Linear Algebra I Final Exam

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$\boxed{1}$ Let $A = \begin{bmatrix} 1 & 0 & -3 \\ -3 & 1 & 6 \\ 2 & -2 & -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$.

(a) Find $det\,A$

(b) Find A^{-1}.

(c) Solve the equation $A\mathbf{x} = \mathbf{b}$.

(a):

$det A = 1\times 1\times(-1)+0\times 6\times 2+(-3)\times(-3)\times(-2)-((-3)\times 1\times 2+1\times 6\times(-2)+0\times(-3)\times(-1)) = -1$

(b):

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ -3 & 1 & 6 & 0 & 1 & 0 \\ 2 & -2 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & -2 & 5 & -2 & 0 & 1 \end{array}\right] \quad \begin{array}{l} 2.row + 3\times 1.row \\ 3.row - 2\times 1.row \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 0 & -1 & 4 & 2 & 1 \end{array}\right] \quad 3.row + 2\times 2.row$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 3 & 1 & 0 \\ 0 & 0 & 1 & -4 & -2 & -1 \end{array}\right] \quad (-1)\times 3.row$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & -6 & -3 \\ 0 & 1 & 0 & -9 & -5 & -3 \\ 0 & 0 & 1 & -4 & -2 & -1 \end{array}\right] \quad \begin{array}{l} 1.row + 3\times 3.row \\ 2.row + 3\times 3.row \end{array}$$

Thus we have: $A^{-1} = \begin{bmatrix} -11 & -6 & -3 \\ -9 & -5 & -3 \\ -4 & -2 & -1 \end{bmatrix}$.

(c): $A\mathbf{x} = \mathbf{b} \Leftrightarrow \mathbf{x} = A^{-1}\mathbf{b}$. **Thus**

$$\mathbf{x} = \begin{bmatrix} -11 & -6 & -3 \\ -9 & -5 & -3 \\ -4 & -2 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} = \quad \cdots$$

$\boxed{2}$ Let $A = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ where b represents a number.

(a) Decide for which b the matrix A is invertible.

(b) Find A^{-1} for such b that A is invertible.

(c) Decide for which b the equation $A\mathbf{x} = \mathbf{0}$ has a non trivial solution.

(a):

A **is invertible** $\Leftrightarrow det\,A \neq 0 \Leftrightarrow 1\times 1\times 1 - b\times 1\times 1 \neq 0 \Leftrightarrow b \neq 1$

(b):

$$\left[\begin{array}{ccc|ccc} 1 & 0 & b & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & b & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1-b & -1 & 0 & 1 \end{array}\right] \quad 3.row - 1.row$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -\dfrac{b}{1-b} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & \dfrac{1}{1-b} & 0 & \dfrac{1}{1-b} \end{array}\right] \quad \begin{array}{l} 1.row - \dfrac{b}{1-b}\times 3.row \\ \dfrac{1}{1-b}\times 3.row \end{array}$$

Thus

$$A^{-1} = \begin{bmatrix} 1 & 0 & -\dfrac{b}{1-b} \\ 0 & 1 & 0 \\ \dfrac{1}{1-b} & 0 & \dfrac{1}{1-b} \end{bmatrix}$$

(c): $A\mathbf{x} = \mathbf{0}$ **has at least one non trivial solution** $\Leftrightarrow$ A **is not invertible** $\Leftrightarrow$ $b = 1$

$\boxed{3}$ Find the standard matrices of the following linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$:

(a) Reflection through the x_2-axis.

(b) Reflection through the origin.

(c) Horizontal expansion by factor k (i.e. $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} kx \\ y \end{bmatrix}$)

(a): $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **(b):** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **(c):** $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$

$\boxed{4}$ Find the standard matrix of the linear transformation

$$\varphi : \mathbb{R}^2 \to \mathbb{R}^3; \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix}.$$

Since $\varphi(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$ **and** $\varphi(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}$**, the standart matrix of** φ **is** $M_\varphi = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix}$**.**

$\boxed{5}$ Show that, for an $n \times n$ matrix A, 0 is one of its eigen values if and only if A is not invertible.

0 **is an eigen value of** $A \Leftrightarrow A\mathbf{x} = 0\mathbf{x}$ **(i.e.** $A\mathbf{x} = \mathbf{0}$**) has a non-trivial solution** $\Leftrightarrow$ A **is not invertible.**

$\boxed{6}$ Suppose that λ_1 and λ_2 are two distinct eingen values of a matrix A and $\mathbf{u}_1$ and $\mathbf{u}_2$ are eigen vectors corresponding to them respectively. Show that $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent.

We have $A\mathbf{u}_1 = \lambda_1\mathbf{u}_1$ **and** $A\mathbf{u}_2 = \lambda_2\mathbf{u}_2$**.**
Suppose that

(1) $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 = \mathbf{0}$

for $\alpha_1, \alpha_2 \in \mathbb{R}$**.**
We want to show that $\alpha_1 = \alpha_2 = 0$**.**
By applying A **on both sides of (1) we obtain**

(2) $\alpha_1\lambda_1\mathbf{u}_1 + \alpha_2\lambda_2\mathbf{u}_2 = A(\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2) = \mathbf{0}$**.**

By subtracting $\lambda_1 \times$ **(1) from (2) we obtain**

(3) $\alpha_2(\lambda_2 - \lambda_1)\mathbf{u}_2 = \mathbf{0}$**.**

Since $\mathbf{u}_2 \neq \mathbf{0}$ **(**$\mathbf{u}_2$ **was an eigen vector) and** $\lambda_2 - \lambda_1 \neq 0$ **by assumption, it follows that** $\alpha_2 = 0$**. By substituting** $\alpha_2 = 0$ **in (1) (and noting that** $\mathbf{u}_1 \neq \mathbf{0}$**) we also obtain** $\alpha_1 = 0$**. Thus we have** $\alpha_1 = \alpha_2 = 0$ **as desired.**