Axiomatic set theory and the foundation of mathematics

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Contents

1 Naïve axiomatic set theory

naive

In the following, The axiom system ZFC (Zermelo-Fraenkel Set Theory with Axiom of Choice) of set theory is formulated with certain redundancy. This redundant formulation

(0) The present text is a lecture note of the course I gave in the summer semester 2019 at Kobe university. The text is further extended and worked-out when I reused it in a course in Katowice in October 2019.

The most up-to-date version of this text is downloadable as:

https://fuchino.ddo.jp/kobe/logic-ss2019.pdf

At present, this text is still a work in progress. It will be constantly updated during and after the courses in 2019. Any comments and/or questions in connection with this text are appreciated, and to be sent to the email address above.

is chosen on purpose so that some important subtheories of ZFC become subsystem of the axiom system ZFC.

The axioms of ZFC are built on the single predicate "$\in$" where "$x \in y$" means x belongs to y as an element.

We do not introduce a predicate for expressing "x is a set" since everything is a set in set theory.[(1)] If we say "for all x ..." in an axiom of set theory, what is meant is "for all set x, ...".

1.1 Zermelo Set Theory and the Axiom of Choice

zermelo

The following axioms make the axiom system of Zermelo Set Theory (Z) which is a subset of the axiom system of ZFC introduced in the next subsection.

(Extensionality) If we have $u \in x$ if and only if $u \in y$ for any u, then $x = y$.

(Empty Set) There is x such that $u \notin x$ for any u.

By the Axiom of Extensionality, the set x as in the Axiom of Empty Set exists uniquely. We denote this unique set by $\emptyset$.

(Pairing Axiom) For x and y there is z such that, for any u, $u \in z$ if and only if $u = x$ or $u = y$.

Again by the Axiom of Extensionality, the set z as in the Pairing Axiom for each x and y is unique. We shall denote this unieque z up to x and y by $\{x, y\}$. If $x = y$, we write $\{x, x\} = \{x\}$ and call it the *singleton* x.

(Axiom of Union) For any x, there is y such that, for any u, $u \in y$ if and only if $u \in z$ for some $z \in x$ (notation: $y = \bigcup x$).

The next axiom is actually an axiom scheme in the sense that, for each property Φ, an instance of the axiom is made.

(Axiom of Separation) If $\Phi(\cdot)$ is a property expressed by using only "$\in$" and "$=$" then for any x there is y such that $u \in y$ if and only if $u \in x$ and $\Phi(u)$ holds.

The set y in the Axiom of Separation is often denoted by $y = \{u \in x : \Phi(u)\}$.

By Pairing axiom and Axiom of Union, we can construct $\bigcup\{x, y\}$ for any x and y. This set is denoted by $x \cup y$. Clearly we have $u \in x \cup y$ if and only if $u \in x$ or $u \in y$ for any u.

(Axiom of Infinity) There is x such that

(1.1) $\emptyset \in x$ and $y \cup \{y\} \in x$ for any any $y \in x$.[(2)] naive-a

(Power-Set Axiom) For any x, there is y such that, for any u, $u \in y$ if and only if all elements of u are elements of x.

The set y in the Power-set Axiom is denoted by $y = \mathcal{P}(x)$ and y is said to be a it power-set of x. The relation "$u \in y$ then $u \in x$ for all u" is denoted by $y \subseteq x$ and y is said to be a subset of x. With this notation, we can write $\mathcal{P}(x) = \{y : y \subseteq x\}$. The construction of power-set seen in this way is outside the scope of the Axiom of Separation since a set which would delimit the range of y in $\{y : y \subseteq x\}$ (for example $\mathcal{P}(x)$) can only be obtained using the Power-Set Axiom.

Ordered pair of sets a and b can be treated as the following set.

(1.2) $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$. naive-0

. This definition is a reasonable one beacuse of:

L-naive-0

Lemma 1.1 *For any a, b and a', b' $\langle a, b\rangle = \langle a', b'\rangle$ if and only if $a = a'$ and $b = b'$.* ❑

L-naive-1

Lemma 1.2 *For any a and b the is c such that, for any u, $u \in c$ if and only if there are $v \in a$ and $w \in b$ such that $u = \langle v, w\rangle$.*

Proof. Recalling (1.2), such c can be introduced as

(1.3) $\{u \in \mathcal{P}(\mathcal{P}(a \cup b)) : u = \langle v, w\rangle$ for some $v \in a$ and $w \in b\}$. naive-1

There is such a set because of the Axiom of Power-set and the Axiom of Separation.

❑ (Lemma 1.2)

For a, b, the set c as in Lemma 1.2 is uniquely determined. We call such c a (Cartesian) product of a and b, and denote it by $a \times b$.

For a, and b, $f \subseteq a \times b$ is said to be a *function from a to b*, if, for any $v \in a$, there is a unique $w \in b$ such that $\langle v, w\rangle \in f$ (notation: $f : a \to b$). For $v \in a$ the unique $w \in b$ with $\langle v, w\rangle \in f$ is denoted by $f(v) = w$ and called the value of f at v.

(Axiom of Choice) For any x such that $\emptyset \notin x$, there is a function $f : x \to \bigcup x$ such that $f(u) \in u$ for all $u \in x$.

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(1) That is, except proper classes which are just reformulations of properties of sets and hence not objects in set theory.

(2) Not that the set $y \cup \{y\}$ is the set of all elements of y and y itself. In [Zermelo 1908], the Axiom of Infinity was formulated with (1.1) replaced by "$\emptyset \in x$ and $\{y\} \in x$ for any $y \in x$". The original form of the Axiom of Infinity is equivalent to our Axiom of Infinity in ZF but the equivalence is not provable in the Zermelo Set Theory Z. The formulation of the Axiom of Infinite with (1.1) is choosen here to make the collecgtion of all natural numbers ω a set under the standard definition of natural numers due to von Neumann possible in Z.

Axiom of Choice is abbreviated as AC. The axiom system obtained by adding the Axiom of Choice to the Zermelo Set Theory Z is denoted by ZC and called the Zermelo Set Theory with AC.

1.2 Zermelo-Fraenlel Set Theory

zermelo-fraenkel

The axiom system of Zermelo-Fraenlel Set Theory (ZF) is obtained by adding the following two axioms to the axiom systemo of Zermelo Set Theory.

The first one of the two additional axioms is actually an axiom scheme like Axiom of Separation:

(Replacement) For any x, if $\Phi(\cdot,\cdot)$ is a property expressed by using only "$\in$" and "$=$" such that, for any $u \in x$ there is the unique v such that $\Phi(u,v)$ holds, then there is y such that $v \in y$ if and only if $\Phi(u,v)$ for some $u \in x$ (notation: $y = \{v : \Phi(u,v) \text{ for some } u \in x\}$).

The significance of the next axiom of ZF becomes clear in connection with the theory of the transfinite induction.

(Foundation) For any non empty x, there is $y \in x$ such that there is no $y' \in x$ with $y' \in y$.

2 Predicate Logic

pred-logic

2.1 Predicate logic in meta-mathematics

meta

The naïve approach to the axiomatization of set theory we saw in the previous section was inacurate in many respects. One of the main problem was that we could not specify what are "set theoretic properties" in the definition of the Axiom of Separation and Axiom of Replacement.

We want to re-introduce the axioms of set theory on the basis of the formal logic which will be introduced below. When we introduce the logic, we do it before introducing the axionms of set theory. We are in a world outside set theory. This means in particular that, at this stage, we cannot use the fictive notion of infinity, like the set of all natural numbers. If we write nevertheless some thing like "For $n \in \mathbb{N}$, ...", then this is merely abbreviation of the statement "For a concretely given expression (numeral) $\underline{n}$ of a natural number, ...".

The mathematics available at this stage must be the arguments on finite concrete objects, like symbols and finite sequences of symbols, with concrete operations, like basic operations on digits, operations on finite sequences (concatenation of sequences, for example), etc. This standpoint is often called *meta-mathematics* from which most the following arguments in this section are done. In summary, the metha-mathematics is the world of finite sequences of symbols on an imaginary paper.[3]

The notion of meta-mathematics was introduced by David Hilbert and he used the word „finitär" to describe the concrete operations on concrete objects. The word is then translated to the invented word "finitary" in English. This word "finitary" is very often confused with the word "finite". All the objects treated in the finitary satnd point are at most potentially infinite and we do not assume the existence of infinite sets like the set of all natural numbers.

In meta-mathematics, we try to establish general facts about concretely given objects, like a concretely given expressions for numbers, say 1, 2457, etc., or a concretely given sequence of symbols "abc", "$x \equiv y$" etc., and concrete operations on these object. An infinite collection can be also concretely given. For example, an infinite "subset" S of $\mathbb{N}$ is concretely given if an algorithm is given according to which we can decide whether an arbitrary concretely given number n belongs to S.

Thus, if we begin an argument declaring that something is "concretely given", this signalizes that the following discussion is done in meta-mathematics and we are talking about concrete operations on the object which is executable (at least theoretically[4]) once it is really concretely given. Actually, we are going to write this expression "something be concretely given" many times in the following, since we transfer back and forth between meta-mathematics and the mathematics inside a formal system, and since it would be awkward to say a phrase like "now we work again in meta-mathematics" each time when we return to a discussion in meta-mathematics.

To define our formal system of predicate logic, which we call K^*,[5], we first set the collection of symbols which are to be used in the system.

The following symbols are mandatory for all instances of the formal theories:

(2.1) variable symbols: 'x_0', 'x_1', 'x_2',... [6] meta-a-0

(2.2) logical symbols: '$\rightarrow$', '$\neg$', '$\exists$'

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[3] We say here an "imaginary paper" since we do not place any restriction on the size of the paper or on the physical feasibility of the operations done on the paper: The paper can be so large that we can write more symbols than the number of the elementary particles in the whole universe and operations undertaken on the symolbs written on the paper may be repeated more than the times the fastest CPU clock will tick from the begin of the universe till now.

[4] That is, if we ignore the physical limitations on time and resource for the execution of the manipulation.

[5] The choice of the name K^* for the system we are introducing in the following is quite arbitrary except that the letter K is chosen because of the German expression „logischer Kalkül" which means a logical deduction system.

[6] We assume that all variables are formally of the form "x_i". Although, for readability, we shall also use some other letters like 'y', 'z', 'y'' 'z'' etc. to represent variables.

(2.3) equality symbol: '$\equiv$'

(2.4) auxiliary symbols: '(', ')', ',' meta-a-1

Some of the following symbols are used depending on the instance of the formal theory:

(2.5) constant symbols: 'c_0', 'c_1', 'c_2' ,...

(2.6) function symbols: 'f_0', 'f_1', 'f_2' ,...

(2.7) relation symbols: 'r_0', 'r_1', 'r_2' ,... (7)

We assume that all the symbols we introduced are distinct from each other.

To be more concrete, and to make the translation of notions of meta-mathematics into set theory smoother, we assume here that all these symbols are pairs of numerals. To be concrete, let us assume that

(2.8) the variable symbols 'x_0', 'x_1', 'x_2',... are the pairs $\langle \underline{0}, \underline{0} \rangle$, $\langle \underline{0}, \underline{1} \rangle$, $\langle \underline{0}, \underline{2} \rangle$,... resp.,

(2.9) the logical symbols '$\rightarrow$', '$\neg$', '$\exists$' are the pairs $\langle \underline{1}, \underline{0} \rangle$, $\langle \underline{1}, \underline{1} \rangle$, $\langle \underline{1}, \underline{2} \rangle$ resp.,

(2.10) the equality symbol '$\equiv$' is the pair $\langle \underline{2}, \underline{0} \rangle$,

(2.11) the auxiliary symbols '(', ')', ',' are pairs $\langle \underline{3}, \underline{0} \rangle$, $\langle \underline{3}, \underline{1} \rangle$, $\langle \underline{3}, \underline{2} \rangle$ resp.,

(2.12) the constant sysmbols 'c_0', 'c_1', 'c_2' ,... are the pairs $\langle \underline{4}, \underline{0} \rangle$, $\langle \underline{4}, \underline{1} \rangle$, $\langle \underline{4}, \underline{2} \rangle$,... resp.,

(2.13) the function sysmbols 'f_0', 'f_1', 'f_2' ,... are the pairs $\langle \underline{5}, \underline{0} \rangle$, $\langle \underline{5}, \underline{1} \rangle$, $\langle \underline{5}, \underline{2} \rangle$,... resp.,

and

(2.14) the relation sysmbols 'r_0', 'r_1', 'r_2' ,... are the pairs $\langle \underline{6}, \underline{0} \rangle$, $\langle \underline{6}, \underline{1} \rangle$, $\langle \underline{6}, \underline{2} \rangle$,... resp.

We then fix a computable function $ar : \mathbb{N} \to \mathbb{N} \setminus \{0\}$ such that, for each $k \in \mathbb{N} \setminus \{0\}$, there are infinitely many $n \in \mathbb{N}$ such that $ar(n) = k$ and we assume the function sysmbol f_n (i.e. the pair $\langle \underline{5}, \underline{n} \rangle$) and the relation symbol r_n (i.e. the pair $\langle \underline{6}, \underline{n} \rangle$) have the arity $ar(n)$.

A collection $\mathcal{L}$ of constant, function, and/or relation symbols is called a *language*. We fix a language for each theory we would like to develop over the logical system K^*. To specify a language, we often fix some finite index set I, J, $K \subseteq \mathbb{N}$ and write

(2.15) $\mathcal{L} = \{c_i, f_j, r_k\}_{i \in I, j \in J, k \in K}$. meta-0

(7) We assume that a fixed arity (the number of the parameters the symbol accept) is attached to each of the function and relation symbols. We assume that we have unbounded number of function and relation symbols in stock for each given arity.

Similarly to variables, these symbols are officially fixed as written here. In spite of this strict setting we often call some of these symbols differently to make the formal expressions more connected to the conventional notation. For example the binary relation symbol in the language of set theory which should represent the element relation is written as ε instead of r_k for some $k \in \mathbb{N}$.

Some/all of the index sets I, J, K may be empty.

For example, in case of Zermelo-Fraenkel set theory in the non-naïve axiomatization, which will be introduced in the next section, we use a language with a single binary relation symbol which should represent the element relation. This relation symbol (r_n for some n with $ar(n) = 2$) is denoted as ε and the language of the (Zermelo-Fraenkel) set theory is denoted by $\mathcal{L}_\varepsilon$. Thus $\mathcal{L}_\varepsilon = \{\varepsilon\}$.

For a given language $\mathcal{L}$, let $Seq_\mathcal{L}$ be the "set" of all finite sequences of symbols in (2.1) $\sim$ (2.4) as well as symbols in $\mathcal{L}$.[8] For a sequence $s \in Seq_\mathcal{L}$, $\ell n(s)$ denotes the length of the sequence.

