On models of ZFC

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* Section 3 and Appendices A,B may be obsolete. For the latest version of the materials in these sections see: Sakaé Fuchino, "Axiomatic set theory and the foundation of mathematics",
https://fuchino.ddo.jp/kobe/logic-ss2019.pdf

Section 1 of the following note is based on my talk I gave at the Kobe Logic Colloquium on 26. December 2013. The material of the section is largely based on a note by Andrés Caicedo [1] although the presentation I chose here lay more stress on distinction between metamathematics and the mathematics inside an axiom system of set theory. In particular, the formulation of the Woodin's Lemma (Proposition 1.3) I chose here answers a question of Makoto Kikuchi at the talk. I learned Proposition 1.7 from David Asperó who was present at the talk.

Section 2 gives a complete answer to one of the questions Wojciech Bielas asked me during my stay in Katowice in October 2014. Nothing in the section should be new.

In Section 3 we consider several variations of Gödel's Speed-up Theorem and analyze their impacts to set theory.

In Appendix 1, we introduce a Gödel numbering in the framework of set theory and see how syntax and semantics of first-order logic can be developed using it. In Appendix 2, we give a proof of the Diagonal Lemma on basis of the Gödel numbering given in Appendix 1.

A part of the materials in this note will be later reused in a textbook of the author in preparation on basics of set-theory and forcing.

The most up-to-date version of this note is downloadable as:

https://fuchino.ddo.jp/notes/woodin-incompl-e.pdf

1 Woodin's set-theoretic proof of the Second Incompleteness Theorem

Let us first review the Diagonal Lemma which is formulated here in the framework of ZFC.

Let $\mathcal{L}_\varepsilon$ denote the language of ZFC. $\mathcal{L}_\varepsilon$ consists merely of the binary relation symbol "ε". Let $\mathcal{L}_{\mathsf{ZFC}} = \{\varepsilon, \emptyset, \{\cdot\}, \cup, \cap, \setminus , ...\}$ be the expansion of $\mathcal{L}_\varepsilon$ with all the names of the definable constants and functions we introduce in course of our discussions. Let ZFC* be the natural conservative extension of ZFC in $\mathcal{L}_{\mathsf{ZFC}}$ obtained from ZFC by adding the intended definitions of the new symbols in $\mathcal{L}_{\mathsf{ZFC}}$. We often call ZFC* simply as "ZFC" if it would not make any confusion: Note that, for any $\mathcal{L}_{\mathsf{ZFC}}$-formula φ, there is an $\mathcal{L}_\varepsilon$-formula φ^- such that

$$\mathsf{ZFC}^* \vdash \varphi \leftrightarrow \varphi^-.$$

We assume that the Gödel numbering is fixed so that it gives an algorithm which calculates a closed $\mathcal{L}_{\mathsf{ZFC}}$-term $\ulcorner\varphi\urcorner$ of an element in V_ω to each given $\mathcal{L}_\varepsilon$-formula φ. Note that the Gödel numbering $\varphi \mapsto \ulcorner\varphi\urcorner$ is a meta-mathematical operation. We can canonically define ($\mathcal{L}_{\mathsf{ZFC}}$-terms for) subsets $\ulcorner\ulcorner Fml_{\mathcal{L}_\varepsilon}\urcorner\urcorner$, $\ulcorner\ulcorner Sent_{\mathcal{L}_\varepsilon}\urcorner\urcorner$ and $\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$ of V_ω which represent $Fml_{\mathcal{L}_\varepsilon}$, $Sent_{\mathcal{L}_\varepsilon}$ and ZFC in the meta-mathematics in such a way that e.g., we have $\mathsf{ZFC} \vdash \ulcorner\varphi\urcorner \in \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$ for any concretely given axiom φ of ZFC and $\mathsf{ZFC} \vdash \ulcorner\varphi\urcorner \notin \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$ for any concretely given $\mathcal{L}_\varepsilon$-sentence φ which does not belong to the axiom system of ZFC.

Lemma 1.1 (Diagonal Lemma) *For an arbitrary $\mathcal{L}_\varepsilon$-formula $\psi = \psi(x)$, there is an $\mathcal{L}_\varepsilon$-sentence σ such that*

(1.1) $\mathsf{ZFC} \vdash \sigma \leftrightarrow \psi(\ulcorner\sigma\urcorner)$. □ diag-a-0

See Appendix B for a proof of Lemma 1.1. Note that this lemma is actually a meta-theorem formulated for each (concretely given) $\mathcal{L}_\varepsilon$-formula ψ. A standard proof of the lemma like the one in Appendix B gives an algorithm to compose an $\mathcal{L}_\varepsilon$-formula σ as above for a given $\mathcal{L}_\varepsilon$-formula $\psi = \psi(x)$.

In the following, we often consider the situation (in ZFC) that, for some sets M and E, "$\langle M, E\rangle$ is a model of ZFC". This actually means that (ZFC proves) $\langle M, E\rangle \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$ (i.e., $\forall x \in \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner (\langle M, E\rangle \models x)$).

For an $\mathcal{L}_\varepsilon$-structure M and an $\mathcal{L}_\varepsilon$-formula φ (in meta-mathematics) we often write simply $M \models \varphi$ in place of $M \models \ulcorner\varphi\urcorner$.

In ZFC (or in some extension of ZFC), let $M = \langle M, E\rangle$ be an $\mathcal{L}_\varepsilon$-structure with $M \models \ulcorner\ulcorner \mathsf{ZFC} \urcorner\urcorner$ and let m, $e \in M$ be such that

$$M \models \text{“}\langle m, e\rangle \text{ is a } \mathcal{L}_\varepsilon\text{-structure”}.$$

Then let $m^* = \langle m^*, e^*\rangle$ be the $\mathcal{L}_\varepsilon$-structure with

$$(1.2)\qquad m^* = \{x \in M \,:\, x\, E\, m \ (\Leftrightarrow\ M \models x\, \varepsilon\, m)\} \text{ and}$$ star-0

$$e^* = \{\langle x, y\rangle \in (m^*)^2 \,:\, M \models \text{“}\, x\, e\, y\text{”}\ (\Leftrightarrow\ M \models \langle x, y\rangle\, \varepsilon\, e)\}.$$

The following can be proved by induction on the construction of $\varphi \in \ulcorner\ulcorner \mathit{Fml}_{\mathcal{L}_\varepsilon} \urcorner\urcorner$:

