{ZFC} \vdash^{P_0'} x \not\equiv x$ from a given proof P_0 of CH from ZFC.

This means that CH is unprovable from ZFC as far as ZFC itself is consistent.

Likewise we can also show that for each finite fragment T of ZFC there is another finite fragment T^{**} of ZFC containing T such that, for any countable transitive model M of T^{**}, we can construct, by means of forcing method, a countable transitive model

(1.3) $M^{**} \supseteq M$ of T such that $M^{**} \models \mathrm{CH}$. meta-2

With the same argument as above we can show that if there is a proof of CH from ZFC then we can find a proof of $x \not\equiv x$ from ZFC.

Thus,we can conclude that neither CH nor $\neg$CH are provable from it as far as ZFC is consistent.

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certain sets) inside the axiom system. When we say that such a fragment is "sufficiently large", we mean such a fragment of ZFC that contains all the instances of these axiom schemes which are needed in the following arguments (where only finitely many instances of Separation and Replacement are used in any case). Sometimes we consider, in place of a finite fragment, the full axiom system of ZFC with Powerset Axiom replaced by a weaker axiom only guaranteeing that only sets with certain cardinalities have their powerset (note that $\mathcal{H}(\chi)$ for a large enough regular χ satisfies such an axiom system).

[3] The main clause of this sentence is a statement in metamathematics while the statement following this "such that" is meant to be formulated in ZFC.

[4] A set a is said to be transitive if $y \in a$ holds for any $x \in a$ and $y \in x$.

[5] We see later that the algorithm for finding T^* for each given T is uniform to each finite fragment T of ZFC.

2 Generic filters and generic extensions

generic

In this section, we shall see some more technical details of the construction of M^* and M^{**} of the last section.

In ZFC, a set $\mathbb{P}$ with a binary relation $\leq_{\mathbb{P}}$ is said to be a preordering if $\leq_{\mathbb{P}}$ is transitive and reflexive, i.e.

(2.1) $p \leq_{\mathbb{P}} q$ and $q \leq_{\mathbb{P}} r$ imply $p \leq_{\mathbb{P}} r$ for all p, q, $r \in \mathbb{P}$; po-1

(2.2) $p \leq_P p$ for all $p \in \mathbb{P}$. po-2

A preordering $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ is a poset (or a forcing notion) if $\mathbb{P}$ has a maximal element[6], i.e. such an element $p \in \mathbb{P}$ that, for all $q \in \mathbb{P}$, $q \leq_{\mathbb{P}} p$ holds. For a poset $\mathbb{P}$ we fix a maximal element and denote it by $\mathbb{1}_{\mathbb{P}}$. We also write $\mathbb{P} = (\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ to indicate explicitly which partial order the poset $\mathbb{P}$ is endowed with and which maximal element of $\mathbb{P}$ is chosen.

A subset of a poset $\mathbb{P}$ is a *filter* if

(2.3) $\mathbb{1}_{\mathbb{P}} \in G$; po-3

(2.4) For any $p, q \in \mathbb{P}$, if $p \in G$ and $p \leq_{\mathbb{P}} q$, then $q \in \mathbb{P}$; po-4

(2.5) For any p, $q \in G$, there is $r \in G$ such that $r \leq_{\mathbb{P}} p$, q. po-5

A subset D of a poset $\mathbb{P}$ is said to be *dense* if, for any $p \in \mathbb{P}$, there is some $q \in D$ such that $q \leq_{\mathbb{P}} p$.

We fix a sufficiently large finite fragment ZFC_0 of ZFC. Actually we can fix ZFC_0 only when we have written down all the following arguments in this article. This is like labels in a LaTeX file, whose value is set to be "**???**" in the first run of LaTeX on the source file. In the second run the values are fixed as some sequence of symbols.[7]

In any case what we can produce as a mathematical theory is a finite object so that after we have written down a discourse we can fix ZFC_0 as a sufficiently large finite set containing all the axioms of ZFC which were used in the proofs in the present manuscript and proofread the written material from the beginning again to check if it makes sense.

Suppose now that M is a countable transitive set with $\langle M, \in \rangle \models \mathrm{ZFC}_0$ [8] and $\mathbb{P} \in M$ is a poset with $\mathbb{P} = (\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$.[9]

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(6) We have to talk about "a" maximal element since, by the absence of anti-symmetry of $\leq_{\mathbb{P}}$, there may be more than one maximal elements.

(7) The situation with LaTeX is actually worse since, for each natural number n, the is a LaTeX source file which we have to compile more than n times to get the correct final dvi file (Exercise).

(8) If M is a countable transitive set with $\langle M, \in \rangle \models T$, we shall also say that M is a countable transitive model of T.

(9) If we say $\mathbb{P} \in M$ for a poset with $\mathbb{P} = (\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$, then we mean $\mathbb{P} \in M$, $\leq_{\mathbb{P}} \in M$ and $\mathbb{1}_{\mathbb{P}} \in M$ where the last condition follows from the first if M is transitive.

Note that for a transitive M, $\mathbb{P}$ is a poset if and only if $M \models$ "$\mathbb{P}$ is a poset". A filter $G \subseteq \mathbb{P}$ is said to be $(M, \mathbb{P})$-generic if, for any dense $D \subseteq \mathbb{P}$ with $D \in M$, we have $G \cap D \neq \emptyset$.

In most of the cases, an $(M, \mathbb{P})$-generic filter is not an element of M. To formulate this more precisely, we need the following definitions. For a poset $\mathbb{P}$ and p, $q \in \mathbb{P}$, p and q are *compatible* (in $\mathbb{P}$) if there is $r \in \mathbb{P}$ such that $r \leq_{\mathbb{P}} p$, q. Otherwise p and q are said to be *incompatible* (in $\mathbb{P}$). A poset $\mathbb{P}$ is said to be *atomless* if, for any $p \in \mathbb{P}$, there are q, $q' \in \mathbb{P}$ such that q, $q' \leq_{\mathbb{P}} p$ and, q and q' are incompatible.

Lemma 2.1 *For a transitive model M of* ZFC_0 *and a poset $\mathbb{P} \in M$, if $\mathbb{P}$ is atomless and G is a $(\mathbb{P}, M)$-generic filter, then $G \notin M$.*

Proof. Since M is transitive we have $\mathbb{P} \subseteq M$. Suppose that $G \subseteq \mathbb{P} \subseteq M$ is a filter and $G \in M$. We show that G is not $(M, \mathbb{P})$-generic. Let $D = \mathbb{P} \setminus G$. Then we have $D \in M$[10]. D is dense in $\mathbb{P}$: For any $p \in \mathbb{P}$, let q, $r \in \mathbb{P}$ be such that q, $r \leq_{\mathbb{P}} p$ and, q and r are incompatible. By (2.5), it is impossible that both of q and r are in G. Say $q \notin G$. Then we have $q \leq_{\mathbb{P}} p$ and $q \in D$.

