$\mathcal{L}_{\infty\kappa}$-Cohen algebras

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Abstract

A Boolean algebra A is called $\mathcal{L}_{\infty\kappa}$-Cohen if A is $\mathcal{L}_{\infty\kappa}$-elementarily equivalent to the Cohen algebra $\mathbb{C}_\kappa$ with π-weight κ. In this paper we study the class of $\mathcal{L}_{\infty\kappa}$-Cohen algebras for various κ's.

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0 Introduction

Let κ be an infinite cardinal. Let us recall that $\mathcal{L}_{\infty\kappa}$ is the logic whose formulas are constructed recursively just like in first order logic with the difference that conjunction and disjunction of any set of formulas as well as quantification over a block of free variables of cardinality $< \kappa$ is allowed.

$\mathbb{C}_\kappa$ is the completion of the free Boolean algebra $\mathrm{Fr}\,\kappa$ on free generators κ. A complete Boolean algebra of the form $\mathbb{C}_\kappa$ is called a Cohen algebra. A Boolean algebra A is called $\mathcal{L}_{\infty\kappa}$-Cohen algebra when A is $\mathcal{L}_{\infty\kappa}$-elementarily equivalent to the Cohen algebra $\mathbb{C}_\kappa$. In this paper we shall study $\mathcal{L}_{\infty\kappa}$-Cohen algebras for various κ's.

A Boolean algebra B is called $\mathcal{L}_{\infty\kappa}$-free if B is $\mathcal{L}_{\infty\kappa}$-elementarily equivalent to the free Boolean algebra $\mathrm{Fr}\,\kappa$ (see [9]). In Section 2 we give an algebraic characterization of $\mathcal{L}_{\infty\kappa}$-Cohen algebras for regular κ (Theorem 2.3) which is quite similar to Kueker's algebraic characterization of $\mathcal{L}_{\infty\kappa}$-free Boolean algebras (see Theorem 1.5 below). From this characterization of $\mathcal{L}_{\infty\kappa}$-Cohen algebras it follows that the σ-completion of an $\mathcal{L}_{\infty\kappa}$-free Boolean algebra is an $\mathcal{L}_{\infty\kappa}$-Cohen algebra (Corollary 2.4). We still do not know if there are $\mathcal{L}_{\infty\kappa}$-Cohen algebras which are not σ-completion of $\mathcal{L}_{\infty\kappa}$-free Boolean algebras. Since every $\mathcal{L}_{\infty\kappa}$-free Boolean algebra B for $\kappa > \aleph_1$ satisfies the ccc, the σ-completion of B is actually the completion of B. Hence for $\kappa > \aleph_1$, the completion of any $\mathcal{L}_{\infty\kappa}$-free Boolean algebra is $\mathcal{L}_{\infty\kappa}$-Cohen. However this is not true for $\kappa = \aleph_1$ (see the remark at the end of Section 2).

In Section 3 we show that, for a singular λ, every $\mathcal{L}_{\infty\lambda}$-Cohen algebra of π-weight λ is isomorphic to $\mathbb{C}_\lambda$ (Theorem 3.1). The proof of the theorem is similar to the proof of a special case of Shelah's Singular Compactness Theorem ([23]) given in Hodges [11].

In [9] it was shown that the existence of $\mathcal{L}_{\infty\aleph_1}$-free nonfree Boolean algebras of cardinality $\aleph_1$ satisfying the ccc as well as $\mathcal{L}_{\infty\aleph_2}$-free nonfree Boolean algebras of cardinality $\aleph_2$ is provable in ZFC. However the construction given there does not guarantee that the completion of the Boolean algebra constructed is not isomorphic to a Cohen algebras. In Section 4 we give a modified construction (in ZFC) of $\mathcal{L}_{\infty\aleph_1}$-free Boolean algebras in $\aleph_1$ which satisfy the ccc but whose completions are not isomorphic to $\mathbb{C}_{\aleph_1}$ (Corollary 4.2 (a)). Under $V = L$ the same construction can be also applied to every non weakly compact regular κ to get $\mathcal{L}_{\infty\kappa}$-free Boolean algebras in κ whose completions are not isomorphic to the Cohen algebra $\mathbb{C}_\kappa$. Using a result by L. B. Shapiro we also construct (in ZFC) an $\mathcal{L}_{\infty\aleph_2}$-free Boolean algebra of π-weight $\aleph_2$ whose completion is not isomorphic to any Cohen algebra (Corollary 4.8).

Complete $\mathcal{L}_{\infty\aleph_1}$-Cohen algeras are already quite similar to Cohen algebras in the following sence: if C is a complete $\mathcal{L}_{\infty\aleph_1}$-Cohen algebra in a ground model V then every real in V^C is obtained in the generic extension of V by adding a Cohen real (Corollary 2.7). This means that we cannot distinguish complete $\mathcal{L}_{\infty\aleph_1}$-Cohen algebras from Cohen algebras only by looking at indivisual reals added by the algebras. Yet the global structure of a complete $\mathcal{L}_{\infty\kappa}$-Cohen algebra can be quite different from that of a Cohen algebra: in Section 5 we shall show that there exist complete $\mathcal{L}_{\infty\aleph_1}$-Cohen algebras having no factor isomorphic to $\mathfrak{C}_{\aleph_1}$ (Theorem **??**).

1 Preliminaries

The notation used here is standard. For the basic facts about Boolean algebras the reader may consult [15], and [12, 18] for set theory. The basic facts about the logic $\mathcal{L}_{\infty\kappa}$ can be found in [1].

For Boolean algebra A, A^+ denotes the partial ordering $A\setminus\{0\}$ and $\overline{A}$ the completion of A. We always assume that A is a dense subalgebra of $\overline{A}$. For Boolean algebras A, B we denote by $A \oplus B$ the free product of A and B. In forcing, the completion of $A \oplus B$ corresponds to the product of partial orderings A^+ and B^+. Without loss of generality we may assume that $A \oplus B \leq A' \oplus B'$ holds for A, A', B, B' such that $A \leq A'$ and $B \leq B'$.

If A is a subalgebra of a Boolean algebra B, this is denoted by $A \leq B$. A subalgebra A of a Boolean algebra B is *relatively complete in* B ($A \leq_{\rm rc} B$) if for every $b \in B^+$ there exists the greatest element a of A such that $a \leq b$. We shall denote such a by $\mathrm{pr}^B_A(b)$. If $A \leq B$ and if A is not a relatively complete subalgebra of B, this is denoted by $A \leq_{\neg{\rm rc}} B$.

A subalgebra A of a Boolean algebra B is said to be a *regular subalgebra of* B ($A \leq_{\rm reg} B$) if, for every $S \subseteq A$ such that $\sum^A S$ exists, $\sum^B S$ also exists and $\sum^A S = \sum^B S$ holds. (Equivalently, if every maximal antichain in A is maximal in B.) If $A \leq_{\rm rc} B$ holds then it follows that $A \leq_{\rm reg} B$. If B is a complete Boolean algebra then a subalgebra A of B is a *complete subalgebra* if for every $S \subseteq A$, $\sum^B S \in A$. Thus A is a complete subalgebra of B if and only if it is a regular subalgebra of B and complete.

