数学ノート (2018-)

渕野 昌 (Sakaé Fuchino)

主な更新日: 19.05.20(Mo22:15(JST)) 18.08.11(Sa02:18(JST))

2024 年 2 月 22 日 (17:54) 版

目次

1 α-stationarity in $\mathcal{P}_\kappa(\lambda)$

For a cardinal κ and a set X, we use the expressions $\mathcal{P}_\kappa(X)$ and $[X]^{<\kappa}$ interchangeably to denote $\{a \subseteq X : |a| < \kappa\}$ except that when we write $\mathcal{P}_\kappa(X)$ for a cardinal κ, we always assume that $\kappa \subseteq X$. More generally, for sets X, Y with $Y \subseteq X$ we write $\mathcal{P}_Y(X)$ to denote $[X]^{<|Y|}$. alpha-stat

For $\alpha \in \mathsf{On} \setminus 1$, α-stationarity of subsets of $\mathcal{P}_Y(X)$ is defined by recursion on α as follows: $S \subseteq \mathcal{P}_Y(X)$ is *1-stationary* if $|Y|$ is regular uncountable and S is stationary in the sense of Jech (i.e. if it intersects with each club subset of $\mathcal{P}_Y(X)$).

For $\alpha \in \mathsf{On} \setminus 2$, S is *α-stationary* in $\mathcal{P}_Y(X)$ if, for any $0 < \beta < \alpha$ and β-stationary $T \subseteq \mathcal{P}_Y(X)$, there is $a \in S$ such that $T \cap \mathcal{P}_{Y\cap a}(a)$ is β-stationary in $\mathcal{P}_{Y\cap a}(a)$.

Similarly we say that $S \subseteq \mathcal{P}_Y(X)$ is *diagonally 1-stationary* if $|Y|$ is regular uncountable and S is stationary in the sense of Jech (i.e. if it intersects with each club subset of $\mathcal{P}_Y(X)$).

For $\alpha \in \mathsf{On} \setminus 2$, S is *diagonally α-stationary* in $\mathcal{P}_Y(X)$ if, for any $0 < \beta < \alpha$ and for any family $\langle T_x \subseteq \mathcal{P}_Y(X) : x \in X \rangle$ of sets such that each T_x is β-stationary subset of $\mathcal{P}_Y(X)$, there is $a \in S$ such that $T_x \cap \mathcal{P}_{Y \cap a}(a)$ is β-stationary in $\mathcal{P}_{Y\cap a}(a)$ for all $x \in a$.

Lemma 1.1 *For any $\alpha = 1$ or 2, $Y \subseteq X$ with regular uncountable $|Y|$ and $S \subseteq \mathcal{P}_Y(X)$, if S is diagonally α-stationary in $\mathcal{P}_Y(X)$ then S is α-stationary in $\mathcal{P}_Y(X)$.* P-a-stat-a-0

Proof. For $\alpha = 1$, 1-stationarity and diagonal 1-stationarity coincide with stationarity by definition. Suppose that $S \subseteq \mathcal{P}_Y(X)$ is diagonally 2-stationary. For 1-stationary $T \subseteq \mathcal{P}_Y(X)$, let $\langle T_x : x \in X \rangle$ be defined by $T_x = T$ for all $x \in X$. Note that all T_x, $x \in X$ are diagonally 1-stationary. By the diagonal 2-stationarity of S, there is $a \in S$ such that $T_x \cap \mathcal{P}_{Y\cap a}(a)$ is diagonally 1-stationary for all $x \in a$. By the definition of T_x, $x \in X$, it follows that $T \cap \mathcal{P}_{Y\cap a}(a)$ is 1-stationary. This shows that S is 2-stationary. □ (Lemma 1.1)

Lemma 1.2 *Suppose that κ is a supercompact cardinal. Then for any $\lambda \geq \kappa$ and $\alpha \in \kappa \setminus 1$, $\mathcal{P}_\kappa(\lambda)$ is α-stationary and diagonally α-stationary (in itself).* P-a-stat-0

Proof. We prove that $\mathcal{P}_\kappa(\lambda)$ is diagonally α-stationary for all $\alpha \in \kappa \setminus 1$. The proof for α-stationarity is similar.

For $\alpha = 1$, the assertion is trivial. So assume that $\alpha > 1$, $\beta \in \alpha \setminus 1$. Let $\lambda' = \lambda^\kappa$ and let $j : V \overset{\prec}{\rightarrow} M$ be such that $crit(j) = \kappa$, $j(\kappa) > \lambda'$ and

$$(1.1) \qquad {}^{\lambda'}M \subseteq M. \qquad \text{a-stat-0}$$

Then $j''\lambda \in M$ by (1.1).

Claim 1.2.1 *If $|x| < \kappa$ then $j(x) = j''x$.*

$\vdash$ Let $f : \delta \to x$ be a surjection for some $\delta < \kappa$. We have $M \models$ "$j(f)''j(\delta) \equiv j(x)$" by elementarity. Since $j(\delta) = \delta$, $M \models j(f)''\delta \equiv j(x)$. For $\alpha < \delta$, $M \models j(f)(\alpha) \equiv j(f(\alpha))$ by elementarity. Thus $j(x) = \{j(y) : y \in x\} = j''x$. $\dashv$ (Claim 1.2.1)

Suppose that $\vec{T} = \langle T_\xi : \xi < \lambda \rangle$ is a sequence of diagonally β-stationary subsets in $\mathcal{P}_\kappa(\lambda)$. Since

$$(1.2) \qquad (\mathcal{P}_\kappa(j''\lambda))^M = (\mathcal{P}_\kappa(j''\lambda))^V \text{ [1]}, \qquad \text{a-stat-0-0}$$

we have

$$(1.3) \qquad M \models j(\vec{T})_{j(\xi)} \cap \mathcal{P}_{j(\kappa)\cap j''\lambda}(j''\lambda) = j(T_\xi) \cap \mathcal{P}_\kappa(j''\lambda) = \{j(x) : x \in T_\xi\} \qquad \text{a-stat-1}$$

[1] $(\mathcal{P}_\kappa(j''\lambda))^M \subseteq (\mathcal{P}_\kappa(j''\lambda))^V$ is clear. To prove $(\mathcal{P}_\kappa(j''\lambda))^M \supseteq (\mathcal{P}_\kappa(j''\lambda))^V$, let $x \in (\mathcal{P}_\kappa(j''\lambda))^V$ and let $\langle \beta_\xi : \xi < \delta \rangle$ be an emumeration of x for some $\delta < \kappa$. Let $\alpha_\xi = j^{-1}(\beta_\xi)$ for all $\xi < \delta$. $x = j(\{\alpha_\xi : \xi < \delta\}) \in M$ by Claim 1.2.1.

for all $\xi \in \lambda$. Since $\mathcal{P}(\mathcal{P}_\kappa(j''\lambda)) \subseteq M$ by (1.2) and (1.1), we have $\mathcal{P}^M(\mathcal{P}_\kappa(j''\lambda)) = \mathcal{P}^V(\mathcal{P}_\kappa(j''\lambda))$ and $\left({}^{\lambda\prime}\mathcal{P}(\mathcal{P}_\kappa(j''\lambda))\right)^V \in M$ by (1.1).

It follows that

$$(1.4)\qquad M \models \exists a \in \mathcal{P}_{j(\kappa)}(j(\lambda))\,\forall \xi \in a\,\left(j(\vec{T})_\xi \cap \mathcal{P}_{j(\kappa)\cap a}(a) \text{ is diagonally } \beta\text{-stationary}\right).$$

By elementarity and since $\beta < \kappa$

$$(1.5)\qquad V \models \exists a \in \mathcal{P}_\kappa(\lambda)\,\forall \xi \in a\,(T_\xi \cap \mathcal{P}_{\kappa\cap a}(a) \text{ is diagonally } \beta\text{-stationary}).$$

This shows that $\mathcal{P}_\kappa(\lambda)$ is diagonally α-stationary. □ (Lemma 1.2)

Lemma 1.3 (1) *If $\mathcal{P}_\kappa(\lambda)$ is 2-stationary, then κ is a limit cardinal.* *P-a-stat-1*

(2) *If $S \subseteq \mathcal{P}_\kappa(\lambda)$ is 2-stationary then for any stationary $T \subseteq \mathcal{P}_\kappa(\lambda)$ there are stationarily many $r \in S$ such that $T \cap \mathcal{P}_{\kappa\cap r}(r)$ is stationary.*

(3) *If $\mathcal{P}_\kappa(\lambda)$ is 2-stationary, then κ is a weakly Mahlo cardinal.*

Proof. (1): Suppose that $\kappa = \mu^+$ then $C = \{a \in \mathcal{P}_\kappa(\lambda) : |a| = \mu\}$ is a club and hence stationary. But, for any $r \in \mathcal{P}_\kappa(\lambda)$, $|\kappa \cap r| \leq \mu$ and hence $C \cap \mathcal{P}_{\kappa\cap r}(r) = \emptyset$.

(2): Suppose that $S \subseteq \mathcal{P}_\kappa(\lambda)$ is 2-stationary and $T \subseteq \mathcal{P}_\kappa(\lambda)$ is stationary. Let $C \subseteq \mathcal{P}_\kappa(\lambda)$ be a club. We have to show that there is $r \in S \cap C$ such that $T \cap \mathcal{P}_{\kappa\cap r}(r)$ is stationary.

Let $f : {}^{\omega>}\lambda \to \lambda$ be such that

$$(1.6)\qquad C_f = \{a \in \mathcal{P}_\kappa(\lambda) : \kappa \cap a \in \kappa \text{ and } a \text{ is closed with respect to } f\} \subseteq C.$$

Since C_f is a club, $T \cap C_f$ is stationary. Let $r \in S$ be such that $(T \cap C_f) \cap \mathcal{P}_{\kappa\cap r}(r)$ is stationary in $\mathcal{P}_{\kappa\cap r}(r)$. We have

Claim 1.3.1 $\kappa \cap r \in \kappa$. *Cl-a-stat-0*

$\vdash$ Otherwise, there is an $\alpha \in \sup(\kappa \cap r) \setminus r$. Let $Y = \{b \in \mathcal{P}_{\kappa\cap r}(r) : \sup(\kappa \cap b) > \alpha\}$. Y is club in $\mathcal{P}_{\kappa\cap r}(r)$ but $Y \cap C_f = \emptyset$. Thus $(T \cap C_f) \cap \mathcal{P}_{\kappa\cap r}(r) \cap Y = \emptyset$. This is a contradiction to the stationarity of $(T \cap C_f) \cap \mathcal{P}_{\kappa\cap r}(r)$ in $\mathcal{P}_{\kappa\cap r}(r)$. $\dashv$ (Claim 1.3.1)

Claim 1.3.2 *r is closed with respect to f.* *Cl-a-stat-1*

$\vdash$ This is clear since $T \cap C_f$ is cofinal in $\mathcal{P}_{\kappa\cap r}(r)$ (with respect to $\subseteq$) and elements of $T \cap C_f$ are closed with respect to f. $\dashv$ (Claim 1.3.2)

From the Claims above it follows that $r \in C_f \subseteq C$ is as desired.

(3): Let $T = \{a \in \mathcal{P}_\kappa(\lambda) : \kappa \cap a \in \kappa\}$. T is a club and hence stationary. Let $r \in \mathcal{P}_\kappa(\lambda)$ be such that

$$(1.7)\qquad T \cap \mathcal{P}_{\kappa\cap r}(r) \text{ is stationary in } \mathcal{P}_{\kappa\cap r}(r).$$ a-stat-2

Similarly to Claim 1.3.1, we can show that $\kappa \cap r \in \kappa$.

Claim 1.3.3 *$\kappa \cap r$ is a cardinal.* Cl-a-stat-2

$\vdash$ Otherwise there is $\mu < \kappa \cap r$ such that $|\kappa \cap r| = \mu$. But then the set $\{a \in \mathcal{P}_{\kappa\cap r}(r) : \sup(a \cap \kappa) \geq \mu\}$ is a club in $\mathcal{P}_{\kappa\cap r}(r)$ disjoint from T. $\dashv$ (Claim 1.3.3)

Claim 1.3.4 *$\kappa \cap r$ is a regular cardinal.* Cl-a-stat-3

$\vdash$ Otherwise there is an $s \subseteq \kappa \cap r$ cofinal in $\kappa \cap r$ with $|s| < \kappa \cap r$. But then the set $\{a \in \mathcal{P}_{\kappa\cap r}(r) : \sup(a \cap \kappa) \supseteq s\}$ is a club in $\mathcal{P}_{\kappa\cap r}(r)$ disjoint from T. $\dashv$ (Claim 1.3.4)

Since there are stationarily many r with (1.7) by (2), it follows from Claim 1.3.4 that κ is weakly Mahlo. ❑ (Lemma 1.3)

A regular cardinal κ is said to be *c.c.c.-generically supercompact* if for every $\lambda \geq \kappa$ there is a c.c.c. poset $\mathbb{P}$ such that for a $(V, \mathbb{P})$-generic $\mathbb{G}$ there is an inner model $M \subseteq V[\mathbb{G}]$ with

(1.8) $\quad V[\mathbb{G}] \models$ "${}^{\lambda}M \subseteq M$" a-stat-2-0

and j with

(1.9) $\quad V[\mathbb{G}] \models$ "$j : V \xrightarrow{\prec} M$, $crit(j) = \kappa$ and $j(\kappa) > \lambda$". a-stat-2-1

This definition is stronger than the generic supercompactness by ccc posets in [9].

Lemma 1.4 *Suppose that M is an inner model of V with* P-a-stat-1-a

(1.10) $\quad {}^{\lambda}M \subseteq M$, *and* a-stat-2-2

(1.11) $\quad \mathbb{P} \in M$ *is a λ^+-c.c. poset.* a-stat-2-3

If $\mathbb{G}$ is a $(V, \mathbb{P})$-generic filter then, in $V[\mathbb{G}]$, we have ${}^{\lambda}M[\mathbb{G}] \subseteq M[\mathbb{G}]$.

Proof. Suppose that $\langle a_\alpha : \alpha < \lambda\rangle \in V[\mathbb{G}]$ is a sequence of elements of $M[\mathbb{G}]$. For each $\alpha < \lambda$, let $\underset{\sim}{a}_\alpha \in M$ be a $\mathbb{P}$-name such that $\underset{\sim}{a}_\alpha[\mathbb{G}] = a_\alpha$ such that $\langle \underset{\sim}{a}_\alpha : \alpha < \lambda\rangle \in M$.

We need the λ^+-c.c. of $\mathbb{P}$ to realize the last condition: Let $\underset{\sim}{f}$ be a $\mathbb{P}$-name of $\langle \underset{\sim}{a}_\alpha : \alpha < \lambda\rangle$ such that all properties of the sequence needed below is forced already by $\mathbb{1}_{\mathbb{P}}$ (maximal principle!). For each $\alpha < \lambda$, let $\langle \underset{\sim}{a}_{\alpha,\mathbb{p}} : \mathbb{p} \in S_\alpha\rangle$ where S_α is a maximal antichain such that, for each $\mathbb{p} \in S_\alpha$, we have $\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\alpha) = \underset{\sim}{a}_{\alpha,\mathbb{p}}$" and each $\underset{\sim}{a}_{\alpha,\mathbb{p}}$ is a $\mathbb{P}$-name in M. By the λ^+-c.c. of $\mathbb{P}$, each S_α, $\alpha < \lambda$ haves size $\leq \lambda$. Hence, by (1.10), $\langle\langle \underset{\sim}{a}_{\alpha,\mathbb{p}} : \mathbb{p} \in S_\alpha\rangle : \alpha < \lambda\rangle \in M$. The sequence $\langle \underset{\sim}{a}_\alpha : \alpha < \lambda\rangle \in M$ as desired can be conctructed easily from this sequence.

By the assumption on M, we have $\langle \underset{\sim}{a}_\alpha : \alpha < \lambda\rangle \in M$. Thus

(1.12) $\quad \underset{\sim}{s} = \{\langle op_{\mathbb{P}}(\underset{\sim}{a}_\alpha, \check{\alpha}), \mathbb{1}_{\mathbb{P}}\rangle : \alpha \in \lambda\} \in M$.

Since $\underset{\sim}{s}$ is a $\mathbb{P}$-name and $\underset{\sim}{s}[\mathbb{G}] = \langle \underset{\sim}{a}_\alpha[\mathbb{G}] : \alpha < \lambda\rangle = \langle a_\alpha : \alpha < \lambda\rangle$, it follows that $\langle a_\alpha : \alpha < \lambda\rangle \in M[G]$. ❑ (Lemma 1.4)

Corollary 1.5 *If the statement “there exists a supercompact cardinal” is consistent (over* ZFC*) then so is the statement “$2^{\aleph_0}$ is c.c.c.-generically supercompact.”* P-a-stat-1-a-0

Proof. Let κ be supercompact. Then, letting $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$, we have

(1.13) $\quad V[\mathbb{G}] \models$ “$2^{\aleph_0} = \kappa$ and κ is c.c.c.-generically supercompact”

for any $(V, \mathbb{P})$-generic $\mathbb{G}$. □ (Corollary 1.5)

Lemma 1.6 *Suppose that μ, κ, λ are regular uncountable cardinals with $\mu \leq \kappa \leq \lambda$.* P-a-stat-1-0

(1) *If $\mathbb{P}$ is a μ-cc poset and $\underset{\sim}{C}$ is a $\mathbb{P}$-name of a club subset of $\mathcal{P}_\kappa(\lambda)$ (in $V[\mathbb{G}]$), then there is $C \subseteq \mathcal{P}_\kappa(\lambda)$ (in V) such that C is club in $\mathcal{P}_\kappa(\lambda)$ and $\Vdash_{\mathbb{P}}$ “$C \subseteq \underset{\sim}{C}$”.*

(2) *If $S \subseteq \mathcal{P}_\kappa(\lambda)$ is stationary and $\mathbb{P}$ is a μ-cc poset, then we have $\Vdash_{\mathbb{P}}$ “$\check{S}$ is stationary in $\mathcal{P}_\kappa(\lambda)$”.*

Proof. (1): Let $C = \{x \in \mathcal{P}_\kappa(\lambda) : \Vdash_{\mathbb{P}}$ “$\check{x} \in \underset{\sim}{C}$”$\}$. It is easy to show that C is closed (with respect to $\subseteq$-increasing sequence of length $< \kappa$). To show that C is cofinal in $\mathcal{P}_\kappa(\lambda)$ (with respect to $\subseteq$), suppose $a \in \mathcal{P}_\kappa(\lambda)$. Let $\langle a_n, \underset{\sim}{a}_n : n \in \omega \rangle$ be a sequence such that

(1.14) $\quad a_0 = a$; a-stat-3

(1.15) $\quad \underset{\sim}{a}_n$ is a $\mathbb{P}$-name with $\Vdash_{\mathbb{P}}$ “$\check{a}_n \subseteq \underset{\sim}{a}_n \in \underset{\sim}{C}$”; a-stat-4

(1.16) $\quad a_{n+1} \in \mathcal{P}_\kappa(\lambda)$ and $\Vdash_{\mathbb{P}}$ “$\underset{\sim}{a}_n \subseteq \check{a}_{n+1}$”. a-stat-5

Note that (1.16) is possible by the μ-cc of $\mathbb{P}$. Let $b = \bigcup_{n \in \omega} a_n$. Then $a \subseteq b$ by (1.14) and $b \in C$ by (1.15) and (1.16).