Lemma 1.2 *The following assertion is provable in* ZFC*: Suppose that $M = \langle M, E\rangle$ is an $\mathcal{L}_\varepsilon$-structure with $M \models \ulcorner\ulcorner \mathsf{ZFC} \urcorner\urcorner$, and $m, e \in M$ are such that $M \models$ “$\langle m, e\rangle$ is a $\mathcal{L}_\varepsilon$-structure”. Then, for any $\varphi \in \ulcorner\ulcorner \mathit{Fml}_{\mathcal{L}_\varepsilon} \urcorner\urcorner$ with $\varphi = \varphi(\vec{x})$ and $\vec{a} \in M$ such that $\ell(\vec{x}) = \ell(\vec{a})$ and $M \models$ “components of $\vec{a}$ are all $\varepsilon\, m$”, we have*

$$(1.3)\qquad M \models \langle m, e\rangle \models \varphi^M(\vec{a}) \ \Leftrightarrow\ \langle m^*, e^*\rangle \models \varphi(\vec{a})$$

where φ^M is the element of $(\ulcorner\ulcorner \mathit{Fml}_{\mathcal{L}_\varepsilon} \urcorner\urcorner)^M$ which corresponds to φ. [1] ❏

A property (i.e. a concretely given $\mathcal{L}_\varepsilon$-formula in meta-mathematics) $P(\cdot)$ is said to be *hereditary* if

$$(1.4)\qquad \mathsf{ZFC} \vdash \forall M\, (P(M) \to M \models \ulcorner\ulcorner \mathsf{ZFC} \urcorner\urcorner)$$
$$\wedge\, \forall M \forall m\, ((P(M) \wedge m \in M \wedge M \models P(m)) \to P(m^*)).$$ [2]

By Lemma 1.2, the property “M is a model of $\ulcorner\ulcorner \mathsf{ZFC} \urcorner\urcorner$” is hereditary.

Proposition 1.3 (H. Woodin) *For hereditary $P(\cdot)$, we have*

$$\mathsf{ZFC} \vdash \forall M\, (P(M)$$
$$\to (M \models \forall m \neg P(m) \vee \exists m \in M (P(m^*) \wedge m^* \models \forall n \neg P(n)))).$$

Proof. Let

$$(1.5)\qquad \mathrm{Th}_P = \{\varphi \in \ulcorner\ulcorner \mathit{Sent}_{\mathcal{L}_\varepsilon} \urcorner\urcorner \,:\, \forall N\, (P(N) \to N \models \varphi)\}.$$ diag-a

By Diagonal Lemma, there is an $\mathcal{L}_\varepsilon$-sentence η_P such that

$$(1.6)\qquad \mathsf{ZFC} \vdash \eta_P \leftrightarrow (\ulcorner \neg\eta_P \urcorner \in \mathrm{Th}_P).$$ [3] diag-0

[1] Note that we have to make this distinction since M is not necessarily a transitive $\in$-model. In particular the ω in M may be non-standard and not isomorphic to the ω in the universe.

[2] Here, “$M \models P(m)$” is an abbreviation of $M \models \mathit{subst}(\ulcorner P(x) \urcorner, x, m)$. See the footnote (1).

[3] Note that “$\ulcorner \neg\eta_P \urcorner \in \mathrm{Th}_P$” can be rewritten as “$\neg \ulcorner \eta_P \urcorner \in \mathrm{Th}_P$” where the last ‘$\neg$’ denotes the function (symbol in ZFC^*) $\neg : \mathit{Fml}_{\mathcal{L}_\varepsilon} \to \mathit{Fml}_{\mathcal{L}_\varepsilon}$ representing metamathematical operation of taking the negation of formulas.

Claim 1.3.1 $\mathsf{ZFC} \vdash \exists N\, P(N) \rightarrow \exists N\, (P(N) \land N \models \eta_P)$.

$\vdash$ We work in ZFC. Let N be such that $P(N)$. If $N \models \eta_P$ then we are done. Suppose $N \not\models \eta_P$. Then, since $N \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$, we have $N \models \ulcorner\neg\eta_P\urcorner \notin \mathrm{Th}_P$ by (1.6). Hence, by (1.5), there is $n \in N$ such that $N \models P(n) \land n \models \eta_P$. By the hereditarity of P and by Lemma 1.2, we have $P(n^*) \land\ n^* \models \eta_P$. $\dashv$ (Claim 1.3.1)

Claim 1.3.2 $\mathsf{ZFC} \vdash \forall M\, (P(M) \land M \models \eta_P \rightarrow M \models \forall n \neg P(n))$.

$\vdash$ We work in ZFC. Suppose $P(M)$ and $M \models \eta_P$. Then, since $P(M)$ implies $M \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$, we have $M \models \neg\eta_P \in \mathrm{Th}_P$ by $M \models \eta_P$ and (1.6). Thus

$$M \models \forall n(P(n) \rightarrow n \models \neg\eta_P). \tag{1.7}$$ diag-1

If we had $M \models \exists n P(n)$, it follows by Claim 1.3.1 that $M \models \exists n(P(n) \land n \models \eta_P)$. This is a contradiction to (1.7). Thus, we should have $M \models \forall n \neg P(n)$. $\dashv$ (Claim 1.3.2)

Working further in ZFC, suppose $P(M)$. If $M \models \forall n \neg P(n)$ then we are done. Assume otherwise. That is, $M \models \exists n P(n)$. By Claim 1.3.2, we have then $M \not\models \eta_P$. By $M \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$ and by (1.6), it follows that $M \models \ulcorner\neg\eta_P\urcorner \notin \mathrm{Th}_P$. Hence, there is $m \in M$ such that $M \models P(m) \land m \models \eta_P$. Since P is hereditary and by Lemma 1.2, we have $P(m^*)$ and $m^* \models \eta_P$. Thus, we have $m^* \models \forall n \neg P(n)$ by Claim 1.3.2. ❑ (Proposition 1.3)

Since the Completeness Theorem can be proved in ZFC, we have

$$\mathsf{ZFC} \vdash \mathit{consis}(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner) \leftrightarrow \exists M\, (M \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner). \tag{1.8}$$

Here, $\mathit{consis}(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)$ is the $\mathcal{L}_\varepsilon$-formula saying

$$\forall\mathcal{P}\, (\text{if } \mathcal{P} \text{ is a proof from } \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner \text{ then the conclusion of } \mathcal{P} \text{ is not } \emptyset \not\equiv \emptyset).$$

Lemma 1.4 *The property $P(\cdot)$ defined by "$P(M) \ \leftrightarrow\ M \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$" is hereditary.*

Proof. Suppose that $P(M)$ and $M \models$ "$P(m)$" for $m \in M$. For $\varphi \in \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$, we have $M \models$ "$\varphi^M\ \varepsilon\ \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$". Thus $M \models$ "$m \models \varphi^M$". By Lemma 1.2, it follows that $m^* \models \varphi$. ❑ (Lemma 1.4)

Note that $P(m^*) \land m^* \models \forall n \neg P(n)$ means for this $P(\cdot)$ that

$$m^* \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner \text{ and} \tag{1.9}$$ star-1

$$m^* \models \forall n\, (n \not\models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner). \tag{1.10}$$ star-2

By Completeness Theorem, (1.10) is equivalent to

$$m^* \models \neg\mathit{consis}(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner) \tag{1.11}$$ star-3

under (1.9).