Thus G is not $(M, \mathbb{P})$-generic since $G \cap D = \emptyset$. ❑ (Lemma 2.1)

An $(M, \mathbb{P})$-generic filter does exist for a countable transitive model M of ZFC_0:

Lemma 2.2 *If M is a countable transitive model of* ZFC_0*, then for any poset $\mathbb{P} \in M$ there is an $(M, \mathbb{P})$-generic filter G.* existence-of-generic

Proof. Since M is countable the set $\mathcal{D} = \{D \in M : D \text{ is a dense subset of } \mathbb{P}\}$ is countable as well. Let $\mathcal{D} = \{D_n : n \in \omega\}$. Let $\langle p_n : n \in \omega \rangle$ be a descending sequence (with respect to $\leq_{\mathbb{P}}$) of elements of $\mathbb{P}$ such that $p_n \in D_n$ for all $n \in \omega$. The construction of such a sequence is possible since each D_n $(n \in \omega)$ is dense in $\mathbb{P}$.

Let

$$G = \{p \in \mathbb{P} : p_n \leq_{\mathbb{P}} p \text{ for some } n \in \omega\}. \tag{2.6}$$

Then G is a $(M, \mathbb{P})$-generic filter. ❑ (Lemma 2.2)

For a poset $\mathbb{P}$, we define the class $V^{\mathbb{P}}$ recursively by

$$\underset{\sim}{x} \in V^{\mathbb{P}} \Leftrightarrow \text{ all elements of } \underset{\sim}{x} \text{ are of the form } \langle \underset{\sim}{y}, p \rangle \text{ where } \underset{\sim}{y} \in V^{\mathbb{P}} \text{ and } p \in \mathbb{P}. \tag{2.7}$$

[11] po-6

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[10] Here, we assume that the instance of Axiom of Separation, which is needed to prove that $D = \{p \in \mathbb{P} : p \notin G\}$ is a set, is in ZFC_0.

[11] This recursive definition can be realized as follows: Let $\langle f_\alpha : \alpha \in \mathrm{On} \rangle$ be an class sequence defined by

(2.8) $f_\alpha : V_\alpha \to 2$ for all $\alpha \in \mathrm{On}$;

Elements of $V^{\mathbb{P}}$ are called $\mathbb{P}$-names. As we already have done, $\mathbb{P}$-names are denoted here by alphabets with undertilde[(12)].

For a transitive model M of ZFC_0, and a poset $\mathbb{P} \in M$, let

$$M^{\mathbb{P}} = (V^{\mathbb{P}})^M = \{x \in M \,:\, M \models x \text{ is a } \mathbb{P}\text{-name}\}. \tag{2.11}$$ po-7

By the definition of $\mathbb{P}$-names (in particular by (2.8), (2.9) and (2.10)) The statement "x is a $\mathbb{P}$-name" is *absolute* over M, i.e., for any $x \in M$, x is a $\mathbb{P}$-name if and only if $M \models x$ is a $\mathbb{P}$-name. [(13)]

For a poset $\mathbb{P}$ and a filter $G \subseteq \mathbb{P}$, we define the interpretation $\underset{\sim}{x}^G$ of a $\mathbb{P}$-name $\underset{\sim}{x}$ under G recursively[(14)] by

$$\underset{\sim}{x}^G = \{\underset{\sim}{y}^G \,:\, \langle \underset{\sim}{y}, p\rangle \in \underset{\sim}{x} \text{ for some } p \in G\}. \tag{2.15}$$ po-7-0

For a transitive M, a poset $\mathbb{P} \in M$ and a filter $G \subseteq M$, let

$$M[G] = \{\underset{\sim}{x}^G \,:\, \underset{\sim}{x} \in M^{\mathbb{P}}\}. \tag{2.16}$$ po-8-0

$M[G]$ is called a generic extension of M by $\mathbb{P}$.

For a set x let $\check{x}$ be the $\mathbb{P}$-name defined recursively by

$$\check{x} = \{\langle \check{y}, \mathbb{1}_{\mathbb{P}}\rangle \,:\, y \in x\}. \tag{2.17}$$ [(15)] po-9-0

For a poset $\mathbb{P}$, let

$$\underset{\sim}{G}_{\mathbb{P}} = \{\langle \check{p}, p\rangle \,:\, p \in \mathbb{P}\}. \tag{2.18}$$ po-10-0

Clearly $\underset{\sim}{G}_{\mathbb{P}}$ is a $\mathbb{P}$-name. We shall simply write $\underset{\sim}{G}$ instead of $\underset{\sim}{G}_{\mathbb{P}}$, if it is clear which poset $\mathbb{P}$ is meant.

L-0

(2.9) $f_{\alpha+1}(x) = 1 \;\Leftrightarrow\;$ all elements of x are of the form $\langle y, p\rangle$ such that $f_\alpha(y) = 1$ and $p \in \mathbb{P}$ for all $\alpha \in \mathrm{On}$ and $x \in V_{\alpha+1}$;

(2.10) $f_\gamma = \bigcup_{\alpha<\gamma} f_\alpha$ if γ is a limit ordinal.

By induction on $\alpha \in \mathrm{On}$ we can show easily that $\langle f_\alpha \,:\, \alpha \in \mathrm{On}\rangle$ is increasing with respect to $\subseteq$. We then define $V^{\mathbb{P}}$ as $\{x \,:\, f_{rank(x)+1}(x) = 1\}$. By (2.9), this $V^{\mathbb{P}}$ then satisfies (2.7).

(12) There are also texts in which $\mathbb{P}$-names are denoted by dotted alphabets, like $\dot{x}$, $\dot{y}$, $\dot{z}$, etc.

(13) Here we assume again that all instances of Axiom of Separation needed to make this hold are included in ZFC_0.

(14) This recursion can be carried out via defining an increasing class sequence $\langle f_\alpha \,:\, \alpha \in \mathrm{On}\rangle$ by

(2.12) $f_\alpha : V^{\mathbb{P}} \cap V_\alpha \to V$ for all $\alpha \in \mathrm{On}$;

(2.13) $f_{\alpha+1}(\underset{\sim}{x}) = \{f_\alpha(\underset{\sim}{y}) \,:\, \langle y, p\rangle \in \underset{\sim}{x} \text{ for some } p \in G\}$ for all $\alpha \in \mathrm{On}$ and $\underset{\sim}{x} \in V^{\mathbb{P}} \cap V_{\alpha+1}$;

(2.14) $f_\gamma = \bigcup_{\alpha<\gamma} f_\alpha$ for all limit $\gamma \in \mathrm{On}$

and letting $\underset{\sim}{x}^{\mathbb{P}} = f_{rank(x)+1}(\underset{\sim}{x})$.

(15) Note that $\check{\emptyset} = \emptyset$.

Lemma 2.3 (1) *For any filter G on a poset $\mathbb{P}$, and any set x, we have $(\check{x})^G = x$.*

(2) *For any filter G on a poset $\mathbb{P}$, we have $\underset{\sim}{G}^G = G$.*

Proof. (1): By $\in$-induction on x: We have to show that if $(\check{y})^G = y$ for all $y \in x$ then $(\check{x})^G = x$.

So suppose that $(\check{y})^G = y$ for all $y \in x$. By (2.15) and since $\mathbb{1}_{\mathbb{P}} \in G$ for any filter G, we have $\check{x}^G = \{\check{y}^G : y \in \mathbb{P}\} = \{y : y \in x\} = x$.