For a complete Boolean algebra B and $X \subseteq B$, $\langle X\rangle^{cm}_B$ denotes the subalgebra of B completely generated by X. In general $\langle X\rangle^{cm}_B$ is not equal to the completion of the subalgebra $[X]_B$ of B generated by X. However it *is* the case when $[X]_B$ is a regular subalgebra of B. $\langle X\rangle^{\sigma\text{-}cm}_B$ is the smallest σ-complete subalgebra of B containing X. If B satisfies the ccc then we have $\langle X\rangle^{\sigma\text{-}cm}_B = \langle X\rangle^{cm}_B$.

For any set X, $\mathrm{Fr}\,X$ denotes the free Boolean algebra over the free generators X. For X, Y such that $X \subseteq Y$ we always regard $\mathrm{Fr}\,X$ as a subalgebra of $\mathrm{Fr}\,Y$ by the canonical embedding. For a cardinal κ, $\mathbb{C}_\kappa$ is the completion of $\mathrm{Fr}\,\kappa$. A complete Boolean algebra of the form $\mathbb{C}_\kappa$ is called a Cohen algebra. We always consider that $\mathrm{Fr}\,\kappa$ is a dense subalgebra of $\mathbb{C}_\kappa$. For $X \subseteq \kappa (\subseteq \mathrm{Fr}\,\kappa)$ the complete subalgebra $\langle X\rangle^{cm}_{\mathbb{C}_\kappa}$ of $\mathbb{C}_\kappa$ is denoted by $\mathbb{C}_X$. Also for $\lambda \leq \kappa$ we assume that $\mathbb{C}_\lambda$ is a complete subalgebra of $\mathbb{C}_\kappa$ in the above sense.

For a Boolean algebra B and $b \in B$, $B\restriction b$ denotes the Boolean algebra $\{a \in B : a \leq b\}$ with the partial ordering induced from the partial ordering of B. For Boolean algebras A, B such that $A \leq B$ and $b \in B$, $A \cdot b$ denotes the subalgebra of $B\restriction b$ with the underlying set $\{a \cdot b : a \in A\}$.

For a Boolean algebra B the π-*weight of* B is defined by $\pi(B) = \min\{|X| : X \subseteq B,\ X$ is dense in $B\}$. A Boolean algebra B is π-*homogeneous* if $\pi(B\restriction b) = \pi(B)$ for every $b \in B^+$. Note that $\pi(B) = \pi(\overline{B})$ holds. The following lemma is well-known:

Lemma 1.1 *A complete Boolean algebra B is isomorphic to $\mathbb{C}_{\aleph_0}$ if and only if B is atomless and $\pi(B) = \aleph_0$.* ∎

For Boolean algebras A, B such that $A \leq_{\rm reg} B$, $\pi(B/A) = \min\{\,|\,X\,|\;:\; X \subseteq B,\ A[X]$ is dense in $B\,\}$ is called the *π-weight of B over A*. B is π-homogeneous over A if $\pi(B\restriction b/A\cdot b) = \pi(B/A)$ for every $b \in B^+$. Note that, if B satisfies the ccc and $\pi(\overline{B}/\overline{A}) \geq \aleph_0$, then $\pi(B/A) = \pi(\overline{B}/\overline{A})$ holds and B is π-homogeneous over A if and only if $\overline{B}$ is π-homogeneous over $\overline{A}$.

A complete Boolean algebra B and its complete subalgebra A correspond to a two step iteration of generic extensions in the following sense (Solovay and Tennenbaum [24]): If A is a complete Boolean algebra and $\dot{C}$ is an A-name of a complete Boolean algebra in the generic extension then $A * \dot{C}$ is the complete Boolean algebra (in the ground model) with the underlying set D which is a maximal set of A-names with the property that

$$\text{if } \dot{c} \in D \text{ then } \Vdash_A \text{“}\dot{c} \in \dot{C}\text{”; if } \dot{c}_1,\, \dot{c}_2 \in D \text{ and } \dot{c}_1 \neq \dot{c}_2, \text{ then } \not\Vdash_A \text{“}\dot{c}_1 = \dot{c}_2\text{”}.$$

The partial ordering of D is defined by: $\dot{c}_1 \leq \dot{c}_2 \Leftrightarrow \Vdash_A$ “$\dot{c}_1 \leq \dot{c}_2$” for $\dot{c}_1$, $\dot{c}_2 \in D$. For $a \in A \setminus \{\,0_A, 1_A\,\}$ let $e(a)$ be the unique element $\dot{d}$ of D such that $a\Vdash_A$ “$\dot{d} = 1_{\dot{C}}$” and $-a\Vdash_A$ “$\dot{d} = 0_{\dot{C}}$”, and $e(0_A)$ and $e(1_A)$ be the uniquely determined elements $\dot{d}_0$, $\dot{d}_1$ of D such that $\Vdash_A$ “$\dot{d}_0 = 0_{\dot{C}}$” and $\Vdash_A$ “$\dot{d}_1 = 1_{\dot{C}}$”. The mapping e then embeds A regularly into $A * \dot{C}$. We shall always identify $a \in A$ with $e(a)$ and consider that $A \leq_{\rm reg} A * \dot{C}$.

Conversely, for complete Boolean algebras A, B such that $A \leq_{\rm reg} B$, let $\dot{G}^B_A$ be an A-name of the filter on B generated from the generic filter on A and $(B : A)$ an A-name of “$B/\dot{G}^B_A$”. Then we have $\Vdash_A$ “$(B : A)$ is a complete Boolean algebra”.

For Boolean algebras A, B_1 and B_2 such that $A \leq B_1$ and $A \leq B_2$ we say that B_1 and B_2 are isomorphic over A (notation: $B_1 \cong_A B_2$) if there is an isomorphism from B_1 to B_2 extending the identity mapping on A.

Lemma 1.2 (a) *For a complete Boolean algebra A and A-names $\dot{C}$, $\dot{C}'$ of complete Boolean algebras $\Vdash_A$ “$\dot{C} \cong \dot{C}'$” holds if and only if $A*\dot{C}$ and $A*\dot{C}'$ are isomorphic over A.*

(b) *For complete Boolean algebras A, B such that $A \leq_{\rm reg} B$, B and $A * (B : A)$ are isomorphic over A.*

(c) *For a complete Boolean algebra A and an A-name $\dot{C}$ of a complete Boolean algebra we have that $\Vdash_A$ “$(A * \dot{C} : A) \cong \dot{C}$”.*

(d) *For complete Boolean algebras A, B, C such that $A \leq_{\rm reg} B$ if $\Vdash_A$ “$(B : A) \cong$* the completion of C” *then $B \cong_A \overline{A \oplus C}$ holds.*

(e) *For complete Boolean algebras A and B it holds that $\Vdash_A$ “$(\overline{A \oplus B}^V : A) \cong \overline{B}$”.* ∎

For more details see e.g. pp 232 – 237 in [12].

