On Reflection Theorems of Paracompactness

Sakaé Fuchino (渕野 昌)

Dept. of Computer Sciences
Kobe University
(神戸大学大学院 システム情報学研究科)

http://kurt.scitec.kobe-u.ac.jp/~fuchino/

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A reflection theorem on meta-Lindelöfness

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- The Main results are slight improvements of theorems in:

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Assume the Fodor-type Reflection Principle (FRP).

For a locally separable countably tight space X if all subspaces of X of cardinality $\leq \aleph_1$ are meta-Lindelöf then X is meta-Lindelöf.

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FRP: For any regular $\lambda \geq \aleph_2$, stationary $E \subseteq E^{\lambda}_{\omega}$ and for any $g : E \to [\lambda]^{\aleph_0}$ s.t. $g(\alpha)$ is a cofinal subset of α for all $\alpha \in E$ (i.e. g is a ladder system), there is $\eta < \lambda$ of cofinality ω_1 s.t. $\{x \in [\eta]^{\aleph_0} : \sup(x) \in E, g(\sup(x)) \subseteq x\}$ is stationary in $[\eta]^{\aleph_0}$.

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Axiom R, Rado's Conjecture and FRP

- FRP is shown to be equivalent to many mathematical reflection theorems over ZFC (S.F., H. Sakai, L. Soukup and T. Usuba (201?)).
- Some of the mathematical reflection theorems are previously known to be consequences of Axiom R.
- Axiom R implies FRP.
- Axiom R implies $2^{\aleph_0} \leq \aleph_2$.
- FRP imposes practically no restriction on the size of the continuum. More exactly, FRP is preserved by c.c.c. generic extension. In particular the reflection statements of Theorem 1 and many other theorems are consistent, say with $2^{\aleph_0} = \aleph_{2012}$.
- Rado's Conjecture implies FRP. Thus the reflection statements of Theorem 1 and many other theorems are also consequences of Rado's Conjecture. Rado's Conjecture also imply $2^{\aleph_0} \leq \aleph_2$ but Axiom R and Rado's Conjecture are independent.

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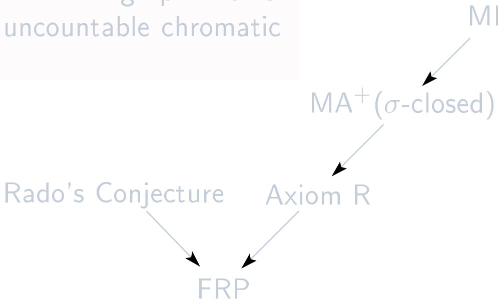
FRP, Axiom R, Rado's Conjecture etc.

Axiom R: For any cardinal $\lambda \geq \aleph_2$, for any stationary $S \subseteq [\lambda]^{\aleph_0}$ and for any ω_1-club $C \subseteq [\lambda]^{\aleph_1}$, there is $I \in C$ s.t. $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

Rado's Conjecture: For any linear ordering I and any intersection graph G consisting of some intervals of I, if G has uncountable chromatic number then there is a subgraph H of G of cardinality $\aleph_1$ with uncountable chromatic number.



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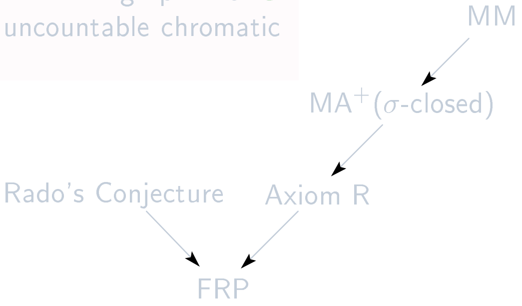