Thus, we obtained the following by applying Proposition 1.3 to this $P(\cdot)$:

Corollary 1.5 (a) $\mathsf{ZFC} \vdash \forall M\,(M \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner \rightarrow (M \models \neg consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner) \vee \exists m \in M\,(m^* \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner \wedge m^* \models \neg consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner))))$.

(b) $\mathsf{ZFC} \vdash consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner) \rightarrow \exists M\,(M \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner \wedge M \models \neg consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner))$. ❑

The Second Incompleteness Theorem for ZFC can be derived immediately from (b) of the Corollary above:

Corollary 1.6 (Incompleteness Theorem for ZFC) *Suppose that* ZFC *is consistent. Then we have* $\mathsf{ZFC} \nvdash consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)$.

Proof. Suppose, toward a contradiction, that $\mathsf{ZFC} \vdash consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)$. By Corollary 1.5, (b), it follows that

$$(1.12)\quad \mathsf{ZFC} \vdash \exists M\,(M \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner \wedge M \models \neg consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)).$$ diag-2

Let T be a (concretely given) finite subset of ZFC and P a (also concretely given) proof with $T \vdash^P consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)$.

In ZFC, let M be an $\mathcal{L}_\varepsilon$-structure with $M \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$ and $M \models \neg consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)$. But, since $M \models T$, we have $M \models consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)$. This is a contradiction. ❑ (Corollary 1.6)

The next proposition is interesting in contrast to Corollary 1.5, (a):

Proposition 1.7 $\mathsf{ZFC} \vdash \forall M\,(M \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner \rightarrow \exists m \in M(m^* \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner))$.

Proof. We work in ZFC: Suppose that $M \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$. If M is an ω-model (that is, if "$\omega^M \cong \omega$"[(4)]), then we have $\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner = \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner^M$.[(5)] Hence $M \models consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)$.[(6)] Thus, by Completeness Theorem, there is $m \in M$ such that $M \models m \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$. It follows that $m^* \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$ by Lemma 1.2 (see also Lemma 1.4).

If M is not an ω-model then ω^M contains a non-standard number $n^\dagger$. By Lévy's Reflection Principle in M, we have

$$(1.13)\quad M \models \text{“}\exists m\,(m \models \varphi \text{ for all } \varphi \in \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner \text{ with } rank(\varphi) < n^\dagger)\text{”}.$$

Let $m \in M$ be one of such models. For all $\varphi \in \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$, we have $M \models$ "$\varphi^M \in \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$ and $rank(\varphi^M) < n^\dagger$" and hence, by Lemma 1.2, $m^* \models \varphi$. This shows that $m^* \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$. ❑ (Proposition 1.7)

In the proof above $m \in M$ such that $m^* \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$ does not necessarily satisfy $M \models m \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$. Actually this is impossible in general by Corollary 1.5.

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(4) More precisely: $((\omega^M)^*, (<^M)^*) \cong (\omega, <)$ where "$*$" is in the sense of (1.2)

(5) More precisely, we mean here that there is a natural one-to-one correspondence between elements of $\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$ and $(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner^M)^*$.

(6) If M is an ω-model of $\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$ and $M \models \neg consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)$, then the proof of inconsistency from $(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)^M$ in M can be translated to a proof of inconsistency from $\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$ in V.

In the arguments above the Completeness Theorem is always applied to countable theories for which AC is not necessary. Thus ZFC there can be replaced by ZF. The arguments clearly work also for any recursive T containing ZF. Actually we can also treat recursive extensions of certain weak set-theory with the same arguments. Thus almost the full extent of the Second Incompleteness Theorem can be reestablished by the proof as above.

2 Existence of models of ZFC with additional properties

We already noticed that the Completeness Theorem tells us that

$$\mathsf{ZFC} \vdash consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner) \leftrightarrow \exists m(m \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner).$$

It follows that

(2.1) the axiom system $\mathsf{ZFC} + consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)$ is equivalent to the axiom system $\mathsf{ZFC} + \exists m\,(m \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)$. equiv-0

Let us consider the following axiom systems:

(a) ZFC,

(b) $\mathsf{ZFC} + \exists m\,(m \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)$,

(c) $\mathsf{ZFC} + \exists m\,(m \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner \wedge m$ is an ω-model$)$,

(d) $\mathsf{ZFC} + \exists m\,(m \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner \wedge m$ is a transitive $\in$-model$)$,

(e) $\mathsf{ZFC} + \exists \alpha \in \mathrm{On}\,(V_\alpha \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)$,

(f) $\mathsf{ZFC} + \exists \kappa\,(\kappa$ is an inaccessible cardinal$)$.

Clearly, we have (f) $\Rightarrow$ (e) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a).[(7)] We show that none of the implications is convertible.

(a) $\not\Leftarrow$ (b) (under the assumption of the consistency of (a)): This follows from (2.1) and the Second Incompleteness Theorem.

(b) $\not\Leftarrow$ (c) (under the assumption of the consistency of (b)): Assume otherwise. Then we have

$$\text{(b)} \vdash \exists m\,(m \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner \wedge m \text{ is an } \omega\text{-model}).$$

Working in the axiom system (b), let M be an ω-model of $\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$. Then

$$M \models consis(\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner)$$

(see footnote (6)). Thus

(7) "$(\alpha) \Rightarrow (\beta)$" here means that $(\alpha) \vdash \varphi$ for all sentence φ in (β).

$$(\mathrm{b}) \vdash \mathit{consis}(\ulcorner\ulcorner(\mathrm{b})\urcorner\urcorner).$$

By the Second Incompleteness Theorem (see the remark after Proposition 1.7), it follows that the axiom system (b) is inconsistent. This is a contradiction to our assumption.

(c) $\not\Rightarrow$ (d) (under the assumption of the consistency of (c)): Assume otherwise. Then we have

$$(\mathrm{c}) \vdash \exists m\,(m \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner \land m \text{ is a transitive } \in\text{-model}).$$

Working in the axiom system (c), let M be a transitive $\in$-model of $\ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$. By Löwenheim-Skolem Theorem and Mostowski's Theorem, we may assume that M is countable.