A complete subalgebra A of a complete Boolean algebra B is said to be a *factor* of B if there exists a complete Boolean algebra C such that $B \cong_A \overline{A \oplus C}$. Note that if A is a factor of B then every automorphism of A can be extended to an automorphism on B. For complete Boolean algebras A, B isomorphic to Cohen algebras such that $A \leq B$ we write $A \parallel B$ if $B \cong_A \overline{A \oplus \mathbb{C}_\kappa}$ for some κ and $A \not\parallel B$ if this is not the case for any κ.

Lemma 1.3 *Let A be a complete subalgebra of a complete Boolean algebra B. If B is π-homogeneous over A and $\pi(B/A)=\aleph_0$ then $B\cong_A\overline{A\oplus\mathfrak{C}_{\aleph_0}}$ holds.*

Proof By the assumption we have that $\Vdash_A$ "$(B:A)$ is π-homogeneous and $\pi((B:A))=\aleph_0$". By Lemma 1.1 it follows that $\Vdash_A$ "$(B:A)\cong\mathfrak{C}_{\aleph_0}$". Hence by Lemma 1.2 (d) it holds that $B\cong_A\overline{A\oplus\mathfrak{C}_{\aleph_0}}$. ▮ (Lemma 1.3)

Lemma 1.4 *Let A and C be complete Boolean algebras. If A satisfies the ccc and $\overline{A\oplus C}\cong_A\overline{A\oplus\mathfrak{C}_\kappa}$ for a cardinal κ, then it follows that $C\cong\mathfrak{C}_\kappa$.*

Proof By Lemma 1.2 (a) we have that $\Vdash_A$ "$(\overline{A\oplus C}^V:A)\cong(\overline{A\oplus\mathfrak{C}_\kappa{}^V}^V:A)$". By Lemma 1.2 (e) it follows that $\Vdash_A$ "$\overline{C}\cong\overline{\mathfrak{C}_\kappa{}^V}$". Since A satisfies the ccc it follows that $C\cong\mathfrak{C}_\kappa$ (see [7]). ▮ (Lemma 1.4)

It is easily seen that the assumption of the ccc of A can not be omitted from the lemma above: e.g. let A be the completion of $Fn(\aleph_1,\aleph_2,\aleph_1)$. We have that $\Vdash_A$ "$\mathfrak{C}_{\aleph_1}{}^V\cong\mathfrak{C}_{\aleph_2}{}^V$". Hence by Lemma 1.2 (a) it follows that $\overline{A\oplus\mathfrak{C}_{\aleph_1}}\cong_A\overline{A\oplus\mathfrak{C}_{\aleph_2}}$.

By Kueker's theorem, $\mathcal{L}_{\infty\kappa}$-free algebras enjoy a purely algebraic characterization ([17], see also [4]). For Boolean algebras the theorem can be formulated as follows. For a subalgebra A of a Boolean algebra B we write $A\mid B$ if $B\cong_A A\oplus\mathrm{Fr}\,\lambda$ for some cardinal λ.

Theorem 1.5 (D. Kueker) *Let κ be a regular cardinal. A Boolean algebra C is $\mathcal{L}_{\infty\kappa}$-free if and only if there exists a set S of subalgebras of C such that*

(0′) Every $A\in S$ is a relatively complete subalgebra of B, A is free and of cardinality $<\kappa$;

(1′) For every $S'\subseteq S$ such that $\mid S'\mid<\kappa$ there exists a $B\in S$ such that $A\mid B$ for every $A\in S'$;

(2′) $\bigcup S=C$. ▮

In Section 4, some rudiments of the theory of projective algebras are used. We shall summarize here the facts needed in the section. For the proof the reader may consult [16]. A Boolean algebra B is *projective over a subalgebra A of B* ($A\leq_{\rm proj}B$) if $B\oplus\mathrm{Fr}\,\kappa\cong_A A\oplus\mathrm{Fr}\,\kappa$ holds for $\kappa=\mid B\mid+\aleph_0$. Note, that this definition differs from the original one of the projectivity. A Boolean algebra B is *countably generated over a subalgebra A* if there exists a countable set $X\subseteq B$ such that $B=A[X]$ holds.

Lemma 1.6 ([16]) *For a Boolean algebra B, $A\leq B$ and $b\in B$, if $A\leq_{\rm rc}B$ then $A(b)\leq_{\rm rc}B$.* ▮

Theorem 1.7 (Haydon, see [16]) *For a Boolean algebra B and $A\leq B$, the following are equivalent:*

(*a*) *B is projective over A;*

(*b*) *There exist an ordinal* ρ *and a continuously increasing sequence* $(B_\alpha)_{\alpha<\rho}$ *of subalgebras of B such that:* $B_0 = A$, $B_\alpha \leq_{\mathrm{rc}} B_{\alpha+1}$, $B_{\alpha+1}$ *is countably generated over* B_α *for every* $\alpha < \rho$ *and* $\bigcup_{\alpha<\rho} B_\alpha = B$. ∎

2 Characterization of $\mathcal{L}_{\infty\kappa}$-Cohen algebras

In this section we shall give an algebraic characterization of $\mathcal{L}_{\infty\kappa}$-Cohen algebras which is quite similar to the characterization of $\mathcal{L}_{\infty\kappa}$-free algebras mentioned in the last section.

For Boolean algebras A, B such that $A \leq B$, (B, A) denotes the structure B with a unary relation consisting of all elements of A. Note that $(B, A) \cong (B', A')$ if and only if there exists an isomorphism of B and B' extending an isomorphism of A and A'.

Lemma 2.1 *Let λ, κ be cardinals such that $\lambda < \kappa$ and let $B \leq \mathbb{C}_\kappa$. If $(\mathbb{C}_\kappa, B) \equiv_{\mathcal{L}_{\infty\kappa}} (\mathbb{C}_\kappa, \mathbb{C}_\lambda)$ then we have $(\mathbb{C}_\kappa, B) \cong (\mathbb{C}_\kappa, \mathbb{C}_\lambda)$.*

Proof Let $\varphi((x_\alpha)_{\alpha<\lambda})$ be an $\mathcal{L}_{\infty\kappa}$-formula with free variables $(x_\alpha)_{\alpha<\lambda}$ such that for any sequence $(c_\alpha)_{\alpha<\lambda}$ of elements of $\mathbb{C}_\kappa$ we have $\mathbb{C}_\kappa \models \varphi[(c_\alpha)_{\alpha<\lambda}]$ if and only if $(\mathbb{C}_\kappa, \alpha)_{\alpha<\lambda} \equiv_{\mathcal{L}_{\infty\kappa}} (\mathbb{C}_\kappa, c_\alpha)_{\alpha<\lambda}$ holds where λ in $(\mathbb{C}_\kappa, \alpha)_{\alpha<\lambda}$ is seen as the set of free generators of $\mathrm{Fr}\,\lambda \subseteq \mathbb{C}_\lambda$. We have that $\mathbb{C}_\kappa \models \psi[(\alpha)_{\alpha<\lambda}]$ where

$$\psi((x_\alpha)_{\alpha<\lambda}) \quad \longleftrightarrow$$

$$\forall (y_\alpha)_{\alpha<\lambda} \exists (z_\alpha)_{\alpha<\lambda} \quad [\quad \varphi((z_\alpha)_{\alpha<\lambda}) \wedge \text{“}\, \{\, y_\alpha \,:\, \alpha < \lambda \,\} \text{ and the subalgebra } A \text{ completely generated by } \{\, x_\alpha \,:\, \alpha < \lambda \,\} \text{ are included in the subalgebra } A' \text{ completely generated by } \{\, z_\alpha \,:\, \alpha < \lambda \,\} \text{ and } A' \cong_A \overline{A \oplus \mathbb{C}_\lambda} \text{ holds”} \quad].$$