(2): Suppose that $\not\Vdash_{\mathbb{P}}$ “$\check{S}$ is stationary in $\mathcal{P}_\kappa(\lambda)$”. Then there is $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}}$ “$\check{S}$ is non stationary in $\mathcal{P}_\kappa(\lambda)$”. Let $\underset{\sim}{C}$ be a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}}$ “$\underset{\sim}{C}$ is a club subset of $\mathcal{P}_\kappa(\lambda)$ and $\underset{\sim}{C} \cap \check{S} = \emptyset$”. By (1) (applied to $\mathbb{P} \downarrow p$), there is a club $C \subseteq \mathcal{P}_\kappa(\lambda)$ in V such that $p \Vdash_{\mathbb{P}}$ “$\check{C} \subseteq \underset{\sim}{C}$”. It follows that $C \cap S = \emptyset$. But this is a contradiction to the stationarity of S. □ (Lemma 1.6)

Lemma 1.7 *If κ is c.c.c.-generically supercompact then $\mathcal{P}_\kappa(\lambda)$ is diagonally 2-stationary for all $\lambda \geq \kappa$.* P-a-stat-2

Proof. In V, let $\vec{S}\langle S_\xi : \xi \in \lambda \rangle$ be a sequence of stationary subsets of $\mathcal{P}_\kappa(\lambda)$. Let $\mathbb{P}$, $\mathbb{G}$, M, j be as in the definition of the c.c.c.-genric supercomactness.

In $V[\mathbb{G}]$ we have $j''\lambda \in M$ by (1.8). Also by (1.8) and by (1.9), we have

(1.17) $\quad \mathcal{P}_\kappa(j''\lambda)^{V[\mathbb{G}]} = \mathcal{P}_\kappa(j''\lambda)^M = \mathcal{P}_{j(\kappa) \cap j''\lambda}(j''\lambda)^M$ and

(1.18) $\quad j(\vec{S})_{j(\xi)} \cap \mathcal{P}_{j(\kappa) \cap j''\lambda}(j''\lambda)^M = \{j''x : x \in S_\xi\}$ for all $\xi \in \lambda$.

Since $V[\mathbb{G}] \models$ "S_ξ is stationary in $\mathcal{P}_\kappa(\lambda)$" by Lemma 1.6, (2), we have $M \models$ "S_ξ is stationary in $\mathcal{P}_\kappa(\lambda)$".

Thus we have $M \models \forall\xi \in {}''\lambda\,(j((\vec{S})_\xi \mathcal{P}_{j(\kappa)\cap j''\lambda}(j''\lambda))$ is stationary) and

(1.19) $\quad M \models \exists a \in \mathcal{P}_{j(\kappa)}(j(\lambda))\,\forall\xi \in a\,\Big(j(\vec{S})_\xi \cap \mathcal{P}_{j(\kappa)\cap a}(a) \text{ is stationary}\Big).$

By elementarity, it follows that

(1.20) $\quad V \models \exists a \in \mathcal{P}_\kappa(\lambda)\,\forall\xi \in a\ (S_\xi \cap \mathcal{P}_{\kappa\cap a}(a) \text{ is stationary}).$

□ (Lemma 1.7)

For regular cardinals κ, λ, λ' with $\kappa \le \lambda \le \lambda'$ and $S' \subseteq \mathcal{P}_\kappa(\lambda')$, let

(1.21) $\quad S'_{\cap\lambda} = \{a \cap \lambda : a \in S'\}.$ a-stat-6

For $S \subseteq \mathcal{P}_\kappa(\lambda)$, let

(1.22) $\quad S^{\cup\lambda'\setminus\lambda} = \{a \cup b : a \in S,\, b \in \mathcal{P}_\kappa(\lambda' \setminus \lambda)\}.$ a-stat-7

Lemma 1.8 (0) *For all $\alpha \in \mathsf{On}$ if $S \subseteq \mathcal{P}_\kappa(\lambda)$ is (diagonally) α-stationary then any $\tilde{S} \subseteq \mathcal{P}_\kappa(\lambda)$ with $S \subseteq \tilde{S}$ is (resp., diagonally) α-stationary.* *P-a-stat-3*

(1) *For all $\alpha \in \mathsf{On}$, $S \subseteq \mathcal{P}_\kappa(\lambda)$ is (diagonally) α-stationary in $\mathcal{P}_\kappa(\lambda)$ if and only if $S^{\cup\lambda'\setminus\lambda}$ is (resp., diagonally) α-stationary in $\mathcal{P}_\kappa(\lambda')$.*

(2) *For all $\alpha \in \mathsf{On}$, if $S' \subseteq \mathcal{P}_\kappa(\lambda')$ is (diagonally) α-stationary in $\mathcal{P}_\kappa(\lambda')$ then $S'_{\cap\lambda}$ is (resp./ diagonally) α-stationary in $\mathcal{P}_\kappa(\lambda)$.*

Proof. (0): This can be proved by straightforward induction by $\alpha \in \mathsf{On} \setminus 1$.

(1): We prove the statement for diagonal stationarity.

(2): The assertion follows from (1) and (2). Suppose that $S' \subseteq \mathcal{P}_\kappa(\lambda')$ is (diagonally) α-stationary. Let $S = S'_{\cap\lambda}$. Then we have $S' \subseteq S^{\cup\lambda'\setminus\lambda}$. By (1), $S^{\cup\lambda'\setminus\lambda}$ is (diagonally) α-stationary in $\mathcal{P}_\kappa(\lambda')$. Thus, by (2), $S = S'_{\cap\lambda}$ is (diagonally) α-stationary in $\mathcal{P}_\kappa(\lambda)$.

□ (Lemma 1.8)

2 Consistency of theories

The following Theorem 2.1 answers a question Andrjei Kucharski asked during my stay in Katowice in 2018. Since the theorem is about the non-existence of a proof in meta-mathematics, it is a meta-(meta-mathematical) theorem. consis

The meta-(meta-mathemementical) Theorem must be a well-known fact. I thank Taishi Kurahashi for giving me a hint for the proof of the theorem.

Theorem 2.1 *Assume that T is a weak set theory in a broad sense (including, e.g., the case "$T = \mathsf{PA}$") in which the Second Incompleteness Theorem can be formulated and proved. Let T' be a theory extending T and such that $T' \vdash consis(\ulcorner\ulcorner T\urcorner\urcorner)$.* P-consis-0

If there is a (meta-mathematical) proof of the consistency of T' from the assumption that T is consistent, then we can obtain a proof of the contradiction from T.

In other words, there is no (meta-mathematical) proof of the consistency of T' if T is consistent.

Proof. Assume that there would be

(2.1) a (meta-mathematical) proof of the consistency of T' from the assumption that T is consistent. consis-a

Then we should be able to translate this proof to a proof of $consis(\ulcorner\ulcorner T\urcorner\urcorner) \rightarrow consis(\ulcorner\ulcorner T'\urcorner\urcorner)$ from T. Thus, we have

(2.2) $T \vdash\ consis(\ulcorner\ulcorner T\urcorner\urcorner) \rightarrow consis(\ulcorner\ulcorner T'\urcorner\urcorner)$. consis-1

Note that, since our meta-mathematics should be strictly constructive, the proof (2.1) must give

(2.3) an algorithm $\mathcal{A}$ such that, given a proof $\mathcal{P}$ of the contradiction from T', $\mathcal{A}$ gives us a proof $\mathcal{A}(\mathcal{P})$ of the contradiction from T. consis-1-0

By (2.2), it follows that

(2.4) $T\ +\ consis(\ulcorner\ulcorner T\urcorner\urcorner) \vdash\ consis(\ulcorner\ulcorner T'\urcorner\urcorner)$. consis-2

By the assumption on T', we have $T' \vdash T\ +\ consis(\ulcorner\ulcorner T\urcorner\urcorner)$. It follows that

$$\vdash consis(\ulcorner\ulcorner T'\urcorner\urcorner)\ \rightarrow\ consis(\ulcorner\ulcorner T\ +\ consis(\ulcorner\ulcorner T\urcorner\urcorner)\urcorner\urcorner).$$

From this and (2.4), we obtain

(2.5) $T\ +\ consis(\ulcorner\ulcorner T\urcorner\urcorner) \vdash\ consis(\ulcorner\ulcorner T\ +\ consis(\ulcorner\ulcorner T\urcorner\urcorner)\urcorner\urcorner)$. consis-2-0

By the Second Incompleteness Theorem (applied to the theory $T\ +\ consis(\ulcorner\ulcorner T\urcorner\urcorner)$), it follows that $T\ +\ consis(\ulcorner\ulcorner T\urcorner\urcorner)$ is inconsistent. Since $T\ +\ consis(\ulcorner\ulcorner T\urcorner\urcorner)$ is a sub-theory of T', T' is also inconsistent. Hence, by the algorithm $\mathcal{A}$ in (2.3), we obtain a proof of the contradiction from T. ❑ (Theorem 2.1)

The theorem above applies to many situations in set-theory. For example:

There is no meta-mathematical proof of the consistency of ZFC + *"there is an inaccessible cardinal" from the assumption of the consistency of* ZFC *(provided that* ZFC *is consistent);*

There is no meta-mathematical proof of the consistency of ZFC + *"there is a measurable cardinal" from the assumption of the consistency of* ZFC *or even of the consistency of* ZFC + *"there is an inaccessible cardinal" (provided that the theory* ZFC *or* ZFC + *"there is an inaccessible cardinal" respectively is consistent), etc.*

Theorem 2.2 *Assume that* (2.6) $\mathsf{ZFC} + \mathit{consis}(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)$ *is consistent. Then there is no proof in* ZFC *of* P-consis-1 x-consis-a-0

$$\mathit{consis}(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner) \rightarrow \mathit{consis}(\ulcorner\ulcorner\mathsf{ZFC} + \exists\kappa(\kappa \textit{ is inaccessible})\urcorner\urcorner).$$

Proof. Suppose, toward a contradiction, that

$$\mathsf{ZFC} \vdash \mathit{consis}(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner) \rightarrow \mathit{consis}(\ulcorner\ulcorner\mathsf{ZFC} + \exists\kappa(\kappa \text{ is inaccessible})\urcorner\urcorner). \tag{2.7}$$ x-consis-0

Then

$$\mathsf{ZFC} + \exists\kappa(\kappa \text{ is inaccessible}) \vdash \mathit{consis}(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner) \rightarrow \mathit{consis}(\ulcorner\ulcorner\mathsf{ZFC} + \exists\kappa(\kappa \text{ is inaccessible})\urcorner\urcorner).$$

Since $\mathsf{ZFC} + \exists\kappa(\kappa \text{ is inaccessible}) \vdash \mathit{consis}(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)$, it follows that

$$\mathsf{ZFC} + \exists\kappa(\kappa \text{ is inaccessible}) \vdash \mathit{consis}(\ulcorner\ulcorner\mathsf{ZFC} + \exists\kappa(\kappa \text{ is inaccessible})\urcorner\urcorner).$$

By the Second Incompleteness Theorem, there is a proof Q such that

$$\mathsf{ZFC} + \exists\kappa(\kappa \text{ is inaccessible}) \vdash^{Q} 0 \equiv 1.$$

Then $\ulcorner Q\urcorner$ witnesses

$$\mathsf{ZFC} \vdash \neg\, \mathit{consis}(\ulcorner\ulcorner\mathsf{ZFC} + \exists\kappa(\kappa \text{ is inaccessible})\urcorner\urcorner).$$

From this and (2.7), it follows that $\mathsf{ZFC} \vdash \neg\, \mathit{consis}(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)$. This is a contradiction to the assumption of (2.6). ❑ (Theorem 2.2) Scan_2020-10-18–15.52-math-notes - annotated.pdf p.84

3 σ-linked partial orderings

sigma-1

Proposition 3.1 *Any separative σ-linked partial ordering has cardinality $\leq 2^{\aleph_0}$.*

Proof. Suppose otherwise and assume that $P = \langle P, \leq_P\rangle$ is a separartive partial ordering of size $\geq (2^{\aleph_0})^+$ such that P is partitioned into linked sets P_n, $n \in \omega$. Without loss of generality, we may assume that P is the positive elements of a Boolean algebra B (separativity is needed for this) and $|P| = (2^{\aleph_0})^+$.

Let $\sqsubset$ be a linear ordering on P and let $f : [P]^2 \rightarrow \omega^2$; $\{p, q\} \mapsto (k_0, k_1)$ be such that k_0 is such that $p - q \in P_{k_0-1}$ if $p - q \neq 0_B$, $k_0 = 0$ otherwise; k_1 is such that $q - p \in P_{k_1-1}$ if $q - p \neq 0_B$, $k_1 = 0$ otherwise.

By Erdős-Rado Theorem, there is an f-homogeneous $H \in [P]^{\aleph_1}$.

Let $f''[H]^2 = \{\langle k_0^*, k_1^*\rangle\}$. $k_0^* = k_1^* = 0$ does not hold, since otherwise, all p, $q \in H$ would be identical.

Suppose, that $k_0^* \neq 0$, then for any three distinct elements p, q, $r \in H$ with $p \sqsubset q \sqsubset r$, we have $p - q \in P_{k_0^*-1}$ and $q - r \in P_{k_0^*-1}$. Since $p - q$ and $q - r$ are disjoint, this is a contradiction.

Similarly, also $k_1^* \neq 0$ would lead to a contradiction. □ (Proposition 3.1)

4 Generic large cardinals

gen-large

Lemma 4.1 *Any generically measurable cardinal is regular.* *L-gen-large-a*

Proof. Suppose that $j : \mathsf{V} \overset{\preceq}{\rightarrow} M \subseteq \mathsf{V}[\mathbb{G}]$ is such that $crit(j) = \kappa$.

If κ were singular, there would be a sequence $\vec{\kappa} = \langle \kappa_\alpha : \alpha < \mu\rangle$ in V with $\mu < \kappa$ and $\kappa_\alpha < \kappa$ such that $\kappa = \sup_{\alpha<\mu} \kappa_\alpha$. Since $j(\vec{\kappa}) = \vec{\kappa}$ by elementarity and $\kappa = crit(j)$, it follows that $j(\kappa) = \sup_{\alpha<\mu} \kappa_\alpha$ by elementarity. This is a contradiction to the assumption that $\kappa = crit(j)$. □ (Lemma 4.1)

The following Lemma will be yet improved significantly in section 9 (see Lemma 9.3 and Theorem 9.8).

Lemma 4.2 *Suppose that κ is generically measurable by a c.c.c. poset $\mathbb{P}$. Then κ is weakly Mahlo.* *L-gen-large-0*

Proof. Suppose that $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter, M an inner model in $\mathsf{V}[\mathbb{G}]$ and $j : \mathsf{V} \overset{\preceq}{\rightarrow} M$ is such that $crit(j) = \kappa$.

Claim 4.2.1 *$M \models$ "κ is a regular cardinal".* *Cl-gen-large-0*

⊢ κ is a regular cardinal by Lemma 4.1. By the c.c.c. of $\mathbb{P}$, κ is also a regular cardinal in $\mathsf{V}[\mathbb{G}]$. It follows that κ is a regular cardinal in $M \subseteq \mathsf{V}[\mathbb{G}]$. ⊣ (Claim 4.2.1)

Claim 4.2.2 *κ is weakly Mahlo.* *Cl-gen-large-2*

⊢ Suppose that $C \subseteq \kappa$ is a club. We have to show that there is an $\alpha \in C$ such that α is a regular cardinal. By elementarity $M \models$ "$j(C)$ is a club in $j(\kappa)$". $j(C) \cap \kappa = C$ and $\kappa = \sup C$. It follows that $M \models \kappa \in j(C)$. By Claim 4.2.1, it follows that

(4.1) $M \models$ "there is an $\alpha \in j(C)$ such that α is a regular cardinal".

By elementarity

(4.2) $\mathsf{V} \models$ "there is an $\alpha \in C$ such that α is a regular cardinal".

⊣ (Claim 4.2.2)

□ (Lemma 4.2)

The lemma above can be still improved (see also Theorem 9.8):

Lemma 4.3 (1) *Suppose that κ is generically measurable for a poset $\mathbb{P}$ and j, $M \subseteq \mathsf{V}[\mathbb{G}]$ for a $(\mathsf{V},\mathbb{P})$-generic $\mathbb{G}$ such that M is an inner model of $\mathsf{V}[\mathbb{G}]$ $j : V \overset{\preceq}{\to} M$, $crit(j) = \kappa$. Then, in $V[\mathbb{G}]$,* L-lt-conti-1-1

$$F = \{a \in (\mathcal{P}(\kappa))^{\mathsf{V}} : \kappa \in j(a)\} \tag{4.3}$$ lt-conti-2-2-0

is a V-normal ultrafilter on (the Boolean algebra) $(\mathcal{P}(\kappa))^V$.

(2) *If $\mu < \kappa$ and κ is generically measurable for a μ-cc poset $\mathbb{P}$ then there is a μ-saturated normal ideal over κ (in V). In particular, κ is κ-weakly Mahlo.*

Proof. See [9] (L-lt-conti-1-1). See also Proposition 16.8 in [14]. □ (Lemma 4.3)

Lemma 4.4 *Suppose that κ is generically measurable by a $<\mu$-closed poset for a regular uncountable cardinal $\mu < \kappa$. Then κ is a regular cardinal and $2^{<\mu} < \kappa$.* L-gen-large-1

Proof. Suppose that κ is generically measurable by a $<\mu$-closed poset $\mathbb{P}$ and let $\mathbb{G}$ be a $(\mathsf{V},\mathbb{P})$-generic filter with transitive $M \subseteq \mathsf{V}[\mathbb{G}]$ and elementary embedding $j : \mathsf{V} \overset{\preceq}{\to} M$ such that $crit(j) = \kappa$.