We define $\mathcal{L}$-terms $\subseteq Seq_\mathcal{L}$ recursively as follows:

(2.16) If x is a variable symbol, then the sequence of length 1 consisting of x is an $\mathcal{L}$-term;[9] meta-1

(2.17) If c is a constant symbol in $\mathcal{L}$ then the sequence of length 1 consisting of c is an $\mathcal{L}$-term;[10] meta-2

(2.18) If f is an k-ary function symbol in $\mathcal{L}$ and $t_0, \ldots, t_{k-1}$ are $\mathcal{L}$-terms, then $f(t_0, \ldots, t_{k-1})$ is also an $\mathcal{L}$-term;[11] meta-3

(2.19) Nothing else. meta-4

In case of the language $\mathcal{L}_\varepsilon$ of set theory, since it does not have any constant or function symbol, $\mathcal{L}_\varepsilon$-terms are just sequences of length 1 consisting of a variable symbol.

For an $\mathcal{L}$-term t, $free(t)$ denotes the set of all variable symbols which appear in t. If $free(t) \subseteq \{v_0, \ldots, v_{k-1}\}$, for some variable symbols $v_0, \ldots, v_{k-1}$ (each of these symbols is of the form x_n for some natural number n), we write $t = t(v_0, \ldots, v_{k-1})$. Note that, by this notation, there may be some variables in the list $v_0, \ldots, v_{k-1}$ which do not appear at all in t. Such variables are sometimes called "dummy variables". The possibility of dummy variables is included here on purpose to enable some details of induction proofs easier to formulate.

For a language $\mathcal{L}$, the $\mathcal{L}$-formulas $\subseteq Seq_\mathcal{L}$ are now defined recursively as follows:

(2.20) If t_0, t_1 are $\mathcal{L}$-terms then $t_0 \equiv t_1$ is an $\mathcal{L}$-formula;[12] meta-5

(2.21) If $t_0, \ldots, t_{k-1}$ are $\mathcal{L}$-terms and r is a k-ary relation symbol in $\mathcal{L}$, then $r(t_0, \ldots,$ meta-6

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[8] We treat the symbol $Seq_\mathcal{L}$ similarly to $\mathbb{N}$ above.

[9] Sometimes we write simply x to denote this sequence "x".

[10] Also we write simply c to denote this sequence.

[11] With "$f(t_0, \ldots, t_{k-1})$", we denote the sequence obtained by concatenating all of these symbols and sequences in this order. The same convention will be used hereafter without mention.

[12] Similarly to the footnote (11), we mean with $t_0 \equiv t_1$ the concatenation of sequence t_0, symbol $\equiv$ and sequence t_1. We simply drop further notices of similar kind (e.g. in (2.21), (2.22), etc).

$t_{k-1})$ is an $\mathcal{L}$-formula;[13]

(2.22) If φ_0 and φ_1 are $\mathcal{L}$-formulas then $(\varphi_0 \to \varphi_1)$ is also an $\mathcal{L}$-formula; meta-7

(2.23) If φ_0 is an $\mathcal{L}$-formula, then $\neg\varphi_0$ is also an $\mathcal{L}$-formula; meta-8

(2.24) If φ_0 is an $\mathcal{L}$-formula and x a variable symbol, then $\exists x\, \varphi_0$ is also an $\mathcal{L}$-formula; meta-9

(2.25) Nothing else. meta-10

$\mathcal{L}$-formulas of the form (2.20) or (2.21) are called *atomic formulas*. A variable symbol x in an $\mathcal{L}$-formula φ is said to be free if it is not in the scope of a quantification "$\exists x$" in φ (see for exact definition below). $\mathcal{L}$-formula φ without free variables is called an $\mathcal{L}$-*sentence*. We shall define the notion of subformula and free variable more precisely below.

The intended reading of these formulas is as follows:

(2.26) meta-16

$t_0 \equiv t_1$:	"t_0 and t_1 are identical".
$r(t_0, \ldots, t_{k-1})$:	"$t_0, \ldots, t_{k-1}$ stand in the relation r".
$(\varphi_0 \to \varphi_1)$:	"φ_0 implies φ_1".
$\neg\varphi_0$:	"φ_0 does not hold".
$\exists x\, \varphi_0$:	"there exists an x such that φ_0 holds".

For readability, we often use the following abbreviations:

(2.27) meta-17

abbreviation	actual sequence; explanations
$t_0 \not\equiv t_1$	$\neg t_0 \equiv t_1$
$t_0\ f\ t_1$	$f(t_0, t_1)$; for some binary function symbols f, like "$t_0 + t_1$", "$t_0 \cdot t_1$", etc.
$t_0\ r\ t_1$	$r(t_0, t_1)$; for some binary relation symbols r, like "$t_0\ \varepsilon\ t_1$", "$t_0 < t_1$", etc.
$t_0 \not r\ t_1$	$\neg r(t_0, t_1)$; for some binary relation symbols r, like "$t_0 \not\varepsilon\ t_1$", "$t_0 \not< t_1$", etc.
$(\varphi_0 \vee \varphi_1)$	$(\neg\varphi_0 \to \varphi_1)$
$(\varphi_0 \wedge \varphi_1)$	$\neg(\varphi_0 \to \neg\varphi_1)$
$(\varphi_0 \leftrightarrow \varphi_1)$	"$((\varphi_0 \to \varphi_1) \wedge (\varphi_1 \to \varphi_0))$"; "$\wedge$" in the expression must be further resolved.
$\forall x\, \varphi_0$	$\neg\exists x \neg\varphi_0$
$\exists x!\, \varphi_0$	"$(\exists x\, \varphi_0 \wedge \forall x_0 \forall x_1\, ((\varphi_0(x_0/x) \wedge \varphi_0(x_1/x)) \to x_0 \equiv x_1))$"; x_0 and x_1 are variables not appearing in φ_0, "$\forall$" and "$\wedge$" in the expression must be further resolved.

[13] For binary relation symbol r we often write $t_0\ r\ t_1$ instead of $r(t_0, t_1)$. In particular, in case of $\mathcal{L}_\varepsilon$, we write $x\ \varepsilon\ y$ instead of $\varepsilon(x, y)$ see (2.27) for similar shorthands.

According to the intended interpretation of $\mathcal{L}$-formulas (2.26), these abbreviations can be read as follows:

(2.28) $t_0 \not\equiv t_1$: t_0 and t_1 are not identical. meta-18
$t_0 \not r\, t_1$: $r(t_0, t_1)$ does not hold.
$(\varphi_0 \vee \varphi_1)$: at least one of φ_0 and φ_1 holds.
$(\varphi_0 \wedge \varphi_1)$: both of φ_0 and φ_1 hold.
$(\varphi_0 \leftrightarrow \varphi_1)$: φ_0 is equivalent to φ_1.
$\forall x\, \varphi_0$: φ_0 holds for all x.
$\exists x!\, \varphi_0$: there exists exactly one x satisfying φ_0.

For an $\mathcal{L}$-formula φ with $\ell n(\varphi) = \ell$, let s_i denotes the ith symbol in the sequence φ for $i < \ell$. For $i < j \leq \ell$ Let $\varphi_{\langle i,j\rangle}$ be the subsequence "$s_i s_{i+1} \cdots s_{j-1}$" of φ. We define the set $Sub_\varphi \subseteq \{\langle i,j\rangle : i < j \leq \ell\}$ of the positions of subformulas recursively as follows:

(2.29) If φ is an atomic formula and $\ell = \ell n(\varphi)$, then $Sub_\varphi = \{\langle 0, \ell\rangle\}$; meta-11

(2.30) If φ is $(\varphi_0 \rightarrow \varphi_1)$ for $\mathcal{L}$-formulas φ_0, φ_1, $\ell = \ell n(\varphi)$, $\ell_0 = \ell n(\varphi_0)$ then $Sub_\varphi = \{\langle i+1, j+1\rangle : \langle i,j\rangle \in Sub_{\varphi_0}\} \cup \{\langle i+\ell_0+2, j+\ell_0+2\rangle : \langle i,j\rangle \in Sub_{\varphi_1}\} \cup \{\langle 0,\ell\rangle\}$; meta-12

(2.31) If φ is $\neg\varphi_0$ and $\ell = \ell n(\varphi)$, then $Sub_\varphi = \{\langle i+1, j+1\rangle : \langle i,j\rangle \in Sub_{\varphi_0}\} \cup \{\langle 0,\ell\rangle\}$; meta-13

(2.32) If φ is $\exists x\, \varphi_0$ and $\ell = \ell n(\varphi)$, then $Sub_\varphi = \{\langle i_2, j+2\rangle : \langle i,j\rangle \in Sub_{\varphi_0}\} \cup \{\langle 0,\ell\rangle\}$. meta-14

Suppose that x is a variable symbol and it is a kth symbol of an $\mathcal{L}$-formula φ. Then we say that x appears freely in φ as the kth symbol of φ if there is no $\langle i,j\rangle \in Sub_\varphi$ such that $i \leq k < j$, and $\varphi_{\langle i,j\rangle}$ is of the form $\exists x\, \psi$. For an $\mathcal{L}$-formula φ with $\ell = \ell n(\varphi)$, let

(2.33) $free(\varphi) = \{x : x$ is a variable symbol and x appears in φ freely as kth symbol of φ for some $k < \ell\}$. meta-15

For an $\mathcal{L}$-formula φ, if $free(\varphi) \subseteq \{v_0, ..., v_{k-1}\}$, for some variable symbols $v_0, ..., v_{k_1}$, we write $\varphi = \varphi(v_0, ..., v_{k-1})$. $\mathcal{L}$-formula φ with $free(\varphi) = \emptyset$ is called an *$\mathcal{L}$-sentence*.

2.2 Axiomatic set theory over the predicate logic

axiomatic-set-theory

Let $\mathcal{L}_\varepsilon$ be the language consisting solely of a single binary relation symbol ε. We can reformulate the axioms introduced in vernacular language of mathematics in Section 1 as a collection of $\mathcal{L}_\varepsilon$-sentences.

The following $\mathcal{L}_\varepsilon$-sentences make up the axiom system of Zermelo set theory. We say the axiom system, since we have to take infinitely many $\mathcal{L}_\varepsilon$-sentences as axioms of the Zermelo set theory.

In the following, we shall use freely the abbreviations introduced in (2.27) as well as other abbreviations which will be introduced below.

(Extensionality) $\forall x \forall y\,(\forall z\,(z \in x \leftrightarrow z \in y) \to x \equiv y)$.[14]

(Empty Set) $\exists x \forall y\, y \notin x$.

In the following we use the expressions

$$\text{(2.34)}\quad \emptyset \,\varepsilon\, x, \quad x \,\varepsilon\, \emptyset, \qquad x \equiv \emptyset, \quad \emptyset \equiv x \qquad \text{zf-0}$$

as abbreviations of

$$\text{(2.35)}\quad \exists y\,(\forall z\, z \notin y \wedge y \,\varepsilon\, x), \quad \exists y\,(\forall z\, z \notin y \wedge x \,\varepsilon\, y),^{(15)} \qquad \exists y\,(\forall z\, z \notin y \wedge x \equiv y), \quad \exists y\,(\forall z\, z \notin y \wedge y \equiv x) \qquad \text{zf-1}$$

respectively.

By the Axiom of Extensionality, these formulas are equivalent to

$$\text{(2.36)}\quad \exists! y\,(\forall z\, z \notin y \wedge y \,\varepsilon\, x), \quad \exists! y\,(\forall z\, z \notin y \wedge x \,\varepsilon\, y), \qquad \exists! y\,(\forall z\, z \notin y \wedge x \equiv y), \quad \exists! y\,(\forall z\, z \notin y \wedge y \equiv x), \qquad \text{zf-2}$$

respectively[16] .

By the Axiom of Empty-set, these formulas are equivalent to

$$\text{(2.37)}\quad \forall y\,(\forall z\, z \notin y \to y \,\varepsilon\, x), \quad \forall y\,(\forall z\, z \notin y \to x \,\varepsilon\, y), \qquad \forall y\,(\forall z\, z \notin y \to x \equiv y), \quad \forall y\,(\forall z\, z \notin y \to y \equiv x), \qquad \text{zf-3}$$

respectively.

(Pairing Axiom) $\forall x \forall y \exists z \forall u\,(u \,\varepsilon\, z \leftrightarrow (u \equiv x \vee u \equiv y))$.

In the following we use the abbreviations

$$\text{(2.38)}\quad z \,\varepsilon\, \{x, y\}, \quad \{x, y\} \,\varepsilon\, z, \qquad \{x, y\} \equiv z, \quad z \equiv \{x, y\} \qquad \text{zf-4}$$

to denote the formulas

$$\text{(2.39)}\quad \exists u\,(\forall v\,(v \varepsilon u \leftrightarrow (v \equiv x \vee v \equiv y) \wedge z \,\varepsilon\, u), \ \exists u\,(\forall v\,(v \varepsilon u \leftrightarrow (v \equiv x \vee v \equiv y) \wedge u \,\varepsilon\, z), \qquad \text{zf-5}$$

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(14) In the next subsection, when we have introduced the system K^* of logical deduction, we can show that $\forall x \forall y\,(x \equiv y \to \forall z\,(z \in x \leftrightarrow z \in y))$ is a provable sentence. Actually this formula is almost identical with an instance of one of the axiom schemas which guarantee that $\equiv$ is interpreted as the equality.

It follows that $\forall x \forall y\,(\forall z\,(z \in x \leftrightarrow z \in y) \leftrightarrow x \equiv y)$ is a theorem provable form the Axiom of Extensionality.

(15) In the next subsection, when we have introduced the system K^* of logical deduction, we can prove that this formula is equivalent to the formula "$x \not\equiv x$" in K^*.

(16) What we really mean here is that, for example, the formula

$$(\exists y\,(\forall z\, z \notin y \wedge y \,\varepsilon\, x) \leftrightarrow \exists! y\,(\forall z\, z \notin y \wedge y \,\varepsilon\, x))$$

can be proved in the deduction system K^* which is going to be introduced in the next subsection, under the Axiom of Extensionality. Similar remarks apply when we talk about implication, equivalence, or provability in the following.

$$\exists u\,(\forall v\,(v\varepsilon u\ \leftrightarrow\ (v\equiv x\vee v\equiv y)\wedge z\equiv u),\ \exists u\,(\forall v\,(v\varepsilon u\leftrightarrow\ (v\equiv x\vee v\equiv y)\wedge u\equiv z),$$

respectively. Similarly to the case of "$\emptyset$", by the Axiom of Exensionality, we can replace the outermost quantification $\exists u$ in the formulas above with $\exists! u$. By the Pairing Axiom, we can prove the equivalence (in the system K^* introduced below) of the formulas in (2.39) with the formulas

$$\text{(2.40)}\quad \forall u\,(\forall v\,(v\varepsilon u\leftrightarrow (v\equiv x\vee v\equiv y)\rightarrow z\ \varepsilon\ u),\ \forall u\,(\forall v\,(v\varepsilon u\leftrightarrow (v\equiv x\vee v\equiv y)\rightarrow u\ \varepsilon\ z),$$
$$\forall u\,(\forall v\,(v\varepsilon u\leftrightarrow (v\equiv x\vee v\equiv y)\rightarrow z\equiv u),\ \forall u\,(\forall v\,(v\varepsilon u\leftrightarrow (v\equiv x\vee v\equiv y)\rightarrow u\equiv z),$$

zf-6

respectively.

As in (1.2), we introduce the ordered pair $\langle x,y\rangle$ as the abbreviation of

(1.2)$'$ $\{\{x\},\{x,y\}\}$

where "$\{x\}$" is used as an abbreviation of "$\{x,x\}$".