Hence we have

$$(*) \quad \mathbb{C}_\kappa \models \forall (x_\alpha)_{\alpha<\lambda} [\, \varphi((x_\alpha)_{\alpha<\lambda}) \rightarrow \psi((x_\alpha)_{\alpha<\lambda}) \,].$$

By the assumption it follows that B is completely generated by some $\{\, b_\alpha \,:\, \alpha < \lambda \,\}$ such that $\mathbb{C}_\kappa \models \varphi[(b_\alpha)_{\alpha<\lambda}]$. By $(*)$ we can construct an increasing sequence $(B_n)_{n\in\omega}$ of subalgebras of $\mathbb{C}_\kappa$ and an increasing sequence $(X_n)_{n\in\omega}$, $X_n \in [\kappa]^\lambda$, such that $B_0 = B$ and; $B_n \subseteq \mathbb{C}_{X_n} \subseteq B_{n+1}$, $|\, X_{n+1} \setminus X_n \,| = \lambda$, $(\mathbb{C}_\kappa, B_n) \equiv_{\mathcal{L}_{\infty\kappa}} (\mathbb{C}_\kappa, \mathbb{C}_\lambda)$ and $B_{n+1} \cong_{B_n} \overline{B_n \oplus \mathbb{C}_\lambda}$ for every $n \in \omega$ hold.

Let $X = \bigcup_{n\in\omega} X_n$. Then $\bigcup_{n\in\omega} B_n = \bigcup_{n\in\omega} \mathbb{C}_{X_n}$ is dense in $\mathbb{C}_X$. By the construction we have that $(\bigcup_{n\in\omega} B_n, B) \cong (\bigcup_{n\in\omega} \mathbb{C}_{X_n}, \mathbb{C}_{X_0})$. It follows that $(\mathbb{C}_X, B) \cong (\mathbb{C}_X, \mathbb{C}_{X_0})$. Since $\mathbb{C}_X$ is a factor of $\mathbb{C}_\kappa$ it follows that $(\mathbb{C}_\kappa, B) \cong (\mathbb{C}_\kappa, \mathbb{C}_{X_0})$. Since $(\mathbb{C}_\kappa, \mathbb{C}_{X_0}) \cong (\mathbb{C}_\kappa, \mathbb{C}_\lambda)$ we obtain that $(\mathbb{C}_\kappa, B) \cong (\mathbb{C}_\kappa, \mathbb{C}_\lambda)$. ▮ (Lemma 2.1)

Lemma 2.2 *Let B, C be complete Boolean algebras such that $B \leq C \leq \mathbb{C}_\kappa$, $(\mathbb{C}_\kappa, B) \equiv_{\mathcal{L}_{\infty\kappa}} (\mathbb{C}_\kappa, \mathbb{C}_\lambda)$ and $(\mathbb{C}_\kappa, C) \equiv_{\mathcal{L}_{\infty\kappa}} (\mathbb{C}_\kappa, \mathbb{C}_\mu)$ for some $\lambda \leq \mu < \kappa$. Then it holds $B \parallel \overline{C \oplus \mathbb{C}_\mu}$. If $\lambda < \mu$ then we have $B \parallel C$.*

Proof By Lemma 2.1 we have $B \parallel \mathbb{C}_\kappa$ and $C \parallel \mathbb{C}_\kappa$. Let $\{\, u_\alpha \,:\, \alpha < \kappa \,\}$ and $\{\, v_\alpha \,:\, \alpha < \kappa \,\}$ be free generators of dense subalgebras of $\mathbb{C}_\kappa$ such that $B = \langle \{\, u_\alpha \,:\, \alpha < \lambda \,\} \rangle^{cm}_{\mathbb{C}_\kappa}$ and

$C = \langle\{ v_\alpha : \alpha < \mu \}\rangle^{cm}_{\mathfrak{C}_\kappa}$ hold. Let $(X_n)_{n\in\omega}$ and $(Y_n)_{n\in\omega}$ be increasing sequences in $[\lambda]^\mu$ such that $X_0 = Y_0 = \mu + \mu$ and

$$\langle\{ u_\alpha : \alpha \in X_n \}\rangle^{cm}_{\mathfrak{C}_\kappa} \subseteq \langle\{ v_\alpha : \alpha \in Y_n \}\rangle^{cm}_{\mathfrak{C}_\kappa} \subseteq \langle\{ u_\alpha : \alpha \in X_{n+1} \}\rangle^{cm}_{\mathfrak{C}_\kappa}$$

holds for every $n \in \omega$. Let $X = \bigcup_{n\in\omega} X_n$ and $Y = \bigcup_{n\in\omega} Y_n$. We have $|\, X \,| = |\, Y \,| = \mu$ and $\langle\{ u_\alpha : \alpha \in X \}\rangle^{cm}_{\mathfrak{C}_\kappa} = \langle\{ v_\alpha : \alpha \in Y \}\rangle^{cm}_{\mathfrak{C}_\kappa}$. Let $D = \langle\{ u_\alpha : \alpha \in X \}\rangle^{cm}_{\mathfrak{C}_\kappa}$. Then $D \cong \mathfrak{C}_\mu$, $B \parallel D$ and $D \cong_C \overline{C \oplus \mathfrak{C}_\mu}$. It follows that $B \parallel \overline{C \oplus \mathfrak{C}_\mu}$.

If $\lambda < \mu$ then there exists a subalgebra C' of C such that $B \le C'$, $(\mathfrak{C}_\kappa, C') \equiv_{\mathcal{L}_{\infty\kappa}} (\mathfrak{C}_\kappa, \mathfrak{C}_\mu)$ and $C \cong_{C'} \overline{C' \oplus \mathfrak{C}_\mu}$. Hence from the first part of the lemma it follows that $B \parallel C$. ▮ (Lemma 2.2)

Theorem 2.3 *Let κ be a regular cardinal. A Boolean algebra C is $\mathcal{L}_{\infty\kappa}$-Cohen if and only if there exists a set S of subalgebras of C such that*

(*0*) *Every $A \in S$ is a regular subalgebra of B, A is isomorphic to $\mathfrak{C}_\lambda$ for some $\lambda < \kappa$;*

(*1*) *For every $S' \subseteq S$ such that $|\, S' \,| < \kappa$ there exists a $B \in S$ such that $A \parallel B$ for every $A \in S'$;*

(*2*) *$\bigcup S = C$.*

Proof "$\Leftarrow$": Let S be a set of subalgebras of C satisfying the properties (*0*), (*1*) and (*2*). Then

$$\mathcal{F} = \{ f : f \text{ is an isomorphism from an } A \in S \text{ to } \mathfrak{C}_X \text{ for some } X \in [\kappa]^{<\kappa} \}$$

is a family of partial isomorphisms from C to $\mathfrak{C}_\kappa$ satisfying the back-and-forth property relevant to the $\mathcal{L}_{\infty\kappa}$-elementary equivalence.