κ is regular by Lemma 4.1.

Suppose now, toward a contradiction, that $2^{<\mu} \geq \kappa$. Since κ is a regular cardinal it follows that there is a $\mu_0 < \mu$ such that $2^{\mu_0} \geq \kappa$. Let $\lambda = 2^{\mu_0}$ and let $f : \lambda \to \mathcal{P}(\mu_0)$ be a bijection. By elementarity,

$$M \models \text{“} j(f) : j(\lambda) \to \mathcal{P}(\mu_0) \text{ and } j \text{ is a bijection”}. \tag{4.4}$$ gen-large-0

Since $\mathbb{P}$ is $<\mu$-closed, we have $\mathcal{P}(\mu_0)^{\mathsf{V}} = \mathcal{P}(\mu_0)^{\mathsf{V}[\mathbb{G}]} \supseteq \mathcal{P}(\mu_0)^M$. By (4.4), it follows that $\mathcal{P}(\mu_0)^{\mathsf{V}} = \mathcal{P}(\mu_0)^M$. Thus

$$M \models \text{“} j(f) : j(\lambda) \to \mathcal{P}(\mu_0)^{\mathsf{V}} \text{ and } j \text{ is a bijection”}. \tag{4.5}$$ gen-large-1

We have $j(f)(j(\alpha)) = j(f(\alpha)) = f(\alpha)$ for each $\alpha \in \lambda$. Thus $j(f)''\lambda = \mathcal{P}(\mu_0)^{\mathsf{V}}$. Since $j(f)$ should be an injection, $\kappa \in j(\lambda) \setminus j''\lambda$ cannot be assigned to any element of $\mathcal{P}(\mu_0)^{\mathsf{V}}$ by $j(f)$. This is a contradiction. □ (Lemma 4.4)

4.1 Two dimensional Laver genericity

The materials in this subsection have been copied to https://fuchino.ddo.jp/papers/SDLS-III-x.pdf. The version in this paper may be more up-to-date than what you find in this subsection. Laver-gen2

For properties $\mathfrak{P}$ and $\mathfrak{Q}$ of posets, a cardinal κ is *storngly Laver-generically supercompact for* $(\mathfrak{P}, \mathfrak{Q})$ if, for any poset $\mathbb{P}$ with $\mathbb{P} \models \mathfrak{P}$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of a poset with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \models \mathfrak{Q}$" and $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$ with $\mathbb{G} \subseteq \mathbb{H}$ such that there are $j, M \subseteq \mathsf{V}[H]$ with

(4.6) M is a transitive class in $\mathsf{V}[\mathcal{H}]$, gen-large-1-0

(4.7) $j : \mathsf{V} \overset{\prec}{\to} M$, gen-large-2

(4.8) $crit(j) = \kappa$ and $j(\kappa) > \lambda$, gen-large-3

(4.9) $\mathbb{P}, \mathbb{H} \in M$, gen-large-4

(4.10) $j''\lambda \in M$, and gen-large-5

(4.11) $([M]^{\aleph_0})^{\mathsf{V}[\mathbb{H}]} \subseteq M$. gen-large-6

Lemma 4.5 *Suppose that $\mathbb{P} \models \mathfrak{P}_0 \to \mathfrak{P}_1$ and $\mathbb{P} \models \mathfrak{Q}_1 \to \mathfrak{Q}_0$ hold for all posets $\mathbb{P}$. If κ is strongly Laver-generically supercompact for $(\mathfrak{P}_1, \mathfrak{Q}_1)$, then κ is strongly Laver-generically supercompact for $(\mathfrak{P}_0, \mathfrak{Q}_0)$.* ❑ P-gen-large-0

Proposition 4.6 *Suppose that $\mathfrak{P}$ is "$<\kappa$-closed" and $\mathfrak{Q}$ is "proper". If κ is strongly Laver-generically supercompact for $(\mathfrak{P}, \mathfrak{Q})$, then, for any $\mu \geq \kappa$, and for $\mathbb{P} = \mathrm{Col}(\kappa, \mu)$, we have* P-gen-large-1

(4.12) $\Vdash_{\mathbb{P}}$ "*κ is generically supercompact by proper posets*". gen-large-7

Proof. Note that $\mathbb{P} \models \mathfrak{P}$. Let $\mathbb{G}$ be an arbitrary $(\mathsf{V}, \mathbb{P})$-generic filter. We have to show that $\mathsf{V}[\mathbb{G}] \models$ "κ is generic supercompact for proper".

Let $\lambda \geq \mu^{<\kappa}$ and let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}}$ is a proper poset" such that there are a $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic filter $\mathbb{H}$ with $\mathbb{G} \subseteq \mathbb{H}$, and j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that (4.6) $\sim$ (4.11) hold.

By (4.9), we have $\mathbb{G} \in M$. Let $\mathbb{R} = j(\mathbb{P})$. By elementarity,

(4.13) $M \models$ "$\mathbb{R}$ is directedly $< j(\kappa)$-closed". gen-large-7-0

By (4.8), $M \models |j''\mathbb{G}| < j(\kappa)$. Hence, there is $\mathbb{r} \in \mathbb{R}$ such that $M \models \mathbb{r} \leq_{\mathbb{R}} j''\mathbb{G}$. Let $\mathbb{K}$ be a $(\mathsf{V}[\mathbb{H}], \mathbb{R})$-generic filter with $\mathbb{r} \in \mathbb{K}$. Then

(4.14) $\tilde{j} : \mathsf{V}[\mathbb{G}] \overset{\prec}{\to} M[\mathbb{K}]$; $\underset{\sim}{a}^{\mathbb{G}} \mapsto j(\underset{\sim}{a})^{\mathbb{K}}$ gen-large-8

is well-defined and $j \subseteq \tilde{j}$. In particular we have $\kappa = crit(\tilde{j})$, $\tilde{j} > \lambda$ and $\tilde{j}''\lambda \in M[\mathbb{K}]$.

By (4.13) and (4.11), we have $\mathsf{V}[\mathbb{H}] \models$ "$\mathbb{R}$ is σ-closed". Thus, for $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$-name $\underset{\sim}{\mathbb{R}}$ corresponding to $\mathbb{R}$ with $\Vdash_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}}$ "$\underset{\sim}{\mathbb{R}}$ is σ-closed", $\mathbb{P} * \underset{\sim}{\mathbb{Q}} * \underset{\sim}{\mathbb{R}}$ is proper and it induces a generic elementary embedding for generic λ-supercompactness. ❑ (Proposition 4.6)

Corollary 4.7 (1) *Suppose that κ is strongly Laver-generically spuercompact for $(\mathfrak{P}, \mathfrak{Q})$ where $\mathfrak{P}$ is "$<\kappa$-closed" and $\mathfrak{Q}$ is "proper". Suppose further that $\lambda > \kappa$ is a supercompact and let $\mathbb{P} = \mathrm{Col}(\kappa, \lambda)$. Then* P-gen-large-2

(a) $\Vdash_{\mathbb{P}}$ "*κ is generically supercompact by proper pos*"*; and*

(b) $\Vdash_{\mathbb{P}}$ "*κ^+ is generically supercompact by $<\kappa$ -closed posets*"

(2) *For κ, λ, $\mathbb{P}$ as in (1), we have*

(a) $\Vdash_{\mathbb{P}}$ "$\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$"*; and*

(b) $\Vdash_{\mathbb{P}}$ "$\mathsf{GRP}^{<\kappa}(\leq\kappa)$".

Proof. "(1), (a)": By Proposition 4.6. "(1), (b)": By Lemma 4.10 in [8].

"(2), (a)": By (1), (a) and, Theorem 2.10 and Propositions 3.1 in [9]. "(2), (b)": By (1), (b) above and Lemma 4.11 in [8]. ❑ (Corollary 4.7)

5 Axiom of Choice in Zermelo's set theory

When Zermelo proved his „Wohlordnungssatz“ (Well-ordering Theorem), there was not yet the Zermelo-Fraenkel axiom system ZF but at most the axiom system Zermelo introduced in his 1908 paper [20]. The axiom system Z (the system obtained by dropping the Axiom of Replacement and Axiom of Regularity from ZF) corresponds to the axiom system in [20] but it differs from Zermelo's system in the treatment of the axiom of infinity. Since the modern treatment of ordered pairs and functions were not yet introduced in Zermelo's 1908 paper, it is not so clear whether he proved his „Wohlordnungssatz“ really in the framework of the axiomatics of his 1908 paper. Also in all modern textbooks, the equivalence of AC with the „Wohlordnungssatz“ and Zorn's Lemma is proved in the framework of ZF. Thus, it is rather difficult to find an easy to read source to check that this equivalence can be established already in Z. In the following we will check this in a most self-contained manner so that the text will be also accessible for students who just began to learn the axiomatic set-theory. AC-Z

In this section we are working in the Zermelo's set theory Z if not mentioned otherwise. For a set X, a binary relation R on X, that is, $R \subseteq X^2$ is said to be a *well-ordering on X* if R is a linear ordering, that is, if it is irreflexive, anti-symmetric and transitive relation satisfying

$$(5.1) \qquad \forall a, b \in X\ (a\,R\,b\ \vee\ a = b\ \vee\ b\,R\,a),$$

such that, reading "$a\ R\ b$" as "a is smaller than b with respect to R", every non empty $Y \subseteq X$ has the minimal element with respect to R.

If R is a linear-ordering (a well-ordering, resp.) on X, then we also say that $\langle X, R\rangle$ is a linearly-orderd set (a well-ordered set, resp.), or also simply a linear ordering (a well-ordering, resp.). For such a structure $\langle X, R\rangle$, if it is clear which R is attatched to X, we call $\langle X, R\rangle$ simply as X and, to declare that we call $\langle X, R\rangle$ simply as X, we write: $X = \langle X, R\rangle$.

For a linear ordering $\langle X, R\rangle$, $Y \subseteq X$ is said to be an *initial segment of X* if Y is downward closed in X with respect to R, that is, if for all $a \in Y$ and $b \in X$ with $b\,R\,a$ we always have $b \in Y$. $Y \subseteq X$ is an *end segment of X* if $X \setminus Y$ is an initial segment of X.

For $Y \subseteq X$ and $R \subseteq X^2$, $R \restriction Y$ denotes the binary relation $R \cap Y^2$ on Y. For linear orderings $X_0 = \langle X_0, R_0 \rangle$ and $X_1 = \langle X_1, R_1 \rangle$, we say that X_1 is a *end-extension* of X_0 if $X_0 \subseteq X_1$, X_0 is an initial segment of X_1 with respect to R_1 and $R_0 = R_1 \restriction X_0$.

Lemma 5.1 (1) *Suppose that $\langle u, r \rangle$ is a linear ordering (a well-orderting, resp.) and $u_0 \subseteq u$. Then $\langle u_0, r \restriction u_0 \rangle$ is also a linear ordering (a well-ordering, resp.).* L-AC-Z-1

(2) *If $\langle u, r \rangle$ is a linear ordering (a well-ordering, resp.) and $a \notin u$ then $\langle u', r' \rangle$ defined by $u' = u \cup \{a\}$ and $r' = r \cup (u \times \{a\})$ is also a linear ordering (a well-ordering, resp.) and $\langle u', r' \rangle$ is an end-extension of $\langle u, r \rangle$.*

(3) *Suppose that $\mathcal{F}$ is a family (set) of linear orderings (of well-orderings, resp.) such that, for any $\langle u_0, r_0 \rangle$, $\langle u_1, r_1 \rangle \in \mathcal{F}$,*

(5.2) *either $\langle u_1, r_1 \rangle$ is an end-extension of $\langle u_0, r_0 \rangle$ or $\langle u_0, r_0 \rangle$ is an end-extension of $\langle u_1, r_1 \rangle$.* AC-Z-0

Then $\langle U, R \rangle$ for

(5.3) $U = \bigcup\{u : \langle u, r \rangle \in \mathcal{F} \text{ for some } r\}$ *and* $R = \bigcup\{r : \langle u, r \rangle \in \mathcal{F} \text{ for some } u\}$

is also a linear ordering (a well-ordering, resp.) and, for all $\langle u, r \rangle \in \mathcal{F}$, $\langle U, R \rangle$ is an end extension of $\langle u, r \rangle$.

Proof. Exercise. ❑ (Lemma 5.1)

Axiom of Choice (AC) is defined as the following assertion:

AC: For any A with $\emptyset \notin A$, there is a mapping $f : A \to \bigcup A$ such that $f(a) \in a$ for all $a \in A$.

A mapping f as above is called a *choice funcion for A*.

Theorem 5.2 (Zermelo, 1904) *The following are equivalent over* Z*:* T-AC-Z-0
(a) AC. (b) (Well-ordering Theorem) *For any X there is a well-ordering R on X.*

Proof. (b) ⇒ (a): Suppose that (b) holds and let A be a set with $\emptyset \notin A$. Let R be a well-ordering on $\bigcup A$. Then

(5.4) $f : A \to \bigcup A$; $a \mapsto$ the minimal element of $a \subseteq \bigcup A$ with respect to R AC-Z-2

is a choice function for A.

(a) ⇒ (b): Suppose that AC holds. Let X be an arbitrary set. If $X = \emptyset$ then $R = \emptyset$ is a well-ordering on X. Hence we may assume that $X \neq \emptyset$.

Let $A = \mathcal{P}(X) \setminus \{\emptyset\}$ and let $f : A \to \bigcup A$ $(= X)$ be a choice function on A. We show that there is a well-ordering R on X such that, for any non-empty $Y \subseteq X$, $f(Y)$ is the minimal element of Y with respect to R.

Let

(5.5) $\mathcal{F} = \{\langle u, r\rangle : u \subseteq X,\ r \subseteq u^2,\ r$ is a well-ordering on u and, for any non-empty end segment v of u with respect to r, the minimal element of v with respect to r is $f(X \setminus (u \setminus v))\}$. AC-Z-3

Claim 5.2.1 *$\mathcal{F} \neq \emptyset$.* *Cl-AC-Z-0*

$\vdash$ $\langle\emptyset, \emptyset\rangle \in \mathcal{F}$. $\dashv$ (Claim 5.2.1)

Claim 5.2.2 *For any $\langle u_0, r_0\rangle$, $\langle u_1, r_1\rangle \in \mathcal{F}$,* *Cl-AC-Z-1*

(5.6) *either $\langle u_1, r_1\rangle$ is an end-extension of $\langle u_0, r_0\rangle$ or $\langle u_0, r_0\rangle$ is an end-extension of $\langle u_1, r_1\rangle$.* *AC-Z-4*

$\vdash$ Suppose that $\langle u_0, r_0\rangle$, $\langle u_1, r_1\rangle \in \mathcal{F}$ are a counter-example to the assertion of the Claim. Let

(5.7) $\mathcal{F}_0 = \{v : v \subseteq u_0 \cap u_1,\ r_0 \restriction v = r_1 \restriction v$ v is an initial segment of u_0 with respect to r_0 and v is an initial segment of u_1 with respect to $r_1\}$ AC-Z-6

Let $v^* = \bigcup \mathcal{F}_0$. Then $v^* \subseteq u_0 \cap u_1$, $r_0 \restriction v^* = r_1 \restriction v^*$ v^* is an initial segment of u_0 with respect to r_0 and v^* is an initial segment of u_1 with respect to r_1. Thus v^* is the maximal element of $\mathcal{F}_0$ with respect to $\subseteq$.

By the choice of $\langle u_0, r_0\rangle$, $\langle u_1, r_1\rangle \in \mathcal{F}$, we have $v^* \subsetneqq u_0$ and $v^* \subsetneqq u_1$. By definition of $\mathcal{F}$, we have

(5.8) the minimal element of $u_0 \setminus v^*$ with respect to r_0 $= f(X \setminus v^*) =$ the minimal element of $u_1 \setminus v^*$ with respect to r_1. AC-Z-7

Let a be this element. Then $v^* \cup \{a\} \in \mathcal{F}_0$. This is a contradiction to the maximality of v^*. $\dashv$ (Claim 5.2.2)

Let

(5.9) $X_0 = \bigcup\{u \in \mathcal{P}(X) : \langle u, r\rangle \in \mathcal{F}$ for some $r \subseteq u\}$ and $R_0 = \bigcup\{r \in \mathcal{P}(X^2) : \langle u, r\rangle \in \mathcal{F}$ for some $u \in \mathcal{P}(X)\}$. AC-Z-8

Claim 5.2.3 *R_0 is a well-ordering on X_0.* *Cl-AC-Z-2*

$\vdash$ By Claim 5.2.2 and Lemma 5.1, (3). $\dashv$ (Claim 5.2.3)

Thus, $\langle X_0, R_0\rangle$ is the maximal element of $\mathcal{F}$ with respect to componentwise inclusion. The following Claim finishes the proof:

Claim 5.2.4 *$X_0 = X$.* *Cl-AC-Z-3*

$\vdash$ Suppose not and let $a_0 = f(X \setminus X_0)$. Let $X_1 = X_0 \cup \{a_0\}$ and $R_1 = R_0 \cup (X_0 \times \{a_0\})$. Then $\langle X_1, R_1 \rangle \in \mathcal{F}$ by Lemma 5.1, (2). This is a contradiction to the maximality of $\langle X_0, R_0 \rangle$. $\dashv$ (Claim 5.2.4)

$\Box$ (Theorem 5.2)

A pair $\langle P, < \rangle$ for a set P and a binary relation $<$ is said to be a *partial ordering* if $<$ is irreflexive, anti-symmetric and transitive relation. A subset C of P for a partial ordering $\langle P, < \rangle$ is a *chain* if $< \restriction C$ is a linear ordering on C. For $X \subseteq P$, $a \in P$ is an *upper-bound of X* (with respect to $<$) if $b = a$ or $b < a$ holds for all $b \in X$. $a \in P$ is a *maximal element* (with respect to $<$) if there is no $b \in P$ such that $a < b$.

Zorn's Lemma is the following assertion:

Zorn's Lemma: Suppose that $\langle P, < \rangle$ is a partial ordering such that

(5.10) any chain has an upper-bound in P. AC-Z-10

Then P has at least one maximal element.

Theorem 5.3 *The following are equivalent over* Z*:* *T-AC-Z-1*

(a) AC.

(b) *Zorn's Lemma.*

Proof. By Theorem 5.2, it is enough to show that Zorn's Lemma is equivalent to Well-ordering Theorem over Z.