(Axiom of Union) $\forall x\exists y\forall z\,(z\ \varepsilon\ y\leftrightarrow \exists u\,(u\ \varepsilon\ x\wedge z\ \varepsilon\ u))$.

In connection with the Axiom of Union, we introduce the abbreviating notation of the union:

$$\text{(2.41)}\quad y\ \varepsilon \bigcup x,\quad \bigcup x\ \varepsilon\ y,$$
$$y\equiv\bigcup x,\quad \bigcup x\equiv y$$

zf-7

are abbreviations of

$$\text{(2.42)}\quad \exists z\,(\forall u\,(u\ \varepsilon\ z\leftrightarrow \exists v\,(v\ \varepsilon\ x\wedge u\ \varepsilon\ v))\ \wedge\ y\ \varepsilon\ x),^{(17)}$$
$$\exists z\,(\forall u\,(u\ \varepsilon\ z\leftrightarrow \exists v\,(v\ \varepsilon\ x\wedge u\ \varepsilon\ v))\ \wedge\ z\ \varepsilon\ y),$$
$$\exists z\,(\forall u\,(u\ \varepsilon\ z\leftrightarrow \exists v\,(v\ \varepsilon\ x\wedge u\ \varepsilon\ v))\ \wedge\ y\equiv z),$$
$$\exists z\,(\forall u\,(u\ \varepsilon\ z\leftrightarrow \exists v\,(v\ \varepsilon\ x\wedge u\ \varepsilon\ v))\ \wedge\ y\equiv z)$$

zf-8

respectively. The same kind of remarks about necessary renaming of the bounded variables in the formulas similar to the remarks after (2.35) or (2.39) and remarks on equivalent formulas like the ones in (2.36) and (2.37) also apply here.

The formulation of the Axiom of Separation in the naïve axiomatic setting was ambiguous in that we could not specify exactly what the "property" was. We can now introduce this axiom without any ambiguity as a scheme of axioms in the following way:

For each $\mathcal{L}_\varepsilon$-formula $\varphi=\varphi(x,x_0,...,x_{n-1})$, we consider the formula of the following form as an instance of the axiom scheme of the Axiom of Separation:

(Axiom of Separation)$_\varphi$ $\forall x_0\cdots\forall x_{n-1}\forall y\exists u\forall x\,(x\ \varepsilon\ u\leftrightarrow (x\ \varepsilon\ y\wedge\varphi(x,x_0,...,x_{n-1})))$.

(17) By the Axiom of Union this formula will be proved (in K^* to be equivalent with $\exists z\,(z\ \varepsilon\ x\wedge y\ \varepsilon\ z)$)

(Axiom of Infinity) $\exists x\,(\emptyset \mathrel{\varepsilon} x \;\wedge\; \forall y\,(y \mathrel{\varepsilon} x \rightarrow \bigcup\{y,\{y\}\}))$.

(Power-set Axiom) $\forall x \exists y \forall u\,(u \mathrel{\varepsilon} y \leftrightarrow u \subseteq x)$.

Here "$u \subseteq x$" is an abbreviation of "$\forall v\,(v \mathrel{\varepsilon} u \rightarrow v \mathrel{\varepsilon} x)$".

The axioms introduced so far make up the axiom system of Zermelo set theory (denoted by Z) the axiom system contains infinitely many sentences because of the Axiom (Scheme) of Separation.

The Axiom of Choice can be now formulated as an $\mathcal{L}_\varepsilon$-sentence as follows:

(Axiom of Choice)

$$\forall x\,(\emptyset \not\mathrel{\varepsilon} x \;\rightarrow\; \exists y\,(\forall u\,(u \mathrel{\varepsilon} y \rightarrow \exists v \exists w\,(v \mathrel{\varepsilon} x \wedge (w \mathrel{\varepsilon} x \wedge \langle v,w\rangle \equiv u)) \\ \wedge\; \forall v\,(v \mathrel{\varepsilon} x \rightarrow \exists! w\,(\langle v,w\rangle \mathrel{\varepsilon} y)))))$$

Note that the subformulas of the formula of the Axiom of Choice express that "y is a choice function on x":

$$\forall x\,(\emptyset \not\mathrel{\varepsilon} x \;\rightarrow\; \exists y\,(\underbrace{\forall u\,(u \mathrel{\varepsilon} y \rightarrow \exists v \exists w\,(v \mathrel{\varepsilon} x \wedge (w \mathrel{\varepsilon} v \wedge \langle v,w\rangle \equiv u)))}_{y \text{ consists of elements of the form } \langle v,w\rangle \text{ such that } v\in x \text{ and } w\in v} \\ \wedge\; \underbrace{\forall v\,(v \mathrel{\varepsilon} x \rightarrow \exists! w\,(\langle v,w\rangle \mathrel{\varepsilon} y))}_{y \text{ is a function on } x})). \tag{2.43}$$ zf-9

The axiom system consisting of axioms of Z and the Axiom of Choice is denoted by ZC.

The axiom system of Zermelo-Fraenkel set theory (notation: ZF) consists of axioms of Z and the following two kinds of axioms. The first one, Axiom of Replacement is again an axiom scheme and consists of infinitely many $\mathcal{L}_\varepsilon$-sentences.

For each $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(y,z,x_0,\ldots,x_{n-1})$, let

(Axiom of Replacement)$_\varphi$

$$\forall x_0 \cdots \forall x_{n-1} \forall x\,(\forall y\,(y \mathrel{\varepsilon} x \rightarrow \exists! z\,\varphi(y,z,x_0,\ldots,x_{n-1})) \\ \rightarrow\; \exists u \forall v\,(v \mathrel{\varepsilon} u \leftrightarrow \exists y\,(y \mathrel{\varepsilon} x \wedge \varphi(y,v,x_0,\ldots,x_{n-1})))).$$

(Axiom of Regularity) $\forall x\,(x \not\equiv \emptyset \;\rightarrow\; \exists y\,(y \mathrel{\varepsilon} x \wedge \forall z\,(z \mathrel{\varepsilon} x \rightarrow \neg z \mathrel{\varepsilon} y)))$.

The axiom system of Zermelo-Fraenkel set theory with Axiom of Choice (notation: ZFC) is the axiom system consisting of the axioms of ZF and the Axiom of Choice.

2.3 The deduction system K^*

ded-sys

logic-10-2019-05-15.pdf

In the following we are going to introduce a deduction system for the predicate logic which we call K^*.[18]

[18] This is just a quite arbitrary name and has no special meaning in it except that the letter "K" is chosen because of the German expression "logisches Kalkül" which means logical deduction system.

For this, we have to introduce first the propositional logic and the notion of tautology. In the following we are still working in meta-mathematics.

Let A_0, A_1, A_2 ,... be (a potentially infinite collection of) symbols which we call *propositional symbols*. Formulas of the *propositional logic* are sequences of symbols defined inductively by:

(2.44) Each sequence of length 1 consisting of a propositional symbol is a formula of the propositional logic; prop-logic-0

(2.45) If φ and ψ are formulas of the propositional logic, then $(\varphi \to \psi)$, $\neg\psi$ are formulas of the propositional logic; prop-logic-1

(2.46) Nothing else. prop-logic-2

For example $(\neg((A_0 \to A_1) \to \neg A_4) \to A_5)$ is a formula of the propositional logic. For a formula φ of the propositional logic, we write $\varphi = \varphi(A_0, ..., A_{n-1})$ if all the propositional symbols appearing in φ are among $A_0, ..., A_{n-1}$. For example, if we denote the formula of the propositional logic given in the example above as φ, we have $\varphi = \varphi(A_0, A_1, A_2, A_3, A_4, A_5)$ but we can also write $\varphi = \varphi(A_0, A_1, A_4, A_5)$.

Let $\mathit{2} = \{0, 1\}$ and[(19)] $(\mathit{2})^n = \underbrace{\mathit{2} \times \cdots \times \mathit{2}}_{n} = \{\langle i_0, ..., i_{n-1}\rangle \; : \; i_0, ..., i_{n-1} \in \mathit{2}\}$ for $n \in \mathbb{N}$.

For formulas $\varphi = \varphi(A_0, ..., A_{n-1})$ of propositional logic, we define their functional interpretation $f_{\varphi(A_0,...,A_{n-1})} : (\mathit{2})^n \to \mathit{2}$ recursively as follows:

(2.47) If φ is the sequence "A_k" for some $k < n$ then prop-logic-3

$$f_{\varphi(A_0,...,A_{n-1})}(\langle i_0, ..., i_{n-1}\rangle) = \begin{cases} 1, & \text{if } i_k = 1; \\ 0, & \text{otherwise} \end{cases}$$

for all $\langle i_0, ..., i_{n-1}\rangle \in (\mathit{2})^n$;

(2.48) If $\varphi = (\varphi_0 \to \varphi_1)$,[(20)] then prop-logic-4

$$f_{\varphi(A_0,...,A_{n-1})}(\langle i_0, ..., i_{n-1}\rangle) = \begin{cases} 1, & \text{if } f_{\varphi_0(A_0,...,A_{n-1})}(\langle i_0, ..., i_{n-1}\rangle) = 0 \text{ or} \\ & \quad f_{\varphi_1(A_0,...,A_{n-1})}(\langle i_0, ..., i_{n-1}\rangle) = 1; \\ 0, & \text{otherwise} \end{cases}$$

for all $\langle i_0, ..., i_{n-1}\rangle \in (\mathit{2})^n$;

(2.49) If $\varphi = \neg\varphi_0$, then prop-logic-5

$$f_{\varphi(A_0,...,A_{n-1})}(\langle i_0, ..., i_{n-1}\rangle) = \begin{cases} 1, & \text{if } f_{\varphi_0(A_0,...,A_{n-1})}(\langle i_0, ..., i_{n-1}\rangle) = 0; \\ 0, & \text{otherwise} \end{cases}$$

for all $\langle i_0, ..., i_{n-1}\rangle \in (\mathit{2})^n$.

[(19)] The intended interpretation of 0 and 1 are "false" and "true".

[(20)] Note that we have $\varphi_0 = \varphi_0(A_0, ..., A_{n-1})$ and $\varphi_1 = \varphi_1(A_0, ..., A_{n-1})$.

A formula $\varphi = \varphi(A_0, ..., A_{n-1})$ in propositional logic is a *tautology* if $f_{\varphi(A_0,...,A_{n-1})}$ takes always the value 1. This is well defined: we can prove easily that whether φ is tautology or not does not depend on the choice of the list of variables $A_0, ..., A_{n-1}$. For a language $\mathcal{L}$, $\mathcal{L}$-formula ψ (in predicate logic) is said to be a *tautology* if there is a tautology $\varphi = \varphi(A_0, ..., A_{n-1})$ and $\mathcal{L}$-formulas $\varphi_0, ..., \varphi_{n-1}$ such that ψ is obtained from φ by replacing $A_0, ..., A_{n-1}$ by $\varphi_0, ..., \varphi_{n-1}$ respectively.

By the definition of functional interpretation of formulas of propositional logic given by (2.47) through (2.49), tautologies in predicate logic should be formulas which are true in any interpretation as long as the logical connectives ”→” and ”¬” are interpreted as logical “then” and “not”.

For each language $\mathcal{L}$, we introduce the notion of the deduction in the system K^* of predicate logic by introducing Axioms and Deduction Rules and defining proofs in K^* (of an $\mathcal{L}$-formula form a set T of $\mathcal{L}$-formulas).

In the following, we write $\varphi_0 \to \cdots \varphi_{k-1} \to \varphi$ for formulas $\varphi_0, ..., \varphi_{k-1}, \varphi$ to denote

$$(2.50)\quad (\varphi_0 \to (\varphi_1 \to (\cdots (\varphi_{k-2} \to (\varphi_{k-1} \to \varphi))\cdots)) \qquad \text{pred-logic-5-0}$$

As a special case of this notation, we often drop the parentheses in $(\varphi \to \psi)$ and write $\varphi \to \psi$ as far as no confusion occurs.

Axioms of K^* in $\mathcal{L}$:

- All $\mathcal{L}$-formulas which are tautologies.
- (Axioms of Equality) All $\mathcal{L}$-formulas of the following form

$$(2.51)\quad x \equiv x; \qquad \text{pred-logic-6}$$

$$(2.52)\quad x \equiv y \to y \equiv x; \qquad \text{pred-logic-7}$$

$$(2.53)\quad x \equiv y \to y \equiv z \to x \equiv z; \qquad \text{pred-logic-8}$$

$$(2.54)\quad u_0 \equiv v_0 \to \cdots \to u_{m-1} \equiv v_{m-1} \to f(u_0, ..., u_{m-1}) \equiv f(v_0, ..., v_{m-1}) \qquad \text{pred-logic-9}$$

where f is an m-ary function symbol in $\mathcal{L}$;

$$(2.55)\quad u_0 \equiv v_0 \to u_1 \equiv v_1 \to \cdots \to u_{n-1} \equiv v_{n-1} \to r(u_0, ..., u_{n-1}) \to r(v_0, ..., v_{n-1}) \qquad \text{pred-logic-10}$$

where r is an n-ary relation symbol in $\mathcal{L}$.

- (Axiom of Substitution) All $\mathcal{L}$-formulas of the form

$$(2.56)\quad \varphi(t/x) \to \exists x\varphi \qquad \text{pred-logic-11}$$

where t is substitutable for x in φ.[21]

---

[21] $\varphi(t/x)$ denotes the formula obtained form φ by substituting t for all free appearance of x in φ. t is *substitutable* for x in φ if all free appearances of x are not in the scope of the quantification of any of the variables appearing in t.

Deduction rules of K^* in $\mathcal{L}$: All diagrams with $\mathcal{L}$-formulas of the following form:

- (Modus Ponens)

  $\dfrac{\varphi,\ \varphi \rightarrow \psi}{\psi}$

- (Rule of Existential Quantification)

  $\dfrac{\varphi \rightarrow \psi}{\exists x \varphi \rightarrow \psi}$

  where x does not appear in ψ as a free variable.

For a concretely given collection Γ of $\mathcal{L}$-formulas and an $\mathcal{L}$-formula φ, a *proof* of φ from Γ (in K^*) is a finite sequence $\varphi_0, ..., \varphi_n$ such that

(2.57) each φ_i, $i \leq n$ is either an Axiom of K^*, or one of the formulas in Γ, or there are i_0, $i_1 < i$ such that $\dfrac{\varphi_{i_0},\ \varphi_{i_1}}{\varphi_1}$ is an instance of Modus Ponens, or there is $i_0 < i$ such that $\dfrac{\varphi_{i_0}}{\varphi_i}$ is an instance of the Rule of Existential Quantification; and pred-logic-12

(2.58) $\varphi_n = \varphi$. pred-logic-13

If $\mathcal{P}$ is a proof of φ from Γ, we write $\Gamma \vdash^{K^*,\mathcal{P}} \varphi$. If there is some proof $\mathcal{P}$ of φ from Γ, we write $\Gamma \vdash^{K^*} \varphi$. Some times we drop the superscript K^* and write simply $\Gamma \vdash^{\mathcal{P}} \varphi$ or $\Gamma \vdash \varphi$.