"$\Rightarrow$": Assume that C is an $\mathcal{L}_{\infty\kappa}$-Cohen algebra. Let

$$S^* = \{ A : A \le \mathfrak{C}_\kappa,\ (\mathfrak{C}_\kappa, \mathfrak{C}_\lambda) \equiv_{\mathcal{L}_{\infty\kappa}} (\mathfrak{C}_\kappa, A) \text{ for some } \lambda < \kappa \}.$$

Clearly S^* satisfies the conditions (*0*) and (*2*). S^* also satisfies the condition (*1*): Let $S' \subseteq S^*$ be such that $|\, S' \,| < \kappa$. Let $X \in [\kappa]^{<\kappa}$ be such that $\bigcup S' \subseteq \mathfrak{C}_X$. Let $Y \subseteq \kappa \setminus X$ be such that $|\, X \,| = |\, Y \,|$. Since $\mathfrak{C}_{X\cup Y} \cong_{\mathfrak{C}_X} \overline{\mathfrak{C}_X \oplus \mathfrak{C}_{|X|}}$, it follows by Lemma 2.2 that $A \parallel \mathfrak{C}_{X\cup Y}$ holds for every $A \in S'$. Since we can express "$S^* \models$(0), (1), (2)" in an $\mathcal{L}_{\infty\kappa}$-sentence (in the language of Boolean algebra) we conclude that

$$S = \{ A : A \le C,\ (\mathfrak{C}_\kappa, \mathfrak{C}_\lambda) \equiv_{\mathcal{L}_{\infty\kappa}} (C, A) \text{ for some } \lambda < \kappa \}$$

also satisfies the conditions (*0*), (*1*) and (*2*). ▮ (Theorem 2.3)

Corollary 2.4 *For regular κ if A is an $\mathcal{L}_{\infty\kappa}$-free Boolean algebra then the σ-completion of A is an $\mathcal{L}_{\infty\kappa}$-Cohen algebra.*

Proof Let A be an $\mathcal{L}_{\infty\kappa}$-free Boolean algebra and C be the σ-completion of A. Let S be a set of subalgebras of A which satisfies the conditions (*0′*), (*1′*) and (*2′*) in Theorem 1.5. Let

$$\bar{S} = \{ \langle B \rangle_C^{cm} : B \in S \}.$$

(Note that $\langle B \rangle_C^{cm} = \langle B \rangle_C^{\sigma\text{-}cm}$ holds since $B \in S$ satisfies the ccc.) $\bar{S}$ satisfies the conditions (*0*), (*1*) and (*2*) in Theorem 2.3. ▮ (Corollary 2.4)

Theorem 2.5 *A complete Boolean algebra C is $\mathcal{L}_{\infty\aleph_1}$-Cohen algebra if and only if*

(*) $\min\{ \pi(C\restriction c) : c \in C^+ \} \geq \aleph_1$ *and*

($^*_*$) *Every countably generated complete subalgebra of C has π-weight $\leq \aleph_0$.*

Proof "⇒": Suppose that C is a complete $\mathcal{L}_{\infty\aleph_1}$-Cohen algebra. Since the statement: "any countable set of positive elements below a positive element is not dense below this element" can be formulated in $\mathcal{L}_{\infty\aleph_1}$ and is satisfied in $\mathbb{C}_{\aleph_1}$, this is also true in C. Hence (*) holds.

Similarly, since the statement ($^*_*$) can be formulated in $\mathcal{L}_{\infty\aleph_1}$ and is satisfied in $\mathbb{C}_{\aleph_1}$, it holds also in C.

"⇐": Let C be a complete Boolean algebra satisfying (*) and ($^*_*$). Let

$$S = \{ B : B \leq_{\text{reg}} C,\ B \cong \mathbb{C}_{\aleph_0} \}.$$

S satisfies the conditions (*0*), (*1*) and (*2*) in Theorem 2.3: (*0*) follows immediately from the definition. By ($^*_*$), (*2*) also holds. To show (*1*) let $S' \subseteq S$ be a countable set. For each $B' \in S'$ let $U_{B'}$ a countable dense subset of B'. Let $U = \bigcup_{B' \in S'} U_{B'}$. By ($^*_*$) there exists a countable regular subalgebra D of C such that $B = \langle D \rangle_C^{cm}$ includes U (and hence every $B' \in S'$). By (*) we can choose D so that, for every $d \in D^+$ and for every finite $U \subseteq D$, $\{ u \cdot d : u \in U \}$ is not dense in $B\restriction d$. Then for every $B' \in S'$, B is π-homogeneous over B' with $\pi(B/B') = \aleph_0$. By Lemma 1.3 it follows that $B' \parallel B$ for every $B' \in S'$. ▮ (Theorem 2.5)

Corollary 2.6 *A complete Boolean algebra C is a complete subalgebra of a complete $\mathcal{L}_{\infty\aleph_1}$-Cohen algebra if and only if C satisfies the condition ($^*_*$) of Theorem 2.5.*

Proof If C is a complete subalgebra of a complete $\mathcal{L}_{\infty\aleph_1}$-Cohen algebra $\tilde{A}$ then every countably generated complete subalgebra A of C is a countably generated complete subalgebra of $\tilde{C}$. By Theorem 2.5 it follows that $\pi(A) \leq \aleph_0$. Conversely if C satisfies the condition ($^*_*$) then $\overline{C \oplus \mathbb{C}_{\aleph_1}}$ satisfies the conditions (*) and ($^*_*$). ▮ (Corollary 2.6)

Theorem 2.5 can be also formulated in the language of forcing. Let V be the model of set theory in which we are "working". Let G be a V-generic filter over a complete Boolean algebra B in V. $r \in \mathbb{R}^{V[G]} \setminus \mathbb{R}^V$ is said to be almost Cohen over V if there exists a V-generic filter H over ${\mathbb{C}_{\aleph_0}}^V$ such that $V[r] = V[H]$. Note that for any V-generic filter G over a Cohen algebra (in V) every $r \in \mathbb{R}^{V[G]} \setminus \mathbb{R}^V$ is almost Cohen over V.

Note that the condition $(^*_*)$ in Theorem 2.5 is equivalent to the assertion that , for any V-generic filter G over C, every $r \in \mathbb{R}^{V[G]} \setminus \mathbb{R}^V$ is almost Cohen over V.

Corollary 2.7 *A complete Boolean algebra C is $\mathcal{L}_{\infty\aleph_1}$-Cohen algebra if and only if* $\min\{\, \pi(C\restriction c) \,:\, c \in C^+ \,\} \geq \aleph_1$ *and, for any V-generic filter G over C, every $r \in \mathbb{R}^{V[G]} \setminus \mathbb{R}^V$ is almost Cohen over V.* ∎

It is still open if every π-homogeneous complete subalgebra of a Cohen algebra is isomorphic to a Cohen algebra. The following corollary says that a π-homogeneous complete subalgebra of a Cohen algebra of uncountable π-weight is $\mathcal{L}_{\infty\aleph_1}$-Cohen.