(2): By (2.15), (2.18) and (1), we have

$$\text{(2.19)} \quad \underset{\sim}{G}^G = \{\check{p}^G : p \in G\} = \{p : p \in G\} = G.$$

❏ (Lemma 2.3)

L-1

Lemma 2.4 *Suppose that M is a transitive model of* ZFC_0 *and $\mathbb{P} \in M$ is a poset. Suppose further that G is a filter over M. Then:*

(1) *$M[G]$ is a transitive set.*

(2) *$M \subseteq M[G]$ and $G \in M[G]$.*

(3) *For all $\underset{\sim}{x} \in M^{\mathbb{P}}$, we have $rank(\underset{\sim}{x}^G) \leq rank(\underset{\sim}{x})$.*

(4) *$M[G] \cap \mathrm{On} = M \cap \mathrm{On}$.*

Proof. (1): Suppose that $x \in M[G]$. By the definition (2.16) of $M[G]$, there is a $\mathbb{P}$-name $\underset{\sim}{x} \in M$ such that $x = \underset{\sim}{x}^G$. If $y \in x$ then by (2.15), $y = \underset{\sim}{y}^G$ for some $\underset{\sim}{y} \in V^{\mathbb{P}}$ and $p \in G$ such that $\langle \underset{\sim}{y}, p \rangle \in \underset{\sim}{x}$. Since $\underset{\sim}{y} \in trcl(\underset{\sim}{x}) \subseteq M$ by the transitivity of M, it follows that $\underset{\sim}{y} \in M$ and thus $y \in M[G]$.

(2): For all $x \in M$ $\check{x} \in M$ by $M \models \mathrm{ZFC}_0$. Thus $x = \check{x}^G \in M[G]$ for all $x \in M$. Since $M \models \mathrm{ZFC}_0$ and $\mathbb{P} \in M$, we have $\underset{\sim}{G}_{\mathbb{P}} \in M$ and thus $G = (\underset{\sim}{G}_{\mathbb{P}})^G \in M[G]$.

(3): This can be proved by induction of $rank(\underset{\sim}{x})$. Suppose that the inequality is established for all names $\underset{\sim}{y}$ with $rank(\underset{\sim}{y}) < rank(\underset{\sim}{x})$. Then

$$\text{(2.20)} \quad \begin{aligned} rank(\underset{\sim}{x}^G) &= \sup\{rank(\underset{\sim}{y}^G) : \langle \underset{\sim}{y}, p \rangle \in \underset{\sim}{x} \text{ and } p \in G\} \ . \\ &\leq \sup\{rank(\underset{\sim}{y}) : \langle \underset{\sim}{y}, p \rangle \in \underset{\sim}{x} \text{ and } p \in G\} \\ &\leq \sup\{rank(\underset{\sim}{y}) : \langle \underset{\sim}{y}, p \rangle \in \underset{\sim}{x}\} \\ &\leq rank(\underset{\sim}{x}) \end{aligned}$$

(4): Note that $\mathrm{On} \cap M$, $\mathrm{On} \cap M[G] \in \mathrm{On}$ since M and $M[G]$ are transitive (for $M[G]$ this is by (1)).

Since $M \subseteq M[G]$ by (2), we have $\mathrm{On} \cap M \subseteq \mathrm{On} \cap M[G]$. Thus we have $\mathrm{On} \cap M \leq \mathrm{On} \cap M[G]$

For $\alpha \in M[G]$, if $\alpha = \underset{\sim}{x}^G$ for a $\underset{\sim}{x} \in M^{\mathbb{P}} \subseteq M$, then by (3), we have

$$\text{(2.21)} \quad \alpha = rank(\alpha) = rank(\underset{\sim}{x}^G) \leq rank(\underset{\sim}{x}) < \mathrm{On} \cap M.$$

Thus we have $\mathrm{On} \cap M[G] \leq \mathrm{On} \cap M$. ❑ (Lemma 2.4)

To prove the next two Theorems we need a very much sophisticated and profound technical tool called "forcing relation" which will be introduced in the following sections. For now we assume these Theorem and going to see how the ideas sketched in Section 1 can be accomplished.

generic-ZFC

Theorem 2.5 *If T is a finite fragment of* ZFC*, there is a large enough fragment T^* of* ZFC *containing T such that, if M is a transitive model of T^*, $\mathbb{P} \in M$ a poset and G an $(M, \mathbb{P})$-generic filter, Then $M[G] \models T$.* ❑

A poset $\mathbb{P}$ is said to satisfy the ccc (countable chain condition) if any pairwise incompatible subset[16] $A \subseteq \mathbb{P}$ is countable.

ccc

Theorem 2.6 *For a finite fragment T of* ZFC *containing* ZFC_0 *and T^* as in Theorem 2.5, suppose that M is a transitive model of T^*, $\mathbb{P} \in M$ is a poset with*

(2.22) $M \models$ "*$\mathbb{P}$ satisfies the ccc*"

and G an $(M, \mathbb{P})$-generic filter. Then we have $\mathrm{Card}^M = \mathrm{Card}^{M[G]}$. [17] ❑

For $\kappa \in \mathrm{On}$, we regard

(2.23) $\mathrm{Fn}(\kappa, 2) = \{p : p : x \to 2 \text{ for some } x \in [\kappa]^{<\aleph_0}\}$. [18]

[19] as a partial ordering with the order $\leq$ defined by

(2.24) $p \leq q \;\Leftrightarrow\; p \subseteq q$

for all p, $q \in \mathrm{Fn}(\kappa, 2)$. $\emptyset$ is the maximal element of this partially ordered set and thus we obtain the poset

(2.25) $(\mathrm{Fn}(\kappa, 2), \leq, \emptyset)$

which we also denote simply by $\mathrm{Fn}(\kappa, 2)$. The proof of the following Lemma is not very much involved but we shall also postpone its proof.

cohen-ccc

Lemma 2.7 *For any $\kappa \in \mathrm{On}$ the poset* $\mathrm{Fn}(\kappa, 2)$ *satisfies the ccc.* ❑

L-2

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[16] $A \subseteq \mathbb{P}$ is said to be pairwise incompatible if any distinct p, $q \in \mathbb{P}$ are incompatible in $\mathbb{P}$. A pairwise incompatible subset A of a poset $\mathbb{P}$ is also called an antichain in $\mathbb{P}$.

[17] For a class $\mathcal{C}$ introduced by an $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(x)$ as $\mathcal{C} = \{x : \varphi(x)\}$ we denote with $\mathcal{C}^M$ the set $\{x \in M : M \models \varphi(x)\}$. For a transitive M if $M \models$ "α is an ordinal then α is always really an ordinal. However, in general, $M \models$ "α is a cardinal" does not imply that α is a cardinal. In fact for large enough fragment T, if M is a countable model of ZFC_0 then M thinks there is $\aleph_1$. But what M thinks is $\aleph_1$ $(= (\aleph_1)^M)$ is actually a countable ordinal since $\mathrm{On} \cap M$ is a countable ordinal and hence also $(\aleph_1)^M \in \mathrm{On} \cap M$.

[19] For a set X we denote with $[X]^{<\aleph_0}$ the set of all finite subsets of X.