If Γ is empty set, we write $\vdash^{K^*,\mathcal{P}} \varphi$, $\vdash^{K^*} \varphi$, $\vdash^{\mathcal{P}} \varphi$, $\vdash \varphi$ instead of $\Gamma \vdash^{K^*,\mathcal{P}} \varphi$, $\Gamma \vdash^{K^*} \varphi$, $\Gamma \vdash^{\mathcal{P}} \varphi$, $\Gamma \vdash \varphi$ respectively.

bbd-all: p.37- L-pred-logic-0

Lemma 2.1 *For a collection Γ of $\mathcal{L}$-formulas and $\mathcal{L}$-formulas $\varphi_0, ..., \varphi_{k-1}$, φ, if*

(2.59) $\Gamma \vdash \varphi_0 \rightarrow \varphi_1 \rightarrow \cdots \rightarrow \varphi_{k-1} \rightarrow \varphi$ *and* pred-logic-14

(2.60) $\Gamma \vdash \varphi_0$, $\Gamma \vdash \varphi_1, ..., \Gamma \vdash \varphi_{k-1}$, pred-logic-15

then $\Gamma \vdash \varphi$.

Proof. Let $\mathcal{P}$ be a proof of (2.59) and $\mathcal{P}_0, ..., \mathcal{P}_{k-1}$ the proofs of (2.60). Then

(2.61) $\mathcal{P} ^\frown \mathcal{P}_0 ^\frown \cdots ^\frown \mathcal{P}_{k-1} ^\frown \langle \varphi_0 \rightarrow \cdots \rightarrow \varphi_{k-1} \rightarrow \varphi \rangle ^\frown \langle \varphi_1 \rightarrow \cdots \rightarrow \varphi_{k-1} \rightarrow \varphi \rangle ^\frown \cdots$
$^\frown \langle \varphi_{k-1} \rightarrow \varphi \rangle ^\frown \langle \varphi \rangle$

is a proof of φ from Γ. Note that, for example, $\varphi_1 \rightarrow \cdots \rightarrow \varphi_{k-1} \rightarrow \varphi$ is obtained from the last formulas of $\mathcal{P}$ and $\mathcal{P}_0$ by Modus Ponens. ❑ (Lemma 2.1)

In the following, we shall often cite Lemma 2.1 simply as Modus Ponens.

L-pred-logic-0-a

Lemma 2.2 $\vdash \exists x\, x \equiv x$.

Proof. By the Axiom of Substitution, we have $\vdash x \equiv x \rightarrow \exists x\, x \equiv x$. By the Axiom of Equality (2.51), we have $\vdash x \equiv x$. Hence, by Modus Ponens (Lemma 2.1), it follows that $\vdash \exists x\, x \equiv x$. ❑ (Lemma 2.2)

Theorem 2.3 (Deduction Theorem) (1) *For any collection of $\mathcal{L}$-formulas Γ and $\mathcal{L}$-fromulas φ, ψ, if $\Gamma \vdash \varphi \to \psi$ then $\Gamma, \varphi \vdash \psi$.*[22] T-pred-logic-0

(2) *For Γ and φ, ψ as in (1), if φ is a $\mathcal{L}$-sentence then $\Gamma \vdash \varphi \to \psi \Leftrightarrow \Gamma, \varphi \vdash \psi$.*

Proof. (1): If $\Gamma \vdash \varphi \to \psi$, then we have $\Gamma, \varphi \vdash \varphi \to \psi$. We also have $\Gamma, \varphi \vdash \varphi$ (Note that $\langle\varphi\rangle$ is a proof[23]). Thus, by Lemma 2.1, we have $\Gamma, \varphi \vdash \psi$.

(2): By (1), it is enough to show "$\Rightarrow$". We show

(2.62) If $\Gamma, \varphi \vdash^{\mathcal{P}} \psi$, then, starting from $\mathcal{P}$, we can construct a proof $\mathcal{Q}$ with $\Gamma \vdash^{\mathcal{Q}} \varphi \to \psi$ pred-logic-16

by induction on the number n of formulas in $\mathcal{P}$ (that is, by the length of $\mathcal{P}$ as a sequence).

If $n = 1$, then ψ is either (i) a logical axiom or (ii) a formula in Γ or (iii) φ. If (i) or (ii) holds then, since $\psi \to \varphi \to \psi$ is a tautology, the sequence $\langle\psi,\ \psi \to \varphi \to \psi,\ \varphi \to \psi\rangle$ is a proof of $\varphi \to \psi$ from Γ. If (iii) holds then $\varphi \to \psi$ is identical with $\psi \to \psi$ which is a tautology. Thus $\langle\varphi \to \psi\rangle$ is a proof of $\varphi \to \psi$ form Γ.

Now assume that (2.62) holds for all proofs with less than n formulas and assume that $\Gamma, \varphi \vdash^{\mathcal{P}} \psi$ where $\mathcal{P}$ as a sequence has the length n.

Case I. ψ is either a logical axiom or one of the formulas among Γ, φ. In this case, we can prove $\varphi \to \psi$ from Γ just as in the case of $n = 1$.

Case II. ψ is deduced by Modus Ponens from some formulas in $\mathcal{P}$. Then there is some $\mathcal{L}$-fromula η such that η and $\eta \to \psi$ appear in $\mathcal{P}$ and ψ is deduced from these two formulas. The initial segments of $\mathcal{P}$ to each of these formulas are shorter than $\mathcal{P}$ hence by indcuction hypothesis there are proofs $\mathcal{P}_0$, $\mathcal{P}_1$ such that $\Gamma \vdash^{\mathcal{P}_0} \varphi \to \eta$ and $\Gamma \vdash^{\mathcal{P}_1} \varphi \to (\eta \to \psi)$. Since $(\varphi \to \eta) \to (\varphi \to (\eta \to \psi)) \to (\varphi \to \psi)$ is a tautology, denoting this formula θ, and the formula $(\eta \to \psi)) \to (\varphi \to \psi)$, the sequence $\mathcal{P}_0{}^\frown\mathcal{P}_1{}^\frown\langle\theta, \theta_0, \varphi \to \psi\rangle$ is a proof of $\varphi \to \psi$ by Lemma 2.1.

Case III. ψ is of the form $\exists x\, \eta \to \xi$ where x does not appear in ξ as free variable and $\eta \to \xi$ appears in $\mathcal{P}$. Let $\mathcal{P}'$ be the initial segment of $\mathcal{P}$ ending with $\eta \to \xi$. Since $\Gamma, \varphi \vdash^{\mathcal{P}'} \eta \to \xi$ and $\mathcal{P}'$ is shorter than $\mathcal{P}$, we have $\Gamma \vdash \varphi \to \eta \to \xi$. Since $(\varphi \to \eta \to \xi) \to (\eta \to \varphi \to \xi)$ is a tautology, we get $\Gamma \vdash \eta \to \varphi \to \xi$. Remembering that φ is a sentence, we may apply the Rule of Existential Quantification to obtain $\Gamma \vdash \exists x\, \eta \to \varphi \to \xi$. Now since $(\exists x \eta \to \varphi \to \xi) \to (\varphi \to \exists x \eta \to \xi)$ is a tautology, we get $\Gamma \vdash \varphi \to \underbrace{\exists x\, \eta \to \xi}_{=\psi}$. ❑ (Theorem 2.3)

further "proof theory": logic-12-2019-05-22.pdf

A (concretely given) collection of $\mathcal{L}$-sentences T is called a(n $\mathcal{L}$-)*theory*.

L-pred-logic-0-0

Lemma 2.4 *For any language $\mathcal{L}$, $\mathcal{L}$-theory T and $\mathcal{L}$-formulas φ, ψ,*

$T \vdash \varphi$ and $T \vdash \psi \Leftrightarrow T \vdash (\varphi \wedge \psi)$.

---

[22] For a collection Γ of $\mathcal{L}$-fromulas and $\mathcal{L}$-formulas φ, ψ, we simply write $\Gamma, \varphi \vdash \psi$ in place of $\Gamma \cup \{\varphi\} \vdash \psi$.

[23] In abuse of notation, we denote with $\langle a_0, ..., a_{n-1}\rangle$ the sequence whose ith component is a_i for $i < n$. Strictly speaking, we have to write $\{\langle 0, a_0\rangle\}$ instead of $\langle a_0\rangle$ and $\{\langle 0, a_0\rangle, \langle 1, a_1\rangle\}$ instead of $\langle a_0, a_1\rangle$ etc.

Proof. Suppose that $T \vdash \varphi$ and $T \vdash \psi$. Since $\varphi \to \psi \to (\varphi \wedge \psi)$ is a tautology, we have $T \vdash \varphi \to \psi \to (\varphi \wedge \psi)$. If follows from Lemma 2.1 that $T \vdash (\varphi \wedge \psi)$.

Suppose now that $T \vdash (\varphi \wedge \psi)$. Since $(\varphi \wedge \psi) \to \varphi$ and $(\varphi \wedge \psi) \to \psi$ are tautologies, we have $T \vdash (\varphi \wedge \psi) \to \varphi$ and $T \vdash (\varphi \wedge \psi) \to \psi$. By Lemma 2.1, it follows that $T \vdash \varphi$ and $T \vdash \psi$. ❑ (Lemma 2.4)

An $\mathcal{L}$-theory T is *inconsistent* if T proves any $\mathcal{L}$-formula. Otherwise T is *consistent*.

L-pred-logic-1

Lemma 2.5 *For any language $\mathcal{L}$ and $\mathcal{L}$-theory T the following are equivalent:*

(a) *T is inconsistent;*

(b) *For some $\mathcal{L}$-formula φ, we have $T \vdash \varphi$ and $T \vdash \neg\varphi$;*

(c) *$T \vdash (\varphi \wedge \neg\varphi)$ for some $\mathcal{L}$-formula φ;*

(d) *$T \vdash x \not\equiv x$.*

Proof. (a) $\Rightarrow$ (b): Clear by definition.

(b) $\Leftrightarrow$ (c): By Lemma 2.4.

(c) $\Rightarrow$ (d): Asuume that $T \vdash (\varphi \wedge \neg\psi)$ for some $\mathcal{L}$-formula φ. Since $(\varphi \wedge \neg\varphi) \to x \not\equiv x$ is a tautology, we have $T \vdash (\varphi \wedge \neg\varphi) \to x \not\equiv x$. By Lemma 2.1 it follows that $T \vdash x \not\equiv x$.

(d) $\Rightarrow$ (a): Assume $T \vdash x \not\equiv x$. Since $x \equiv x$ is one of the Axioms of Equality, we have $T \vdash x \equiv x$. Hence, by Lemma 2.4, it follows that $T \vdash (x \equiv x \wedge x \not\equiv x)$. For any $\mathcal{L}$-formula φ, $(x \equiv x \wedge x \not\equiv x) \to \varphi$ is a tautology. Thus $T \vdash (x \equiv x \wedge x \not\equiv x) \to \varphi$ and $T \vdash \varphi$ by Lemma 2.1. ❑ (Lemma 2.5)

L-pred-logic-2

Lemma 2.6 *For any language $\mathcal{L}$, collection of $\mathcal{L}$-formula T, $\mathcal{L}$-fromula φ and variables $v_0, \ldots, v_{n-1}$, we have*

$$T \vdash \varphi \Leftrightarrow T \vdash \forall v_0 \cdots \forall v_{n-1} \varphi$$

Proof. It is enough to prove the asserion for $n = 1$. Recall that $\forall v\, x$ is an abbreviation of $\neg\exists v \neg\varphi$.

($\Rightarrow$): Suppose that $T \vdash \varphi$ and let ϕ be an arbitrary $\mathcal{L}$-sentence which is a tautology. Since $T, \psi \vdash \varphi$, we have $T \vdash \psi \to \varphi$ by the Deduction Theorem (Theorem 2.3). Since $(\psi \to \varphi) \to (\neg\varphi \to \neg\psi)$ is a tautology, we obtain $T \vdash \neg\varphi \to \neg\psi$ by Lemma 2.1. Thus, by the Rule of Existential Quantification, it follows that

(2.63) $\quad T \vdash \exists v \neg\varphi \to \neg\psi.$[24] pred-logic-17

Now, since $(\exists v \neg\varphi \to \neg\psi) \to (\psi \to \neg\exists v \neg\varphi)$ is a tautology, we obtain

(2.64) $\quad T \vdash \psi \to \neg\exists v \neg\varphi$ pred-logic-18

[24] Remember that ψ is a sentence and hence does not contain x as a free variable!

by Lemma 2.1. Since ψ is a tautology, we have $T \vdash \psi$. Thus, again by Lemma 2.1, $T \vdash \neg\exists v \neg\varphi$, that is, $T \vdash \forall v\, \varphi$.

($\Leftarrow$): This can be proved similarly to "$\Rightarrow$". (Exercise). □ (Lemma 2.6)

Lemma 2.7 (Substitution Theorem) *Suppose that $\mathcal{L}$ is a language and T a collection of $\mathcal{L}$-formulas. If $T \vdash \varphi$ for an $\mathcal{L}$-fromula and t is a $\mathcal{L}$-term substitutable in the variable x in φ, Then $T \vdash \varphi(t/x)$.* L-pred-logic-3

Proof. By $T \vdash \neg\varphi \rightarrow \neg\varphi$. By the Axiom of Substitution, we have $T \vdash \neg\varphi(t/x) \rightarrow \exists x \neg\varphi$. Since $(\neg\varphi(t/x) \rightarrow \exists\neg\varphi) \rightarrow (\neg\exists x\neg\varphi \rightarrow \varphi(t/x))$ is a tautology,

$$(2.65) \quad \vdash \neg\exists x\neg\varphi \rightarrow \varphi(t/x)$$ pred-logic-19

by Lemma 2.1. Since $T \vdash \varphi$, we have

$$(2.66) \quad T \vdash \neg\exists x\neg\varphi$$ pred-logic-20

by Lemma 2.6. Thus, by (2.65), (2.66) and Lemma 2.1, we obtain $T \vdash \varphi(t/x)$. □ (Lemma 2.7)

2.4 Predicate logic in Z and the Completeness Theorem

comp

2.5 Some semantic proofs of proof theoretic assertions

katowice-03-2019-10-14

The following are applications of Soundness Theorem and Completeness Theorem. Both of the following theorems can be proved proof-theoretically (i.e., in meta-mathematically without appealing to any amount of set-theory). The proof-theoretical proofs are much more difficult than the ones presented here. For example, a possible proof of Theorem 2.8 is that we first establish the equivalence of our system K^* to a Gentzen style deduction system and applying the Cut Elimination Theorem of Gentzen. The model theoretic proofs via Soundness and Completeness Theorems[(25)] are just a sort of first approximations to the corresponding proof-theoretic proofs which can be much more complicated. !!!!

T-pred-logic-1

Theorem 2.8 *Suppose that T is an $\mathcal{L}$-theory, φ an $\mathcal{L}$-fromula, and $\mathcal{L}'$ is a language extending $\mathcal{L}$. Then we have*

$T \vdash^{\mathcal{L}} \varphi$ *if and only if* $T \vdash^{\mathcal{L}'} \varphi$[(26)]

Proof. If $T \vdash^{\mathcal{L}} \varphi$ then we clearly have $T \vdash^{\mathcal{L}'} \varphi$. Suppose $T \nvdash^{\mathcal{L}} \varphi$. Then, by Completeness Theorem, there is an $\mathcal{L}$-structure $\mathfrak{A}$ such that $\mathfrak{A} \models T$ but $\mathfrak{A} \not\models \varphi$.