Corollary 2.8 *Let C be a complete $\mathcal{L}_{\infty\aleph_1}$-Cohen algebra. Then every complete subalgebra B of C with* $\min\{\, \pi(B\restriction b) \,:\, b \in B^+ \,\} \geq \aleph_1$ *is an $\mathcal{L}_{\infty\aleph_1}$-Cohen algebra. In particular every complete subalgebra B of a Cohen algebra with* $\min\{\, \pi(B\restriction b) \,:\, b \in B^+ \,\} \geq \aleph_1$ *is an $\mathcal{L}_{\infty\aleph_1}$-Cohen algebra.*

Proof B satisfies the condition $(^*_*)$ in Theorem 2.5. ∎ (Corollary 2.8)

From Corollary 2.7 it follows that every complete $\mathcal{L}_{\infty\aleph_1}$-Cohen algebra C does not collapse $\aleph_1$: If $V[G] \models |\, \omega_1^V \,| = \aleph_0$ for a V-generic filter over C, there exists a real r in $V[G]$ which codes the order type ω_1^V. Such r cannot be almost Cohen.

In [9] it is shown that for any ω_1-tree T the algebra $B = \mathrm{Treealg}(T) \oplus \mathrm{Fr}\,\aleph_1$ is an $\mathcal{L}_{\infty\aleph_1}$-free Boolean algebra where Treealg(T) denotes the tree algebra over T (see [15]). By Corollary 2.4 the σ-completion of such B is an $\mathcal{L}_{\infty\aleph_1}$-Cohen algebra. However the full completion of B in general is not an $\mathcal{L}_{\infty\aleph_1}$-Cohen algebra. For example if T is a special Aronszajn tree, the completion $\overline{B}$ of $B = \mathrm{Treealg}(T) \oplus \mathrm{Fr}\,\aleph_1$ collapses $\aleph_1$. Thus, by the remark above, $\overline{B}$ is not an $\mathcal{L}_{\infty\aleph_1}$-Cohen algebra.

3 A Singular Compactness Theorem

By Shelah's Singular Compactness Theorem it holds that, for a singular λ, every $\mathcal{L}_{\infty\lambda}$-free Boolean algebra of cardinality λ is isomorphic to $\operatorname{Fr}\lambda$. In this section we shall show that a similar theorem holds for $\mathcal{L}_{\infty\lambda}$-Cohen algebras:

Theorem 3.1 *If λ is singular then every $\mathcal{L}_{\infty\lambda}$-Cohen algebra C of π-weight λ is isomorphic to $\mathfrak{C}_\lambda$.*

The proof is a modification of the proof of Shelah's Singular Compactness Theorem in Hodges [11]. The following lemma is the set-theoretic core of the proof.

Lemma 3.2 *Let λ be a singular cardinal with $\operatorname{cof}\lambda = \kappa$. Let A be a complete Boolean algebra which satisfies the following condition:*

(*0*) *For any $X \subseteq A$ it holds that $\pi(\langle X\rangle_A^{cm}) \leq |\, X\,| + \aleph_0$.*

Let $(\lambda_\alpha)_{\alpha<\kappa}$ be a continuously increasing sequence of cardinals $< \lambda$ such that $\kappa < \lambda_0$ and $\sup\{\,\lambda_\alpha \,:\, \alpha < \kappa\,\} = \lambda$ hold. Further let $(X_\alpha)_{\alpha<\kappa}$ be a sequence of subsets of A such that $|\, X_\alpha\,| \leq \lambda_\alpha$ for $\alpha < \kappa$ holds. Then there exists an increasing sequence $(A_\alpha)_{\alpha<\kappa}$ of subalgebras of A such that

(*1*) *$X_\alpha \subseteq A_\alpha$ for $\alpha < \kappa$;*

(*2*) *$\pi(A_\alpha) \leq \lambda_\alpha$;*

(*3*) *$\bigcup_{\alpha<\gamma} A_\alpha$ is dense in A_γ for every limit $\gamma < \kappa$.*

Proof By induction on $n \in \omega$ we construct sequences $(A^n_\alpha)_{\alpha<\kappa}$ and $\{\, a^n_{\alpha,\beta}\,\}_{\beta<\lambda_\alpha}$ for $\alpha < \kappa$ so that

(*a*) for every $n \in \omega$, $(A^n_\alpha)_{\alpha<\kappa}$ is an increasing sequence of complete subalgebras such that $\tau(A^n_\alpha) \leq \lambda_\alpha$ for $\alpha < \kappa$;

(*b*) $X_\alpha \subseteq A^0_\alpha$ for $\alpha < \kappa$;

(*c*) $\{\, a^n_{\alpha,\beta} \,:\, \beta < \lambda_\alpha\,\}$ is a dense subset of $(A^n_\alpha)^+$;

(*d*) $A^{n+1}_\alpha \supseteq \{\, a^n_{\alpha',\beta} \,:\, \alpha' < \kappa,\, \beta < \lambda_\alpha\,\} \cup A^n_\alpha$.

For $\alpha < \kappa$ let $A_\alpha = \bigcup_{n\in\omega} A^n_\alpha$. We show that $(A_\alpha)_{\alpha<\kappa}$ is as desired: (*1*) follows from (*b*). For (*2*) let

$$Y_\alpha = \{\, a^n_{\alpha,\beta} \,:\, \beta < \lambda_\alpha,\, n \in \omega\,\}.$$

Then Y_α is dense in A_α and $\langle Y_\alpha\rangle_A^{cm} \supseteq A_\alpha$. For (*3*) let $\gamma < \kappa$ be a limit ordinal and let $a \in A_\gamma$. By (*c*) there exist $n \in \omega$ and $\beta < \lambda_\gamma$ such that $a^n_{\gamma,\beta} \leq a$. By the continuity of $(\lambda_\alpha)_{\alpha<\kappa}$, there exists $\alpha < \gamma$ such that $\beta < \lambda_\alpha$. By (*d*) it holds that $a^n_{\gamma,\beta} \in A^{n+1}_\alpha \subseteq A_\alpha$.
▮ (Lemma 3.2)

Proof of Theorem 3.1: Let $X = \{ a_\beta : \beta < \lambda \}$ be a dense subset of C. Let $\kappa = \mathrm{cof}\,\lambda$ and let $(\lambda_\alpha)_{\alpha<\kappa}$ be a continuously increasing sequence of cardinals $< \lambda$ such that $\lambda_0 > \kappa$ and $\bigcup_{\alpha<\kappa} \lambda_\alpha = \lambda$. By induction on $n \in \omega$ we define a sequences $(A_n^\alpha)_{\alpha<\kappa}$, $(B_n^\alpha)_{\alpha<\kappa}$, $(C_n^\alpha)_{\alpha<\kappa}$, $(X_n^\alpha)_{\alpha<\kappa}$ such that