Assume first that Well-ordering Theorem holds and $\langle P, < \rangle$ is a partial ordering satisfying (5.10). Let R be a well-ordering on the set P and let

(5.11) $\mathcal{P} = \{u \in \mathcal{P}(P) : < \restriction u$ is a well-ordering on u, for any proper initial segment v of u with respect to $< \restriction u$, the minimal element of $u \setminus v$ with respect to $< \restriction u$ is the minimal element of $\{p \in P : p$ is an upper-bound of v with respect to $<\}$ with respect to $R\}$. AC-Z-11

Similarly to the proof of (a) $\Rightarrow$ (b) of Theorem 5.2, we can prove that elements of $\mathcal{P}$ as linear orderings ordered by $<$ or equivalently R restricted to them are linearly ordered with respect to end-extention. Hence we have $U^* = \bigcup\mathcal{P} \in \mathcal{P}$ and U^* is the maximal element of $\mathcal{P}$ with respect to end-extension of elements of $\mathcal{P}$. U^* is a chain in P with respect to $<$ by Lemma 5.1, (3). Let $p^* \in P$ be an upper-bound of U^*. Then $p^* \in U^*$, that is p^* must be the maximal element of U^* with respect to $< \restriction U^*$. Also p^* must be a maximal element of P with respect to $<$.

Suppose now that Zorn's Lemma holds and let X be a set. Let

(5.12) $\mathcal{F} = \{\langle u, r \rangle \in \mathcal{P}(X) \times \mathcal{P}(X^2) : r \subseteq u^2$ and r is a well-orderting on $u\}$. AC-Z-12

For $\langle u_0, r_0 \rangle$, $\langle u_1, r_1 \rangle \in \mathcal{F}$, let

(5.13) $\langle u_0, r_0\rangle \sqsubset \langle u_1, r_1\rangle \ \Leftrightarrow \langle u_1, r_1\rangle$ is an end extension of $\langle u_0, r_0\rangle$. AC-Z-13

Then $\mathcal{F} = \langle \mathcal{F}, \sqsubset\rangle$ is a partial ordering and, by Lemma 5.1, (3), every chain C in $\mathcal{F}$ has the least upper-bound $\langle U, R\rangle$ where

(5.14) $U = \bigcup\{u \in \mathcal{P}(X) : \langle u, r\rangle \in C \text{ for some } r \subseteq u^2\}$ and
$R = \bigcup\{r \in \mathcal{P}(X^2) : \langle u, r\rangle \in C \text{ for some } u \in \mathcal{P}(X)\}$.

By Zorn's Lemma, it follows that $\mathcal{F}$ has a maximal element $\langle U_0, R_0\rangle$. The following Claim finishes the proof:

Claim 5.3.1 $U_0 = X$. *Cl-AC-Z-4*

$\vdash$ Suppose otherwise and let $a \in X \setminus U_0$. Letting $U_1 = U_0 \cup \{a\}$ and $R_1 = R_0 \cup (U_0 \times \{a\})$, we have $\langle U_1, R_1\rangle \in \mathcal{F}$ and $\langle U_0, R_0\rangle$ F $\langle U_1, R_1\rangle$ by Lemma 5.1, (2). This is a contradiction to the assumption that $\langle U_0, R_0\rangle$ is a maximal element. $\dashv$ (Claim 5.3.1) ❑ (Theorem 5.3)

Theorem 5.4 (Cantor-Bernstein Theorem) *In* Z*, if there are 1-1 mappings from M to N and also from N to M, then there is a bijection from M to N.*

Proof. See Chapter 19 in [1] where the proof given there is claimed to be one by Julius Kőnig.

General Topology by John L. Kelley also contains an "intuitively elegant form of the proof of Theorem 0.20" due to G. Birkhoff and S. MacLane. ❑ (Theorem 5.4)

6 Hilbert's Paradox and Grothendieck universe

The following Theorem 6.1 appears in Hilbert's course note of 1905 (see V. Peckhaus and R. Kahle [18], see also A. Kanamori, [15]): Gu

Theorem 6.1 (in Z, Hilbert's Paradox) *There is no set X such that* *P-Gu-a-0*

(6.1) *X is closed with respect to powerset operation, that is, for any $a \in X$, $\mathcal{P}(a) \in X$; and* *Gu-a-0*

(6.2) *$\bigcup Y \in X$, for any set $Y \subseteq X$, .* *Gu-a-1*

Note that there are infinite sets with the property (6.1) alone and infinite sets with the property (6.2) alone. For example: $V_\gamma \models$ (6.1) for any limit ordinal γ. $\mathcal{P}(X) \models$ (6.2) for any set X.[(2)]

---

(2) If $a \in V_\gamma$ then $a \in V_\alpha$ for some $\alpha < \gamma$. Since $a \subseteq V_\alpha$ by transitivity of V_α, we have $b \subseteq V_\alpha$ for all $b \subseteq a$. Thus $\mathcal{P}(a) \subseteq V_{\alpha+1}$ and $\mathcal{P}(a) \in V_{\alpha+2} \subseteq V_\gamma$.

If $Y \subseteq \mathcal{P}(X)$ then $\bigcup Y \subseteq X$ and hence $\bigcup Y \in \mathcal{P}(X)$.

Proof of Theorem 6.1. Suppose that there is a set X with $X \models$ (6.1), (6.2). Then $\bigcup X \in X$ and $\mathcal{P}(\bigcup X) \in X$. It follows that $\mathcal{P}(\bigcup X) \subseteq \bigcup X$: for any $a \in \mathcal{P}(\bigcup X)$, since a is an element of an element of X (namely $\mathcal{P}(\bigcup X)$), we have $a \in \bigcup X$. But this is a contradiction to Cantor's Theorem which states that, for any set a, there is no surjection from a to $\mathcal{P}(a)$. □ (Theorem 6.1)

The class $\mathsf{V} = \{x : x = x\}$ clearly satisfies (6.1) and (6.2). On the other hand, a class X satisfying (6.1) and (6.2) does not need to be equal to V:

Lemma 6.2 *There are proper subclasses X of V with $X \models$ (6.1), (6.2).* P-Gu-a-0-0

Proof. Let $X = \{a : |b| \leq |a| \text{ for any } b \in \mathit{trcl}(a)\}$. Clearly $X \neq \mathsf{V}$ but $X \models$ (6.1), (6.2).

In ZF, $X = \{V_\alpha : \alpha \in \mathsf{On}\}$ is another example. □ (Lemma 6.2)

The second example in the proof of Lemma 6.2 is quite symptomatic in ZF (see Proposition 6.3 below). Actually, the properties of Hilbert's (non-existent) set can be slightly extended to obtain a characterization of V:

Proposition 6.3 (ZF) *If a class X satisfies (6.1) and (6.2), then $V_\alpha \in X$ for all $\alpha \in \mathsf{On}$.* P-Gu-a-0-2

Proof. Since $\emptyset = \bigcup \emptyset$ and $\emptyset \subseteq X$, we have $V_0 = \emptyset \in X$ by (6.2).

If $V_\alpha \in X$ then $V_{\alpha+1} = \mathcal{P}(V_\alpha) \in X$ by (6.1). If $V_\beta \in X$ for all $\beta < \gamma$ for a limit ordinal then $V_\gamma = \bigcup \{V_\beta : \beta < \gamma\} \in X$ by (6.2). □ (Proposition 6.3)

Corollary 6.4 (ZF) *If a class X is transitive and satisfies (6.1) and (6.2), then $X = \mathsf{V}$.* P-Gu-a-0-1

Proof. We have $V_\alpha \in X$ for all $\alpha \in X$. Since X is transitive, it follows that $X \supseteq \mathsf{V}$.

□ (Corollary 6.4)

By modifying further the properties of the non-existent set in Hilbert's paradox, we obtain the definition of Grothendieck universes (which are sets!). An uncountable Grothendieck universe U exists if and only if an inaccessible cardinal exists. In terms of set theory, an uncountable Grothendieck universe U is simply $\mathcal{H}(\kappa)$ for an inaccessible κ. A weakening of the notion of Grothendieck universe (called here weakly Grothendieck universe) characterizes the families $\mathcal{H}(\kappa)$ of hereditarily of cardinality $<\kappa$ sets for regular κ (see Lemma 6.10, (8) and (9)). Note that such $\mathcal{H}(\kappa)$'s are still "universes" in that they are models of ZFC − Powerset Axiom.

A set U is said to be a *Grothendieck universe* if

(6.3) U is transitive; Gu-0

(6.4) $\emptyset \in U$; (under the Axiom of Foundation, this follows from (6.3)). Gu-0-0

(6.5) U is closed with respect to pairing operation, i.e., for any a, $b \in U$, we have $\{a, b\} \in U$; Gu-1

(6.6) U is closed with respect to power set operation, i.e., for any $a \in U$, we have $\mathcal{P}(a) \in U$; and, Gu-2

(6.7) for any $I \in U$ and $f \in {}^I U$, we have $\bigcup_{i \in I} f(i) \in U$.[(3)] Gu-3

The definition of Grothendieck universes makes sense under ZF without Axiom of Foundation or as a matter of fact even without Separation and Replacement although the most natural setting for this notion seems to be the full ZFC with the Axiom of Foundation (see e.g. Lemma 6.7, (8) and Lemma 6.10, (10) below). The following properties of Grothendieck universes can be proved in a weak fragrment of ZF:

Lemma 6.5 *Let U be a Grothendieck universe.* *P-Gu-a-0-3*

(1) *For any $a \in U$, we have $\bigcup a \in U$.*

(2) *Let $\kappa = \mathsf{On} \cap U$. Then, for any $a \in [U]^{<\kappa}$, we have $a \in U$.*

Proof. (1): Let $id_a : a \to U;\ a \ni b \mapsto b \in a \subseteq U$. By (6.7), $\bigcup a = \bigcup_{b \in a} id_a(b) \in U$.

(2): Suppose $a \in [U]^{<\kappa}$ and let $\mu = |a|$. Then $\mu \in U$. Let $f : \mu \to U$ be an enumeration of a and let f^* be a mapping on μ defined by $f^*(\xi) = \{f(\xi)\}$. We have $f^* \in {}^\mu U$ by (6.5). Thus $a = \bigcup_{\xi \in \mu} f^*(\xi) \in U$. ❑ (Lemma 6.5)

Under the full axiom system of ZFC with the Axiom of Foundation and the Axiom of Choice, we can say much more than this (see Lemma 6.7,(7)).

Grothendieck universes obtain certain regularity even under the set theory without the Axiom of Foundation:

Lemma 6.6 (in ZF - Axiom of Foundation) *If U is a Grothendieck universe then $U \notin U$. There is no $\in$-descending sequence starting and ending with U.* *P-Gu-a-1*

Proof. Let U be a Grothendieck universe. Toward a contradiction, suppose that $U \in U$. Let $\kappa = \mathsf{On} \cap U$.

Claim 6.6.1 *κ is a limit ordinal.*

$\vdash$ Suppose that $\alpha \in \kappa$. Then $\alpha + 1 = \alpha \cup \{\alpha\} = \bigcup\{\{\alpha\}, \{\{\alpha\}\}\} \in U$ by (6.5) and (6.7). Hence $\alpha + 1 \in \kappa$. $\dashv$ (Claim 6.6.1)

Let $f : U \to \kappa \subseteq U$ be defined by

(6.8) $$f(x) = \begin{cases} x, & \text{if } x \in \mathsf{On}; \\ \emptyset & \text{otherwise.} \end{cases}$$

---

(3) Actually (6.5) is redundant: If $x \in U$ then $\{x\} \in \mathcal{P}(\mathcal{P}(x)) \in U$. Thus by (6.3), we have $\{x\} \in U$. By (6.4), it follows from this that $2 = \{\emptyset, \{\emptyset\}\} = \mathcal{P}(\{\emptyset\}) \in U$. Thus, for any $a, b \in U$, we have $f = \{\langle 0, \{a\}\rangle, \langle 1, \{b\}\rangle\} \in {}^2U$ and $\{a, b\} = \bigcup f \in U$.

Then $\kappa = \bigcup \kappa = \bigcup_{x\in U} f(i) \in U$ by our assumption $U \in U$ and (6.7). (Note that the first equality holds because κ is a limit ordinal.) Thus $\kappa \in \kappa$. This is a contradiction.

Since U is transitive, if there were a $\in$-descending sequence starting and ending with U, then this would imply $U \in U$. ❑ (Lemma 6.6)

Lemma 6.7 *Work in* ZF*. Suppose that U is a Grothendieck universe and $\kappa = \mathsf{On} \cap U$.* P-Gu-0

(1) *If $a \in U$ and $b \subseteq a$ then $b \in U$.*

(2) *$\omega \subseteq U$ (hence $\kappa \geq \omega$) and $\mathcal{H}(\kappa) \subseteq U$.*

(3) *If $a, b \in U$ then $a \cup b \in U$.*

(4) *If $a, b \in U$ then $a \times b \in U$.*

(5) *If $f \in {}^{I}U$ for some $I \in U$ then $f''I \in U$.*

(6) *For any $\langle a, r\rangle \in U$, if r is a well ordering on a, then we have $otp(\langle a, r\rangle) \in U$.*

(7) *If $\kappa > \omega$ then κ is a regular cardinal. If we assume* AC*, then κ is an inaccessible cardinal.*

(8) *Under* AC*, we have $U = \mathcal{H}(\kappa)$. If $\kappa > \omega$, $U \models$* ZF *and further, if* AC *holds, then we have $U \models$* ZFC*.*

(9) *Conversely to (7), $\mathcal{H}(\kappa)$ is a Grothendieck universe if either $\kappa = \omega$ or κ is inaccessible.*

Proof. (1): This follows directly from (6.6) and (6.3). We shall show the assertion here without appealing to (6.6):

Suppose that $a \in U$ and $b \subseteq a$. Let $f : a \to U$ be defined by

(6.9) $$f(x) = \begin{cases} \{x\}, & \text{if } x \in b; \\ \emptyset, & \text{otherwise} \end{cases}$$ Gu-4-0

for $x \in a$. f is well-defined by (6.5). We have $b = \bigcup_{x\in a} f(x) \in U$ by (6.7).

(2): We first show that $\omega \subseteq U$.

$\emptyset \in U$ by (6.4). By (6.5), it follows that $\{\emptyset\} \in U$ and $\{\emptyset, \{\emptyset\}\} \in U$. Thus $\kappa \geq 3$. By (6.7), it follows that,

(6.10) $$\text{for any } a \in U,\ a \cup \{a\} = \bigcup \underbrace{\{\{a\}, \{\{a\}\}\}}_{\in [U]^{<\kappa}} \in U \text{ by Lemma 6.5, (2) and (1).}$$ Gu-5

Since ω is characterized as the minimal set W (with respect to $\subseteq$) satisfying $\emptyset \in W$ and $a \cup \{a\} \in W$ for all $a \in W$, it follows that $\omega \subseteq U$. In particular, we also have $\kappa \geq \omega$.

To prove $\mathcal{H}(\kappa) \subseteq U$ by induction on the rank of elements of $\mathcal{H}(\kappa)$, it is enough to show: If $a \in \mathcal{H}(\kappa)$ and $a \subseteq U$ then $a \in U$. This follows from Lemma 6.5, (2).

(3): Suppose that $a, b \in U$.

(6.11) $$a \cup b = \bigcup \underbrace{\{\{a\}, \{b\}\}}_{\in\ [U]^{<\kappa} \text{ by (6.5)}} \in U \quad \text{by Lemma 6.5, (2) and (1).}$$ Gu-7 by (6.7).

(4): $a \times b \subseteq \underbrace{\mathcal{P}(\mathcal{P}(a \cup b))}_{\in U \text{ by (3) and (6.6)}}$. Thus $a \times b \in U$ by (1).

(5): Suppose $f \in {}^I U$ for some $I \in U$. Let $f^* : I \to U; i \mapsto \{f(i)\}$. f^* is well-defined by (6.5). Then $f''I = \bigcup_{i \in I} f^*(i) \in U$ by (6.7).

(6): Suppose, toward a contradiction, that $\langle a, r\rangle \in U$ be such that r is a well-ordering on a but $otp(\langle a, r\rangle) \notin U$. Suppose furhter that $\langle a, r\rangle$ is chosen such that $otp(\langle a, r\rangle)$ is minimal among the order-type of such pairs.

$otp(\langle a, r\rangle)$ is not a successor ordinal: if $otp(\langle a, r\rangle) = \alpha + 1$, then let I be the initial segment of a below the maximal element of a with respect to r. By (6.3), (1) and (6.5), we have $\langle I, r \restriction I\rangle \in U$ and $otp(\langle I, r \restriction I\rangle) = \alpha$. By the minimality of $otp(\langle a, r\rangle)$, it follows that $\alpha \in U$. Hence $otp(\langle a, r\rangle) = \alpha + 1 = \alpha \cup \{\alpha\} \in U$. This is a contradiction to the choice of $\langle a, r\rangle$.

Hence we may assume that $otp(\langle a, r\rangle)$ is a limit ordinal. For each $b \in a$, let I_b be the initial segment of a consisting of all elements of a smaller than b with respect to r and let $\alpha_b = otp(\langle I_b, r \restriction I_b\rangle)$. Since $\langle I_b, r \restriction I_b\rangle \in U$, $\alpha_b \in U$ by the minimality of $otp(\langle a, r\rangle)$. Thus $otp(\langle a, r\rangle) = \bigcup\{\alpha_b : b \in a\} \in U$ by (6.7). This is a contradiction to the choice of $\langle a, r\rangle$.

(7): We first show that κ is a regular cardinal. Suppose otherwise and let $\delta = cf(\kappa) < \kappa$ and $\langle \alpha_\xi : \xi\rangle$ be a sequence of ordinals $< \kappa$ such that $\sup_{\xi<\delta} \alpha_\xi = \kappa$. Then $\kappa = \bigcup\{\{\alpha_\xi\} : \xi \in \delta\} \in U$ by (6.7) and thus $\kappa \in \kappa$. This is a contradiction.

For $\delta < \kappa$, we have $2^\delta = |\mathcal{P}(\delta)| = otp(\langle \mathcal{P}(\delta), r\rangle) \in U$ by (6) where r is the well-ordering on $\mathcal{P}(\delta)$ of order-type $|\mathcal{P}(\delta)|$.

(8): We have shown in (2) that $\mathcal{H}(\kappa) \subseteq U$ holds. To show that $U \subseteq \mathcal{H}(\kappa)$, suppose otherwise and let $a \in U \setminus \mathcal{H}(\kappa)$ be with the minimal possible rank. Thus $b \in \mathcal{H}(\kappa)$ holds for all $b \in a$. By AC, and (6), $\lambda = |a| \in U$. It follows that $\lambda < \kappa$ and hence $a \in \mathcal{H}(\kappa)$. This is a contradiction to the choice of a.