---

(25) Such proofs are also often called semantical proofs in contrast to the syntactical or proof-theoretical proofs which can be done in the meta-mathematical frame-work without utilizing the notion of interpretation of formulas in models, and Soundness and Completeness Theorems.

(26) We denote with $T \vdash^{\mathcal{L}} \varphi$ the statement "there is a proof $\mathcal{P}$ of φ from T such that all the components of $\mathcal{P}$ are $\mathcal{L}$-formulas."

Let $\tilde{\mathfrak{A}}$ be an arbitrary expansion of $\mathfrak{A}$ to a $\mathcal{L}'$-structure. We still have $\tilde{\mathfrak{A}} \models T$ and $\tilde{\mathfrak{A}} \not\models \varphi$. Thus we have $T \not\vdash^{\mathcal{L}'} \varphi$ by Soundness Theorem. ❑ (Lemma 2.8)

For a language $\mathcal{L}$ and $\mathcal{L}$-theory T, an $\mathcal{L}'$-theory $\tilde{T}$ with $\tilde{T} \supseteq T$ for a language $\mathcal{L}' \supseteq \mathcal{L}$ is said to be a *conservative extension of* T if, for any $\mathcal{L}$-sentence φ, $\tilde{T} \vdash \varphi$ implies $T \vdash \varphi$.

T-pred-logic-2

Theorem 2.9 *Suppose that T is an $\mathcal{L}$-theory.*

(1) *If*

$$T \vdash \exists! x \varphi \tag{2.67}$$ pred-logic-21

for an $\mathcal{L}$-fromula $\varphi = \varphi(x)$, then the theory $\tilde{T} = T \cup \{\varphi(\underset{\sqcup}{c})\}$ is a conservative extension of T where $\underset{\sqcup}{c}$ is a new constant symbol not appearing in $T \cup \{\varphi\}$.

(2) *If*

$$T \vdash \forall x_0 \cdots \forall x_{k-1} \exists! x\, \varphi \tag{2.68}$$ pred-logic-22

for an $\mathcal{L}$-formula $\varphi = \varphi(x, x_0, ..., x_{k-1})$, then the theory $T \cup \{\forall x_0 \cdots \forall x_{k-1} \varphi(\underset{\sqcup}{f}(x_0, ..., x_{k-1}), x_0, ..., x_{k-1})\}$ is a conservative extension of T where $\underset{\sqcup}{f}$ is a new k-aray function symbol not appearing in $T \cup \{\varphi\}$.

(3) *For any $\mathcal{L}$-formula $\varphi = \varphi(x_0, ..., x_{k-1})$, the theory $T \cup \{\forall x_0 \cdots \forall x_{k-1} (\underset{\sqcup}{r}(x_0, ..., x_{k-1f}) \leftrightarrow \varphi(x_0, ..., x_{k-1}))\}$ is a conservative extension of T where $\underset{\sqcup}{r}$ is a new k-ary relation symbol not appearing in $T \cup \{\varphi\}$.*

Proof. We prove (2). Other assertions of the theorem can be treated similarly. Let T and φ be as in (2) and let $\tilde{T} = T \cup \{\forall x_0 \cdots \forall x_{k-1} \varphi(\underset{\sqcup}{f}(x_0, ..., x_{k-1}), x_0, ..., x_{k-1})\}$. Let ψ be an $\mathcal{L}$-formula and suppose $\tilde{T} \vdash \psi$. We have to show that $T \vdash \psi$. By the Completeness Theorem, it is enough to show that $T \models \psi$.

Suppose that $\mathfrak{A} = \langle A, ...\rangle$ be an $\mathcal{L}$-structure with $\mathfrak{A} \models T$. Let

$$\underset{\sqcup}{f}^{\mathfrak{A}'} = \{\langle\langle a_0, ..., a_{k-1}, b\rangle\rangle : \mathfrak{A} \models \varphi(b, a_0, ..., a_{k-1})\}. \tag{2.69}$$ pred-logic-23

By (2.68), $\underset{\sqcup}{f}^{\mathfrak{A}'} : A^k \to A$. Let $\mathfrak{A}' = \langle \underbrace{A, ...,}_{\mathfrak{A}} \underset{\sqcup}{f}^{\mathfrak{A}'}\rangle$. Then $\mathfrak{A}' \models \tilde{T}$. By Soundness Theorem, it follows that $\mathfrak{A}' \models \psi$. Hence $\mathfrak{A} = \mathfrak{A}' \restriction \mathcal{L} \models \psi$. ❑ (Theorem 2.9)

3 Incompleteness Theorems and Speed-up Theorems

incompl

3.1 Ehrenfeucht-Mycielski Speed-up Theorem

ehrenfeucht-mycielski

logic-15-2019-06-10.pdf

The meta-mathematical objects, like symbols, (finite) sequences of symbols, (finite) sequences of (finite) sequences of symbols, etc. correspond elements of V_ω.

To make the coding of languages and logic in set theory easier, we expand the language $\mathcal{L}_\varepsilon$ of set thoery with a new constant symbol $\emptyset$ and two function symbols $\{\cdot, \cdot\}$ and $\cdot \cup \cdot$.

Let $\mathcal{L}_{\{\}}$ be the extension of $\mathcal{L}_\varepsilon$ obtained by adding the constant symbol $\emptyset$ and the binary function symbols $\{\cdot,\cdot\}$, $\cdot\cup\cdot$. Let $\mathsf{Z}_{\{\}}$ be the $\mathcal{L}_{\{\}}$-theory obtained from Z by adding the axioms which claim that the expected interpretation of the new symbols is valid.

Thus $\mathsf{Z}_{\{\}}$ is a conservative extension of Z and, e.g., $\forall x\,(x \not\varepsilon\, \emptyset)$ is one of the axioms of $\mathsf{Z}_{\{\}}$. We introduced the theory Z_0 as a concretely given subtheory of $\mathsf{Z}_{\{\}}$ which is strong enough for coding the meta-mathematical objects and notions in it. In the following, T is a concretely given (consistent) theory in $\mathcal{L}_{\{\}}$ which extends Z_0. Usually what we have in mind is such a theory T that is compatible with ZF or ZFC but we do not assume this unless we mention it explicitly.

We can express "all" concrete hereditarily finite sets by closed $\mathcal{L}_{\{\}}$-terms. In particular, for any concretely given number n, we have a closed $\mathcal{L}_{\{\}}$-term $\underline{n}$ which corresponds to the number n in $\mathsf{Z}_{\{\}}$: For 0, 1, 2, 3, etc., we set $\underline{0}$, $\underline{1}$, $\underline{2}$, $\underline{3}$ etc. to be $\emptyset$, $(\emptyset\cup\{\emptyset,\emptyset\})$, $((\emptyset\cup\{\emptyset,\emptyset\})\cup\{(\emptyset\cup\{\emptyset,\emptyset\}),(\emptyset\cup\{\emptyset,\emptyset\})\})$, etc. In general, we define $\underline{n+1}$ to be $\underline{n}\cup\{\underline{n},\underline{n}\}$.

By induction on n (in meta-mathematics), we can show that for any $n \in \mathbb{N}$

(3.1) $Z_0 \vdash$ "$\emptyset$ is the minimal element of ω", incompl-a

(3.2) $\mathsf{Z}_0 \vdash$ "$\underline{n}\ \varepsilon\ \omega$" and incompl-0

(3.3) $\mathsf{Z}_0 \vdash$ "$\underline{n+1}$ is the successor of $\underline{n}$". incompl-1

Recall that we assumed that symbols of meta-mathematics are elements of $\mathbb{N}^2$. The collection of meta-mathematical finite sequences of symbols is denoted by *Seq*. We identify the pair $\langle m,n\rangle$ with the sequence of length 1 with the entry $\langle m,n\rangle$ and regard $\mathbb{N}^2$ as a subset of *Seq* with this identification.

$\ulcorner s\urcorner$: corner quote of s

We assign a closed $\mathcal{L}_{\{\}}$-term $\ulcorner s\urcorner$ with $\mathsf{Z}_0 \vdash$ "$\ulcorner s\urcorner \in {}^{\omega>}(\omega^2)$" to each symbol or finite sequence of symbols s such that if s is a sequence of length ℓ and its kth symbol is (coded by) the ordered pair $\langle m,n\rangle$ then $\mathsf{Z}_0 \vdash$ "the length of $\ulcorner s\urcorner$ is $\underline{\ell}$" and $\mathsf{Z}_0 \vdash$ "the $\underline{k}$th symbol of $\ulcorner s\urcorner$ is $\langle\underline{m},\underline{n}\rangle$". Similarly $\ulcorner\cdot\urcorner$ is extended to finite sequences of finite sequences of symbols, finite sequences of finite sequences of finite sequences of symbols etc.

We cite the following theorem on a characterization of recursiveness without a proof.

Theorem 3.1 (Representability) *For any $S \subseteq Seq$, S is recursive (i.e. the validity of "$s \in S$" is computable) if and only if there is an $\mathcal{L}_{\{\}}$-formula $\psi = \psi(x)$ such that for any sequence s of symbols,* T-incompl-a

(3.4) *if $s \in S$ then $\mathsf{Z}_0 \vdash \psi(\ulcorner s\urcorner)$ and*

(3.5) *if $s \notin S$ then $\mathsf{Z}_0 \vdash \neg\psi(\ulcorner s\urcorner)$.* ❑

チェックする: inc:req:int: defn:representable-fn in https://builds.openlogicproject.org/open-logic-debug.pdf

The following theorem will be proved later as Theorem 3.14:

Theorem 3.2 (Diagonal Lemma) *For an arbitrary formula ψ in a language $\mathcal{L}$ extending $\mathcal{L}_{\{\}}$, there is an $\mathcal{L}$-formula σ such that* T-incompl-0

(3.6) $\quad free(\sigma) \subseteq free(\psi) \setminus \{x_0\}$, *and* incompl-2

(3.7) $\quad \mathsf{Z}_0 \vdash \sigma \leftrightarrow \psi(\ulcorner\sigma\urcorner/x_0)$. ❑ incompl-3

Tarski's Undefinability of the Truth is an easy application of the Diagonal Lemma:

Corollary 3.3 (Tarski, Undefinability of the Truth) *For any language $\mathcal{L}$ extending $\mathcal{L}_{\{\}}$ and a consistent theory T in $\mathcal{L}$ with $T \supseteq \mathsf{Z}_0$, there is no $\mathcal{L}$-formula $\chi = \chi(x)$ such that $T \vdash \varphi \leftrightarrow \chi(\ulcorner\varphi\urcorner)$ for all $\mathcal{L}$-sentences φ.*[(27)] T-incompl-0-0

Proof. For any $\mathcal{L}$-formula χ, letting ψ to be the formula $\neg\chi$, there is a $\mathcal{L}_{\{\}}$-sentence φ such that $Z_0 \vdash \varphi \leftrightarrow \psi(\varphi)$ by Diagonal Lemma (Theorem 3.2). It follows that $T \vdash \varphi \leftrightarrow \neg\chi(\varphi)$. This shows that χ is not a definition of the truth for T. ❑ (Corollary 3.3)

A weak form of the first incompleteness theorem follows immediately from Tarski's theorem:

T-incompl-0-0-0

Corollary 3.4 *For any language $\mathcal{L}$ extending $\mathcal{L}_{\{\}}$ and a concretely given consistent theory T in $\mathcal{L}$, there is no proof of the completeness of T (in metamathematics).*

Proof. For contradiction, suppose that there would be a metamathematical proof of the completeness of T. The proof would be then reformulated in the formalism and we obtain:

(3.8) $\quad T \vdash$ "$\ulcorner\ulcorner T\urcorner\urcorner$ is complete". x-incompl-0

By Corollary 3.3, the predicate $provable_{\ulcorner\ulcorner T\urcorner\urcorner}(\cdot)$ is not a definition of truth. This means that there is an $\mathcal{L}$-sentence φ such that $T \nvdash \varphi \leftrightarrow provable_{\ulcorner\ulcorner T\urcorner\urcorner}(\ulcorner\varphi\urcorner)$.

If $T \vdash \varphi$, then we would have $T \vdash provable_{\ulcorner\ulcorner T\urcorner\urcorner}(\varphi)$. Thus it would follow that $T \vdash \varphi \leftrightarrow provable_{\ulcorner\ulcorner T\urcorner\urcorner}(\ulcorner\varphi\urcorner)$. This is a contradiction to the choice of φ.

Suppose $T \nvdash \varphi$, then

(3.9) $\quad T \vdash \neg\varphi$ x-incompl-1

by the assumption of the completeness of T. It follows that $T \vdash provable_{\ulcorner\ulcorner T\urcorner\urcorner}(\neg\varphi)$. By (3.8), it follows that

(3.10) $\quad T \vdash \neg provable_{\ulcorner\ulcorner T\urcorner\urcorner}(\varphi)$. x-incompl-2

Now, it follows from (3.9) and (3.10) that $T \vdash \varphi \leftrightarrow provable_{\ulcorner\ulcorner T\urcorner\urcorner}(\ulcorner\varphi\urcorner)$. This is again a contradition to the choice of φ. ❑ (Corollary 3.4)

---

(27) Such χ (if it would exist) is called a definition of the truth for T.

Remember that a proof $\mathcal{P}$ in K^* is a sequence of finite sequences of symbols. We define the length $L(\mathcal{P})$ of a proof $\mathcal{P}$ to be the length of the finite sequence obtained by concatenating all the components of the sequence $\mathcal{P}$.

For an $\mathcal{L}_{\{\}}$-theory $T \supseteq \mathsf{Z}_0$, and $\tau \in \mathsf{Th}(T)$, let

(3.11) $W_T(\tau) =$ the smallest possible length $L(\mathcal{P})$ of a proof $\mathcal{P}$ of τ. incompl-7

Theorem 3.5 (Ehrenfeucht and Mycielski Speed-up Theorem, [Ehrenfeucht-Mycielski 1971]) *Suppose that T is a concretely given consistent theory in a finite language $\mathcal{L}$ extending $\mathcal{L}_{\{\}}$ with $T \supseteq \mathsf{Z}_0$ and φ_0 is independent from T. Then, there is no recursive function $S : \mathbb{N} \to \mathbb{N}$ such that* E-M-thm

(3.12) $W_T(\tau) \leq S(W_{T\cup\{\varphi_0\}}(\tau))$ *for all* $\tau \in \mathsf{Th}(T)$. incompl-8

In other words, for any recursive $S : \mathbb{N} \to \mathbb{N}$, there is a $\tau \in \mathsf{Th}(T)$ such that $S(W_{T\cup\{\varphi_0\}}(\tau)) < W_T(\tau)$.

Theorem 3.5 follows from the next two Lemmas. Note that, in connection with Theorem 3.1, $F : \mathbb{N} \to \mathbb{N}$ is recursive if and only if F as a graph $F \subseteq \mathbb{N}^2 \subseteq Seq$ is recursive.