Lemma 2.8 *Suppose that M is a transitive model of* ZFC_0 *and* $\kappa \in \mathrm{On} \cap M$ $(= \mathrm{On}^M)$ *is a limit ordinal. Let* $\mathbb{P}$ *be the poset* $\mathrm{Fn}(\kappa, 2)$*. Then for any* $(M, \mathbb{P})$*-generic filter* G *over* $\mathbb{P}$*, letting* $g = \bigcup G$*, we have*

(1) $g : \kappa \to 2$*;*

(2) *Letting* $a_\gamma = \{n \in \omega : g(\gamma + n) = 1\}$ *for all limit ordinal* $\gamma < \kappa$*,* a_γ*,* $\gamma \in \mathrm{Lim} \cap \kappa$ *are pairwise distinct.*

Proof. (1): Note that $\kappa \subseteq M$. $\bigcup G$ is a mapping since any p, $q \in G$ are compatible ($\Leftrightarrow$ compatible as functions).

For an arbitrary $\alpha \in \kappa$, the set

$$(2.26) \quad D_\alpha = \{p \in \mathbb{P} : \alpha \in \mathrm{dom}(p)\}$$

is dense in $\mathbb{P}$ and $D \in M$.[20] Hence, by genericity of G, there is $p \in G \cap D$. Thus $\alpha \in \mathrm{dom}(p) \subseteq \mathrm{dom}(g)$.

(2): Let γ, $\gamma' \in \mathrm{Lim} \cap \kappa$ with $\gamma \neq \gamma'$. We show that $a_\gamma \neq a_{\gamma'}$. Let

$$(2.27) \quad D_{\gamma,\gamma'} = \{p \in \mathbb{P} : \text{there is } n \in \omega \text{ such that } \gamma + n,\ \gamma' + n \in \mathrm{dom}(p) \text{ and } p(\gamma + n) \neq p(\gamma' + n).\}$$

It is easy to see that $D_{\gamma,\gamma'}$ is dense in $\mathbb{P}$ and $D_{\gamma,\gamma'} \in M$. By genericity of G, there is $p \in G \cap D_{\gamma,\gamma'}$. Since $p \subseteq \bigcup G = g$, it follows that a_γ and $a_{\gamma'}$ are different. ❑ (Lemma 2.8)

With these preparations we can show that there are T^* and M^* as in (1.1) for each given finite fragment T of ZFC.

Let T be a finite fragment of ZFC. We may assume that T contains all the axioms of ZFC_0. Let T^* be the extension of T in Theorem 2.5. Let M be a countable transitive model of T^* and let $\kappa \in \mathrm{On}$ be such that $M \models \kappa = \aleph_2$.[21] Let $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$. Since $\mathbb{P} \in M$ there is an $(M, \mathbb{P})$-generic filter G by Lemma 2.2. By the choice of T^*, $M[G] \models T$ and by Lemma 2.7 and Theorem 2.6, We have $\mathrm{Card}^M = \mathrm{Card}^{M[G]}$. In particular, $M[G] \models \kappa = \aleph_2$. Since

$$(2.28) \quad M[G] \models \text{“there are at least } \kappa \text{ distinct subsets of } \omega\text{”}$$

by Lemma 2.8, (2), it follows that $M[G] \models \neg\mathrm{CH}$.

For the construction of M^{**} in (1.3), we proceed similarly with the poset $\mathbb{P} \in M$ with $M \models \mathbb{P} = \mathrm{Fn}(\omega_1, 2, \aleph_1)$ where

$$(2.29) \quad \mathrm{Fn}(\omega_1, 2, \aleph_1) = \{p : p : x \to 2 \text{ for some } x \in [\omega_1]^{<\aleph_1}\}$$

with the partial order defined in the same way as for $\mathrm{Fn}(\kappa, 2)$:

$$(2.30) \quad p \leq q \;\Leftrightarrow\; q \subseteq p$$

[20] This is also the place where we have to assume that a certain instance of Axiom of Separation is included in ZFC_0.

[21] Here we are assuming that ZFC_0 contains enough axioms to prove the assertion “There is $\aleph_2$”.

for p, $q \in \mathrm{Fn}(\omega_1, 2, \aleph_1)$ and the maximal element $\emptyset$.

That the generic extension $M[G]$ by this $\mathbb{P}$ is as desired can be seen in the following Lemma:

L-3

Lemma 2.9 *Suppose that M is a transitive model of* ZFC_0*. Let $\mathbb{P}$ be the poset defined by $M \models \mathbb{P} = \mathrm{Fn}(\omega_1, 2, \aleph_1)$ and G an $(M, \mathbb{P})$-generic filter over $\mathbb{P}$. Letting $g = \bigcup G$, we have*

(0) $(\omega_1)^M = (\omega_1)^{M[G]}$.

(1) $g : \omega_1^M \to 2$

(2) $(\mathcal{P}(\omega))^M = (\mathcal{P}(\omega))^{M[G]}$.

(3) *There is an $f \in M$ which is is a surjection from ω_1^M to $\mathcal{P}(\omega)^M$.*

Proof. (0) and (2): can be proved fairly easily using the forcing relation and some of its basic properties.

(1): can be proved similarly to Lemma 2.8, (1) (Exercise).

(3): For a limit $\gamma \in \mathrm{Lim} \cap \omega_1^M$, let a_γ be defined as before by

$$a_\gamma = \{n \in \omega \,:\, g(\gamma + n) = 1\} \tag{2.31}$$

By (2) $a_\gamma \in \mathcal{P}(\omega)^M$ for all limt $\gamma < \omega_1^M$. By density argument we can prove that

$$f : \omega_1^M \to \mathcal{P}(\omega)^M;\ \alpha \mapsto a_\gamma \text{ where } \gamma \text{ is the } \alpha\text{th limit ordinal in } (\omega_1)^M \tag{2.32}$$

is a surjection.

For this, we only need the following fact: in M, let

$$\begin{aligned} D_a = \{p \in \mathbb{P} \,:\, & \text{there is } \gamma < \omega_1 \text{ such that } \{\gamma + n \,:\, n \in \omega\} \subseteq \mathrm{dom}(p) \\ & \text{and } \{n \in \omega \,:\, p(\gamma + n) = 1\} = a\} \end{aligned} \tag{2.33}$$

for each $a \in \mathcal{P}(\omega)^M$.

Then $D_a \in M$ and D_a is dense in $\mathbb{P}$. $p \in G \cap D_a$ "forces" $f(\gamma) = a$ for $\gamma \in \mathrm{dom}(p)$ as in the definition of D_a. ❑ (Lemma 2.9)

3 Some prerequisites from basic set theory

prerequisites

3.1 Induction and resursive definition on well-founded sets and classes

ind-rec

The most general form of the principle of induction and recursive definition we rely on (in ZFC) is given in Theorem 3.2.

We begin with some definitions. Note that the following is the generalization of the corresponding assertions on sets since a set a can be always seen as the class $\{x \,:\, x \in a\}$.

For a class $\mathcal{X}$ and a binary class relation $\mathcal{R}$ on $\mathcal{X}$, the class relation $\mathcal{R}$ is said to be set-like if, for all $x \in \mathcal{X}$, the class

$$I_{\mathcal{R}}(x) = \{y \in \mathcal{X} \,:\, y \mathrel{\mathcal{R}} x\} \tag{3.1}$$ ind-0

is a set.