(0) $(A_n^\alpha)_{\alpha<\kappa}$, $(B_n^\alpha)_{\alpha<\kappa}$, $(C_n^\alpha)_{\alpha<\kappa}$ are sequences of subalgebras of C such that $\tau^C(A_n^\alpha) = \pi(B_n^\alpha) = \pi(C_n^\alpha) = \lambda_\alpha$ for $\alpha < \kappa$. $X_n^\alpha \subseteq C$ and $|\, X_n^\alpha \,| \leq \lambda_\alpha$;

(1) For $\alpha < \kappa$ and $n \in \omega$, $C_n^\alpha \leq B_n^\alpha \leq A_n^\alpha \leq C_{n+1}^\alpha \leq B_{n+1}^\alpha \leq A_{n+1}^\alpha$ and $X_n^\alpha \subseteq X_{n+1}^\alpha$;

(2) $C_0^\alpha = \langle \{ a_\beta : \beta < \lambda_\alpha \} \rangle_C^{cm}$ for $\alpha < \kappa$;

(3) $B_n^\alpha = \langle X_n^\alpha \rangle_C^{cm}$ and X_n^α is such that for some enumeration $\{ x_\xi \}_{\xi<\lambda_\alpha}$ of X_n^α it holds that $(\mathfrak{C}_\lambda, \xi)_{\xi<\lambda_\alpha} \equiv_{\mathcal{L}_{\infty\lambda}} (C, x_\xi)_{\xi<\lambda_\alpha}$.

(4) $(A_n^\alpha)_{\alpha<\kappa}$ is an increasing sequence and $\bigcup_{\alpha<\gamma} A_n^\alpha$ is dense in A_n^γ for every limit $\gamma < \kappa$.

(5) For $\alpha < \kappa$, $n \in \omega$ and $k \leq n$ there exists a $Y_{n,k}^\alpha \subseteq X_k^{\alpha+1}$ such that $C_{n+1}^\alpha \cap B_k^{\alpha+1} = \langle Y_{n,k}^\alpha \rangle_C^{cm}$.

It is easy to see that the construction above goes through. In particular *(4)* is possible by Lemma 3.2.

For $\alpha < \kappa$ let $C^\alpha = \bigcup_{n\in\omega} A_{\alpha_n}\ (= \bigcup_{n\in\omega} B_{\alpha_n} = \bigcup_{n\in\omega} C_{\alpha_n}\ )$. By *(0)* and *(1)*, $(C^\alpha)_{\alpha<\kappa}$ an increasing sequence of subalgebras of C. By *(2)* $\bigcup_{\alpha<\kappa} C^\alpha$ is dense in C. By *(3)*, Lemma 2.2 and *(5)* we have $(\overline{C^{\alpha+1}}, \overline{C^\alpha}) \cong (\mathfrak{C}_{\lambda_{\alpha+1}}, \mathfrak{C}_{\lambda_\alpha})$. By *(4)*, $\bigcup_{\alpha<\gamma} C^\alpha$ is dense in C^γ for every limit $\gamma < \lambda$. Hence it follows that $C \cong \overline{\bigcup_{\alpha<\kappa} A_\alpha} \cong \mathfrak{C}_\lambda$. ▮ (Theorem 3.1 3.2)

We still do not know if a theorem similar to Theorem 3.1 holds for a weakly compact κ. This is also connected with the following open problem:

Problem 3.3 *Is every π-homogeneous complete subalgebra C of a Cohen algebra isomorphic to a Cohen algebra?*

If $\pi(C) \leq \aleph_1$ the answer is known to be positive ([14]). By *(a)* of the next proposition, we would obtain a theorem similar to Theorem 3.1 if we had a positive answer to Problem 3.3.

Proposition 3.4 *Let κ be a weakly compact cardinal.*

(a) Every $\mathcal{L}_{\infty\kappa}$-Cohen algebra C of cardinality κ can be embedded into $\mathfrak{C}_\kappa$ as a complete subalgebra.

(b) If B is an $\mathcal{L}_{\infty\kappa}$-free Boolean algebra of cardinality κ then the completion $\overline{B}$ of B is isomorphic to $\mathfrak{C}_\kappa$.

Proof *(a)*: Let U be a unary relation symbol and, for each $a \in C$, let c_a be a constant symbol. Let Φ be the set of the following $\mathcal{L}_{\infty\kappa}$ sentences in the language of Boolean algebraswith the new symbols above:

axioms of Boolean algebras;

"$U(\cdot)$ is a set of free generators of a dense subalgebra";

$\{\,\varphi(c_{a_0},\ldots,c_{a_k}) \,:\, \varphi$ is a quantifier-free formula in the language of Boolean algebras, $C \models \varphi[a_0,\ldots,a_k],\ a_0,\ldots,a_k \in C\,\}$;

$\{$ "the sum of $\{\,c_a \,:\, a \in X\,\}$ is 1 " $:\ X \in [C]^{\aleph_0},\ \sum^C X = 1\ \}$.

Φ says that C is embeddable into a Cohen algebra as a regular subalgebra. Since C is $\mathcal{L}_{\infty\kappa}$-Cohen algebra Φ is κ satisfiable. By the weak compactness of κ it follows that Φ is satisfiable and hence C is embeddable into a Cohen algebra as a regular subalgebra.

(b): An argument similar to the proof of (a) shows that B is embeddable into $\mathrm{Fr}\,\kappa$. Since $\overline{B}$ is π-homogeneous and of π-weight κ, it follows from the theorem in [21] that $\overline{B}$ is isomorphic to $\mathfrak{C}_\kappa$. ▮ (Proposition 3.4)

Proposition 3.5 (a) **consis**(*"every $\mathcal{L}_{\infty 2^{\aleph_0}}$-Cohen algebra of π-weight $2^{\aleph_0}$ is isomorphic to $\mathfrak{C}_{2^{\aleph_0}}$"*).

(b) *If* **consis**(*" $\exists$a weakly compact cardinal"*) *then* **consis**(*" $2^{\aleph_0}$ is regular and every $\mathcal{L}_{\infty 2^{\aleph_0}}$-Cohen algebra of π-weight $2^{\aleph_0}$ is embeddable in $\mathfrak{C}_{2^{\aleph_0}}$ as a complete subalgebra"*).

Proof (a): By Theorem 3.1 the assertion holds if e.g. $2^{\aleph_0} = \aleph_{\omega_1}$ holds.

(b): Let κ be a weakly compact cardinal. We show that $\mathfrak{C}_\kappa$ forces the assertion. Note that $\Vdash_{\mathfrak{C}_\kappa}$ " $2^{\aleph_0} = \kappa$ " holds.

Let $\dot{C}$ be a $\mathfrak{C}_\kappa$-name of an $\mathcal{L}_{\infty\kappa}$-Cohen algebra of π-weight κ.