If $\kappa > \omega$ then $\omega \in \kappa \subseteq U$. It follows that U satisfies the Axiom of Infinity. It is also easy to prove that all other Axioms of ZFC hold in U: The Axiom of Extensionality holds in U since U is transitive. The Axiom of Emptyset holds in U by (6.4). $U \models$"The Pairing Axiom" by (6.5). $U \models$"The Axiom of Union" by the remark before Lemma 6.7. The Axiom of Powerset by (6.6). $U \models$"The Axiom of Separation" by (1). $U \models$"The Axiom of Replacement" follows from (5). The Axiom of Foundation holds in any $\in$-model (if we are working in the set theory with the Axiom). If AC holds then for all $a \in U$ with $\emptyset \notin a$ and a choice function $f : a \to \bigcup a$, we have $f \in U$: $a \times (\bigcup a) \in U$ by the remark before Lemma 6.6 and (4). Hence $f \in U$ by $f \subseteq a \times (\bigcup a)$ and (1).

(9): If $\kappa = \omega$ or κ is inaccessible, it is easy to see that $\mathcal{H}(\kappa)$ satisfies the conditions (6.3) $\sim$ (6.7). ❑ (Lemma 6.7)

Since we have $\mathcal{H}(\omega) = V_\omega$ and $\mathcal{H}(\kappa) = V_\kappa$ for an inaccessible κ, (7), (8), (9) in

Lemma 6.7 implies the following. Note that $V_\kappa = \mathcal{H}(\kappa)$ if and only if $\kappa = \omega$ or κ is inaccessible under CH (see e.g. [16] Chapter IV, Lemma 6.3).

Theorem 6.8 (ZFC, [19]) *A set U is a Grothendieck universe if and only if $U = V_\omega = \mathcal{H}(\omega)$ or $U = V_\kappa = \mathcal{H}(\kappa)$ for an inaccessible κ.* ❑ Th-Gu-0

Let us call a set U a *weak Grothendieck universe* if

(6.3) U is transitive;

(6.4) $\emptyset \in U$; (we need this, if we are working in ZF − the Axiom of Foundation).

(6.5) U is closed with respect to pairing operation, i.e., for any a, $b \in U$, we have $\{a, b\} \in U$; and

(6.7) for any $I \in U$ and $f \in {}^I U$, $\bigcup_{i \in I} f(i) \in U$. In particular, for any $a \in [U]^{<\kappa}$, we have $\bigcup a \in U$ where $\kappa = \mathsf{On} \cap U$.

With exactly the same proof as that of Lemma 6.5, we obtain the following:

Lemma 6.9 *Let U be a weak Grothendieck universe.* P-Gu-0-0

(1) *For any $a \in U$, we have $\bigcup a \in U$.*

(2) *Let $\kappa = \mathsf{On} \cap U$. Then, for any $a \in [U]^{<\kappa}$, we have $a \in U$.* ❑

With almost the same proof as that of Lemma 6.7, we also obtain the following Lemma.

Lemma 6.10 *Work in ZF. Suppose that U is a weak Grothendieck universe with $\kappa = \mathsf{On} \cap U$.* P-Gu-1

(1) *If $a \in U$ and $b \subseteq a$ then $b \in U$. Equivalently, if $a \in U$ then $\mathcal{P}(a) \subseteq U$;*(4)

(2) *$\omega \subseteq U$ (hence $\kappa \geq \omega$) and $\mathcal{H}(\kappa) \subseteq U$.*

(3) *If a, $b \in U$ then $a \cup b \in U$.*

(4) *If a, $b \in U$ then $a \times b \in U$.*

(5) *If $f \in {}^I U$ for some $I \in U$ then $f''I \in U$.*

(6) *For any $\langle a, r \rangle \in U$ where r is a well ordering on a, $otp(\langle a, r \rangle) \in U$.*

(7) *If $\kappa > \omega$ then κ is a regular cardinal.*

(8) *Under AC, we have $U = \mathcal{H}(\kappa)$. If $\kappa > \omega$, $U \models \mathsf{ZF}$ − Axiom of Powerset and further if AC holds we have $U \models \mathsf{ZFC}$ − Axiom of Powerset.*

(9) *Conversely to (8), $\mathcal{H}(\kappa)$ is a weak Grothendieck universe if either $\kappa = \omega$ or κ is a regular uncountable cardinal.*

(4) In [17], a set U with this property is said to be *supertransitive*.

Proof. All proofs except that of (4) in Lemma 6.7 work under the assumption that U is a weak Grothendieck universe.

(4): Suppose $a,\ b \in U$. For each $c \in a$, $\{c\} \times b = \bigcup\{\{\{\{c\},\{c,d\}\}\} : d \in b\} \in U$ by (6.7). Thus $a \times b = \bigcup\{\{\{c\} \times b\} : c \in a\} \in U$. □ (Lemma 6.10)

Corollary 6.11 (ZFC) *A set U is a weak Grothendieck universe if and only if $\kappa = \mathsf{On} \cap U$ is a regular cardinal and $U = \mathcal{H}(\kappa)$.* P-Gu-1-a

Proof. By Lemma 6.10, (8) and (9). □ (Corollary 6.11)

Theorem 6.12 (ZFC) *A weak Grothendieck universe U with $\kappa = \mathsf{On} \cap U$ is a Grothendieck universe if and only if either $\kappa = \omega$ or κ is an inaccessible cardinal.* P-Gu-1-0

Proof. By Corollary 6.11 and Theorem 6.8. □ (Theorem 6.12)

From here on we work in ZFC. Let us call a transitive $\in$-model M of ZFC *full* if M is closed with respect to the powerset operation. By (6.6) and Lemma 6.7, (6), any uncountable Grothendieck universe is a full transitive $\in$-model of ZFC. However the existence of a full transitive $\in$-model of ZFC is consistency-wise strictly weaker than the existence of a Grothendieck universe:

Lemma 6.13 *Let us call the assertions "There is a full transitive $\in$-model of ZFC" and "There is an uncountable Grothendieck universe" φ_0 and φ_1 respectively. Then we have* P-Gu-2

(6.12) $\mathsf{ZFC} + \varphi_1 \vdash \varphi_0$ *and* Gu-8

(6.13) $\mathsf{ZFC} + \varphi_1 \vdash consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner + \varphi_0)$. Gu-9

Actually we even have

(6.14) $\mathsf{ZFC} + \varphi_1 \vdash consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner + \varphi_0^*)$. Gu-9-0

where φ_0^* is the assertion "For any set a there is a full transitive $\in$-model M of ZFC with $a \in M$".

Proof. (6.12) is trivial. To show (6.13) we work in $\mathsf{ZFC} + \varphi_1$ and let $U = \mathcal{H}(\kappa)$ be an uncountable Grothendieck universe. By Lemma 6.7, (7), κ is an inaccessible cardinal. Note that we also have $U = V_\kappa$.(5)

For an arbitrary $a \in U$, let $\langle \xi_n : n \in \omega\rangle$ and $\langle M_n : n \in \omega\rangle$ be such that

(6.15) $\langle \xi_n : n \in \omega\rangle$ is a strictly increasing sequence of ordinals below κ; Gu-10

(6.16) $a \in V_{\xi_0}$; Gu-10-0

---

(5) For a regular cardinal κ, $\mathcal{H}(\kappa) = V_\kappa$ if and only if either $\kappa = \omega$ or κ is strongly inaccessible (see e.g. Kunen [16] Ch.IV, Lemma 6.3). The "if" direction of this lemma can be also shown by proving $(V_\alpha)^{\mathcal{H}(\kappa)} = V_\alpha$ for all $\alpha < \kappa$ by induction on $\alpha < \kappa$.

(6.17) $\langle M_n : n \in \omega \rangle$ is a $\subseteq$-increasing sequence of elements of U; Gu-11

(6.18) $V_{\xi_n} \subseteq M_n \prec U$; and Gu-12

(6.19) $M_n \subseteq V_{\xi_{n+1}}$. Gu-13

Let $M_a = \bigcup_{n\in\omega} M_n = \bigcup_{n\in\omega} V_{\xi_n}$. Then $M_a = V_\xi \in U$ where $\xi = \sup_{n\in\omega} \xi_n < \kappa$. $a \in M_a$ by (6.16) and $M_a \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$ since $M_a \prec U$ by (6.17) and (6.18). M_a is full by (6.15). Thus U is a model of $\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner + \varphi_0^*$. It follows that *consis*($\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner + \varphi_0^*$) holds. ❑ (Lemma 6.13)

For a transitive $\in$-model M, we shall call the ordinal $\mathsf{On} \cap M$ the *height* of M.

Theorem 6.14 *If M is a full transitive $\in$-model of* ZFC *with regular height κ then κ is an incaccessible cardinal and $M = \mathcal{H}(\kappa)$. Thus such M is a Grothendieck universe.* *P-Gu-3*

Proof. We first prove that κ is inaccessible. Since the regularity of κ is already assumed, and since we have $\omega < \kappa$ by $M \models$ "the Axiom of Infinity" and transitivity of M, it is enough to show that κ is closed under cardinal exponential. Suppose that $\lambda < \kappa$. Then $\mathcal{P}(\lambda) \in M$ since M is full. Let $\mu \in \kappa$ be such that $M \models$ "$|\mathcal{P}(\lambda)| \equiv \mu$" Then $2^\lambda = |\mu| \le \mu < \kappa$.

$M \supseteq \mathcal{H}(\kappa)$: Suppose otherwise and let $a \in \mathcal{H}(\kappa) \setminus M$ be of the minimal rank among elements of $\mathcal{H}(\kappa)\setminus M$. Then $a \subseteq M$. Since κ is regular, there is $\alpha < \kappa$ such that $a \subseteq (V_\alpha)^M$. Since M is full it follows that $a \in \mathcal{P}((V_\alpha)^M) \in M$. Since M is transitive it follows that $a \in M$. This is a contradiction to the choice of a.

$M \subseteq \mathcal{H}(\kappa)$: Suppose otherwise and let $a \in M \setminus \mathcal{H}(\kappa)$ be of the minimal rank among elements of $M \setminus \mathcal{H}(\kappa)$. Then $a \subseteq \mathcal{H}(\kappa)$. Since $a \notin \mathcal{H}(\kappa)$ we should have $|a| \ge \kappa$. But then $\kappa \le |a| \le |a|^M < \kappa$. A contradiction. ❑ (Theorem 6.14)

Lemma 6.15 (ZF) *If a is transitive then $\emptyset \in a$.* *L-0*

Proof. Let $b \in a$ be with the minimal possible rank. then $b = \emptyset$: Otherwise there would be $c \in b$. By the transitivity of a it follows that $c \in a$. This is a contradiction to the minimality of the rank of b. ❑ (Lemma 6.15)

Some Open problems: The following is an assertion with a wrong proof withdrew from

[https://fuchino.ddo.jp/notes/forcing-outline-katowice-2017.pdf: Lemma {Valpha}]

(6) V_α is $\subseteq$-maximal set X with the property that X is transitive, closed under *trcl*$^-$, union of pairs and subset, and $\alpha \notin X$.

Wrong Proof: (6): For $\alpha = 0$ the assertion is trivial since every non empty transitive set contains $\emptyset$ as an element.

Suppose that the assertion holds for α and X is a transitive set closed under *trcl*$^-$, $\cdot \cup \cdot$ and taking a subset, and $\alpha + 1 \notin X$. We have to show that $X \subseteq V_{\alpha+1}$.

If $\alpha \notin X$ then $X \subseteq V_\alpha \subseteq V_{\alpha+1}$ by induction hypothesis. Thus we may assume that $\alpha \in X$. We have $\{\alpha\} \notin X$ since, otherwise, we would have $\alpha + 1 = \alpha \cup \{\alpha\} \in X$ by the closedness of X with respect to $\cdot \cup \cdot$. In particular, if $a \in X$ then $\alpha \notin a$ since otherwise we would have $\{\alpha\} \in X$ by the closedness of X with respect to subset.

Let $a \in X$ and let $b_0 = \mathit{trcl}^-(a)$. $b_0 \in X$ by the closedness of X with respect to trcl^-. Thus $\alpha \notin b_0$. Let $b_1 = \{\bigcup s : s \in [b_0]^{<\aleph_0}\}$ and $b_2 = \{v : v \subseteq u$ for some $u \in b_1\}$. $a \subseteq b_1$. $b_1 \subseteq X$ since X is transitive and closed with respect to $\cdot \cup \cdot$. Thus $b_2 \subseteq X$ since X is closed with respect to subset. Now b_2 is transitive and closed with respect to trcl^-, $\cdot \cup \cdot$ and subset.

Claim 6.15.1 (1) $\alpha \notin b_1$. (2) $\alpha \notin b_2$. *Cl-ind-0*

$\vdash$ $\dashv$ (Claim 6.15.1)

It follows by induction hypothesis that $b_2 \subseteq V_\alpha$. Since $a \subseteq b_0 \subseteq b_1 \subseteq b_2$, it follows that $a \subseteq V_\alpha$ and $a \in V_{\alpha+1}$. Since a was arbitrary we obtain $X \subseteq V_{\alpha+1}$.

The induction step for limit γ is trivial.

7 Is ω_1 an object in the conventional mathematics?

Most part of the conventional mathematics can be developed in Zermelo's set thoery (or Zermelo's set theory plus Axiom of Choice depending on how the word "conventional" is interpreted). The existence of the ordinal number ω_1 as it is defined in the modern set theory cannot be proved in these systems: V_{ω_1} is a model of Z (or ZC if we work in ZFC) and the statement of the non existence of ω_1. As a matter of fact, (in ZFC) V_α for any limit ordinal $> \omega$ is a model of ZC. Thus we cannot even prove the existence of $\omega + \omega$ in ZC! omega1

In this context, is seems to be a intriguing question whether ω_1 can be considered to be a mathematical object in the scope of the "conventional" mathematics. The following rather trivial theorem can be seen in connection with this question.

Theorem 7.1 (1) (in Z + Countable Choice for countable sets) *Every countable well-ordered set is order-preservingly and continuously embeddable in* $\mathbb{R}$. *T-omega1-0*

(2) (in Z without Choice) *No uncountable well-ordered set is order-preservingly embeddable into* $\mathbb{R}$.

Corollary 7.2 (Countable Choice for countable sets) ω_1 *is the least ordinal which is not order-preservingly embeddable into* $\mathbb{R}$. *C-omega1-0*

❑

Of course, Corollary 7.2 only makes sense in an axiom system which proves the existence of ω_1.

Proof of Theorem 7.1. (1): By induction on the order-type of countable well-ordered set $w = \langle w, \leq_w \rangle$.

For $w = \emptyset$, the assertion is trivial.

Suppose that w has the maximal element m and $w \setminus \{m\}$ is order-preservingly and continuously embeddale in $\mathbb{R}$. Then, without loss of generality, we may assume that $w \setminus \{m\}$ is embeddable in the open interval $(-\infty, 0)$ with the embedding f. We may also assume that, if $w \setminus \{m\}$ does not have a maximal element, then $f''w \setminus \{m\}$ is cofinal in $(-\infty, 0)$. The mapping $f \cup \{\langle m, 0 \rangle\}$ is then an order-preserving continuous embedding of w into $\mathbb{R}$.

Suppose now that w does not have the maximal element and each proper initial segment of w can be embedded order-preservingly and continuously into $\mathbb{R}$. By countabiliy of w we can find a sequence m_i, $i \in \omega$ of elements in w such that $\langle m_i : i \in \omega \rangle$ is strictly increasing and cofinal in w.

Let $w_0 = \{n \in w : n <_w m_0\}$ and $w_{i+1} = \{n \in w : m_i \leq_w n < m_{i+1}\}$ for $i \in \omega$. By the assumption there is a sequence f_i, $i \in \omega$ of order-preserving and continuous embeddings of w_0 to $(-\infty, 0)$ and w_{i+1} to $[i, i+1)$ such that

(7.1) f_{i+1} sends the minimal element of w_{i+1} to i; omega1-0

(7.2) if w_i is cofinal in $\{n \in w : n <_w m_i\}$ (that is, if m_i is a limit in w) then $f''w_i$ is cofinal in $(-\infty, i)$. omega1-1

Note that we need the Countable Choice for countable sets to find such sequence $\langle f_i : i \in \omega \rangle$.

Let $f = \bigcup_{i \in \omega} f_i$. Then f is a order-preserving continuous embedding of w into $\mathbb{R}$.

(2): Toward a conrtadiction, suppose that $w = \langle w, <_w \rangle$ is an uncountable well-ordered set and $f : w \to \mathbb{R}$ is an order-preserving embedding. For each $m \in w$ and the successor m' of m, let $q_m \in \mathbb{Q}$ be such that

(7.3) $f(m) < q_m < f(m')$ omega1-2

(we do not need the Axiom of Choice here to choose q_m's since $\mathbb{Q}$ is well-orderable). Since $\mathbb{Q}$ is countable, there are m, $n \in w$ such that $m \neq n$ but $q_m = q_n$. By the choice (7.3) of q_m and q_n this is a contradiction.(6) ❑ (Theorem 7.1)

8 Absolutensess over $\mathcal{H}(\kappa)$

abs

Theorem 8.1 (Lévy, see p.299 in Kanamori [14]) *For any regular $\kappa > \omega$, $\mathcal{H}(\kappa) \prec_1 \mathsf{V}$.* *T-abs-a*

---

(6) Here, again, we do not need Axiom of Choice since, if there were no m, $n \in w$ with $m \neq n$ and $q_m = q_n$, $w \ni m \mapsto q_m \in \mathbb{Q}$ would be a one to one mapping which would be a contradiction to the assumption of non countability of w.

Proof. Suppose that $a \in \mathcal{H}(\kappa)$ and $\varphi = \varphi(x)$ is a formula of the form $\exists y \psi(x, y)$ where ψ is Δ_0.[(7)]

If $\mathcal{H}(\kappa) \models \varphi(a)$, that is, if $\mathcal{H}(\kappa) \models \exists y \psi(a, y)$, let $b \in \mathcal{H}(\kappa)$ be such that $\mathcal{H}(\kappa) \models \psi(a, b)$. It follows that $\mathsf{V} \models \psi(a, b)$.

Thus $\mathsf{V} \models \exists y \psi(a, y)$.