$\mathcal{R}$ is well-founded if, for all non empty set $a \subseteq \mathcal{X}$, there is an $\mathcal{R}$ minimal element of a (i.e. such $x \in a$ that $y \not\mathrel{\mathcal{R}} x$ for all $y \in a$). Note that in general a minimal element of a is not unique.

For a set-like $\mathcal{R}$ on $\mathcal{X}$ and $x \in \mathcal{X}$, we define

$$I^0_{\mathcal{R}}(x) = \{x\}; \tag{3.2}$$ ind-1

$$I^{n+1}_{\mathcal{R}}(x) = \bigcup\{I_{\mathcal{R}}(y) \,:\, y \in I^n_{\mathcal{R}}(x)\}. \tag{3.3}$$ ind-2

Note that we have in particular $I^1_{\mathcal{R}}(x) = I_{\mathcal{R}}(x)$.

The transitive closure and the weak transitive closure of x with respect to $\mathcal{R}$ is defined by

$$\mathit{trcl}_{\mathcal{R}}(x) = \textstyle\bigcup_{n\in\omega} I^n_{\mathcal{R}}(x) \text{ and} \tag{3.4}$$ ind-3

$$\mathit{trcl}^-_{\mathcal{R}}(x) = \textstyle\bigcup_{n\in\omega\setminus 1} I^n_{\mathcal{R}}(x). \tag{3.5}$$ ind-4

If $\mathcal{R}$ is well-founded, we have $\mathit{trcl}^-_{\mathcal{R}}(x) = \mathit{trcl}_{\mathcal{R}}(x) \setminus \{x\}$.

Note that the class binary relation $\in = \{\langle x, y\rangle \,:\, x \in y\}$ on V is set-like and well-founded (under the Axiom of Foundation).

For the set-like well-founded class relation $\in$, we often abbreviate $\mathit{trcl}_{\in}(x)$ and $\mathit{trcl}^-_{\in}(x)$ as $\mathit{trcl}(x)$ and $\mathit{trcl}^-(x)$ respectively.

L-ind-0

Lemma 3.1 *Suppose that $\mathcal{R}$ is set-like well-founded class relation on the class $\mathcal{X}$. Then any non-empty class $\mathcal{A} \subseteq \mathcal{X}$ has the $\mathcal{R}$-minimal element.*

Proof. Let $a \in \mathcal{A}$ be arbitrary. Then $\mathit{trcl}_{\mathcal{R}}(a)\mathcal{A}$ is a non empty subset of $\mathcal{A}$. An $\mathcal{R}$-minimal element a_0 of this set is an $\mathcal{R}$-minimal element of $\mathcal{A}$. ❑ (Lemma 3.1)

For a class relation $\mathcal{R}$ on $\mathcal{X}$, $\mathcal{A} \subseteq \mathcal{X}$ is said to be (left) closed with respect to $\mathcal{R}$ if for any $x \in \mathcal{A}$, and $y \in \mathcal{X}$ with $y \mathrel{\mathcal{R}} x$ we alway have $y \in \mathcal{A}$. If $\mathcal{R}$ is set-like this simply means that $\mathit{trcl}_{\mathcal{R}}(x) \subseteq \mathcal{A}$ for all $x \in \mathcal{A}$.

Theorem 3.2 (Induction and Recursion Theorem) *Suppose that $\mathcal{R}$ is a set-like and well-founded class binary relation on a class $\mathcal{X}$.* ind-recurs-thm

(1) *For any class $\mathcal{A} \subseteq \mathcal{X}$, if the property*

$$\text{for any } y \in \mathcal{X},\text{ if } I^-_{\mathcal{R}}(x) \subseteq \mathcal{A} \text{ then } y \in \mathcal{A} \tag{3.6}$$ ind-5

holds then $\mathcal{A} = \mathcal{X}$.

(2) *Let*

(3.7) $\mathfrak{F} = \{f \,:\, f$ *is a mapping on a set* D *closed with respect to* $\mathcal{R}\}$.[22] ind-6

If $\mathcal{G}$ *is a class function on* $\mathfrak{F} \times \mathcal{X}$ *then there is a class function* $\mathcal{F}$ *on* $\mathcal{X}$ *such that* $\mathcal{F}(x) = \mathcal{G}(\mathcal{F} \restriction trcl^{-}_{\mathcal{R}}(x), x)$ *holds for all* $x \in \mathcal{X}$.

We shall call the argument based on (1) and (2) of the theorem above as $\mathcal{R}$-induction and $\mathcal{R}$-recursion.

For example, the recursive definition (2.11) of $\mathbb{P}$-names is accomplished by defining the characteristic function $\mathcal{F}$ of $V^{\mathbb{P}}$ by $\mathcal{R}$-recursion on the set-like well-founded relation $\mathcal{R}$ defined by $x \mathrel{\mathcal{R}} y \;\Leftrightarrow\; x \in trcl^{-}(y)$. Note that if $\langle \underset{\sim}{y}, p\rangle \in \underset{\sim}{x}$ then $\underset{\sim}{y} \mathrel{\mathcal{R}} \underset{\sim}{x}$.

Since the class binary relation $\in$ on the class On of ordinals is set-like and well-founded, we can apply the the following Induction and Recursion Theorem for ordinals as a corollary to Theorem 3.2:

Theorem 3.3 (Induction and Recursion Theorem for Ordinals) ind-recurs-ord-thm

(1) *If* $\mathcal{A}$ *is a class of ordinals such that*

(3.8) *for any* $\alpha \in \mathrm{On}$, *if* $\alpha \subseteq \mathrm{On}$ *then* $\alpha \in \mathcal{A}$ ind-7

holds then $\mathcal{A} = \mathcal{X}$.

(2) *Let*

(3.9) $\mathfrak{F} = \{f \,:\, f$ *is a mapping on an ordinal*$\}$.[23]

If $\mathcal{G}$ *is a class function on* $\mathfrak{F}$ *then there is a class function* $\mathcal{F}$ *on* On *such that* $\mathcal{F}(\alpha) = \mathcal{G}(\mathcal{F} \restriction \alpha)$ *holds for all* $\alpha \in \mathrm{On}$. [24] ❑

Using this special case of the Induction and Recursion Theorem, we can define von Neumann's cumulative hierarchy of sets:

Let

(3.10) ind-8

$$V_\alpha = \begin{cases} \emptyset, & \text{if } \alpha = 0; \\ \mathcal{P}(V_\beta), & \text{if } \alpha = \beta + 1 \text{ for some } \beta \in \mathrm{On}; \\ \bigcup_{\beta<\alpha} V_\beta, & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Valpha

Lemma 3.4 (1) *For all* $\alpha, \beta \in \mathrm{On}$ *with* $\alpha \leq \beta$, *we have* $V_\alpha \subseteq V_\beta$.

(2) *For all* $\alpha \in \mathrm{On}$, V_α *is transitive.*

(3) *For all* $\alpha \in \mathrm{On}$, *we have* $\mathrm{On} \cap V_\alpha = \alpha$.

(4) $V = \bigcup_{\alpha \in \mathrm{On}} V_\alpha$. *That is, for any set* x *there is* $\alpha \in \mathrm{On}$ *such that* $x \in V_\alpha$.