L-incompl-0

Lemma 3.6 *If $\mathsf{Th}(T \cup \{\neg\varphi_0\})$ is undecidable*(28) *, there is no recursive $F : \mathbb{N} \to \mathbb{N}$ such that*

(3.13) $W_T(\tau) \leq F(W_T(\varphi_0 \to \tau))$ *for all* $\tau \in \mathsf{Th}(T)$. incompl-9

Proof. Note that we have

(3.14) $T, \neg\varphi_0 \vdash^{K^*} \sigma \;\Leftrightarrow\; T \vdash^{K^*} \neg\varphi_0 \to \sigma \;\Leftrightarrow\; T \vdash^{K^*} (\varphi_0 \vee \sigma)$ incompl-10

by the Deduction Theorem (Theorem 2.3) of K^*.

Suppose, toward a contradiction, that there is a recursive function $F : \mathbb{N} \to \mathbb{N}$ satisfying (3.13). Without loss of generality, we may assume that F is increasing(29) .

Now, by (3.13), we have

(3.15) $W_T(\varphi_0 \vee \sigma) \leq F(W_T(\varphi_0 \to (\varphi_0 \vee \sigma)))$ holds for any $\sigma \in \mathsf{Th}(T)$. incompl-11

Note that $\sigma \in \mathsf{Th}(T)$ implies that $(\varphi_0 \vee \sigma) \in \mathsf{Th}(T)$.

Since $(\varphi_0 \to (\varphi_0 \vee \sigma))$ is a tautology, there is a recursive function (actually a linear function) $G : \mathbb{N} \to \mathbb{N}$ such that

(3.16) $W_T(\varphi_0 \to (\varphi_0 \vee \sigma)) = \ell n(\varphi_0 \to (\varphi_0 \vee \sigma)) = G(\ell n(\sigma))$ incompl-12

Thus

(3.17) $W_T(\varphi_0 \vee \sigma) \underbrace{\leq}_{\text{by (3.15)}} F(W_T(\varphi_0 \to (\varphi_0 \vee \sigma))) \leq F(G(\ell n(\sigma))).$[30] incompl-13

This means that, for each $\mathcal{L}$-sentence σ, we find a proof of $(\varphi_0 \vee \sigma)$ form T in the finite collection of all the proofs in $\mathcal{L}$ of length $\leq F(G(\ell n(\sigma)))$, if $(\varphi_0 \vee \sigma)$ has a proof from T.[31] By (3.14), this is a contradiction to the undecidability of $\mathsf{Th}(T \cup \{\neg\varphi_0\})$.

□ (Lemma 3.6)

logic-2019-bbd-all.pdf, p.55

Lemma 3.7 *There is a recursive function $R : \mathbb{N} \to \mathbb{N}$ such that* L-incompl-1

(3.18) $W_{T\cup\{\varphi_0\}}(\tau) \leq R(W_T(\varphi_0 \to \tau))$ *holds for all* $\tau \in \mathsf{Th}(T \cup \{\varphi_0\})$.[32] L-incompl-2

Proof. The proof of the Deduction Theorem for K^* gives an algorithm to obtain a proof of τ from $T \cup \{\varphi_0\}$ from a given proof of $\varphi \to \tau$ form T. □ (Lemma 3.7)

Proof of Ehrenfeucht-Mycielski Speed-up Theorem 3.5: Suppose, toward a contradiction, that there is a recursive $S : \mathbb{N} \to \mathbb{N}$ satisfying (3.12). Without loss of generality, we may assume that S is increasing. We have

(3.19) $W_T(\tau) \underbrace{\leq}_{\text{by the assumption of (3.12)}} S(W_{T\cup\{\varphi_0\}}(\tau)) \underbrace{\leq}_{\text{by Lemma 3.7}} S(R(W_T(\varphi_0 \to \tau))).$ incompl-14

Since the composition of R and S is recursive, this is a contradiction to Lemma 3.6. Note that $T \cup \{\neg\varphi_0\}$ is consistent and hence undecidable by Theorem 3.16. □ (Lemma 3.7)

The Ehrenfeucht-Mycielski Theorem does not hold if we modify the notion of the length of a proof to be the number of formulas in the proof:

T-length-0

Theorem 3.8 *For any concretely given theory T in an arbitrary language $\mathcal{L}$, there is a recursive theory T^* in $\mathcal{L}$ such that*

(3.20) *T and T^* are equivalent (i.e. they deduce the same sentences);* three-0

(3.21) *Each theorem of T (or equivalently, of T^*) has a proof consisting of exactly three formulas.* three-1

Proof. Let $\langle \psi_n : n \in \mathbb{N} \rangle$ be a recursive enumeration of all the theorems of T. For each $n \in \mathbb{N}$, let

(3.22) $\eta_n = \underbrace{\psi_n \wedge \cdots \wedge \psi_n}_{n+1\text{-times}}.$[33] three-2

---

(28) By Theorem 3.16, this condition is equivalent to the consistency of $T \cup \{\neg\varphi_0\}$.

(29) Otherwise we may replace F by $n \mapsto \max\{F(k) : k \leq n\}$

(30) For sequence s $\ell n(s)$ denotes the length of s.

(31) Note that the finiteness of the correction of proofs (modulo renaming of the variables) follows from the finiteness of $\mathcal{L}$.

(32) Note that by Deduction Theorem (Theorem 2.3), $\tau \in \mathsf{Th}(T \cup \{\varphi_0\})$ if and only if $\varphi_0 \to \tau \in \mathsf{Th}(T)$. Thus, if $\tau \in \mathsf{Th}(T \cup \{\varphi_0\})$, that is, if $W_{T\cup\{\varphi_0\}}$ is defined at τ then, W_T is defined at $\varphi_0 \to \tau$.

Note that $(\eta_n \to \psi_n)$ is a tautology for all $n \in \mathbb{N}$. In particular, all such formulas are axioms of the deduction system K^*.

We show that

$$T^* = \{\eta_n : n \in \mathbb{N}\} \tag{3.23}$$ three-3

is as desired.

T^* is recursive: Suppose that φ is an $\mathcal{L}$-formula of length k if φ is one of $\eta_0,\ldots,\eta_{k-1}$ then φ is an axiom of T^*. Otherwise, it is not.

By the definition of T^*, it is clear that T^* satisfies (3.20). If $T^* \vdash^{K^*} \varphi$ for an $\mathcal{L}$-sentence φ, then $T \vdash^{K^*} \varphi$. Hence, there is $n \in \mathbb{N}$ such that $\varphi = \psi_n$. Then the sequence η_n, $(\eta_n \to \psi_n)$, ψ_n is a proof of φ in T^*. ❑ (Theorem 3.8)

The theorem above is a variation of Craig's Theorem [Craig 1953]. With a similar proof, we can even prove the improvement of Theorem 3.8 in which T may be only recursively enumerable and T^* can be taken to be primitive recursive (Exercise).

3.2 Diagonal lemma

diagonal
logic-17-2019-07-01.pdf

In the following, we work (mainly) in meta-mathematics.

As before, Z denotes the Zermelo set theory in the language $\mathcal{L}_\varepsilon = \{\varepsilon\}$. $\mathsf{Z}_{\{\}}$ is the canonical expansion of Z in $\mathcal{L}_{\{\}} = \{\emptyset, \{\cdot,\cdot\}, \cdot\cup\cdot, \varepsilon\}$. Z_0 is the minimal subset of $\mathsf{Z}_{\{\}}$ containing everything we need in the following discussion, and T, T' etc. are concretely given theories extending Z_0.

In Z_0, V_ω is only a definable class, since the existence of V_ω cannot be proved in Z ([Mathias 2001]). Note that the formula "$x \;\varepsilon\; V_\omega$" can be defined by

$$(\exists n \;\varepsilon\; \omega)(\text{“}x \;\varepsilon\; V_n\text{”}). \tag{3.24}$$ diag-0

Note that, for any closed $\mathcal{L}_{\{\}}$-term t, we have $\mathsf{Z}_0 \vdash t \;\varepsilon\; V_\omega$.

Recall that we assumed that, in meta-mathematics, all the symbols are pairs of numbers, that is, elements of $\mathbb{N}^2$ and thus the collection of finite sequence Seq of symbols in meta-mathematics corresponds to the set ${}^{\omega>}(\omega)^2$ in Z_0. Note that, in contrast to V_ω, ${}^{\omega>}(\omega)^2$ is a set (i.e. $\mathsf{Z}_0 \vdash \exists x(\text{“}x \equiv {}^{\omega>}(\omega^2)\text{”})$).

In meta-mathematics, we also have to consider finite sequences of finite sequences of symbols. Proofs, for example, are such objects. We denote the collection of all finite sequences of finite sequences of symbols by $Seq(Seq)$.

For a finite sequence $s \in Seq$ of symbols, we defined a closed $\mathcal{L}_{\{\}}$-term $\ulcorner s \urcorner$ which represents the set in Z_0 which corresponds to s. If s is a sequence of length ℓ and its kth symbol is (coded by) the ordered pair $\langle m, n\rangle$ then $\mathsf{Z}_0 \vdash$ "the length of $\ulcorner s \urcorner$ is $\underline{\ell}$" and $\mathsf{Z}_0 \vdash$ "the $\underline{k}$th symbol of $\ulcorner s \urcorner$ is $\langle \underline{m}, \underline{n}\rangle$". For $s \in Seq(Seq)$, $\ulcorner s \urcorner$ is defined similarly.

---

(33) "$n+1$" is set here to make η_0 not an empty sequence but η_0 itself.

In Z_0, $\mathit{Term}_{\mathcal{L}_{\{\}}}$ denotes the set of all $\mathcal{L}_{\{\}}$-terms. That is, $\mathit{Term}_{\mathcal{L}_{\{\}}}$ is the set in Z_0 which corresponds to the collection of $\mathcal{L}_{\{\}}$-terms in meta-mathematics. We have $\mathsf{Z}_0 \vdash$ "$\mathit{Term}_{\mathcal{L}_{\{\}}} \subseteq {}^{\omega>}(\omega^2)$". Hence, $\mathit{Term}_{\mathcal{L}_{\{\}}}$ is really a set in the theory Z_0 by the instance of the Axiom of Separation (which should be) available in Z_0. Similarly, let $\mathit{ClTerm}_{\mathcal{L}_{\{\}}}$, $\mathit{Fml}_{\mathcal{L}_{\{\}}}$, and $\mathit{Sent}_{\mathcal{L}_{\{\}}}$ be the set of all closed $\mathcal{L}_{\{\}}$-terms, all $\mathcal{L}_{\{\}}$-formulas, and all $\mathcal{L}_{\{\}}$-sentences, respectively. We do not introduce these as new constant symbols but rather treat them as auxiliary symbols. For example, if we write "$t \in \mathit{Term}_{\mathcal{L}_{\{\}}}$", we consider this as an abbreviation of an $\mathcal{L}_{\{\}}$-formula which expresses that "t is an $\mathcal{L}_{\{\}}$-term".

More generally, for an arbitrary language $\mathcal{L}$, that is, for an arbitrary subset $\mathcal{L}$ of $\{4,5,6\} \times \omega$ extending $\mathcal{L}_{\{\}}$, we let $\mathit{Term}_{\mathcal{L}}$, $\mathit{ClTerm}_{\mathcal{L}}$, $\mathit{Fml}_{\mathcal{L}}$, and $\mathit{Sent}_{\mathcal{L}}$ be the set of all $\mathcal{L}$-terms, all closed $\mathcal{L}$-terms, all $\mathcal{L}$-formulas, and all $\mathcal{L}$-sentences, respectively.

For each concretely given number n, we can define the closed $\mathcal{L}_{\{\}}$-term $\underline{n}$ which is called a *numeral* of n by

(3.25) $\underline{0} = \emptyset$; and diag-1
$\underline{n+1} = \underline{n} \cup \{\underline{n}, \underline{n}\}$ for all $n \in \mathbb{N}$.

In Z_0, we can define a class function $(\cdot)^{V_\omega} : \mathit{ClTerm}_{\mathcal{L}_{\{\}}} \to V_\omega$ such that $(t)^{V_\omega}$ is the interpretation of the closed $\mathcal{L}_{\{\}}$-term t in the structure $\langle V_\omega, \emptyset, \{\cdot,\cdot\}, \cdot \cup \cdot, \in \rangle$. The formula "$(x)^{V_\omega} \equiv y$" can be defined as

(3.26) "$x \mathrel{\varepsilon} \mathit{ClTerm}_{\mathcal{L}_{\{\}}}$ and there are z and f such that $z \subseteq \mathit{ClTerm}_{\mathcal{L}_{\{\}}}$ $x \mathrel{\varepsilon} z$, z is closed with respect to sub-terms, f is a mapping on z, $f($"$\emptyset$"$)$ is the empty set(34), if "$\{t_0, t_1\}$" ε z then $f($"$\{t_0, t_1\}$"$)$ is the set consisting of $f(t_0)$ and $f(t_1)$, if "$t_0 \cup t_1$" ε z then $f($"$t_0 \cup t_1$"$)$ is the union of $f(t_0)$ and $f(t_1)$, and $y \equiv f(x)$. diag-1-a

L-diag-0

Lemma 3.9 (1) *For every closed $\mathcal{L}_{\{\}}$-term t, we have $\mathsf{Z}_0 \vdash t \mathrel{\varepsilon} \mathsf{V}_\omega$.*

(2) $\mathsf{Z}_0 \vdash$ *"$(\forall t \mathrel{\varepsilon} \mathit{ClTerm}_{\mathcal{L}_{\{\}}})((t)^{V_\omega} \mathrel{\varepsilon} V_\omega)$".*

(3) $\mathsf{Z}_0 \vdash$ *"$(\forall x \mathrel{\varepsilon} V_\omega)(\exists t \mathrel{\varepsilon} \mathit{ClTerm}_{\mathcal{L}_{\{\}}})((t)^{V_\omega} \equiv x)$".* ❑

In Z_0, let $\sqsubset$ be a fixed definable well-ordering on ${}^{\omega>}(\omega^2)$ of order type ω. For all $a \in V_\omega$, there is the $\sqsubset$-minimal $t \in \mathit{ClTerm}_{\mathcal{L}_{\{\}}}$ such that $(t)^{V_\omega} = a$ by Lemma 3.9, (3). We denote this t by $\llcorner a \lrcorner$. Actually, we introduced here a class function $\llcorner \cdot \lrcorner : V_\omega \to \mathit{ClTerm}_{\mathcal{L}_{\{\}}}$. We have the following:

L-diag-1

Lemma 3.10 (1) *$Z_0 \vdash (\forall x \mathrel{\varepsilon} V_\omega)\,((\llcorner x \lrcorner)^{V_\omega} \equiv x)$.*

(2) *$Z_0 \vdash (\forall t \mathrel{\varepsilon} \mathit{ClTerm}_{\mathcal{L}_{\{\}}})\,((t)^{V_\omega} \equiv (\llcorner (t)^{V_\omega} \lrcorner)^{V_\omega})$.* ❑

For a concretely given sequence s of symbols, let $\ulcorner s \urcorner$ be the closed $\mathcal{L}_{\{\}}$-term which expresses the element of ${}^{\omega>}(\omega^2)$ corresponding to s, we choose $\ulcorner s \urcorner$ to be the $\sqsubset$-minimal

(34) "$\emptyset$" denotes here the sequence of symbols of length 1 whose single entry is the symbol '$\emptyset$'.

among such closed $\mathcal{L}_{\{\}}$-terms[35] where $\sqsubset$ here is the binary relation on sequences on symbols in meta-mathematics which corresponds to the relation $\sqsubset$ in Z_0. Since $\sqsubset$ in Z_0 is definable, we can use the same definition in the meta-mathematics to introduce the meta-mathematical version of $\sqsubset$.