Suppose now $\mathsf{V} \models \varphi(a)$ and let $b \in \mathsf{V}$ be such that $\mathsf{V} \models \psi(a, b)$. By Lévy-Montague Reflection Theorem, there is $\alpha \in \mathsf{On}$ such that a, $b \in V_\alpha$ and $V_\alpha \models \psi(a, b)$. Let $M \prec V_\alpha$ be such that $trcl(a) \subseteq M$, $b \in M$ and $|M| < \kappa$. Let $f : M \xrightarrow{\cong} trcol(M)$ be the transitive collapse. Then $f(a) = a$ and hence we have $trcol(M) \models \psi(a, f(b))$. Since $trcol(M) \subseteq \mathcal{H}(\kappa)$ it follows that $\mathcal{H}(\kappa) \models \psi(a, f(b))$. Thus $\mathcal{H}(\kappa) \models \varphi(a)$. □ (Theorem 8.1)

Lemma 8.2 *For a limit cardinal λ. $\mathcal{H}(\lambda) = \bigcup_{\mu<\lambda} \mathcal{H}(\mu)$.* L-abs-a-0

Proof. If $a \in \bigcup_{\mu<\lambda} \mathcal{H}(\mu)$, then $a \in \mathcal{H}(\mu_0)$ for some $\mu_0 < \lambda$. Hence $a \in \mathcal{H}(\mu_0) \subseteq \mathcal{H}(\lambda)$.

If $a \in \mathcal{H}(\lambda)$, there is $\mu_0 < \lambda$ such that $|trcl(a)| = \mu_0$. Since $a \in \mathcal{H}((\mu_0)^+)$ and $(\mu_0)^+ < \lambda$, it follows that $a \in \bigcup_{\mu<\lambda} \mathcal{H}(\mu)$. □ (Lemma 8.2)

Corollary 8.3 *$\mathcal{H}(\lambda) \prec_1 \mathsf{V}$ for all uncountable cardinals λ.* L-abs-a-1

Proof. By Theorem 8.1 and Lemma 8.2. □ (Corollary 8.3)

Lemma 8.4 *Suppose that κ is a cardinal. If $\mathcal{H}(\kappa) \prec_2 \mathsf{V}$, then κ is a limit cardinal and*

(8.1) $\quad 2^\mu < \kappa$ *for all* $\mu < \kappa$. abs-a-a-0

In particular if κ is regular and $\mathcal{H}(\kappa) \prec_2 \mathsf{V}$ then κ is an inaccessible cardinal.

Proof. κ must be a limit cardinal: For each $\alpha < \kappa$, we have

$$\mathsf{V} \models \text{“} \exists \beta\, (\beta \text{ is an ordinal } > \alpha \ \wedge \ \forall f\, (f : \alpha \to \beta \ \to \ f \text{ is not a surjection)”}$$

By $\mathcal{H}(\kappa) \prec_2 \mathsf{V}$, and since the formula above is Σ_2, it follows that

$$\mathcal{H}(\kappa) \models \text{“} \exists \beta\, (\beta \text{ is an ordinal } > \alpha \ \wedge \ \forall f\, (f : \alpha \to \beta \ \to \ f \text{ is not a surjection)”}.$$

To show (8.1), suppose that there is a $\mu < \kappa$ such that $2^\mu \geq \kappa$. Then we have $\mathcal{H}(\kappa) \models \varphi(\mu)$ and $\mathsf{V} \models \neg\varphi(\mu)$ where $\varphi = \varphi(x)$ is the Π_2-formula

(8.2) $\quad \forall y \exists z\, (z \subseteq x \wedge z \notin y)$. □ (Lemma 8.4) abs-a-a-1

Theorem 8.5 (Proposition 22.3 in [14]) *Suppose that κ is supercompact. Then $\mathcal{H}(\kappa) = V_\kappa \prec_2 \mathsf{V}$.* T-abs-0

---

(7) Note that, since $\mathcal{H}(\kappa)$ satisfies the Pairing Axiom, and “$x \equiv \langle y, z \rangle$” is Δ_0, this imples the general case with formulas with a block of existential quantification and a block of free variables.

Proof. $\mathcal{H}(\kappa) = V_\kappa$ since κ is inaccessible.

Suppose that $\varphi = \varphi(v_1)$ is Σ_2-formula of the form $\exists v_0 \psi(v_0, v_1)$ where ψ is Π_1.

Suppose $V_\kappa \models \varphi(a)$ for some $a \in V_\kappa$. Then there is $b \in V_\kappa$ such that $V_\kappa \models \psi(b, a)$. By Theorem 8.1, it follows $\mathsf{V} \models \psi(b, a)$. Thus $\mathsf{V} \models \varphi(a)$.

Suppose now that $\mathsf{V} \models \varphi(a)$ for some $a \in V_\kappa$. Then there is some $b \in \mathsf{V}$ such that $\mathsf{V} \models \psi(b, a)$. Let $\lambda_0 \geq \kappa$ be a regular cardinal such that $b \in \mathcal{H}(\lambda_0)$ and let $\lambda = |\mathcal{H}(\lambda_0)|$.

Note that

(8.3) $\mathcal{H}(\lambda_0) \models \psi(b, a)$ abs-a-0

by Theorem 8.1.

Let $j : \mathsf{V} \overset{\prec}{\to} M$ be such that

(8.4) $crit(j) = \kappa,\ j(\kappa) > \lambda,$ abs-0

(8.5) M is an inner model in V and abs-1

(8.6) ${}^{\lambda}M \subseteq M.$ abs-2

By (8.6), we have b, $\mathcal{H}(\lambda_0)^{\mathsf{V}} \in M$ and $\mathcal{H}(\lambda_0)^{\mathsf{V}} \subseteq M$. It follows that $\mathcal{H}(\lambda_0)^{\mathsf{V}} = \mathcal{H}(\lambda_0)^{M}$. Thus

(8.7) $M \models$ "$\mathcal{H}(\lambda_0) \models \psi(b, a)$". abs-3

By Theorem 8.1 (applied in M), it follows that $M \models \psi(b, a)$. by Corollary 8.3 (also applied in M), $M \models$ "$\mathcal{H}(j(\kappa)) \models \psi(b, a)$". Hence

(8.8) $M \models$ "$\exists v_0(\mathcal{H}(j(\kappa)) \models \psi(v_0, a))$". abs-4

Noting $j(a) = a$, we obtain $\mathsf{V} \models$ "$\exists v_0(\mathcal{H}(\kappa) \models \psi(v_0, a))$" by elementarity. This simply means $\mathcal{H}(\kappa) \models \varphi(a)$. ❑ (Theorem 8.5)

In the following, the fact that the satatement "x is a supercompact cardinal" is Δ_2^{ZF} (see [14], p.302) is used.

Proposition 8.6 *Suppose that $\kappa < \lambda$, λ is supercompact and κ is μ-supercompact for all $\mu < \lambda$. Then κ is supercompact.* P-abs-0

Proof. There is a Σ_2-formula $\varphi(x)$ expressing "x is supercompact".

Since $V_\lambda \models \varphi(\kappa)$ and $V_\lambda \prec_2 \mathsf{V}$ by Theorem 8.5, it follows that $\mathsf{V} \models \varphi(\kappa)$. This means that κ is supercompact. ❑ (Proposition 8.6)

Proposition 8.6 implies that "$V_\kappa \prec_2 \mathsf{V}$" in Theorem 8.5 is optimal:

Corollary 8.7 *Suppose that κ is the smallest supercompact cardinal. Then $V_\kappa \not\prec_3 \mathsf{V}$.*

Proof. Let $\varphi = \varphi(u)$ be the Π_2-formula stating that "u is supercompact". Then $\mathsf{V} \models \varphi(\kappa)$ and hence $\mathsf{V} \models \exists u\varphi(u)$. On the other hand, by Proposition 8.6, we have $\mathsf{V}_\kappa \not\models \exists u\varphi(u)$.

❏ (Corollary 8.7)

[14] 23.10 Proposition: If κ is an extendible cardinal, then $V_\kappa \prec_3 \mathsf{V}$.

Lemma 8.8 (1) *For an ordinal α, if $V_\alpha \prec_1 \mathsf{V}$, then $Card \cap \alpha$ is cofinal in α. In particular such α is a limit cardinal.* L-abs-0

(2) *If $V_\kappa \prec_1 \mathsf{V}$, then*

(8.9) $\quad 2^\mu < \kappa$ *holds for all* $\mu < \kappa$. abs-6-0

(3) *If $V_\kappa \prec_1 \mathsf{V}$ then $V_\kappa = \mathcal{H}(\kappa)$.*

Proof. (1): Let $\kappa = \sup(Card \cap \alpha)$ and assume that $\kappa < \alpha$. Then $\kappa \in V_\alpha$. We show that $V_\alpha \not\prec_1 \mathsf{V}$.

If $\alpha = \kappa+1$ then $V_\alpha \models \varphi_0(\kappa)$ and $\mathsf{V} \not\models \varphi_0(\kappa)$ for $\varphi_0 = \varphi_0(x)$ where φ_0 is the Π_1-formula

(8.10) $\quad \forall y\,(\text{"}y \text{ is an ordinal"} \ \rightarrow\ (x \equiv y \vee y \mathrel{\varepsilon} x)).$ abs-4-0

If $\alpha > \kappa+1$ then $a = \mathcal{P}(\kappa) \in V_\alpha$. We have $V_\alpha \models \varphi_1(a)$ but $\mathsf{V} \not\models \varphi_1(a)$ where $\varphi_1 = \varphi_1(x)$ is the Π_1-formula

(8.11) $\quad \forall y\forall z\,((\text{"}y \text{ is an ordinal"} \wedge \text{"}z : y \rightarrow x\text{"}) \ \rightarrow\ \text{"}z \text{ is not surjective"}).$ abs-5

(2): Assume that $V_\kappa \prec_1 \mathsf{V}$. By (1), it follows that κ is a limit cardinal. Suppose, toward a contradiction, that there is $\mu < \kappa$ such that $2^\mu \geq \kappa$. Let $a = \mathcal{P}(\mu)$. Then $a \in V_\kappa$ since κ is a limit ordinal. Thus, for φ_1 as in the proof of (1), we have $V_\kappa \models \varphi_1(a)$ but $\mathsf{V} \not\models \varphi_1(a)$. Since φ_1 is a Π_1-formula, this is a contradiction.

(3): Assume that $V_\kappa \prec_1 \mathsf{V}$. $\mathcal{H}(\kappa) \subseteq V_\kappa$ holds for all infinite cardinal κ (see e.g. [7], Lemma [L-hered-0]). Thus it is enough to show that $V_\kappa \subseteq \mathcal{H}(\kappa)$. Consider the following Σ_1-formula $\varphi_2 = \varphi_2(x)$:

(8.12) $\quad \text{"}\exists y\exists z\exists f\,(y \text{ is transitive} \wedge z \text{ is an ordinal} \wedge f : z \rightarrow y \text{ is a bijection} \wedge x \mathrel{\varepsilon} y\text{"}$ abs-7

For $a \in V_\kappa$, we have $\mathsf{V} \models \varphi_2(a)$. It follows that $V_\kappa \models \varphi_2(a)$. Since $\mathsf{On} \cap V_\kappa = \kappa$, this means that $a \in \mathcal{H}(\kappa)$.

❏ (Lemma 8.8)

Lemma 8.9 *If $\mathcal{H}(\kappa) \prec_2 \mathsf{V}$ for a cardinal κ then $\mathcal{H}(\kappa) = V_\kappa$.* L-abs-1

Proof. Suppose that $\mathcal{H}(\kappa) \prec_2 \mathsf{V}$. Then we have $\kappa > \omega$. Since $\mathcal{H}(\kappa) \subseteq V_\kappa$ always holds, it is enough to show that $V_\kappa \subseteq \mathcal{H}(\kappa)$. For this, it is enough to show that $V_\alpha \in \mathcal{H}(\kappa)$ holds for all $\alpha < \kappa$.

Consider the Σ_2-formula $\varphi = \varphi(x)$:

(8.13) $\exists y \exists f \forall u\,($ "y is an ordinal $\wedge\ x \,\varepsilon\, y \wedge f$ is a function on $y \wedge f(\emptyset) = \emptyset \,\wedge$ abs-8
$(\forall v \,\varepsilon\, y)((v$ is a limit $\rightarrow f(v) = \bigcup_{w \varepsilon v} f(w)) \,\wedge$
$(v$ is a successor of some $v_0 \rightarrow (u \subseteq f(v_0) \leftrightarrow u \,\varepsilon\, f(v))))$".

For $\beta < \kappa$, we have $V \models \varphi(\beta)$. Hence $\mathcal{H}(\kappa) \models \varphi(\beta)$ by $\mathcal{H}(\kappa) \prec_2 \mathsf{V}$. Thus, for $\alpha < \kappa$, letting $\beta + \alpha + 1$ there is a mapping $f \in \mathcal{H}(\kappa)$ on β such that $f(\xi) = V_\xi$ for all $\xi < \beta$. In particular $V_\alpha = f(\alpha) \in \mathcal{H}(\kappa)$. ❑ (Lemma 8.9)

Corollary 8.10 (1) $V_{2^{\aleph_0}} \not\prec_1 \mathsf{V}$. Cor-abs-0

(2) $\mathcal{H}(2^{\aleph_0}) \not\prec_2 \mathsf{V}$.(8)

Proof. (1): $2^{\aleph_0}$ does not satisfy (8.9).

(2): If $\mathcal{H}(2^{\aleph_0}) \prec_2 \mathsf{V}$, then $\mathcal{H}(2^{\aleph_0}) = V_{2^{\aleph_0}}$ by Lemma 8.9. Thus $V_{2^{\aleph_0}} \prec_2 \mathsf{V}$. But this is a conctradiction to (1). Note that this can be seen also directly by $\mathcal{H}(2^{\aleph_0}) \models \varphi(\omega)$ but $\mathsf{V} \models \neg\varphi(\omega)$ where φ is as in (8.2). ❑ (Corollary 8.10)

Theorem 8.11 *For a cardinal κ, the following are equivalent:* (a) $V_\kappa \prec_1 \mathsf{V}$; T-abs-1
(b) $\mathcal{H}(\kappa) = V_\kappa$ *and* $2^\mu < \kappa$ *for all* $\mu < \kappa$; (c) $\mathcal{H}(\kappa) = V_\kappa$.

Proof. (a) ⇒ (b): By Lemma 8.8.

(b) ⇒ (c): Trivial.

(c) ⇒ (a): By Theorem 8.1. ❑ (Theorem 8.11)

Theorem 8.12 *For a regular cardinal κ, the following are equivalent:* (a) $V_\kappa \prec_1 \mathsf{V}$; T-abs-2

(b) *κ is inaccessible;*

(c) $V_\kappa \models \mathsf{ZFC}$;

(d) $V_\kappa = \mathcal{H}(\kappa)$.

Proof. Suppose that κ is a regular cardinal. (a) ⇒ (b): By Lemma 8.8, (2). (b) ⇒ (c): This is well-known.

(c) ⇒ (d): Let $\varphi_2 = \varphi_2(x)$ be the Σ_1-formula in the proof of Lemma 8.8, (3). Since $\mathsf{ZFC} \vdash \forall x \varphi_2$, we have $V_\kappa \models \forall x \varphi_2$. Since $On^{V_\kappa} = \kappa$ it follows that $V_\kappa \subseteq \mathcal{H}(\kappa)$.

(d) ⇒ (a): By Theorem 8.1. ❑ (Theorem 8.12)

Cantor's Attic: Reflecting cardinals

(8) Note that $\mathcal{H}(2^{\aleph_0}) \prec_1 \mathsf{V}$ by Theorem 8.1.

9 Generic elementary embedding

9.1 generic elementary embedding under chain conditions gen

gen-cc

Lemma 9.1 *Suppse that M and N are inner models in V and $j \subseteq \mathsf{V}$ be such that $j : M \overset{\preceq}{\to} N$. Suppose further that $\mathbb{P} \in M$ is a poset in M, $\mathbb{Q} = j(\mathbb{P})$, $\mathbb{G}$ is an $(M, \mathbb{P})$-generic filter and $\mathbb{H}$ is an $(N, \mathbb{Q})$-generic filter such that* L-gen-0

(9.1) $\quad j''\mathbb{G} \subseteq \mathbb{H}$. x-gen-a

Then the class mapping introduced by

(9.2) $\quad$ "$\tilde{j} : M[\mathbb{G}] \to N[\mathbb{H}];\ \underset{\sim}{a}[\mathbb{G}] \mapsto j(\underset{\sim}{a})[\mathbb{H}]$ *for $\mathbb{P}$-name $\underset{\sim}{a}$ in M*" gen-a

is well defined, $j \subseteq \tilde{j}$ and $\tilde{j} : M[\mathbb{G}] \overset{\preceq}{\to} N[\mathbb{H}]$.

Proof. • Well-definedness: Suppose that $\underset{\sim}{a}$, $\underset{\sim}{a}' \in M$ are $\mathbb{P}$-names such that $M[\mathbb{G}] \models$ "$\underset{\sim}{a}[\mathbb{G}] \equiv \underset{\sim}{a}'[\mathbb{G}]$". Then there is a $\mathbb{p} \in \mathbb{G}$ such that $M \models$ "$\mathbb{p} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{a} \equiv \underset{\sim}{a}'$"". By elementarity, it follows that $N \models$ "$j(\mathbb{p}) \Vdash_{\mathbb{Q}}$ "$j(\underset{\sim}{a}) \equiv j(\underset{\sim}{a}')$"". Since $j(\mathbb{p}) \in \mathbb{H}$ by (9.1), it follows that $N[\mathbb{H}] \models$ "$j(\underset{\sim}{a})[\mathbb{H}] \equiv j(\underset{\sim}{a}')[\mathbb{H}]$".