---

(22) The definiton of $\mathcal{F}$ may also contain some parameters.

(23) Also here, the definiton of $\mathcal{F}$ may also contain some parameters.

(24) Note that α is reconstructable from $\mathcal{F} \restriction \alpha$.

Proof. (1), (2) and (3) can be proved easily using Theorem 3.3, (1).

(4): Note that, by the Axiom of Regularity, we may apply Theorem 3.2, (1) for $\mathcal{X} = V$ and $\mathcal{R} = \in$. Thus, we have to show that, if $y \in V_\alpha$ for some $\alpha \in \mathrm{On}$ for all $y \in x$ then $x \in V_\alpha$ for some $\alpha \in \mathrm{On}$. For each $y \in x$, let $\alpha_y = \min\{\alpha \in \mathrm{On} : y \in V_\alpha\}$ and $\alpha^* = \sup\{\alpha_y : y \in x\}$. Then, by (1), we have $x \subseteq V_{\alpha^*}$. By (3.10), it follows that $x \in V_{\alpha^*+1}$. ❑ (Lemma 3.4)

For a set a, let

$$rank(a) = \min\{\alpha \in \mathrm{On} : a \in V_{\alpha+1}\}. \tag{3.11}$$ ind-9

By Lemma 3.4, (4) $rank(a)$ is defined for all sets a.

L-rank

Lemma 3.5 (1) *Fro any sets a and b, if $b \in a$ then $rank(b) < rank(a)$.*

(2) *For any set a we have $rank(a) = \sup\{rank(b) + 1 : b \in a\}$.*

Proof. (1): Since $a \in V_{rank(a)+1}$, we have $a \subseteq V_{rank(a)}$ by (3.10). Thus $b \in V_{rank(a)}$ and $rank(b) < rank(a)$.

(2): Let $\alpha = \sup\{rank(b)+1 : b \in a\}$. Then for all $b \in a$ we have $b \in V_{rank(b)+1} \subseteq V_\alpha$ by Lemma 3.4, (1). It follows that $a \subseteq V_\alpha$ and $a \in V_{\alpha+1}$ thus $rank(a) \leq \alpha$. On the other hand, if $\beta < \alpha$ then there is $b \in a$ such that $rank(b) \geq \beta$ Thus $a \notin V_{\beta+1}$ (If $a \in V_{\beta+1}$ we would have $a \subseteq V_\beta$ and hence $b \in V_\beta$. This is a contradiction to $rank(b) \geq \beta$). This shows that $rank(a) \geq \alpha$. Thus $rank(a) = \alpha = \sup\{rank(b) + 1 : b \in a\}$. ❑ (Lemma 3.5)

By Lemma 3.4, (3) and Lemma 3.5, we have $rank(\alpha) = \alpha$ for all $\alpha \in \mathrm{On}$.

For a regular cardinal κ, let

$$\mathcal{H}(\kappa) = \{a : |\,trcl(a)\,| < \kappa\}. \tag{3.12}$$ ind-10

Elements of $\mathcal{H}(\kappa)$ are said to be *hereditarily of cardinality* $< \kappa$.

Lemma 3.6 *For a regular cardinal κ, we have $\mathcal{H}(\kappa) \subseteq V_\kappa$. In particular $\mathcal{H}(\kappa)$ is a set.*

Proof. Suppose hat $a \in \mathcal{H}(\kappa)$. Then $trcl(a) \in \mathcal{H}(\kappa)$ as well. Now for a set a (which is not necessarily in $\mathcal{H}(\kappa)$) let

$$f_a : trcl(a) \to \mathrm{On};\ x \mapsto rank(x). \tag{3.13}$$ ind-11

Claim 3.6.1 *For any set a, we have $f_a{}''\,trcl^-(a) = rank(a)$.*

$\vdash$ By induction on $rank(a)$. Suppose that the claim holds for all b with $rank(b) < rank(a)$. Note that, in particular, all $b \in a$ satisfy the claim.

By Lemma 3.5, we have $rank(a) = \sup\{rank(b) + 1 : b \in a\}$.

If $\{rank(b) : b \in a\}$ has the largest element α^* then there is a $b^* \in a$ with $rank(b^*) = \alpha^*$ and $\sup\{rank(b) + 1 : b \in a\} = \alpha^* + 1$. Hence $a \subseteq V_{\alpha^*+1}$ and $rank(a) = \alpha^* + 1$.

By the induction hypothesis, $\alpha^* = f_{b^*}{}''\,trcl^-(b^*) = f_a{}''\,trcl^-(b^*) \subseteq f_a{}''\,trcl(a)$. and $\alpha^* = rank(b^*)$ is the largest element of $f_a{}''\,trcl^-(a)$. Thus we have $f_a{}''\,trcl^-(a) = \alpha^* + 1 = rank(a)$.

Suppose that $\{rank(b) : b \in a\}$ does not have the largest element. Then, by the induction hypothesis, we have $rank(a) = \sup\{rank(b)+1 : b \in a\} = \sup\{f_b{}''\,trcl(b) : b \in a\} = \sup\{f_a{}''\,trcl(b) : b \in a\} = f_a{}''\,trcl^-(a)$. $\dashv$ (Claim 3.6.1)

Now, for any $a \in \mathcal{H}(\kappa)$, we have $f_a{}''\,trcl^-(a) = rank(a)$ by Claim 3.6.1. Since $|\,trcl^-(a)\,| < \kappa$ it follows that $rank(a) < \kappa$. Thus $a \in V_\kappa$. ❑ (Lemma 3.6)

A class binary relation $\mathcal{R}$ on a class $\mathcal{X}$ is said to be *extensional* if, for any $a, b \in \mathcal{X}$, $\{c \in \mathcal{X} : c\,\mathcal{R}\,a\} = \{c \in \mathcal{X} : c\,\mathcal{R}\,b\}$ if and only if $a = b$.

Theorem 3.7 (Mostowski's Collapsing Lemma) *Suppose that $\mathcal{R}$ is a class binary relation on a class $\mathcal{X}$ which is set-like, extensional and well-founded. Then there are a transitive classss $\mathcal{M}$ with a class isomorphism $\Pi : (\mathcal{X}, \mathcal{R}) \overset{\cong}{\to} (\mathcal{M}, \in)$. $\mathcal{M}$ and Π as above are uniquely determined by $(\mathcal{X}, \mathcal{R})$.* T-class-mostowski

Proof. We define the class function $\Pi : \mathcal{X} \to V$ by letting

$$(3.14)\qquad \Pi(a) = \{\Pi(b) : b \in I_{\mathcal{R}}(a)\}$$ wf-9

for $a \in \mathcal{X}$.

In connection with Theorem 3.2, (2), we let $\mathfrak{F}$ as in Theorem 3.2, (2) for our $\mathcal{X}$ and the class function G on $\mathfrak{F} \times \mathcal{X}$ is defined by

$$(3.15)\qquad \mathcal{G}(f, a) = \begin{cases} \{f(b) : b\,\mathcal{R}\,a\}, & \text{if } \mathrm{dom}(f) = trcl^-_{\mathcal{R}}(s); \\ \emptyset, & \text{otherwise} \end{cases}$$ wf-10

for each $\langle f, a\rangle \in \mathfrak{F} \times \mathcal{X}$. Our Π can be taken as the unique $\mathcal{F}$ in Theorem 3.2, (2) for this G.