Also we modify the definition of the numerals $\underline{n}$, $n \in \mathbb{N}$ such that $\underline{n}$ is the $\sqsubset$-minimal closed $\mathcal{L}_{\{\}}$-term t with $\mathsf{Z}_0 \vdash \underline{n}_0 \equiv t$ where $\underline{n}_0$ denotes the original numeral for n as defined in (3.25). By (3.25) and the modified definition above, we have

(3.27) $\mathsf{Z}_0 \vdash \underline{0} \equiv \emptyset$; and diag-1-0

$\mathsf{Z}_0 \vdash \underline{n+1} \equiv \underline{n} \cup \{\underline{n}, \underline{n}\}$ for all $n \in \mathbb{N}$.

L-diag-2

Lemma 3.11 *For any concretely given closed $\mathcal{L}_{\{\}}$-term t and number $n \in \mathbb{N}$, we have*

(1) $\mathsf{Z}_0 \vdash \llcorner t \lrcorner \equiv \ulcorner t \urcorner$,

(2) $\mathsf{Z}_0 \vdash \llcorner \underline{n} \lrcorner \equiv \ulcorner \underline{n} \urcorner$,

(3) $\mathsf{Z}_0 \vdash (\ulcorner t \urcorner)^{V_\omega} \equiv t$, *and*

(4) $\mathsf{Z}_0 \vdash (\ulcorner \underline{n} \urcorner)^{V_\omega} \equiv \underline{n}$. ❑

Note that, in (1) above, $\llcorner \cdot \lrcorner$ is an operation in Z_0 while $\ulcorner \cdot \urcorner$ is an "operation" in met-mathematics, and $\ulcorner t \urcorner$ for a concretely given $\mathcal{L}_{\{\}}$-term t is another concretely given $\mathcal{L}_{\{\}}$-term.

Since T is concretely given, we can introduce a predicate "$\varphi \mathrel{\varepsilon} \ulcorner\ulcorner T \urcorner\urcorner$" in $\mathcal{L}_{\{\}}$ which corresponds the meta-mathematical "φ is an $\mathcal{L}_{\{\}}$-sentence in T". Since $\{\varphi \mathrel{\varepsilon} \mathit{Sent}_{\mathcal{L}} : \varphi \mathrel{\varepsilon} \ulcorner\ulcorner T \urcorner\urcorner\} \subseteq {}^{\omega>}(\omega^2)$ is a set, we treat the symbol $\ulcorner\ulcorner T \urcorner\urcorner$ also to denote this set though we do not introduce this symbol formally as a new constant symbol. $\ulcorner\ulcorner T \urcorner\urcorner$ should be a representation of T in the following sense:

L-diag-3

Lemma 3.12 *For any $\mathcal{L}$-sentence φ, (1) if φ is in T, then, we have $\mathsf{Z}_0 \vdash \ulcorner \varphi \urcorner \mathrel{\varepsilon} \ulcorner\ulcorner T \urcorner\urcorner$, and, (2) if φ is not in T, then we have $\mathsf{Z}_0 \vdash \ulcorner \varphi \urcorner \not\mathrel{\varepsilon} \ulcorner\ulcorner T \urcorner\urcorner$.* ❑

In Z_0, let $\mathit{Subst} : \mathit{Fml}_{\mathcal{L}} \times \omega \times \mathit{Term}_{\mathcal{L}_{\{\}}} \to \mathit{Fml}_{\mathcal{L}}$ be the definable function which is introduced by

(3.28) $\mathit{Subst}(\varphi, n, t) \equiv$ "the $\psi \mathrel{\varepsilon} \mathit{Fml}_{\mathcal{L}}$ obtained by replacing all the free appearance of the variable x_n $(= \langle 0, n \rangle)$ in φ with the term t". diag-2

L-diag-4

Lemma 3.13 *Suppose that φ is a concretely given $\mathcal{L}$-formula, s a concretely given sequence of symbols, and $k \in \mathbb{N}$.*

(1) $\mathsf{Z}_0 \vdash \llcorner \ulcorner s \urcorner \lrcorner \equiv \ulcorner (\ulcorner s \urcorner) \urcorner$.

(2) *If s is an $\mathcal{L}_{\{\}}$-term, then $\mathsf{Z}_0 \vdash \mathit{Subst}(\ulcorner \varphi \urcorner, \underline{n}, \ulcorner s \urcorner) \equiv \ulcorner \varphi(s/x_n) \urcorner$.*

---

(35) Note that, for example, $\mathcal{L}_{\{\}}$-terms "$\{\emptyset, \{\emptyset, \emptyset\}\}$", "$\{\{\emptyset, \emptyset\}, \emptyset\}$", "$\{\{\emptyset, \emptyset\}, \{\emptyset, \emptyset\}\} \cup \{\emptyset, \emptyset\}$" are proved to be $\equiv$ to each other in Z_0.

(3) $\mathsf{Z}_0 \vdash \mathit{Subst}(\ulcorner\varphi\urcorner, \underline{n}, \ulcorner\underline{k}\urcorner) \equiv \ulcorner\varphi(\underline{k}/x_n)\urcorner$

(4) $Z_0 \vdash \mathit{Subst}(\ulcorner\varphi\urcorner, \underline{n}, \llcorner\ulcorner s\urcorner\lrcorner) \equiv \ulcorner\varphi(\ulcorner s\urcorner/x_n)\urcorner$.

(5) $Z_0 \vdash \mathit{Subst}(\ulcorner\varphi\urcorner, \underline{n}, \llcorner\underline{k}\lrcorner) \equiv \ulcorner\varphi(\underline{k}/x_n)\urcorner$.

Proof. (1): By Lemma 3.11, (1).

(2): Note that we have $\mathsf{Z}_0 \vdash \ulcorner s\urcorner \;\varepsilon\; \mathit{Term}_{\mathcal{L}_{\{\}}}$. This as well as $\mathsf{Z}_0 \vdash \mathit{Subst}(\ulcorner\varphi\urcorner, \underline{n}, \ulcorner s\urcorner) \equiv \ulcorner\varphi(s/x_n)\urcorner$ can be proved by induction on the definition of $\ulcorner s\urcorner$ along with the construction of s — for the induction step, use: $\varphi(f(t_0, ..., t_\ell)/x_n) = \varphi'(t_0/v_0, ..., t_{\ell-1}/v_{\ell-1})$.

(3): By (2).

(4): By (1) and (2).

(5): By (3) and Lemma 3.11, (2). ❑ (Lemma 3.13)

The following Theorem 3.14, which we already cited as Theorem 3.2, is also known as the Fixed-Point Theorem. This theorem is the essence of the proofs of Gödel's Incompleteness Theorems in [Gödel 1932]. Theorem 3.14 was extracted from the original proofs in [Gödel 1932] of these theorems by R. Carnap in [Carnap 1934a] and [Carnap 1934b].

Theorem 3.14 (Diagonal Lemma) *For an arbitrary formula ψ in a language $\mathcal{L}$ extending $\mathcal{L}_{\{\}}$, there is an $\mathcal{L}$-formula σ such that* diagonal-lemma

(3.29) $\mathit{free}(\sigma) \subseteq \mathit{free}(\psi) \setminus \{x_0\}$, *and* diag-2-0

(3.30) $Z_0 \vdash \sigma \leftrightarrow \psi(\ulcorner\sigma\urcorner/x_0)$. diag-2-1

Proof. If x_0 is not free in ψ, we can simply take σ to be ψ.

Thus, we may assume that $x_0 \in \mathit{free}(\psi)$.

In Z_0, let $f^* : \left({}^{\omega>}(\omega^2)\right)^2 \to {}^{\omega>}(\omega^2)$ be defined by

(3.31) $$f^*(s,t) = \begin{cases} u & \text{if } s \;\varepsilon\; \mathit{Fml}_{\mathcal{L}} \text{ and } u \equiv \mathit{Sbst}(s, 0, \llcorner t\lrcorner) \\ \emptyset & \text{otherwise.} \end{cases}$$ diag-3

Claim 3.14.1 *Suppose that φ is a concretely given $\mathcal{L}$-formula and s a concretely given sequence of symbols. Then $\mathsf{Z}_0 \vdash f^*(\ulcorner\varphi\urcorner, \ulcorner s\urcorner) \equiv \ulcorner\varphi(\ulcorner s\urcorner/x_0)\urcorner$.* Cl-diag-0

$\vdash$ By Lemma 3.13, (1) and (3.31), $\mathsf{Z}_0 \vdash f^*(\ulcorner\varphi\urcorner, \ulcorner s\urcorner) \equiv \mathit{Subst}(\ulcorner\varphi\urcorner, \underline{0}, \ulcorner\ulcorner s\urcorner\urcorner)$. By Lemma 3.13, (2), it follows that $\mathsf{Z}_0 \vdash f^*(\ulcorner\varphi\urcorner, \ulcorner s\urcorner) \equiv \ulcorner\varphi(\ulcorner s\urcorner/x_0)\urcorner$. $\dashv$ (Claim 3.14.1)

In meta-mathematics, let k be the first index such that x_k does not appear in ψ. Let

(3.32) s^* be the closed $\mathcal{L}_{\{\}}$-term $\ulcorner\forall x_k\,(f^*(x_0, x_0) \equiv x_k \;\rightarrow\; \psi(x_k/x_0))\urcorner$, and let diag-4

(3.33) σ be the $\mathcal{L}$-sentence $\forall x_k\,(f^*(s^*, s^*) \equiv x_k \;\rightarrow\; \psi(x_k/x_0))$. diag-5

We show that this σ is as desired. Clearly σ satisfies (3.29).

By (3.32) and Claim 3.14.1, we have

(3.34) $\quad Z_0 \vdash f^*(s^*, s^*) \equiv \underbrace{\ulcorner\forall x_k\,(f^*(s^*,s^*) \equiv x_k \;\rightarrow\; \psi(x_k/x_0))\urcorner}_{=\ulcorner\sigma\urcorner \quad \text{by (3.33)}}.$ diag-6

Thus, by (3.33) and (3.34),

(3.35) $\quad \mathsf{Z}_0 \vdash (\sigma \;\rightarrow\; \psi(\ulcorner\sigma\urcorner/x_0)).$ diag-7

On the other hand, noting that

(3.36) $\quad (\psi(\ulcorner\sigma\urcorner/x_0) \;\rightarrow\; (f^*(s^*,s^*) \equiv x_k \;\rightarrow\; \psi(\ulcorner\sigma\urcorner/x_0)))$ diag-8

is a tautology, and that x_k does not appear in ψ by the choice of k, we have

(3.37) $\quad Z_0 \vdash (\psi(\ulcorner\sigma\urcorner/x_0) \;\rightarrow\; \forall x_k\,(f^*(s^*,s^*) \equiv x_k \;\rightarrow\; \psi(\ulcorner\sigma\urcorner/x_0)))$ diag-9

by Axiom of Substitution and Modus Ponens.

By (3.34), it follows that

(3.38) $\quad Z_0 \vdash (\psi(\ulcorner\sigma\urcorner/x_0) \;\rightarrow\; \underbrace{\forall x_k\,(f^*(s^*,s^*) \equiv x_k \;\rightarrow\; \psi(x_k/x_0))}_{=\sigma}).$ diag-10

Thus

(3.39) $\quad Z_0 \vdash (\psi(\ulcorner\sigma\urcorner/x_0) \rightarrow\; \sigma).$ diag-11

❏ (Theorem 3.14)

In the rest of the present subsection, we shall examine several applications of the Diagonal Lemma.

We already saw that Tarski's Undefinability of Truth is a direct consequence of the Diagonal Lemma (Corollary 3.3). We also have the following semantic version of Corollary 3.3.

For $\mathcal{L}$ and T as in Corollary 3.3, a predicate "$\mathsf{V} \models \cdot$" expressing "$\cdot$ holds in the universe V" should satisfy at least the following two properties:

(3.40) $\quad$ For any extension T' of T in $\mathcal{L}$, if $T' \vdash \varphi$, then $T' \vdash$ "$\mathsf{V} \models \ulcorner\varphi\urcorner$"; incompl-3-0

(3.41) $\quad T \vdash \forall\varphi \mathrel{\varepsilon} \mathit{Fml}_{\mathcal{L}}\,($"$\mathsf{V} \models \neg\varphi$" $\;\rightarrow\; \neg$"$\mathsf{V} \models \varphi$"$)$. incompl-3-1

Theorem 3.15 (Semantic Version of the Undefinability of Truth) *Let $\mathcal{L}$ and T be as in Corollary 3.3. Then there is no $\mathcal{L}$-formula $\chi(x)$ such that* T-incompl-0-1

(3.40′) $\quad$ *For some extension T' of T in $\mathcal{L}$, if $T' \vdash \varphi$, then $T' \vdash \chi(\ulcorner\varphi\urcorner)$;*

(3.41′) $\quad T \vdash \forall\varphi \mathrel{\varepsilon} \mathit{Fml}_{\mathcal{L}}\,(\chi(\ulcorner\neg\varphi\urcorner) \;\rightarrow\; \neg\chi(\ulcorner\varphi\urcorner)).$

Proof. Suppose, toward a contradiction, that there is a $\mathcal{L}$-fromula $\chi(x)$ with (3.40′) and (3.41′). Without loss of generality, we may assume that T' in (3.40′) is identical with T.

Let ψ be the formula $\neg\chi$, and let φ be a fixed point of ψ. That is, φ is an $\mathcal{L}$-sentence with

(3.42) $T \vdash \varphi \leftrightarrow \neg\chi(\ulcorner\varphi\urcorner)$. incompl-3-2

Claim 3.15.1 $T \vdash \varphi$.

$\vdash$ $T, \neg\varphi \vdash \chi(\ulcorner\neg\varphi\urcorner)$ by (3.40′). Thus, $T, \neg\varphi \vdash \neg\chi(\ulcorner\varphi\urcorner)$ by (3.41′) and by Deduction Theorem (Theorem 2.3).

On the other hand, we also have $T, \neg\varphi \vdash \chi(\ulcorner\varphi\urcorner)$ by (3.42) and by Dedcuction Theorem.

Thus, by Lemma 2.4, and again by Dedcuction Theorem, we have $T \vdash (\neg\varphi \rightarrow (\chi(\ulcorner\varphi\urcorner) \wedge \neg\chi(\ulcorner\varphi\urcorner)))$. Since $(\neg\varphi \rightarrow (\chi(\ulcorner\varphi\urcorner) \wedge \neg\chi(\ulcorner\varphi\urcorner))) \rightarrow \varphi$, is a tautology, it follows that $T \vdash \varphi$. $\dashv$ (Claim 3.15.1)

Now by Claim 3.15.1 and (3.40′), we have $T \vdash \chi(\ulcorner\varphi\urcorner)$. On the other hand, by Claim 3.15.1 and (3.42), we have also $T \vdash \neg\chi(\ulcorner\varphi\urcorner)$. This is a contradiction to the assumption that T is consistent. □ (Theorem 3.15)

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$L \prec L$ `if` $0^{\#}$ `exists!` $0^{\#}$ `defines the truth in` L

The next theorem can be proved using Representability Theorem (Theorem 3.1) and Diagonal Lemma (Theorem 3.2).