• $j \subseteq \tilde{j}$: Suppose that $j(a) = b$ for $a \in M$ and $b \in N$. Let $\check{a} \in M$ be such that $M \models$ "$\check{a}$ is a standard $\mathbb{P}$-name for a". By elementarity,

(9.3) $\quad N \models$ "$j(\check{a})$ is a standard $\mathbb{Q}$-name for $j(a)$". a-gen-0-0

Thus

(9.4) $$\tilde{j}(a) \underbrace{=}_{\text{by the def. of } \tilde{j}} j(\check{a})[\mathbb{H}] \overbrace{=}^{\text{by (9.3)}} j(a) = b.$$ gen-0

• $\tilde{j} : M[\mathbb{G}] \overset{\preceq}{\to} N[\mathbb{H}]$: Suppose that $M[\mathbb{G}] \models \varphi(a_0, ..., a_{n-1})$ for an $\mathcal{L}_{\in}$-formula $\varphi = \varphi(x_0, ..., x_{n-1})$ and $a_0, ..., a_{n-1} \in M[\mathbb{G}]$. Then there are $\mathbb{p} \in \mathbb{G}$ and $\mathbb{P}$-names $\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1}$ of $a_0, ..., a_{n-1}$ such that

(9.5) $\quad M \models$ "$\mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, ..., \underset{\sim}{a}_{n-1})$"". gen-1

By elementarity,

(9.6) $\quad N \models$ "$j(\mathbb{p}) \Vdash_{\mathbb{Q}}$ "$\varphi(j(\underset{\sim}{a}_0), ..., j(\underset{\sim}{a}_{n-1}))$"". gen-2

Since $j(\mathbb{p}) \in \mathbb{H}$, it follows form the definition (9.2) of $\tilde{j}$ that

(9.7) $\quad N[\mathbb{H}] \models$ "$\varphi(\tilde{j}(a_0), ..., \tilde{j}(a_{n-1}))$". ❑ (Lemma 9.1) gen-3

Lemma 9.2 *Suppose that $N \subseteq \mathsf{V}$ is an inner model and $\mathbb{Q} \in N$ a poset. If* L-gen-1

(9.8) $\quad {}^{\lambda}N \subseteq N$ *for some cardinal* λ, gen-a-0

(9.9) $\quad \mathbb{Q}$ *satisfies the* λ^+*-c.c.* gen-3-0

and $\mathbb{H}$ *is* $(\mathsf{V}, \mathbb{Q})$*-generic, then* $({}^{\lambda}N[\mathbb{H}])^{\mathsf{V}[\mathbb{H}]} \subseteq N[\mathbb{H}]$.

Proof. See Lemma 1.4. ❑ (Lemma 9.2)

A regular cardinal κ has the *tree property* if there is no κ-Aronszajn tree. This means that κ has the tree property if any κ tree has a κ-branch.

For some more known results about the tree property we are not going to discuss here, see e.g. [5]. In the next lemma, the condition "generically measurable by a ccc poset" can be yet weakened to "generically weakly compact by a ν-cc posets for a $\nu < \kappa$" (see [13]).

Lemma 9.3 *Suppose that* κ *is generically measurable by a ccc poset. then* P-gen-0

(1) κ *has the tree property.*

(2) κ *the stationary limit of weakly inaccessible cardinals (i.e.* κ *is Mahlo)*[9] .

Proof. Suppose that $\mathbb{P}$ is a ccc poset, $\mathbb{G}$ a $(\mathsf{V}, \mathbb{P})$-generic set, and j, M classes in $\mathsf{V}[\mathbb{G}]$ such that $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$.

(1): Suppose that $T \in \mathsf{V}$ is a κ-tree. Without loss of generality, we may assume that $T \subseteq {}^{\kappa>}\kappa$.

Then $M \models$ "$j(T)$ is a $j(\kappa)$-tree" by elementarity. Since $\kappa = \mathit{crit}(j)$, we have $j(T)_{<\kappa} = T$.

Let $t^* \in j(T)_\kappa$, and let $\underset{\sim}{\leq}$ and $\underset{\sim}{t}^*$ be $\mathbb{P}$-names of $j(\leq_T)$ and t_0 respectively. Let

$$T^* = \{t \in T \,:\, \text{there is } \mathbb{p} \in \mathbb{P} \text{ such that } \mathbb{p} \Vdash_{\mathbb{P}} \text{“}t \underset{\sim}{\leq} \underset{\sim}{t_0}\text{”}\}$$

By the ccc of $\mathbb{P}$, T^* is a κ-tree such that each of its levels is countable. Thus, by a theorem of Kurepa (Proposition 7.9 in [14]), there is a κ-branch b in T^* (in V). b is also a κ-branch in T. This shows that κ has the tree property.

(2): Note first that κ is a regular limit cardinal in V (i.e. κ is a weakly inaccessible cardinal — otherwise we obtain a contradiction to $\mathit{crit}(j) = \kappa$). Thus (by the ccc of $\mathbb{P}$) κ is regular in $\mathsf{V}[\mathbb{G}]$ and hence also in M.

For any club $C \subseteq \kappa$ (in V), $M \models$ "$j(C)$ is club in $j(\kappa)$" by elemenarity. $j(C) \cap \kappa = C$ by $\mathit{crit}(j) = \kappa$.

Hence $M \models j(C) \ni \kappa$. Thus, $M \models$ "$j(C)$ contains a weakly inaccessible cardinal".

By elementarity it follows that

$$\mathsf{V} \models \text{“}C \text{ contains a weakly inaccessible cardinal”}.$$

---

(9) This assertion will be yet improved in Theorem 9.8.

Since C was arbitrary, it follows that κ is Mahlo in V. ❑ (Lemma 9.3)

Let us call a cardinal κ *greatly weakly Mahlo* if κ is weakly inaccessible and there exists a κ-complete normal filter $\mathcal{F}$ over κ such that $\{\mu < \kappa : \mu \text{ is a regular cardinal}\} \in \mathcal{F}$ and $\mathcal{F}$ is closed with respect to the Mahlo operation:

(9.10) $$S \mapsto \mathrm{M}\ell\,(S) := \{\alpha \in S : \alpha \text{ has uncountable cofinality and } S \cap \alpha \text{ is stationary in } \alpha\}.$$ x-gen-a-a-a-0

This definition of the Mahlo operation is slightly different from the one given in [4].

For $\alpha \in \mathsf{On}$, we define the notion of α-weakly Mahloness for all cardinals κ by induction on α.

(9.11) κ is *0-weakly Mahlo* if κ is weakly Mahlo; x-gen-a-a-0

(9.12) κ is *1-weakly Mahlo* if κ is weakly Mahlo(10) and $\{\mu < \kappa : \mu \text{ is weakly Mahlo}\}$ is stationary; x-gen-a-0

(9.13) for $1 < \alpha \leq \kappa$, κ is *α-weakly Mahlo* if $\{\mu < \kappa : \mu \text{ is } \beta\text{-weakly Mahlo}\}$ is stationary in κ for all $\beta < \alpha$. x-gen-a-1

(9.14) κ is *hyper-weakly Mahlo* if $\triangle_{\alpha<\kappa}\{\mu < \kappa : \mu \text{ is } \alpha\text{-weakly Mahlo}\}$ is stationary. x-gen-a-2

Lemma 9.4 *For an ordinal κ of uncountable cofinality, if $S \subseteq \kappa$ is a stationary set consisting of regular cardinals, then κ is also regular and hence κ is weakly Mahlo.* P-gen-0-0

Proof. Suppose that S is as above but κ is not regular. Say, $cf\,(\kappa) = \mu < \kappa$. Let $\langle \xi_\alpha : \alpha < \mu \rangle$ be a continuously increasing sequence of ordinals cofinal in κ such that $\xi_0 > \mu$. By the assumption on S, there is $\lambda \in S \cap \{\xi_\alpha : \alpha < \mu\}$. Say, $\lambda = \xi_{\alpha^*}$. Then $cf\,(\lambda) \leq \alpha^* < \mu < \lambda$. This is a contradiction since λ as an element of S must be regular. ❑ (Lemma 9.4)

Lemma 9.5 *Suppose $\alpha \leq \beta \leq \kappa$. If κ is β-weakly Mahlo, then κ is α-weakly Mahlo.* P-gen-0-0-0

Proof. By induction on β. ❑ (Lemma 9.5)

For $S \subseteq \kappa$ and $\alpha < \kappa$, let $\mathrm{M}\ell^{\,\alpha}(X)$ be defined inductively by

(9.15) $\mathrm{M}\ell^{\,0}(X) := X$; x-gen-0

(9.16) $\mathrm{M}\ell^{\,\alpha+1}(X) := \mathrm{M}\ell\,(\mathrm{M}\ell^{\,\alpha}(X))$; x-gen-1

(9.17) for a limit $\gamma < \kappa$, $\mathrm{M}\ell^{\,\gamma}(X) := \bigcap_{\alpha<\gamma} \mathrm{M}\ell^{\,\alpha}(X)$. x-gen-2

Finaly, let

(9.18) $\mathrm{M}\ell^{\,\kappa}(X) := \triangle_{\alpha<\kappa}\mathrm{M}\ell^{\,\alpha}(X)$. x-gen-3

---

(10) Actually, by Lemma 9.4, the weak Mahloness of κ follows from the second condition.

Note that stationary sets are not necessarily closed with respect to intersection of decreasing sequence of short length: Let κ be an uncountable cardinal with $\kappa \geq \omega_\omega$. For $n \in \omega$, let $S_n := \{\alpha < \kappa : \omega_n \leq cf(\alpha) < \omega_\omega\}$. Then each S_n, $n \in \omega$ is stationary. But $\bigcap_{n\in\omega} S_n = \emptyset$.

Lemma 9.6 (1) *For a regular κ, a filter $\mathcal{F}$ over κ is uniform (i.e. every end-segment of κ is in $\mathcal{F}$) and normal, if and only if $\mathcal{F}$ is $<\kappa$-complete and normal.* P-gen-1

(2) *If $\mathcal{F}$ is a uniform normal filter over a regular κ, then $C \in \mathcal{F}$ for all club $C \subseteq \kappa$. It follows that all $S \in \mathcal{F}$ are stationary in κ.*

(3) *If κ is greatly weakly Mahlo and $\mathcal{F}$ is as in the definition of the greatly weak Mahloness of κ, then for all $\alpha < \kappa$ $\{\xi < \kappa : \xi$ is α-Mahlo$\} \in \mathcal{F}$.*

Proof. (1): "$\Leftarrow$" is trivial. For "$\Rightarrow$", suppose that $\delta < \kappa$ and $X_\alpha \in \mathcal{F}$ for all $\alpha < \delta$.

For $\alpha < \kappa$, let

$$S^*_\alpha = \begin{cases} S_\alpha \setminus \delta, & \text{if } \alpha < \delta; \\ \kappa & \text{otherwise.} \end{cases}$$

We have $S^*_\alpha \in \mathcal{F}$ for $\alpha < \delta$ as $\kappa \setminus \delta \in \mathcal{F}$ since $\mathcal{F}$ is uniform.

Then $\mathcal{F} \ni \triangle_{\alpha<\kappa} S^*_\alpha = \bigcap_{\alpha<\kappa} S_\alpha \setminus \delta \subseteq \bigcap_{\alpha<\kappa} S_\alpha$. Thus $\cap_{\alpha<\kappa} S_\alpha \in \mathcal{G}$.

(2): We show first that $Lim(\kappa) = \{\alpha < \kappa : \alpha \text{ is a limit ordinal}\}$ is an element of $\mathcal{F}$. This follows from $Lim(\kappa) = \triangle_{\alpha<\kappa} \kappa \setminus (\alpha + 1) \in \mathcal{F}$.

For a club $C \subseteq \kappa$, let $\langle c_\alpha : \alpha < \kappa \rangle$ be an increasing enumeration of C. Then we have $C \supseteq Lim(\kappa) \cap \triangle_{\alpha<\kappa} \kappa \setminus c_\alpha \in \mathcal{F}$.

For an $S \in \mathcal{F}$, $S \cap C \in \mathcal{F}$ and hence $S \cap C \neq \emptyset$ for all club $C \subseteq \kappa$. Thus S is stationary in κ.

(3): By induction on α. ❑ (Lemma 9.6)

The following Proposition is a variant of Proposition 16.8 in [14].

Proposition 9.7 *Suppose that κ is greatly weakly Mahlo, and let $\mathcal{F}$ be a $<\kappa$-complete normal filter over a regular κ such that* P-gen-0-1

(9.19) $\mathrm{Reg}(\kappa) := \{\mu < \kappa : \mu$ *is regular*$\} \in \mathcal{F}$, *and* x-gen-4-0

(9.20) *$\mathcal{F}$ is closed with respect to the Mahlo operation (as defined in (9.10)).* x-gen-4-1

Then, for any $1 \leq \alpha < \kappa$,

(1) $\mathrm{M}\ell^\alpha(\mathrm{Reg}(\kappa)) \in \mathcal{F}$,

(2) $\mathrm{M}\ell^\alpha(\mathrm{Reg}(\kappa)) = \{\mu < \kappa : \mu$ *is β-weakly Mahlo for all* $\beta < \alpha\}$
$= \{\mu < \kappa : \mu$ *is α_0-weakly Mahlo*$\}$ *for all* $1 \leq \alpha < \omega$ *where α_0 is such that* $\alpha = \alpha_0 + 1$;
$\mathrm{M}\ell^\alpha(\mathrm{Reg}(\kappa)) = \{\mu < \kappa : \mu$ *is β-weakly Mahlo for all* $\beta \leq \alpha\}$
$= \{\mu < \kappa : \mu$ *is α-weakly Mahlo*$\}$ *for all* $\omega \leq \alpha < \kappa$,

(3) *κ is hyper-weakly Mahlo.*

Proof. We first prove (1) and (2) by induction on $1 \leq \alpha < \kappa$. Note that the last equality in both of the cases in (2) follows from Lemma 9.5.

For $\alpha = 1$, we have

$$\begin{aligned}\mathcal{F} \underbrace{\ni}_{\text{by (9.19) and (9.20)}} \mathrm{M}\ell^{1}(\mathrm{Reg}(\kappa)) &= \mathrm{M}\ell(\mathrm{Reg}(\kappa)) \\ &= \{\mu \in \mathrm{Reg}(\kappa) : \mu \cap \mathrm{Reg}(\kappa) \text{ is stationary in } \mu\} \\ &= \{\mu \in \mathrm{Reg}(\kappa) : \mu \text{ is weakly Mahlo}\}. \\ &= \{\mu \in \mathrm{Reg}(\kappa) : \mu \text{ is 0-weakly Mahlo}\}.\end{aligned}$$

Suppose that $\gamma < \kappa$ is a limit ordinal, and (1), (2) hold for all $\alpha < \gamma$. Then

$$\mathrm{M}\ell^{\gamma}(\mathrm{Reg}(\kappa)) \underbrace{=}_{\text{by (9.17)}} \textstyle\bigcap_{\alpha<\gamma} \mathrm{M}\ell^{\alpha}(\mathrm{Reg}(\kappa)) \in \mathcal{F}$$

by the induction hypothesis around (1) and $<\kappa$-completeness of $\mathcal{F}$.

Suppose that $\mu \in \mathrm{M}\ell^{\gamma}(Reg(\kappa))$. Then, by the induction hypothesis around (2), μ is $(\beta+1)$-weakly Mahlo for all $\beta < \delta$. By (9.13), it follows that $\{\xi < \mu : \xi \text{ is } \beta\text{-weakly Mahlo}\}$ is stationary in μ. Thus, again by (9.13), μ is δ-weakly Mahlo.

Conversely, if $\mu < \kappa$ is δ-weakly Mahlo, then, by Lemma 9.5, μ is α-weakly Mahlo for all $\alpha < \delta$. Thus, by the induction hypothesis around (2), $\mu \in \bigcap_{\alpha<\gamma} \mathrm{M}\ell^{\alpha}(\mathrm{Reg}(\kappa)) = \mathrm{M}\ell^{\gamma}(\mathrm{Reg}(\kappa))$. This shows that (2) holds for δ.

Suppose now that (1) and (2) hold for $1 \leq \alpha < \kappa$.

If $\alpha < \omega$, this means in particular that for α_0 such that $\alpha = \alpha_0 + 1$,

$$\mathrm{M}\ell^{\alpha}(\mathrm{Reg}(\kappa)) = \{\mu < \kappa : \mu \text{ is } \alpha_0\text{-weakly Mahlo}\} \in \mathcal{F}.$$

By the definition (9.16) of the iteration of Mahlo operation and (9.20), we have

$$\mathrm{M}\ell^{\alpha+1}(\mathrm{Reg}(\kappa)) = \{\mu < \kappa : \mu \text{ is } \underbrace{(\alpha_0 + 1)}_{=\alpha}\text{-weakly Mahlo}\} \in \mathcal{F}.$$

If $\omega \leq \alpha < \omega$, our assumption is

$$\mathrm{M}\ell^{\alpha}(\mathrm{Reg}(\kappa)) = \{\mu < \kappa : \mu \text{ is } \alpha\text{-weakly Mahlo}\} \in \mathcal{F}.$$

Thus, similarly to above, we obtain

$$\mathrm{M}\ell^{\alpha+1}(\mathrm{Reg}(\kappa)) = \{\mu < \kappa : \mu \text{ is } (\alpha + 1)\text{-weakly Mahlo}\} \in \mathcal{F}.$$

(1) and (2) imply (3):

$$\triangle_{\alpha<\kappa}\{\mu < \kappa : \mu \text{ is } \alpha\text{-weakly Mahlo}\} \underbrace{=}_{\text{by (2)}} \triangle_{\alpha<\kappa}\mathrm{M}\ell^{\alpha}(\mathrm{Reg}(\kappa)) \underbrace{\in}_{\text{by (1) and normality of } \mathcal{F}} \mathcal{F}.$$

In particular $\triangle_{\alpha<\kappa}\{\mu < \kappa : \mu \text{ is } \alpha\text{-weakly Mahlo}\}$ is stationary, and this proves that κ is hyper-weakly mahlo. □ (Proposition 9.7)

Theorem 9.8 *If κ is a generically measurable cardinal by ccc poset, then κ is greatly weakly Mahlo.* P-gen-2

Proof. Let $\mathbb{P}$ be a ccc poset with $(\mathsf{V},\mathbb{P})$-generic $\mathbb{G}$ such that there are classese j, $M \subseteq \mathsf{V}[\mathbb{G}]$ with $j : \mathsf{V} \overset{\preceq}{\to}_\kappa M$.

Note that, since generically large cardinals are definable (see [12]), we may apply forcing theorems in the arguments which involve j and M. In particular, we may assume that $\Vdash_{\mathbb{P}}$ "$j : \mathsf{V} \overset{\preceq}{\to}_\kappa M$".

In $\mathsf{V}[\mathbb{G}]$, let $\tilde{\mathcal{F}} := \{S \subseteq \kappa \,:\, S \in \mathsf{V}, j(S) \ni \kappa\}$ and let $\underset{\sim}{\mathcal{F}}$ be a $\mathbb{P}$-name of $\tilde{\mathcal{F}}$.

In V, let $\mathcal{F} := \{S \subseteq \kappa \,:\, \Vdash_{\mathbb{P}}$ "$\check{S} \mathrel{\varepsilon} \underset{\sim}{\mathcal{F}}$"$\}$. Then

Claim 9.8.1 (1) *$\mathcal{F}$ is a $<\kappa$-complete normal filter.*

(2) $\mathrm{Reg}(\kappa) \in \mathcal{F}$.