The uniqueness of Π follows from Theorem 3.2, (2). Π is injective by the extensionality of $\mathcal{R}$. Let $\mathcal{M} = \Pi''\mathcal{X}$. Then, by (3.14), it is easy to see that $\mathcal{M}$ is transitive and $\Pi : \langle \mathcal{X}, \mathcal{R}\rangle \overset{\cong}{\to} \langle \mathcal{M}, \in\rangle$ (Exercise). ❑ (Theorem 3.7)

3.2 Skolem hull and elementary submodels

skolem

The following is done inside the set theory (i.e. not on the level of meta-mathematics).

Let $\mathcal{L}$ be any language (of the first order logic) and let $\mathcal{L}^{\sqsubseteq}$ be the language obtained from $\mathcal{L}$ by adding a new binary relation symbol $\sqsubseteq$. For any $\mathcal{L}$-structure $\mathfrak{A}$ an expansion $\tilde{\mathfrak{A}}$ of $\mathcal{A}$ to an $\mathcal{L}^{\sqsubseteq}$ structure is said to be a well ordered expansion if $\sqsubseteq^{\tilde{\mathfrak{A}}}$ is a well-ordering on the underlying set A of $\mathfrak{A}$.

Suppose that $\tilde{\mathfrak{A}}$ is a well-ordered expansion of an $\mathcal{L}$-structure $\mathfrak{A}$ with underlying set A. For a subset X of A the Skolem hull $sk_{\tilde{\mathfrak{A}}}(X)$ (or simply $sk(X)$) of X is the closure of the set X with respect to all definable functions in $\tilde{\mathfrak{A}}$. By definition $sk(X)$ is closed with respect to interpretations (in $\mathfrak{A}$) of all function symbols in $\mathcal{L}$ and contains interpretations (in $\mathfrak{A}$) of all constant symbols in $\mathcal{L}$. Thus the restriction of $\mathfrak{A}$ on $sk(X)$ is an $\mathcal{L}$-substructure of $\mathfrak{A}$. We shall denote this substructure of $\mathfrak{A}$ also with $sk(X)$.

T-skolem

Lemma 3.8 *Suppose that $\mathfrak{A}$ is an $\mathcal{L}$-structure with underlying set A and $\tilde{\mathfrak{A}}$ a well-ordered expansion of $\mathfrak{A}$. Then, for any $X \subseteq A$, $sk_{\tilde{\mathfrak{A}}}(X)$ is an elementary substructure of $\mathfrak{A}$.*

Proof. We check that $sk(X)$ satisfies the condition of the Tarski-Vaught test.

Suppose that $\varphi = \varphi(x_0, x_1, ..., x_n)$ is an $\mathcal{L}$-forumla and $b_1^*, ..., b_n^* \in sk(X)$ be such that $\mathfrak{A} \models \exists x_0 \varphi(x_0, b_1^*, ..., b_n^*)$. We have to show that there is $a^* \in sk(X)$ such that $\mathfrak{A} \models \varphi(a^*, b_1^*, ..., b_n^*)$.

Let $f_\varphi : A^n \to A$ be the function defined by

$$f_\varphi(b_1, ..., b_n) = \begin{cases} \text{the } \sqsubseteq^{\tilde{\mathfrak{A}}}\text{-minimal } a \in A \text{ with } \mathfrak{A} \models \varphi(a, b_1, ..., b_n), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if such } a \text{ exists;} \\ \text{the } \sqsubseteq^{\tilde{\mathfrak{A}}}\text{-minimal element of } A, \qquad \text{otherwise.} \end{cases} \tag{3.16}$$

Since f_φ is definable in $\tilde{\mathfrak{A}}$, $sk(X)$ is closed with respect to f_φ. Thus, letting $a^* = f_\varphi(b_1, ..., b_n)$, we have $a^* \in sk(X)$ and $\mathfrak{A} \models \varphi(a^*, b_1^*, ..., b_n^*)$. ❑ (Lemma 3.8)

For a well ordered expansion $\mathfrak{A}^{\sqsubseteq}$ of an $\mathcal{L}$-structure $\mathfrak{A}$ with the underlying set A and $X \subseteq A$, we have $|\, sk(X) \,| \leq \max\{|\, X \,|, |\, \mathcal{L} \,|, \aleph_0\}$. In particular, for a countable $\mathcal{L}$ and infinite $X \subseteq A$, we have $|\, sk(X) \,| = |\, X \,|$. Thus we obtain:

Theorem 3.9 (Downward Löwenheim-Skolem Theorem) *For any $\mathcal{L}$-structure $\mathfrak{A}$ with the underlying set A and $X \subseteq A$ there is an elementary substructure $\mathfrak{B} = \langle B, ... \rangle$ of $\mathfrak{A}$ such that $X \subseteq B$ and $|\, B \,| \leq \max\{|\, \mathcal{L} \,|, |\, X \,|, \aleph_0\}$.* dwLoSko

In particular if $\mathcal{L}$ is countable, then for any $\mathcal{L}$-structure $\mathfrak{A} = \langle A, ... \rangle$ and infinite $X \subseteq A$ there is an elementary substructure $\mathfrak{B} = \langle B, ... \rangle$ of $\mathfrak{A}$ such that $X \subseteq B$ and $|\, A \,| = |\, X \,|$. ❑

3.3 Reflection and absoluteness over transitive models

refl-abs

A class $\mathcal{C}$ of ordinals is said to be *club* (*closed unbounded*) if

(3.17) for all limit $\alpha \in \mathrm{On}$ if $\mathcal{C} \cap \alpha$ is unbounded in α then $\alpha \in \mathcal{C}$ (closed), and club-1

(3.18) for all $\alpha \in \mathrm{On}$ there is $\beta \in \mathcal{C}$ with $\alpha < \beta$ (unbounded). club-2

By (3.18) all club $\mathcal{C} \subseteq \mathrm{On}$ are proper classes. The following is easy to prove:

T-refl-abs-1

Lemma 3.10 *If $\mathcal{C}$, $\mathcal{D} \subseteq \mathrm{On}$ are club then $\mathcal{C} \cap \mathcal{D}$ is also club.* ❑

Any set a can be considered as an $\mathcal{L}_\varepsilon$-structure $\langle a, \in \cap\, a^2\rangle$. This structure is simply denoted by $\langle a, \in\rangle$ or every by a. An $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(x_0, ..., x_{n-1})$ is said to be *absolute* over the $\mathcal{L}_\varepsilon$-structure a if for any $b_0, ..., b_{n-1} \in a$ we have $\langle a, \in\rangle \models \varphi(b_0, ..., b_{n-1})$ if and only if $\varphi(b_0, ..., b_{n-1})$ holds.

The formulas the next theorem mentions are meta-mahtemtical formulas and the theorem is actually a meta-theorem, that is, a collection of theorems for each formula φ.

Theorem 3.11 (Lévy) *For any $\mathcal{L}_\varepsilon$-forumla φ, there is a club $\mathcal{C}_\varphi \subseteq \mathrm{On}$ such that φ is absolute over V_α for each $\alpha \in \mathcal{C}_\varphi$.* T-levy

Proof. By induction on φ.