Now consider the Σ^1_1-sentence φ

$$\begin{aligned}\exists r \in \mathbb{R}\ \exists s \in \mathbb{R}\ (\ & r \text{ codes a countable model } M \text{ of } \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner \\ & \land\ s \text{ codes an isomorphism between } \omega \text{ and } \omega \text{ in } M)\end{aligned}$$

φ is true (in the axiom system (c)). Hence, by Σ^1_2-absoluteness, $M \models \varphi$. Thus $M \models$ (c) and it follows that

$$(\mathrm{c}) \vdash \mathit{consis}(\ulcorner\ulcorner(\mathrm{c})\urcorner\urcorner).$$

By the Second Incompleteness Theorem, the axiom system (c) is inconsistent in contradiction to our assumption.

(d) $\not\Rightarrow$ (e) (under the assumption of the consistency of (d)): Assume otherwise. Then we have

$$(\mathrm{d}) \vdash \exists \alpha \in \mathrm{On}\,(V_\alpha \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner).$$

Working in the axiom system (d), let α be such that $V_\alpha \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$. Then clearly we have $\alpha > \omega$ hence $|\,V_\alpha\,| \geq 2^{\aleph_0}$. Let $M \prec V_\alpha$ be countable then there is an $E \in \mathcal{P}(\omega^2)$ $(\subseteq V_\alpha)$ such that $\langle \omega, E\rangle \cong \langle M, \in\rangle$. Note that $\langle \omega, E\rangle \in V_\alpha$. Since $V_\alpha \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$, it follows that $\langle N, \in\rangle \in V_\alpha$ where N is the Mostowski collapse of M. It follows that

$$V_\alpha \models \exists m\,(m \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner \land m \text{ is a transitive } \in\text{-model}).$$

Thus, as before, we can apply the Second Incompleteness Theorem to the axiom system (d) and conclude that (d) is inconsistent in contradiction to our assumption.

(e) $\not\Rightarrow$ (f) (under the assumption of the consistency of (e)): Assume otherwise. Then we have

$$(2.2) \qquad (\mathrm{e}) \vdash \exists \kappa\,(\kappa \text{ is an inaccessible cardinal}). \qquad \text{equiv-1}$$

Working in the axiom system (e), let κ be the minimal inaccessible cardinal. Then there are $\alpha < \beta < \kappa$ such that $L_\alpha \prec L_\beta \prec L_\kappa \models \ulcorner\ulcorner\mathsf{ZFC}\urcorner\urcorner$. We have $L_\beta \models$ (e) since $L_\alpha \in L_\beta$ but $L_\beta \not\models$ (f) by the minimality of κ. This is a contradiction to (2.2).

3 Speed-up Theorems

Theorem 3.1 *Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a recursive function. Then there is an $\mathcal{L}_{\{\}}$-formula $\varphi(x_1)$ such that, for each $n \in \mathbb{N}$, we have $\mathsf{ZFC}_{\{\}} \vdash \varphi(\underline{n})$ but, if $\mathsf{ZFC}_{\{\}} \vdash^P \varphi(\underline{n})$ for a proof P in $\mathcal{L}_{\{\}}$, then $\mathsf{ZFC}_{\{\}} \vdash rank(\ulcorner P \urcorner) \geq f(\underline{n})$.* (8)

Proof. We may assume that $\mathsf{ZFC}_{\{\}}$ is consistent.

Let $\psi(x_0, x_1)$ be the $\mathcal{L}_{\{\}}$-formula asserting:

$$(3.1) \quad x_0 \in \ulcorner\ulcorner Fml \urcorner\urcorner_{\mathcal{L}_{\{\}}} \\ \wedge \neg\exists p \in V_\omega\big(rank(p) < f(x_1) \wedge proof_{\ulcorner\ulcorner \mathsf{ZFC}_{\{\}}\urcorner\urcorner}(p, Subst(x_0, 1, x_1))\big). \qquad \text{sut-1}$$

By the Diagonal Lemma, there is an $\mathcal{L}_{\{\}}$-formula $\varphi(x_1)$ such that

$$(3.2) \quad \mathsf{ZFC}_{\{\}} \vdash \forall x_1 \in \omega\,(\varphi(x_1) \leftrightarrow \psi(\ulcorner \varphi \urcorner, x_1)). \qquad \text{sut-2}$$

Claim 3.1.1 *For all $n \in \mathbb{N}$ we have $\mathsf{ZFC}_{\{\}} \vdash \varphi(\underline{n})$.*

$\vdash$ Suppose that $\mathsf{ZFC}_{\{\}} \nvdash \varphi(\underline{n})$ for some $n \in \mathbb{N}$. Then, in particular, for all proof P in $\mathcal{L}_{\{\}}$ with $\mathsf{ZFC}_{\{\}} \vdash rank(\ulcorner P \urcorner) < f(\underline{n})$, we have $\mathsf{ZFC}_{\{\}} \nvdash^P \varphi(\underline{n})$. By the definition (3.1) of ψ, it follow that

$$(3.3) \quad \mathsf{ZFC}_{\{\}} \vdash \psi(\ulcorner \varphi \urcorner, \underline{n})$$

and hence $\mathsf{ZFC}_{\{\}} \vdash \varphi(\underline{n})$ by (3.2). This is a contradiction. $\dashv$ (Claim 3.1.1)

A similar argument shows:

Claim 3.1.2 *For all $n \in \mathbb{N}$, there is no proof P in $\mathcal{L}_{\{\}}$ with $\mathsf{ZFC}_{\{\}} \vdash rank(\ulcorner P \urcorner) < f(\underline{n})$ such that $\mathsf{ZFC}_{\{\}} \vdash^P \varphi(\underline{n})$.*

$\vdash$ Suppose that there is $n \in \mathbb{N}$ such that there is a proof P in $\mathcal{L}_{\{\}}$ with

$$(3.4) \quad \mathsf{ZFC}_{\{\}} \vdash rank(\ulcorner P \urcorner) < f(\underline{n}) \text{ and} \qquad \text{sut-2-a-0}$$

$$(3.5) \quad \mathsf{ZFC}_{\{\}} \vdash^P \varphi(\underline{n}). \qquad \text{sut-2-a-1}$$

By (3.2), it follows from (3.5) that

$$(3.6) \quad \mathsf{ZFC}_{\{\}} \vdash \psi(\ulcorner \varphi \urcorner, \underline{n}). \qquad \text{sut-2-0}$$