T-incompl-1

Theorem 3.16 *Suppose that T is a concretely given consistent theory in a language $\mathcal{L}$ extending $\mathcal{L}_{\{\}}$ with $T \supseteq \mathsf{Z}_0$. Then T is not decidable, that is,* $\mathsf{Th}(T) = \{\varphi : \varphi$ is an $\mathcal{L}$-sentence and $T \vdash \varphi\}$ *is not recursive.*

Proof. Assume, toward a contradiction, that T is a concretely given consistent theory in a language $\mathcal{L}$ such that $T \supseteq \mathsf{Z}_0$ and $\mathsf{Th}(T)$ is decidable. By the Representability Theorem 3.1, there is an $\mathcal{L}_{\{\}}$-formula $\psi = \psi(x_0)$ such that, for any $\mathcal{L}$-sentence σ,

(3.43) if $\sigma \in \mathsf{Th}(T)$ then $T \vdash \psi(\ulcorner\sigma\urcorner)$ and incompl-4

(3.44) if $\sigma \notin \mathsf{Th}(T)$ then $T \vdash \neg\psi(\ulcorner\sigma\urcorner)$. incompl-5

By the Diagonal Lemma (Theorem 3.2), there is an $\mathcal{L}_{\{\}}$-sentence σ_0 such that

(3.45) $T \vdash \sigma_0 \;\leftrightarrow\; \neg\psi(\ulcorner\sigma_0\urcorner)$. incompl-6

Claim 3.16.1 $T \nvdash \sigma_0$.

$\vdash$ Suppose $T \vdash \sigma_0$. By (3.45), it follows that $T \vdash \neg\psi(\ulcorner\sigma_0\urcorner)$. Since T is assumed to be consistent, we have $T \nvdash \psi(\ulcorner\sigma_0\urcorner)$. By (3.43), it follows that $T \nvdash \sigma_0$. This is a contradiction. $\dashv$ (Claim 3.16.1)

By Claim 3.16.1 and by (3.44), it follows that $T \vdash \neg\psi(\ulcorner\sigma_0\urcorner)$. Thus, by (3.45), $T \vdash \sigma_0$. This is a contradiction to Claim 3.16.1. □ (Theorem 3.16)

Corollary 3.17 (A version of the First Incompleteness Theorem) *Suppose that T is a concretely given consistent theory in a language $\mathcal{L}$ extending $\mathcal{L}_{\{\}}$ with $T \supseteq \mathsf{Z}_0$. Then T is not complete.* T-incompl-2

Proof. If T were complete then $\mathsf{Th}(T)$ would be recursive (By enumerating the proofs in the language $\mathcal{L}$ of T as $\mathcal{P}_0$, $\mathcal{P}_1$,..., we can decide whether φ is in $\mathsf{Th}(T)$ or not by checking each these proofs one by one if it is a proof of φ or $\neg\varphi$). This would be a contradiction to Theorem 3.16. ❑ (Corollary 3.17)

Note that the Corollary above assumes only the consistency of the theory T, while Gödel's original proof of the First Incompleteness Theorem needed the ω-consistency of the theory. To reduce the assumtipn to the consistency of T, Rosser's trick is necessary if we want to stick to Gödel's original idea of the proof. On the other hand, the Gödel-Rosser proof of the First Incompleteness Theorem is better than the one given above in that the proof gives a concrete example of an independent sentence over T which, intuitively, should be "true".

3.3 Gödel's Speed-up Theorem and Incompleteness Theorems

G-speed-up

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In Z_0, let $\mathit{proof}(x_0, x_1, x_2)$ be the $\mathcal{L}_{\{\}}$-formula: "x_0 is a theory in a language $\mathcal{L} \subseteq \{4,5,6\} \times \omega$, x_2 is an $\mathcal{L}$-formula and x_1 is a proof of x_2 from x_0 in the deduction system K^*".

Let $\mathit{consis}(x_0)$ be the $\mathcal{L}_{\{\}}$-formula: $\neg\exists x_1 \mathit{proof}(x_0, x_1, \ulcorner x \not\equiv x \urcorner)$.

For a concretely given theory T, $\ulcorner\ulcorner T \urcorner\urcorner$ is not a term in the language of Z_0. We merely introduce an abbreviation "$x \mathrel{\varepsilon} \ulcorner\ulcorner T \urcorner\urcorner$" of a corresponding formula which should satisfy:

(3.46) For all concretely given formula φ, if φ belongs to T, then we have $Z_0 \vdash \ulcorner\varphi\urcorner \mathrel{\varepsilon} \ulcorner\ulcorner T \urcorner\urcorner$; if φ does not belong to T, then $Z_0 \vdash \ulcorner\varphi\urcorner \not\mathrel{\varepsilon} \ulcorner\ulcorner T \urcorner\urcorner$. diag-12

The expressions $\mathit{proof}(\ulcorner\ulcorner T \urcorner\urcorner)$ and $\mathit{consis}(\ulcorner\ulcorner T \urcorner\urcorner)$ are to be understood as abbreviations of $\mathcal{L}_{\{\}}$-formulas expressing "for all x_0, if $x_0 = \ulcorner\ulcorner T \urcorner\urcorner$ then $\mathit{proof}(x_0, x_1, x_2)$" and "for all x_0, if $x_0 = \ulcorner\ulcorner T \urcorner\urcorner$ then $\mathit{consis}(x_0)$", respectively.

T-G-speed-up-0

Theorem 3.18 (A variant of Gödel's Speed-up Theorem) *Let $T \supseteq \mathsf{Z}_0$ be a concretely given consistent theory in a finite language $\mathcal{L} \supseteq \mathcal{L}_{\{\}}$. For any recursive function $f : \mathbb{N} \to \mathbb{N}$ there is an $\mathcal{L}_{\{\}}$-formula $\varphi = \varphi(x_1)$ satisfying:*

(3.47) *For any $n \in \mathbb{N}$, $T \vdash \varphi(\underline{n}/x_1)$. For any proof $\mathcal{P}$ in $\mathcal{L}$, if $T \vdash^{\mathcal{P}} \varphi(\underline{n}/x_1)$, then $L(\mathcal{P}) \geq f(n)$.* G-speed-up-0

(3.48) *$T, \mathit{consis}(\ulcorner\ulcorner T \urcorner\urcorner) \vdash (\forall x_1 \mathrel{\varepsilon} \omega)\varphi$.* G-speed-up-1

Remarks. (1) If $T, \mathit{consis}(\ulcorner\ulcorner T \urcorner\urcorner) \vdash^{\mathcal{P}^*} (\forall x_1 \mathrel{\varepsilon} \omega)\varphi$, then there are proofs $\mathcal{P}_n$, $n \in \mathbb{N}$ such that $T, \mathit{consis}(\ulcorner\ulcorner T \urcorner\urcorner) \vdash^{\mathcal{P}_n} \varphi(\underline{n}/x_1)$ and $L(\mathcal{P}_n) \leq p(L(\mathcal{P}^*), n)$ where p is a polynomials of low degree (1 or 2 depending on the proof system. Exercise: find optimal p for our deduction system K^*). Thus, for φ in Theorem 3.18 for fast growing f, we do have a tremendous spead-up, if we go from T to T + $consis(\ulcorner\ulcorner T\urcorner\urcorner)$.

(2) Similarly to footnote (36) , any theory concretely given T with $T \supseteq \mathsf{Z}_0$ has an mutually translatable theory $T' \supseteq \mathsf{Z}_0$ in a finite language.

Proof of Theorem 3.18. Let $\psi = \psi(x_0, x_1)$ be the $\mathcal{L}_{\{\}}$-formula:

$$x_0 \in Fml_{\mathcal{L}_{\{\}}} \wedge \forall p(L(p) < f(x_1) \rightarrow \neg proof(\ulcorner\ulcorner T\urcorner\urcorner, p, Subst(x_0, \underline{1}, \llcorner x_1 \lrcorner)). \tag{3.49}$$ G-speed-up-2

By Diagonal Lemma (Theorem 3.14), there is an $\mathcal{L}_{\{\}}$-formula $\varphi = \varphi(x_1)$ such that

$$\mathsf{Z}_0 \vdash \varphi \leftrightarrow \psi(\ulcorner\varphi\urcorner/x_0). \tag{3.50}$$ G-speed-up-3

We show that this φ is as desired.

Cl-G-speed-up-0

Claim 3.18.1 *For each $n \in \mathbb{N}$, we have $T \vdash \varphi(\underline{n}/x_1)$.*

$\vdash$ Suppose otherwise and assume that $T \nvdash \varphi(\underline{n}/x_1)$ for some $n \in \mathbb{N}$. Then, for any proof $\mathcal{P}$ in $\mathcal{L}$, we have $T \nvdash^{\mathcal{P}} \varphi(\underline{n}/x_1)$. In particular, this holds for all proofs $\mathcal{P}$ in $\mathcal{L}$ with $L(\mathcal{P}) < f(n)$. Since there are only finitely many such $\mathcal{P}$ (upto the renaming of variable symbols), it follows that

$$\mathsf{Z}_0 \vdash \forall p\,(L(p) < f(\underline{n}) \;\rightarrow\; \neg proof(\ulcorner\ulcorner T\urcorner\urcorner, p, Subst(\ulcorner\varphi\urcorner, \underline{1}, \llcorner\underline{n}\lrcorner)). \tag{3.51}$$ G-speed-up-4

Note that $\mathsf{Z}_0 \vdash Subst(\ulcorner\varphi\urcorner, \underline{1}, \llcorner\underline{n}\lrcorner) \equiv \ulcorner\varphi(\underline{n}/x_1)\urcorner$ by Lemma 3.13, (5). Thus, by (3.49) and (3.50), it follows that $\mathsf{Z}_0 \vdash \varphi(\underline{n}, x_1)$. This is a contradiction. $\dashv$ (Claim 3.18.1)

Cl-G-speed-up-1

Claim 3.18.2 *For each $n \in \mathbb{N}$, there is no proof $\mathcal{P}$ in $\mathcal{L}$ with $L(\mathcal{P}) < f(n)$ such that $T \vdash^{\mathcal{P}} \varphi(\underline{n}/x_n)$.*

$\vdash$ Suppose that there is some $n \in \mathbb{N}$ and a proof $\mathcal{P}$ such that $L(\mathcal{P}) < f(n)$ but

$$T \vdash^{\mathcal{P}} \varphi(\underline{n}/x_1). \tag{3.52}$$ G-speed-up-5

We have

$$\mathsf{Z}_0 \vdash \text{“}L(\ulcorner\mathcal{P}\urcorner) < f(\underline{n})\text{”} \;\wedge\; proof(\ulcorner\ulcorner T\urcorner\urcorner, \ulcorner\mathcal{P}\urcorner, \ulcorner\varphi(\underline{n})\urcorner). \tag{3.53}$$ G-speed-up-6

By Lemma 3.13, (5),

$$\mathsf{Z}_0 \vdash Subst(\ulcorner\varphi\urcorner, \underline{1}, \llcorner\underline{n}\lrcorner) \equiv \ulcorner\varphi(\underline{n}/x_1)\urcorner. \tag{3.54}$$ G-speed-up-7

Thus

$$\mathsf{Z}_0 \vdash \neg\psi(\ulcorner\varphi\urcorner/x_0, \underline{n}/x_1). \tag{3.55}$$ G-speed-up-8

On the other hand, (3.52) and (3.50) imply $\mathsf{Z}_0 \vdash \psi(\ulcorner\varphi\urcorner/x_0, \underline{n}/x_1)$. Since $T \supseteq \mathsf{Z}_0$ this is a contradiction to the assumption of the consistency of T. $\dashv$ (Claim 3.18.2)

Claim 3.18.3 $T + \mathit{consis}(\ulcorner\ulcorner T\urcorner\urcorner) \vdash (\forall x_1\ \varepsilon\ \omega)\varphi.$ Cl-G-speed-up-2

$\vdash$ In $T + \mathit{consis}(\ulcorner\ulcorner T\urcorner\urcorner)$, we can translate the meta-mathematical argument of Claim 3.18.1 into a formal proof in K^*. $\dashv$ (Claim 3.18.3)

❏ (Theorem 3.18)

Theorem 3.19 (The Second Incompleteness Theorem) *Suppose that $T \supseteq \mathsf{Z}_0$ is a concretely given consistent thoery. Then we have $T \nvdash \mathit{consis}(\ulcorner\ulcorner T\urcorner\urcorner)$.* T-G-speed-up-1

Proof. We may assume that the language $\mathcal{L}$ of T is finite.[36] Let $f : \mathbb{N} \to \mathbb{N}$ be a fast growing function (e.g. faster than any polynomial function) and let φ be as in Theorem 3.18 for T and this f.

If $T \vdash \mathit{consis}(\ulcorner\ulcorner T\urcorner\urcorner)$, then $T \vdash (\forall x_1 \in \omega)\varphi$ by (3.48). By Remark (1), this is a contradiction to (3.47). ❏ (Theorem 3.19)

At the moment, we do not know any natural mathematical example of speed-up. On the other hand, the next proposition shows that speed-up can be in need to prove any mathematical assertion when it is ill-formulated:

Proposition 3.20 *Supose that $T \supseteq \mathsf{Z}_0$ is a concretely given consistent theory in the language $\mathcal{L}$.*

Then, for any $\mathcal{L}$-sentence φ there is an $\mathcal{L}$-sentence χ such that

(3.56) $T \vdash \varphi \leftrightarrow \chi$, x-G-speed-up-a-0

(3.57) *if χ is provable from T then the length of its shortest proof exceeds any measure of feasibility; but* x-G-speed-up-a

(3.58) *there is a chance that $T + \mathit{consis}(\ulcorner\ulcorner T\urcorner\urcorner)$ proves χ with a feasible proof.*

Proof. By Gödel's Speed-up Theorem, there is an $\mathcal{L}$-sentence ψ such that

(3.59) $T \vdash \psi$ x-G-speed-up-0

but the shortest proof of ψ from T has the length, say, of more than the number of elementary particles in the whole universe, while $T + \mathit{consis}(T)$ proves ψ in a feasible manner.

Let $\chi := \varphi \wedge \psi$. By (3.59), this χ satisfies (3.56).

If φ is provable from T then χ is also provable from T. However since the proof of χ can be modified to form a proof of ψ by adding only few lines, the proof of χ should also have a non-feasible length.

On the other hand, if φ itself has a feasible proof in $T + \mathit{consis}(\ulcorner\ulcorner T\urcorner\urcorner)$, χ also has a feasible proof by the choice of ψ. ❏ (Proposition 3.20)

[36] Since we have numeric $\underline{n}$ for each $n \in \mathbb{N}$ we may replace m_n-ary relation symbol R_n for $n \in \mathbb{N}$, for example, with $R(\underline{n}, \cdot)$ where $R(\underline{n}, \{x_0, ..., x_{m_n-1}\})$ should correspond the relation symbol R_n.

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