For atomic φ, we have $\mathcal{X}_\varphi = \mathrm{On} \setminus \{\emptyset\}$. Thus $\mathcal{C}_\varphi = \mathcal{X}_\varphi$ will do.

The case for $varphi = (\varphi_0 \wedge \varphi_1)$ is clear by Lemma 3.10. The case for $\varphi = \neg\varphi_0$ is trivial.

Thus it is enough show that the assertion of the theorem hols for any forumula φ where φ is of the form $\exists x\, \psi(x, x_0, ..., x_{n-1})$ and the assertion of the theorem holds for ψ.

By the assumption above there is a club $\mathcal{C}_\psi \subseteq \mathcal{X}_\psi$. Let

$$\mathcal{C}_\varphi = \{\alpha \in \mathcal{C}_\psi : \varphi \text{ is absolute over } V_\alpha\}. \tag{3.19}$$

Then we have $\mathcal{C}_\varphi = \mathcal{X}_\varphi \cap \mathcal{C}_\psi$. In particular $\mathcal{C}_\varphi \subseteq \mathcal{X}_\varphi$. Thus it is enough to show that this $\mathcal{C}_\varphi$ is a club.

Ydashed-closed

Claim 3.11.1 *$\mathcal{C}_\varphi$ is closed.*

$\vdash$ Suppose that $\delta \in \mathrm{Lim}$ and $\delta \cap \mathcal{C}_\varphi$ is cofinal in δ. We have to show $\delta \in \mathcal{C}_\varphi$.

By $\mathcal{C}_\varphi \subseteq \mathcal{C}_\psi$, $\delta \cap \mathcal{C}_\psi$ is also cofinal δ. Since $\mathcal{C}_\psi$ is club, it follows that $\delta \in \mathcal{C}_\psi$. For a_0, ..., $a_{n-1} \in V_\delta$, therei is an $\alpha \in \delta \cap \mathcal{C}_\varphi$ such that $a_0, ..., a_{n-1} \in V_\alpha$ since δ is a limit ordinal and $\delta \cap \mathcal{C}_\varphi$ is cofinal δ.

If $\varphi(a_0, ..., a_{n-1})$, then $V_\alpha \models \varphi(a_0, ..., a_{n-1})$ $\alpha \in \mathcal{X}_\varphi$. Thus there is an $a \in V_\alpha$ such that $V_\alpha \models \psi(a, a_0, ..., a_{n-1})$. As $\alpha \in \mathcal{C}_\psi$ we have $\psi(a, a_0, ..., a_{n-1})$. Since $\delta \in \mathcal{C}_\psi$ and $a, a_0, ..., a_{n-1} \in V_\delta$, it follows $V_\delta \models \psi(a, a_0, ..., a_{n-1})$ and hence $V_\delta \models \varphi(a_0, ..., a_{n-1})$.

On the other hand, if $V_\delta \models \varphi(a_0, ..., a_{n-1})$, then there is an $a \in V_\delta$ such that $V_\delta \models \psi(a, a_0, ..., a_{n-1})$. Since $\delta \in \mathcal{C}_\psi$, it follows that $\psi(a, a_0, ..., a_{n-1})$. Thus $\exists x \psi(x, a_0, ..., a_{n-1})$ or $\varphi(a_0, ..., a_{n-1})$ holds.

This shows $\delta \in \mathcal{C}_\varphi$. $\dashv$ (Claim 3.11.1)

Claim 3.11.2 *$\mathcal{C}_\varphi$ is unbounded.*

$\vdash$ For arbitrary $\alpha \in \mathrm{On}$, let α_i, $i \in \omega$ be an increasing sequence of ordinals such that

$$\alpha_0 = \min(\mathcal{C}_\psi \setminus \alpha + 1) \text{ and} \tag{3.20}$$ club-3

$$\alpha_{i+1} = \sup\big(\{\min\{\beta \in \mathcal{C}_\psi \setminus \alpha_i \,:\, V_\beta \models \varphi(a_0, ..., a_{n-1})\} \,:\, a_0, ..., a_{n-1} \in V_{\alpha_i}\} \cup \{\alpha_i\}\big) \tag{3.21}$$ club-4

where we let $\min(\emptyset) = 0$.

For $\delta = \sup(\{\alpha_i \,:\, i \in \omega\})$, we have

$$\alpha < \alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \le \delta \tag{3.22}$$

by (3.20) and (3.21). In particular $\alpha < \delta$.

We show $\delta \in \mathcal{C}_\varphi$.

$\alpha_i \in \mathcal{C}_\psi$, $i \in \omega$ by (3.20) and (3.21). Since $\mathcal{C}_\psi$ is club, it follows that $\delta \in \mathcal{C}_\psi$. For a_0, ..., $a_{n-1} \in V_\delta$, there is $i \in \omega$ such that $a_0, ..., a_{n-1} \in V_{\alpha_i}$.

If $\varphi(a_0, ..., a_{n-1})$ holds then there is an a such that $\psi(a, a_0, ..., a_{n-1})$ and thus there is $\beta \in \mathcal{C}_\psi$ such that $a, a_0, ..., a_{n-1} \in V_\beta$. Let a^*, β^* be such a pair a, β such that β^* is minimal among the β's. By (3.21) we have $\beta^* \le \alpha_{i+1} \le \delta$. Hence $a^* \in V_\delta$. By $\delta \in \mathcal{C}_\psi$ we have $V_\delta \models \psi(a^*, a_0, ..., a_{n-1})$. It follows $V_\delta \models \exists x \psi(x, a_0, ..., a_{n-1})$, that is, $V_\delta \models \varphi(a_0, ..., a_{n-1})$.

If we have $V_\delta \models \varphi(a_0, ..., a_{n-1})$ on the other hand, then we can take an $a \in V_\delta$ such that $V_\delta \models \psi(a, a_0, ..., a_{n-1})$. Since $\delta \in \mathcal{C}_\psi$, it follows that $\psi(a, a_0, ..., a_{n-1})$ holds. Hence we have $\exists x\, \psi(x, a_0, ..., a_{n-1})$, that is, $\varphi(a_0, ..., a_{n-1})$. ⊣ (Claim 3.11.2)

❑ (Theorem 3.11)

The "finite fragment of ZFC" in the following corollary is in the sense of meta-mathematics. The corollary is thus a theorem in ZFC for each fixed (concretely given) finite fragment of ZFC.

ctbl-trans-model

Corollary 3.12 *For any finite fragment T of* ZFC *there is a countably transitive model M of T.*

Proof. Let φ be the $\mathcal{L}_\varepsilon$-sentence $\bigwedge\!\!\!\bigwedge T$ (the conjunction of all formulas in T). By Theorem 3.11, there is an $\alpha \in \mathrm{On}$ such that $V_\alpha \models \varphi$. By Theorem 3.9 there is a countable elementary submodel M of V_α. We have $M \models \varphi$. By Theorem 3.7 there is a transitive N with $M \sim N$. This N is as desired. ❑ (Corollary 3.12)

4 Forcing relation

forcing-rel

5 Infinitary combinatorics

inf-comb

6 Some further application of simple forcing constructions

appl