By the definition (3.1) of ψ this means

$$(3.7) \quad \mathsf{ZFC}_{\{\}} \vdash \ulcorner \varphi \urcorner \in \ulcorner\ulcorner Fml \urcorner\urcorner_{\mathcal{L}_{\{\}}} \\ \wedge \neg\exists p \in V_\omega\big(rank(p) < f(\underline{n}) \wedge proof_{\ulcorner\ulcorner \mathsf{ZFC}_{\{\}}\urcorner\urcorner}(p, Subst(\ulcorner \varphi \urcorner, 1, \underline{n}))\big). \qquad \text{sut-2-1}$$

On the other hand, by (3.4) and (3.5), we have

$$\text{(3.8)} \quad \mathsf{ZFC}_{\{\}} \vdash \ulcorner\varphi\urcorner \in \ulcorner\ulcorner Fml\urcorner\urcorner_{\mathcal{L}_{\{\}}} \;\; \wedge \;\; \ulcorner P\urcorner \in V_\omega$$
$$\wedge \;\; \ulcorner P\urcorner) < f(\underline{n}) \;\; \wedge \;\; proof_{\ulcorner\ulcorner \mathsf{ZFC}_{\{\}}\urcorner\urcorner}(\ulcorner P\urcorner, Subst(\ulcorner\varphi\urcorner, 1, \underline{n})).$$
sut-2-2

From (3.7) and (3.8) we obtain a proof of contradiction from $\mathsf{ZFC}_{\{\}}$.

This is a contradiction to the assumption that $\mathsf{ZFC}_{\{\}}$ is consistent. $\dashv$ (Claim 3.1.2)

$\square$ (Theorem 3.1)

Proposition 3.2 *Suppose that φ is the $\mathcal{L}_{\{\}}$-formula as in the proof of Theorem 3.1. We have $\mathsf{ZFC}_{\{\}} + consis(\ulcorner\ulcorner \mathsf{ZFC}_{\{\}}\urcorner\urcorner) \vdash \forall n \in \omega \; \varphi(n)$.*

Proof. We work in $\mathsf{ZFC}_{\{\}} + consis(\ulcorner\ulcorner \mathsf{ZFC}_{\{\}}\urcorner\urcorner)$. Suppose that $\neg\varphi(n)$ for some $n \in \omega$. Then we have $\neg\psi(\ulcorner\varphi\urcorner, n/x_1)$ by (3.2). Thus there is some $P \in V_\omega$ with $rank(P) < f(n)$ such that

$$\text{(3.9)} \quad proof_{\ulcorner\ulcorner \mathsf{ZFC}_{\{\}}\urcorner\urcorner}(P, \varphi(\llcorner n \lrcorner/x_1)).$$
sut-3

By (3.2), it follows that

$$\text{(3.10)} \quad \exists Q \; proof_{\ulcorner\ulcorner \mathsf{ZFC}_{\{\}}\urcorner\urcorner}(Q, \psi(\ulcorner\varphi\urcorner, \llcorner n \lrcorner/x_1)).$$
sut-3-0

On the other hand (3.9) together with the definition (3.1) of ψ implies

$$\text{(3.11)} \quad \exists R \; proof_{\ulcorner\ulcorner \mathsf{ZFC}_{\{\}}\urcorner\urcorner}(R, \neg\psi(\ulcorner\varphi\urcorner, \llcorner n \lrcorner/x_1)).$$
sut-4

From (3.10) and (3.11), it follows that

$$\text{(3.12)} \quad \neg consis(\ulcorner\ulcorner \mathsf{ZFC}_{\{\}}\urcorner\urcorner)$$

Since we are working in $\mathsf{ZFC}_{\{\}} + consis(\ulcorner\ulcorner \mathsf{ZFC}_{\{\}}\urcorner\urcorner)$, this is a contradiction. $\square$ (Proposition 3.2)

The proofs of Theorem 3.1 and Proposition 3.2 above also apply to the pairs of theories T and $T + consis(T)$ for all strong enough T in place of $\mathsf{ZFC}_{\{\}}$ and $\mathsf{ZFC}_{\{\}} + consis(\ulcorner\ulcorner \mathsf{ZFC}_{\{\}}\urcorner\urcorner)$. Gödel's Speed-up Theorem is Theorem 3.1 and Proposition 3.2 for nth order arithmetic and $(n+1)$st order arithmetic. Note that the $(n+1)$st order arithmetic implies $consis$(nth order arithmetic).

Appendix A Gödel numbering and the model relation in ZFC

Probably the most natural way to code the logic in ZFC is by using sequences in V_ω. For convenience, we shall work in the conservative extension of ZFC which we call $\mathsf{ZFC}_{\{\}}$ in the language $\mathcal{L}_{\{\}} = \{\in, \emptyset, \{\cdot\}, \cup, \cap, \setminus\}$ where $\emptyset$ is a constant symbol, $\{\cdot\}$ a unary function symbol and $\cup$, $\cap$, $\setminus$ binary function symbols.

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(8) Here, "$\cdot \geq f(\cdot)$" is an $\mathcal{L}_{\{\}}$-formula expressing what this notation suggests, formulated according to the definition of f. Note that this is possible since f is recursive.

The extension $\mathsf{ZFC}_{\{\}}$ consists of the original axioms of ZFC together with the definition of new symbols in the intended functionality:

(A.1) $\forall x\ (x \notin \emptyset)$; a-1

$\forall x \forall y\ (x \in \{y\} \leftrightarrow x \equiv y)$;

$\forall x \forall y \forall z\ (x \in y \cup z \leftrightarrow (x \in y \vee x \in z))$;

$\forall x \forall y \forall z\ (x \in y \cap z \leftrightarrow (x \in y \wedge x \in z))$;

$\forall x \forall y \forall z\ (x \in y \setminus z \leftrightarrow (x \in y \wedge x \notin z))$.

Here we work always with ZFC or $\mathsf{ZFC}_{\{\}}$ and their extensions for convenience, but most of the following can be also done without the Axiom of Choice in the formal system.

Note that, in $\mathsf{ZFC}_{\{\}}$, finite sets and ordered pairs are expressible as $\mathcal{L}_{\{\}}$-terms:

(A.2) $\{x_0, ..., x_{n-1}\} :\leftrightarrow \{x_0\} \cup \cdots \cup \{x_{n-1}\}$, a-2

$\langle x, y\rangle :\leftrightarrow \{\{x\}, \{x, y\}\}$.

For each natural number n (in metamathematics) we have a closed $\mathcal{L}_{\{\}}$-term $\underline{n}$ which corresponds to n. For example, 0, 1, 2,... are represented by the closed $\mathcal{L}_{\{\}}$-terms

$\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$,... .

We denote these terms by $\underline{0}$, $\underline{1}$, $\underline{2}$,... .