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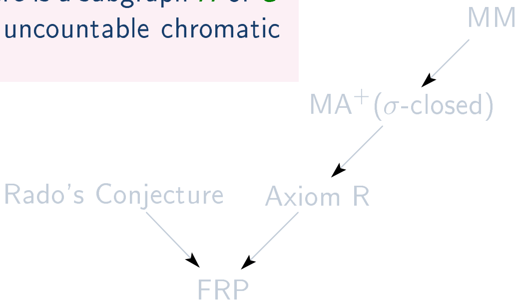

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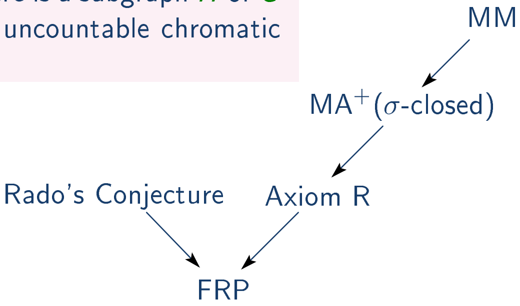
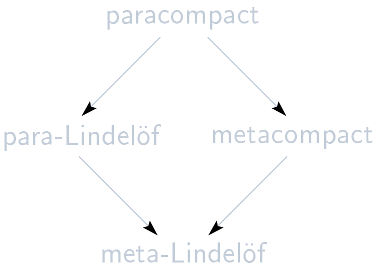


Paracompactness versus meta-Lindelöfness

A space $(X, \mathcal{O})$ is **meta-Lindelöf** if any open covering of X has an open refinement which is point countable.

A space $(X, \mathcal{O})$ is **paracompact** if any open covering of X has an open refinement which is locally finite.



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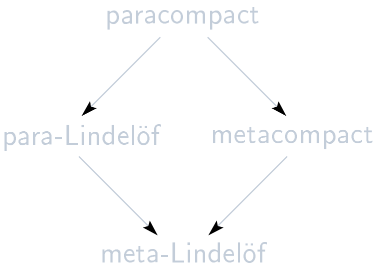

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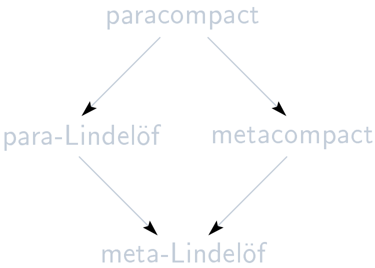

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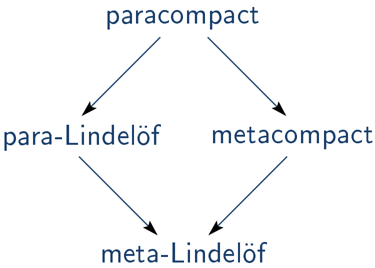

A simple example

Example 3. Let $X = \mathbb{R}$ and $Z = \{\frac{1}{z} : z \in \mathbb{Z}\}$. For $x \in \mathbb{R}$, $x \neq 0$ let $\mathcal{B}(x) = \{(x - \frac{1}{n}, x + \frac{1}{n}) : n \in \omega\}$ and $\mathcal{B}(0) = \{(-\frac{1}{n}, \frac{1}{n}) \setminus Z : n \in \omega\}$. $\mathcal{B}(x)$, $x \in X$ build a nbhd bases of a topology $\mathcal{O}$.
Then we have

(0) $\langle X, \mathcal{O} \rangle$ is separable.
(1) $\langle X, \mathcal{O} \rangle$ is not regular but Hausdorff.
(2) $\langle X, \mathcal{O} \rangle$ is meta-Lindelöf (or even metacompact and para-Lindelöf).
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Locally separable and Lindelöf spaces

Proposition 4. For a locally (separable & Lindelöf) space X, the following are equivalent:

(a) X has an open partition into Lindelöf spaces;
(b) X is paracompact;
(c) X is meta-Lindelöf.

Corollary 5. (to Theorem 1.) (FRP) For a locally (separable & Lindelöf) countably tight space X if all subspaces of X of cardinality $\leq \aleph_1$ are meta-Lindelöf then X is paracompact.