(3) *$\mathcal{F}$ is closed with respect to Mahlo operation.*

$\vdash$ In the following, we use the bullet notation of Asaf Karagila (see e.g. [10]).

(1): Suppose that $\vec{S} := \langle S_\alpha \,:\, \alpha < \mu\rangle \in \mathsf{V}$ for some $\mu < \kappa$ is a sequence of length μ of elements fo $\mathcal{F}$. Then $\Vdash_{\mathbb{P}}$ "$\vec{S}$ is a sequence of elements of $\mathcal{F}$ of length μ". Since $j(\mu) = \mu$ and hence $\Vdash_{\mathbb{P}}$ "$j(\vec{S}) = \langle j(S_\alpha) \,:\, \alpha < \mu\rangle^\bullet$", we have

$$\Vdash_{\mathbb{P}} \text{“}j(\textstyle\bigcap \vec{S})\text{”} = \bigcap j(\vec{S}) = \bigcap\{j(S_\alpha) \,:\, \alpha < \mu\}^\bullet \ni \kappa.$$

Thus $\Vdash_{\mathbb{P}}$ "$\bigcap \vec{S} \in \underset{\sim}{\mathcal{F}}$", and hence $\bigcap\{S_\alpha \,:\, \alpha < \mu\} \in \mathcal{F}$.

If $\vec{S} := \langle S_\alpha \,:\, \alpha < \kappa\rangle$ is a sequence in V of elements of $\mathcal{F}$. then

$$\Vdash_{\mathbb{P}} \text{“}\kappa \in \textstyle\bigcap\{j(S_\alpha) \,:\, \alpha < \kappa\}^\bullet = \bigcap (j(\vec{S}))^\bullet \restriction \kappa\text{”}.$$

Since

$$\Vdash_{\mathbb{P}} \text{“}\kappa \in \textstyle\bigcap (j(\vec{S}))^\bullet \restriction \kappa \;\Leftrightarrow\; \kappa \in \triangle j(\vec{S})^\bullet = j(\triangle \vec{S})\text{”},$$

It follows that $\Vdash_{\mathbb{P}}$ "$\triangle \vec{S} \in \underset{\sim}{\mathcal{F}}$" and $\triangle \vec{S} \in \mathcal{F}$.

(2): Let $R := \mathrm{Reg}(\kappa)$. Then we have $\Vdash_{\mathbb{P}}$ "$j(R) = \mathrm{Reg}(j(\kappa))$" by elementarity (which is assumed to be forced by $\mathbb{1}_{\mathbb{P}}$). Thus $\Vdash_{\mathbb{P}}$ "$\kappa \in j(R)$" and hence $R \in \mathcal{F}$.

(3): If $S \in \mathcal{F}$, then $\Vdash_{\mathbb{P}}$ "$\kappa \in j(S)$" by the definition of $\mathcal{F}$. Thus $\Vdash_{\mathbb{P}}$ "$\mathsf{V} \models S$ is stationary in κ". Since $\mathbb{P}$ is c.c.c., it follows that $\Vdash_{\mathbb{P}}$ "$V^{\mathbb{P}} \models S$ is stationary in κ".

It follows that

$$\Vdash_{\mathbb{P}} \text{“}\kappa \in \mathrm{M}\ell\,(j(S)) = j(\mathrm{M}\ell^{\mathsf{V}}(S))\text{”}.$$

Thus, $\Vdash_{\mathbb{P}}$ "$\mathrm{M}\ell^{\mathsf{V}}(S) \in \underset{\sim}{\mathcal{F}}$" and hence $\mathrm{M}\ell\,(S) \in \mathcal{F}$. $\dashv$ (Claim 9.8.1)

By Proposition 9.7, it follows that κ is greatly weakly Mahlo. ❑ (Theorem 9.8)

The following Lemma corresponds to Proposition 5.15 in [14].

Lemma 9.9 *Suppose that κ is measurable. Then there are stationarily many weakly compact $\mu < \kappa$.* P-gen-3

Proof. Suppose that $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$. Note that we have

(9.21) $\quad {}^{\kappa>}\kappa \subseteq M$ x-gen-5

[If $f \in {}^{\kappa>}\kappa$, $f = j(f) \in M$; if $f \in {}^{\kappa}\kappa$ then $f = f(f) \restriction \kappa \in M$.].

κ is weakly compact in M: [Since κ is strongly inaccessible in V, it is also strongly inaccessible in M by (9.21). The tree property of κ follows from the fact that κ has the tree property in V and (9.21).].

For any club $C \subseteq \kappa$ (in V), $\kappa \in j(C)$: [since $j(C)$ is a club in M by elementarity and $j(C) \cap \kappa = C$ is unbounded below κ.].

Thus

$$M \models \text{“} j(C) \text{ contains a weakly compact cardinal”}.$$

By elementarity, it follows that

$$\mathsf{V} \models \text{“} C \text{ contains a weakly compact cardinal”}.$$

Since C was arbitrary, this shows that $\{\mu < \kappa : \mu \text{ is weakly compact}\}$ is stationary.

□ (Lemma 9.9)

For a class $\mathcal{C}$ of posets a cardinal κ has $\mathcal{C}$*-generic embedding property* if, for any transitive M_0 with $\kappa \in M_0$ and $|M_0| = \kappa$, there is $\mathbb{P} \in \mathcal{C}$ such that $\Vdash_{\mathbb{P}}$ “$\exists j_0 \exists N_0\ (j_0 : M_0 \overset{\prec}{\to}_\kappa N_0)$”.

Lemma 9.10 *Suppose that κ has $\mathcal{C}$-generic embedding property. Then,* P-gen-4

(1) *κ is regular.*

(2) *If elements of $\mathcal{C}$ are $<\kappa$-cc then κ is a limit.*

Proof. (1): Suppose that κ is singular. Let $\mu := cf(\kappa) < \kappa$ and let $\vec{c} := \langle \alpha_\xi : \xi < \mu \rangle$ be an increasing sequence of ordinals cofinal in κ. Let M be a transitive set such that κ, $\bar{c} \in M$ and $|M| = \kappa$. Then, for this M, there cannot be any $\mathbb{P}$ with the property in the definition of $\mathcal{C}$-generic embedding property.

(2): Suppose that $\mathcal{C}$ consists of $<\kappa$-cc posets. Assume, toward a contradiction, that $\kappa = \mu^+$. Let M_0 be a transitive set such that $\kappa \in M_0$, $M_0 \models \kappa = \mu^+$ and $|M_0| = \kappa$. Suppose that $\mathbb{P} \in \mathcal{C}$, $\mathbb{G}$ is $(\mathsf{V}, \mathbb{P})$-generic, and j_0, $N_0 \in \mathsf{V}[\mathbb{G}]$ are such that $j_0 : M_0 \overset{\prec}{\to}_\kappa N_0$. Then $N_0 \models \mu < \kappa < j(\kappa) = \mu^+$. But by the $<\kappa$-cc of κ, $N_0 \models \kappa$ is a cardinal. This is a contraditcion. □ (Lemma 9.10)

Lemma 9.11 *Suppose that $\kappa > \omega_1$ and κ has the $<\mu$-c.c.-generic embedding property for $\mu < \kappa$. Then κ has the tree property.* P-gen-5

Proof. Suppose that κ has the $<\mu$-c.c.-generic embedding property for some $\mu < \kappa$. Then κ is weakly inaccessible by Lemma 9.11.

Suppose that T is a κ-tree. Without loss of generality, $T \subseteq {}^{\kappa>}\kappa$. Let M be a transitive set such that κ, $T \in M$, and $|M| = \kappa$. Let $\mathbb{P}$, $\underset{\sim}{j}$, $\underset{\sim}{N}$ be such that

$$\Vdash_{\mathbb{P}} \text{“} \underset{\sim}{j} : M \overset{\preceq}{\rightarrow}_{\kappa} \underset{\sim}{N} \text{”}.$$

Then $\Vdash_{\mathbb{P}} \text{“} \underset{\sim}{j} \restriction T \equiv id_T \text{”}$. Let $\underset{\sim}{t}^*$ be a $\mathbb{P}$-name of an element of $\underset{\sim}{j}(T)$ of height $\geq \kappa$ and

$$T_0 := \{t \in T : \mathbb{p} \Vdash_{\mathbb{P}} \text{“} t \leq_{j(T)} \underset{\sim}{t}^* \text{”}\}.$$

T_0 is a tree of width $\leq \mu$ and of heigth κ. Thus, by Proposition 7.9 in [14], T_0 has a κ-branch b. Clearly b is also a κ-branch in T. ❏ (Lemma 9.11)

Scan_2021-09-28–17.45 Lücke - annotated.pdf p.9
Does c.c.c.-generic embedding property imply some variant of great weak Mahloness?

Let *Cohen* $:= \{\mathbb{P} : \mathbb{P} \sim \mathrm{Fn}(\kappa, 2, <\aleph_0)\}$.

Theorem 9.12 *P-gen-6*

If κ is Cohen-generically measurable, then κ is stationary limit of cardinals with Cohen-generic embedding proptery.

Proof. Let $\mathbb{P} = \mathrm{Fn}(\lambda, 2, <\aleph_0)$ for some λ such that for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are classes j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that $j : \mathsf{V} \overset{\preceq}{\rightarrow}_{\kappa} M$. It is enough to show that κ has the Cohen-generic empbedding property in M.

In M, Let M_0 be a transitive set with $\kappa \in M_0$ and $|M_0| = \kappa$. Let $\mathbb{Q} := j(\mathbb{P})$. We have $j''\mathbb{P} \lessdot \mathbb{P}$. Let $\mathbb{G}^* = j''\mathbb{G}$. Then $\mathbb{G}^*$ is $(\mathsf{V}, j''\mathbb{P})$-generic and $\mathsf{V}[\mathbb{G}] = \mathsf{V}[\mathbb{G}^*]$. Let $\mathbb{H}^*$ be a $(\mathsf{V}, \mathbb{Q})$-generic filter such that $\mathbb{G}^* \subseteq \mathbb{H}^*$.

Let

$$\tilde{j} : \mathsf{V}[\mathbb{G}] = \mathsf{V}[\mathbb{G}^*] \quad \overset{\preceq}{\rightarrow}_{\kappa} \quad \mathsf{V}[\mathbb{H}^*]; \;\; \underset{\sim}{a}[\mathbb{G}] \;\mapsto\; j(\underset{\sim}{a})[\mathbb{H}^*]$$

for all $\mathbb{P}$-name $\underset{\sim}{a}$ (in V).

❏ (Theorem 9.12)

Corollary 9.13 *Let $\mathcal{C}$ be as in Theorem 9.12 and suppose that κ is tightly $\mathcal{C}$-generically huge, then there are stationarily many $\mu < \kappa$ with tree property.* *P-gen-7*

Proof. By Theorem 9.12 and Lemma 9.11. ❏ (Corollary 9.13)

9.2 Finite support Lévy collapse

Let us begin with recalling some definitions: gen-levy

For classes $\mathcal{C}$, $\mathcal{D}$ of posets, a cardinal κ is said to be *$(\mathcal{C}, \mathcal{D})$-generically supercompact* if, for any $\lambda \geq \kappa$ and $\mathbb{P} \in \mathcal{C}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ such that $\Vdash_{\mathbb{P}} \text{“} \underset{\sim}{\mathbb{Q}} \;\varepsilon\; \mathcal{D} \text{”}$ and, for any

$(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{G} * \mathbb{H}$, there are classes j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, $j(\kappa) > \lambda$, $\mathbb{G} * \mathbb{H} \in M$ and $j''\lambda \in M$.

In the following, we shall denote the conditions above by

$$\Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \text{“}\exists j \exists M\ (j : V \xrightarrow{\preceq}_\kappa M,\ j(\kappa) \geq \lambda,\ \underset{\sim}{\mathbb{G}} * \underset{\sim}{\mathbb{H}} \in M,\ j''M\text{”}. \tag{9.22}$$ x-gen-7

Strictly speaking, we cannot formally quantify classes (and put the “formula” inside the forcing relation). But because of the definability of this situation (see [12]), we can reformulate (9.22) into a legitimate expression.

We call a cardinal κ *tightly $(\mathcal{C}, \mathcal{D})$-generically supercompact* if (9.22) is strengthened to

$$\Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \text{“}\exists j \exists M\ (j : V \xrightarrow{\preceq}_\kappa M,\ j(\kappa) \geq \lambda,\ \underset{\sim}{\mathbb{G}} * \underset{\sim}{\mathbb{H}} \in M,\ |\mathbb{P} * \underset{\sim}{\mathbb{Q}}| \leq j(\kappa),\ j''M\text{”}. \tag{9.23}$$ x-gen-8

A cardinal κ is *Laver-generically supercompact* by $\mathcal{D}$, if, for any $\lambda \geq \kappa$ and $\mathbb{P} \in \mathcal{D}$ there is $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ such that $\Vdash_{\mathbb{P}}$ “$\underset{\sim}{\mathbb{Q}} \in \mathcal{D}$” and

$$\Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \text{“}\exists j \exists M\ (j : V \xrightarrow{\preceq}_\kappa M,\ j(\kappa) \geq \lambda,\ \underset{\sim}{\mathbb{G}} * \underset{\sim}{\mathbb{H}} \in M,\ j''M\text{”}. \tag{9.24}$$ x-gen-9

Similarly as above, we say that a cardinal κ is *tightly Laver-generically supercompact* by $\mathcal{D}$, if for any $\lambda \geq \kappa$ and $\mathbb{P} \in \mathcal{D}$ there is $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ such that $\Vdash_{\mathbb{P}}$ “$\underset{\sim}{\mathbb{Q}} \in \mathcal{D}$” and

$$\Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \text{“}\exists j \exists M\ (j : V \xrightarrow{\preceq}_\kappa M,\ j(\kappa) \geq \lambda,\ \underset{\sim}{\mathbb{G}} * \underset{\sim}{\mathbb{H}} \in M,\ |\mathbb{P} * \underset{\sim}{\mathbb{Q}}| \leq j(\kappa),\ j''M\text{”}. \tag{9.25}$$ x-gen-10

We shall call the poset of the form $\mathrm{Col}(\omega, \lambda)$ (in the notation of [14]) for a regular cardinal λ the finite support Lévy collapse (fs L-c, for short).

Lemma 9.14 *Let $\mathcal{C}$ be the class of all posets and $\mathcal{D}$ be the class of all fs L-c. Then, the following are equivalent:* (a) *κ is (tightly) $(\mathcal{C}, \mathcal{D})$-supercompact.* P-gen-8

(b) *κ is (tightly) Laver-generically supercompact for $\mathcal{D}$.*

Proof. (a) ⇒ (b): is trivial.

(b) ⇒ (a):

❑ (Lemma 9.14)

10 Generalized mixed support iteration along with a Laver function

The materials in this section have been moved to [10]. msi

参考文献

thereferernces

[1] Martin Aigner and Günter M. Ziegler, Proofs from THE BOOK, Fifth Edition, Springer, (2014). 16

[2] Joan Bagaria, Derived topologies on ordinals and stationary reflection, preprint.

[3] ___________, Natural axioms of set theory and the continuum problem, In: Proceedings of the 12-th International Congress of Logic, Methodology, and Philosophy of Science, King's College London (2005), 43–64.

[4] James E. Baumgartner, Alan D. Taylor and Stanley Wagon, On Splitting Stationary Subsets of Large Cardinals, The Journal of Symbolic Logic, Vol.42, No.2 (1977), 203–214. 32

[5] James Cummings and Matthew Foreman, The Tree Property, Advances in Mathematics 133, 1–32 (1998). 31

[6] R. デデキント; 渕野 昌 (訳／解説), "数とは何かそして何であるべきか", (Richard Dedekind, „Stetigkeit und irrationale Zahlen“ (1872) と „Was sind und was sollen die Zahlen“ (1888) の日本語訳および解説), ちくま学芸文庫, (2013), 1 – 336. 40

[7] Sakaé Fuchino, An outline of independence proofs by forcing, lecture note (2017), https://fuchino.ddo.jp/notes/forcing-outline-katowice-2017.pdf 28

[8] Fuchino, Sakaé, Ottenbereit Maschio Rodriques, André, and Sakai, Hiroshi: Strong downward Löwenheim-Skolem theorems for stationary logics, I, submitted.
Extended version of the paper: https://fuchino.ddo.jp/papers/SDLS-x.pdf 12

[9] ________________: Strong downward Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum, preprint.
Extended version of the paper: https://fuchino.ddo.jp/papers/SDLS-II-x.pdf 4, 10, 12

[10] ________________: Strong downward Löwenheim-Skolem theorems for stationary logics, III, to appear.
Extended version of the paper: https://fuchino.ddo.jp/papers/SDLS-III-x.pdf 35, 38

[11] Fuchino, Sakaé, and Rinot, Assaf: Openly generated Boolean algebras and the Fodor-type Reflection Principle, Fundamenta Mathematicae 212, (2011), 261-283.

[12] Fuchino, Sakaé, and Sakai, Hiroshi: The first-order definability of generic large cardinals, submitted.
Extended version of the paper: https://fuchino.ddo.jp/papers/definability-of-glc-x.pdf 35, 38

[13] Fuchino, Sakaé, and Sakai, Hiroshi: Generically supecompact cardinals by c.c.c. forcing, in preparation. 31

[14] Akihiro Kanamori, The Higher Infinite, Springer–Verlag (1994). 10, 25, 26, 27, 28, 31, 33, 35, 37, 38

[15] Akihiro Kanamori, Zermelo and Set Theory, The Bulletin of Symbolic Logic, Vol. 10, no. 4 (2004), 487–553.

[16] Kenneth Kunen, Set Theory, An Introduction to Independence Proofs, Elsevier (1980). 日本語訳: K. キューネン著, 藤田 博司 訳, 集合論 — 独立性証明への案内, 日本評論社 (2008). 16

21, 22

[17] A.R.D. Mathias, Models of Zermelo Set Theory, the Journal of Symbolic Logic, Vol.66, No.2 (2001), 487–496. 21

[18] Volker Peckhaus and Reinhard Kahle, Hilbert's Paradox, Historia Mathematica 29 (2002), 157–175. 16

[19] N.K. Williams, On Grothendieck universes, Compositio Mathematica, Tome 21, No.1, (1969), 1 – 3. 21

[20] Ernst Zermelo, „Untersuchungen über die Grundlagen der Mengenlehre I“, Mathematische Annalen, 65, (1908), 261–281. 日本語訳: “集合論の基礎に関する研究 I”, [6] の日本語訳に付録 B として収録.

12