We define the hierarchy of (metamathematical) formulas $\Sigma_{n,\mathsf{ZFC}_{\{\}}}$ $\Pi_{n,\mathsf{ZFC}_{\{\}}}$, $\Delta_{n,\mathsf{ZFC}_{\{\}}}$ for all natural number n by:

(A.3) $\mathcal{L}_{\{\}}$-formula φ is $\Sigma_{0,\mathsf{ZFC}_{\{\}}} = \Pi_{0,\mathsf{ZFC}_{\{\}}} = \Delta_{0,\mathsf{ZFC}_{\{\}}}$ if there is an $\mathcal{L}_{\{\}}$-formula ψ in prenex normal form with only bounded quantifiers of the form $(\forall x \in y)$ or $(\exists x \in y)$ in the prenex part such that $\mathsf{ZFC}_{\{\}} \vdash \varphi \leftrightarrow \psi$. a-3

(A.4) $\mathcal{L}_{\{\}}$-formula φ is $\Sigma_{n+1,\mathsf{ZFC}_{\{\}}}$ if there is $\Pi_{n,\mathsf{ZFC}_{\{\}}}$-formula ψ such that $\mathsf{ZFC}_{\{\}} \vdash \varphi \leftrightarrow \exists x\, \psi$ for some variable x. a-4

(A.5) $\mathcal{L}_{\{\}}$-formula φ is $\Pi_{n+1,\mathsf{ZFC}_{\{\}}}$ if there is $\Sigma_{n+1,\mathsf{ZFC}_{\{\}}}$-formula such that $\mathsf{ZFC}_{\{\}} \vdash \varphi \leftrightarrow \neg\psi$. a-5

(A.6) $\mathcal{L}_{\{\}}$-formula φ is $\Delta_{n+1,\mathsf{ZFC}_{\{\}}}$ if there is $\Sigma_{n+1,\mathsf{ZFC}_{\{\}}}$-formula ψ_0 and $\Delta_{n+1,\mathsf{ZFC}_{\{\}}}$-formula ψ_1 such that $\mathsf{ZFC}_{\{\}} \vdash \varphi \leftrightarrow \psi_0$ and $\mathsf{ZFC}_{\{\}} \vdash \varphi \leftrightarrow \psi_1$. a-6

As already noticed in Section 1, we can find an $\mathcal{L}_{\varepsilon}$-formula φ^- for all $\mathcal{L}_{\{\}}$-formula φ such that $\mathsf{ZFC}_{\{\}} \vdash \varphi \leftrightarrow \varphi^-$.

If φ^- is $\Sigma_{n,\mathsf{ZFC}}$ [9] then it is easy to see that φ is also $\Sigma_{n,\mathsf{ZFC}_{\{\}}}$. Since all simple atomic formulas (that is, $x \in \{y\}$, $\{x\} \in x$, etc.) in $\mathcal{L}_{\{\}}$ can be expressed by $\Delta_{0,\mathsf{ZFC}}$-formulas, we can also conclude that φ^- is $\Sigma_{n,\mathsf{ZFC}}$ for all $\Sigma_{n,\mathsf{ZFC}_{\{\}}}$ formulas φ.

[9] $\Sigma_{n,\mathsf{ZFC}}$ $\Pi_{n,\mathsf{ZFC}}$, $\Delta_{n,\mathsf{ZFC}}$ are defined for $\mathcal{L}_{\varepsilon}$-formulas on basis of ZFC similarly to $\Sigma_{n,\mathsf{ZFC}_{\{\}}}$ $\Pi_{n,\mathsf{ZFC}_{\{\}}}$, $\Delta_{n,\mathsf{ZFC}_{\{\}}}$ defined for $\mathcal{L}_{\{\}}$-formulas on basis of $\mathsf{ZFC}_{\{\}}$.

We code the symbols appearing in $\mathcal{L}_{\{\}}$-expressions in pairs of natural numbers: We consider

(A.7) variables $x_0, x_1, \ldots$ as pairs $\langle 0,0\rangle$, $\langle 0,1\rangle, \ldots$ and other symbols '$\in$', '$\equiv$', '$\emptyset$', '$\{\cdot\}$', '$\cup$', '$\cap$', '$\setminus$', '$\wedge$', '$\neg$', '$\forall$', '$\exists$', ',', '(', ')' as $\langle 1,0\rangle$, $\langle 1,1\rangle$, ,..., $\langle 1,13\rangle$. a-7

In $\mathsf{ZFC}_{\{\}}$, let $\mathcal{L} = \{\langle 0,n\rangle \,:\, n\in\omega\} \cup \{\langle 1,0\rangle, \langle 1,1\rangle, \ldots, \langle 1,13\rangle\}$ and

$$\mathcal{L}* = \{f \,:\, f : n \to \mathcal{L} \text{ for some } n \in \omega\}.$$

We interpret $f \in \mathcal{L}*$ with $\mathrm{dom}(f) = n$ as a sequence $(f(0), f(1), \ldots, f(n-1))$. Note that, in general, there is no guarantee that $\mathrm{dom}(f)$ corresponds to some (meta-mathematical) concrete number.

For f, $g \in \mathcal{L}*$, $f \frown g \in \mathcal{L}*$ is the concatenation of the sequences f and g. For $a, b \in \mathcal{L}*$, (a) (and (a,b)) denote the sequences $\in \mathcal{L}*$ of length 1 (and 2), with the first (and second) component(s) a (and b).

Note that the $\mathcal{L}_{\{\}}$-formulas representing "$x \in \omega$", "$x \in \mathcal{L}$", "$x \in \mathcal{L}*$" are $\Delta_{0,\mathsf{ZFC}_{\{\}}}$. "$h \equiv f \frown g$" is $\Delta_{1,\mathsf{ZFC}_{\{\}}}$.

If t and t' are closed $\mathcal{L}_{\{\}}$-terms such that $\mathsf{ZFC}_{\{\}} \vdash t, t' \in \mathcal{L}*$, then we can find an $\mathcal{L}_{\{\}}$-term u such that $\mathsf{ZFC}_{\{\}} \vdash u \equiv t \frown t'$. We shall denote such $\mathcal{L}_{\{\}}$-term u also with $t \frown t'$.

Now we define the set $\ulcorner\ulcorner \mathit{Term}_{\mathcal{L}_{\{\}}} \urcorner\urcorner \subseteq \mathcal{L}*$ of all $\mathcal{L}_{\{\}}$-terms by

(A.8) $$x \in \ulcorner\ulcorner \mathit{Term}_{\mathcal{L}_{\{\}}} \urcorner\urcorner \;:\leftrightarrow\; \exists z \exists f\ (z \subseteq \mathcal{L}* \ \wedge z \text{ is closed w.r.t. substrings} \\ \wedge\ x \in z \wedge f : z \to 2 \ \wedge\ \cdots\ \wedge\ f(x) = 1)$$ a-8

where appropriate details corresponding to the recursive definition of $\mathcal{L}_{\{\}}$-terms (in metamathematics) is to be inserted at "$\cdots$".