- The assertion of Corollary 5 is still equivalent to FRP over ZFC.

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Regular locally compact spaces under FRP

Theorem 6. (G. Gruenhage and P. Koszmider, 1996)
Assume $\mathrm{MA}_{\aleph_1}$. For any normal locally compact space X, X is paracompact if $\Leftrightarrow$ X is meta-Lindelöf.

Theorem 7. (S. Watson, 1982)
Assume $\mathrm{MA}_{\aleph_1}(\sigma\text{-centered})$ and that a Suslin tree exists. Then there is a normal locally compact space X which is meta-Lindelöf but not paracompact.

- Thus the following assertion is independent from ZFC:

(*) For any normal locally compact space X,
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Corollary 8. (*) above is independent from ZFC + FRP.

Proof. FRP is presereved by c.c.c. generic extension. □

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Sketch of the proof of Theorem 10. (3/5)

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Sketch of the proof of Theorem 10. (4/5)

- **Case II.:** λ is singular.
- We use the following Fact:

Theorem 11. (S.F. and A. Rinot (2011)) FRP implies SSH.

SSH: $\mathrm{cf}([\kappa]^{\mu}, \subseteq) = \kappa$ for all cardinals μ, κ with $\mu < \mathrm{cf}(\kappa)$.

Claim 2. For sufficiently large regular θ. Suppose that $M \prec \mathcal{H}(\theta)$ s.t. $\omega_1 \subseteq M$, X, $\langle L_\alpha : \alpha \in \lambda \rangle \in M$ and $[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$. Then $\bigcup\{L_\alpha : \alpha \in \lambda \cap M\}$ is a clopen subspace of M.

- Let $\langle M_i : i < \mathrm{cf}(\lambda) \rangle$ e an increasing sequence of elementary submodels of $\mathcal{H}(\theta)$ s.t. $\omega_1 \subseteq M_i$, X, $\langle L_\alpha : \alpha \in \lambda \rangle \in M_i$, $[M_i]^{\aleph_0} \cap M_i$ is cofinal in $[M_i]^{\aleph_0}$ and $\lambda \subseteq \bigcup_{i<\mathrm{cf}(\lambda)} M_i$.
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Sketch of the proof of Theorem 10. (5/5)

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- Let $X_i = \bigcup\{L_\alpha : \alpha \in \lambda \cap M_i\}$.
- By Claim 2, X_i, $i < \mathrm{cf}(\lambda)$ are clopen subspace of X and $L(X_i) < \kappa$. By the induction hypothesis, it follows that X_i, $i < \mathrm{cf}(\lambda)$ are all paracompact. For each $i < \mathrm{cf}(\lambda)$, let $\mathcal{C}_i$ be a locally finite open refinement of the covering $\{L_\alpha : \alpha \in \lambda \cap M_i\}$ of X_i.
- Let $\mathcal{C} = \bigcup_{i<\mathrm{cf}(\lambda)} \mathcal{C}_i$. For A, $B \in \mathcal{C}$, let $A \sim B :\Leftrightarrow A \cap B \neq \emptyset$ and let $\approx$ be the transitive closure of $\sim$.
- It is easy to see that each equivalence class $e \in \mathcal{C}/\approx$ is of cardinality $\leq \mathrm{cf}(\lambda) < \lambda$. Since $\bigcup e$ is a clopen subspace of X it is paracompact again by the induction hypothesis.
- Thus X is partitioned into open paracompact subspaces and hence X is paracompact as well. q.e.d. (Theorem 10.)

Sketch of the proof of Theorem 10. (5/5)

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Sketch of the proof of Theorem 10. (5/5)

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Some open problems

(1) Does the assertion of Theorem 10 imply FRP^R ?
(It does imply FRP)

(2) Is FRP^R really necessary for Theorem 10 ?

(3) Is FRP^R strictly stronger than FRP ?

(4) Does Rado's Conjecture imply FRP^R ?

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