The formula "$x \in \ulcorner\ulcorner \mathit{Term}_{\mathcal{L}_{\{\}}} \urcorner\urcorner$" is $\Delta_{1,\mathsf{ZFC}_{\{\}}}$ since we can represent it also as

(A.9) $$x \in \ulcorner\ulcorner \mathit{Term}_{\mathcal{L}_{\{\}}} \urcorner\urcorner \;\leftrightarrow\; \forall z \forall f\ ((z \subseteq \mathcal{L}* \ \wedge z \text{ is closed w.r.t. substrings} \\ \wedge\ x \in z \wedge f : z \to 2 \ \wedge\ \cdots\ ) \ \to\ f(x) = 1)$$ a-9

Back in metamathematics, we define, for each $\mathcal{L}_{\{\}}$-term t, a closed $\mathcal{L}_{\{\}}$-term $\ulcorner t \urcorner$ which "encodes" the term t as an element of $\ulcorner\ulcorner \mathit{Term}_{\mathcal{L}_{\{\}}} \urcorner\urcorner$ by induction on the construction of t. "$s \frown t$" here means the canonical closed $\mathcal{L}_{\{\}}$-term corresponding to the concatenation of the sequences the closed $\mathcal{L}_{\{\}}$-terms s and t represent.

(A.10) If t is the (string of length 1 consisting of the) variable x_n then $\ulcorner t \urcorner$ is the closed $\mathcal{L}_{\{\}}$-term $\{\langle \underline{0}, \underline{n}\rangle\}$. a-10

(A.11) If t is the (string of length 1 consisting of the) constant symbol $\emptyset$, then $\ulcorner t \urcorner$ is the closed $\mathcal{L}_{\{\}}$-term $\{\langle \underline{1}, \underline{2}\rangle\}$. a-10-0

(A.12) If t is of the form $t = \{t'\}$ for an $\mathcal{L}_{\{\}}$-term t', then $\ulcorner t \urcorner$ is the closed $\mathcal{L}_{\{\}}$-term $(\langle \underline{1}, \underline{3}\rangle, \langle \underline{1}, \underline{12}\rangle) \frown \ulcorner t' \urcorner \frown (\langle \underline{1}, \underline{13}\rangle)$. a-11

(A.13) If t is of the form $(t' \cup t'')$ for some $\mathcal{L}_{\{\}}$-terms t', t'', then $\ulcorner t \urcorner$ is the $\mathcal{L}_{\{\}}$-term $(\langle \underline{1}, \underline{4} \rangle, \langle \underline{1}, \underline{12} \rangle)^\frown \ulcorner t' \urcorner {}^\frown (\langle \underline{1}, \underline{11} \rangle)^\frown \ulcorner t'' \urcorner {}^\frown (\langle \underline{1}, \underline{13} \rangle)$. a-12

(A.14) If t is of the form $(t' \cap t'')$ for some $\mathcal{L}_{\{\}}$-terms t', t'', then $\ulcorner t \urcorner$ is the $\mathcal{L}_{\{\}}$-term $(\langle \underline{1}, \underline{5} \rangle, \langle \underline{1}, \underline{12} \rangle)^\frown \ulcorner t' \urcorner {}^\frown (\langle \underline{1}, \underline{11} \rangle)^\frown \ulcorner t'' \urcorner {}^\frown (\langle \underline{1}, \underline{13} \rangle)$. a-13

(A.15) If t is of the form $(t' \setminus t'')$ for some $\mathcal{L}_{\{\}}$-terms t', t'', then $\ulcorner t \urcorner$ is the $\mathcal{L}_{\{\}}$-term $(\langle \underline{1}, \underline{6} \rangle, \langle \underline{1}, \underline{12} \rangle)^\frown \ulcorner t' \urcorner {}^\frown (\langle \underline{1}, \underline{11} \rangle)^\frown \ulcorner t'' \urcorner {}^\frown (\langle \underline{1}, \underline{13} \rangle)$. a-14

By appropriate realization of "$\cdots$" in (A.8) which is to correspond to (a-10) $\sim$ (a-14), we have $\mathsf{ZFC}_{\{\}} \vdash \ulcorner t \urcorner \in \ulcorner\ulcorner \mathit{Term}_{\mathcal{L}_{\{\}}} \urcorner\urcorner$ for all $\mathcal{L}_{\{\}}$-term t and, conversely, if u is a closed $\mathcal{L}_{\{\}}$-term such that $\mathsf{ZFC}_{\{\}} \vdash u \in \ulcorner\ulcorner \mathit{Term}_{\mathcal{L}_{\{\}}} \urcorner\urcorner$, then there is a closed $\mathcal{L}_{\{\}}$-term t such that $\mathsf{ZFC}_{\{\}} \vdash u \equiv \ulcorner t \urcorner$.

Let us denote the set of all elements of $\subseteq \ulcorner\ulcorner \mathit{Term}_{\mathcal{L}_{\{\}}} \urcorner\urcorner$ corresponding to closed $\mathcal{L}_{\{\}}$-terms by $\ulcorner\ulcorner \mathit{ClTerm}_{\mathcal{L}_{\{\}}} \urcorner\urcorner$.

For each closed $\mathcal{L}_{\{\}}$-term t, we have $\mathsf{ZFC}_{\{\}} \vdash \ulcorner t \urcorner \in \ulcorner\ulcorner \mathit{ClTerm}_{\mathcal{L}_{\{\}}} \urcorner\urcorner$. For a closed $\mathcal{L}_{\{\}}$-term u such that $u = \ulcorner t \urcorner$ for some $\mathcal{L}_{\{\}}$-term t (in the meta-mathematics) let us denote with $\#(u)$ the term t. We have $\mathsf{ZFC}_{\{\}} \vdash \#(u) \in V_\omega$. This gives rise to the definition of the surjection $\#(\cdot) : \ulcorner\ulcorner \mathit{ClTerm}_{\mathcal{L}_{\{\}}} \urcorner\urcorner \to V_\omega$ (the interpretation of $t \in \ulcorner\ulcorner \mathit{ClTerm}_{\mathcal{L}_{\{\}}} \urcorner\urcorner$ as an element $\#(t)$ of V_ω) and its natural inverse $\llcorner \cdot \lrcorner : V_\omega \to \ulcorner\ulcorner \mathit{ClTerm}_{\mathcal{L}_{\{\}}} \urcorner\urcorner$ ($u \in V_\omega$ is related to the canonical term $\llcorner u \lrcorner \in \ulcorner\ulcorner \mathit{ClTerm}_{\mathcal{L}_{\{\}}} \urcorner\urcorner$ representing u) such that we have $\mathsf{ZFC}_{\{\}} \vdash (\forall v \in V_\omega)(\#(\llcorner v \lrcorner) \equiv v)$.

The Gödel numbering $\ulcorner t \urcorner$ of $\mathcal{L}_{\{\}}$-terms t can be similarly extended to Gödel numbering $\ulcorner \varphi \urcorner$ of $\mathcal{L}_{\{\}}$-formulas φ. By appropriate definition of the Gödel numbering and the corresponding definition of the set $\ulcorner\ulcorner \mathit{Fml}_{\mathcal{L}_{\{\}}} \urcorner\urcorner \subseteq \mathcal{L}*$ of $\mathcal{L}_{\{\}}$-formulas, we obtain

$$\mathsf{ZFC}_{\{\}} \vdash \ulcorner \varphi \urcorner \in \ulcorner\ulcorner \mathit{Fml}_{\mathcal{L}_{\{\}}} \urcorner\urcorner$$

for all $\mathcal{L}_{\{\}}$-formula φ and conversely if $\mathsf{ZFC}_{\{\}} \vdash t \in \ulcorner\ulcorner \mathit{Fml}_{\mathcal{L}_{\{\}}} \urcorner\urcorner$ for any closed $\mathcal{L}_{\{\}}$-term t then there is an $\mathcal{L}_{\{\}}$-formula φ such that $\mathsf{ZFC}_{\{\}} \vdash \ulcorner \varphi \urcorner \equiv t$.

For an $\mathcal{L}_{\{\}}$-formula φ, variable symbol x_n and an $\mathcal{L}_{\{\}}$-term t, we can formulate an algorithm A to calculate $\ulcorner \varphi(t/x_n) \urcorner$ (= the Gödel number of the formula obtained by substituting t in x_n in φ) starting from $\ulcorner \varphi \urcorner$ and $\ulcorner t \urcorner$. This renders the substitution function

$$\mathit{Subst} : \ulcorner\ulcorner \mathit{Fml}_{\mathcal{L}_{\{\}}} \urcorner\urcorner \times \omega \times \ulcorner\ulcorner \mathit{Term}_{\mathcal{L}_{\{\}}} \urcorner\urcorner \to \ulcorner\ulcorner \mathit{Fml}_{\mathcal{L}_{\{\}}} \urcorner\urcorner$$

such that we always have $\mathsf{ZFC}_{\{\}} \vdash \mathit{Subst}(\ulcorner \varphi \urcorner, n, \ulcorner t \urcorner) \equiv \ulcorner \varphi(t/x_n) \urcorner$ for any $\mathcal{L}_{\{\}}$-formula φ, number n and $\mathcal{L}_{\{\}}$-term t.

Lemma A.1 *For any $\mathcal{L}_{\{\}}$-formula φ and an expression (i.e. either a formula or term in $\mathcal{L}_{\{\}}$) η we have*

$$\mathsf{ZFC}_{\{\}} \vdash \ulcorner \varphi(\ulcorner \eta \urcorner / x_n) \urcorner \equiv \mathit{Subst}(\ulcorner \varphi \urcorner, \underline{n}, \llcorner \ulcorner \eta \urcorner \lrcorner). \qquad \square$$

[The rest will be written soon.]

Appendix B A Proof of the Diagonal Lemma

Theorem B.1 (Diagonal Lemma, R. Carnap, 1934) *For an arbitrary $\mathcal{L}_{\{\}}$-formula ψ, there is an $\mathcal{L}_{\{\}}$-formula σ such that $free(\sigma) = free(\psi) \setminus \{x_0\}$ and $\mathsf{ZFC}_{\{\}} \vdash \sigma \leftrightarrow \psi(\ulcorner\sigma\urcorner/x_0)$.*

Proof. Let $f^* : (V_\omega)^2 \to V_\omega$ be defined by

(B.1) $$f^*(x,y) = \begin{cases} z, & \text{if } x \in \ulcorner\ulcorner Fml_{\mathcal{L}_{\{\}}}\urcorner\urcorner \text{ and } z = Subst(x, 0, \llcorner y \lrcorner); \\ \emptyset & \text{otherwise.} \end{cases}$$ a2-0

(B.2) Let t^* be the closed $\mathcal{L}_{\{\}}$-term $\ulcorner\forall x_k\,(f^*(x_0, x_0) \equiv x_k \to \psi(x_k/x_0))\urcorner$ a2-1

where the variable x_k is the first variable which does not appear in ψ and let

(B.3) σ be the $\mathcal{L}_{\{\}}$-sentence $\forall x_k\,(f^*(t^*, t^*) \equiv x_k \to \psi(x_k/x_0))$. a2-2

By (B.1) and (B.2), we have

(B.4) $$\mathsf{ZFC}_{\{\}} \vdash f^*(t^*, t^*) \equiv \ulcorner\underbrace{\forall x_k\,(f^*(t^*, t^*)) \equiv x_k \to \psi(x_k/x_0)}_{=\sigma}\urcorner.$$ a2-3

It follows that

(B.5) $$\mathsf{ZFC}_{\{\}} \vdash \forall x_k\,(f^*(t^*, t^*) \equiv x_k \leftrightarrow x_k \equiv \ulcorner\sigma\urcorner).$$ a2-4

Thus, By (B.3) and (B.5),

(B.6) $$\mathsf{ZFC}_{\{\}} \vdash \sigma \to \psi(\ulcorner\sigma\urcorner/x_0).$$ a2-5

Conversely, since

(B.7) $$\psi(\ulcorner\sigma\urcorner/x_0) \to f^*(t^*, t^*) \equiv x_k \to \psi(\ulcorner\sigma\urcorner/x_0)$$ a2-6

Since x_k does not appear in ψ, it follows that

(B.8) $$\mathsf{ZFC}_{\{\}} \vdash (\psi(\ulcorner\sigma\urcorner/x_0) \to \forall x_k\,(f(t^*, t^*) \equiv x_k \to \psi(\ulcorner\sigma\urcorner/k))).$$ a2-7

Hence by (B.5),

(B.9) $$\mathsf{ZFC}_{\{\}} \vdash (\psi(\ulcorner\sigma\urcorner/x_0) \to \forall x_k\,(f(t^*, t^*) \equiv x_k \to \psi(x_k/k))).$$ a2-7-0

Since the right-hand side of the outermost "$\to$" of (B.9) is just σ, what we obtained here is

(B.10) $$\mathsf{ZFC}_{\{\}} \vdash \psi(\ulcorner\sigma\urcorner) \to \sigma.$$ a2

$\square$ (Theorem B.1)

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