Claim 3.5.1 *There exists a complete Boolean algebra D isomorphic to $\mathfrak{C}_\kappa$ such that $\mathfrak{C}_\kappa * \dot{C} \leq_{\mathrm{reg}} D$ and $\mathfrak{C}_\kappa \parallel D$.*

Proof of Claim 3.5.1 Let U be a unary relation symbol and, for each $\dot{a} \in \mathfrak{C}_\kappa * \dot{C}$, let $c_{\dot{a}}$ be a constant symbol. Let Ψ be the following set of $\mathcal{L}_{\infty\kappa}$ sentences in the language of Boolean algebras with the new symbols above:

axioms of Boolean algebras;

"$U(\cdot)$ is a set of free generators of a dense subalgebra";

$\{\,U(c_\alpha) \,:\, \alpha \in \kappa\,(\subseteq \mathrm{Fr}\,\kappa \subseteq \mathfrak{C}_\kappa \subseteq \mathfrak{C}_\kappa * \dot{C})\,\}$;

$\{\,\varphi(c_{\dot{a}_0},\ldots,c_{\dot{a}_k}) \,:\, \varphi$ is a quantifier-free formula in the language of Boolean algebras, $\mathfrak{C}_\kappa * \dot{C} \models \varphi[\dot{a}_0,\ldots,\dot{a}_k],\ \dot{a}_0,\ldots,\dot{a}_k \in \mathfrak{C}_\kappa * \dot{C}\,\}$;

$\{$ "the sum of $\{\,c_{\dot{a}} \,:\, \dot{a} \in X\,\}$ is 1 " $:\ X \in [\mathfrak{C}_\kappa * \dot{C}]^{\aleph_0},\ \sum^{\mathfrak{C}_\kappa * \dot{C}} X = 1\,\}$.

Ψ is κ-satisfiable: Let $Y \subseteq \mathbb{C}_\kappa * \dot{C}$ such that $|\, Y \,| < \kappa$. Let Ψ' be the subset of Ψ consisting of formulas of Ψ which contain only constant symbols of the form $c_{\dot{a}}$, $\dot{a} \in Y$ and 0, 1. We show that Ψ' is satisfiable. Since $\Vdash_{\mathbb{C}_\kappa}$ "$\dot{C}$ is an $\mathcal{L}_{\infty\kappa}$-Cohen algebra" there exists a $\mathbb{C}_\kappa$-name $\dot{A}$ such that $\Vdash_{\mathbb{C}_\kappa}$ "$\dot{A} \leq_{\rm reg} \dot{C}$", $\Vdash_{\mathbb{C}_\kappa}$ "$\dot{A}$ is isomorphic to $\mathbb{C}_\lambda$" for some $\lambda < \kappa$ and $\Vdash_{\mathbb{C}_\kappa}$ "$\dot{a} \in \dot{A}$" for all $\dot{a} \in Y$. By Lemma 1.2 (*a*) and (*c*) we have that $\mathbb{C}_\kappa * \dot{A} \cong_{\mathbb{C}_\kappa} \overline{\mathbb{C}_\kappa \oplus \mathbb{C}_\lambda}$. Hence we can expand $\mathbb{C}_\kappa * \dot{A}$ to a model of Ψ'.

By weak compactness of κ there exists a model D of Ψ of cardinality κ. By identifying each $\dot{a} \in \mathbb{C}_\kappa * \dot{C}$ with ${c_{\dot{a}}}^D$ this D is as desired. ▮ (Claim 3.5.1)

Let D be as in Claim 3.5.1. For any generic filter G over $\mathbb{C}_\kappa$ we have $\dot{C}^G \leq_{\rm reg} (D : \mathbb{C}_\kappa)^G$. But since $\mathbb{C}_\kappa \parallel D$ it follows that $(D : \mathbb{C}_\kappa)^G$ is isomorphic to a Cohen algebra. ▮ (Proposition 3.5)

Similarly to Theorem 2.8 in [9] we have the following:

Proposition 3.6 *Let κ be a weakly compact cardinal. If a Boolean algebra A of cardinality κ is represented as the union of continuously increasing sequence $(A_\alpha)_{\alpha<\kappa}$ of subalgebras of A such that $\overline{A_\alpha}$ is a Cohen algebra of π-weight $< \kappa$ for every $\alpha < \kappa$, then $\overline{A}$ is isomorphic to $\mathbb{C}_\kappa$.*

Proof Let κ, A and $(A_\alpha)_{\alpha<\kappa}$ be as above. By modifying the sequence $(A_\alpha)_{\alpha<\kappa}$ we may assume that $|\, A_\alpha \,| \leq |\, \alpha \,| + \aleph_0$ holds for every $\alpha < \kappa$.

Assume that $\overline{A}$ is not isomorphic to $\mathbb{C}_\kappa$. Let

$$\mathcal{S}_\alpha = \{\, \beta < \kappa \,:\, \alpha \leq \beta,\, \overline{A_\alpha} \not\parallel \overline{A_\beta} \,\}$$

for $\alpha < \kappa$.

Claim 3.6.1 *$\mathcal{S} = \{\, \alpha < \kappa \,:\, \mathcal{S}_\alpha$ is stationary $\}$ is stationary subset of κ.*

Proof of Claim 3.6.1 Otherwise there would be a club set

$$\mathcal{C} \subseteq \{\, \alpha < \kappa \,:\, \{\, \beta < \kappa \,:\, \overline{A_\alpha} \parallel \overline{A_\beta} \,\} \text{ contains a club subset} \,\}.$$

Hence we can choose a continuously increasing sequence $(\gamma_\alpha)_{\alpha<\kappa}$ of ordinals $\in \mathcal{C}$ such that $\overline{A_{\gamma_\alpha}} \parallel \overline{A_{\gamma_\beta}}$ holds for every $\alpha < \gamma < \kappa$. It follows that $\overline{A}$ is isomorphic to $\mathbb{C}_\kappa$. This is a contradiction to our assumption. ▮ (Lemma 3.6)

By weak compactness of κ there exists a $\lambda < \kappa$ such that $\mathcal{S}$ and every $\mathcal{S}_\alpha$ for $\alpha \in \mathcal{S} \cap \lambda$ are stationary below λ. It follows that $\overline{A_\lambda}$ is not isomorphic to a Cohen algebra. This is a contradiction. ▮ (Proposition 3.6)

4 $\mathcal{L}_{\infty\kappa}$-Cohen algebras constructed from $\mathcal{L}_{\infty\kappa}$-free Boolean algebras

In [9] it is shown that for every cardinal κ there exists an $\mathcal{L}_{\infty\aleph_1}$-free Boolean algebra B which does not satisfy the κ-cc. The σ-completion C of B is an $\mathcal{L}_{\infty\aleph_1}$-Cohen algebra by Corollary 2.4 but C does not satisfy the κ-cc. If T is a Suslin tree and Treealg(T) is its tree algebra then $B = \text{Treealg}(T) \oplus \text{Fr}\,\aleph_1$ is an $\mathcal{L}_{\infty\aleph_1}$-free Boolean algebra ([9]). B satisfies the ccc but is not absolutely ccc. Let C be the completion of B. Then again C is an $\mathcal{L}_{\infty\kappa}$-Cohen algebra by Corollary 2.4. C has the π-weight $\aleph_1$ and satisfies the ccc. But since C is not absolutely ccc, C is not isomorphic to $\mathbb{C}_{\aleph_1}$.

We shall show that ccc $\mathcal{L}_{\infty\aleph_1}$-Cohen algebras of π-weight $\aleph_1$ not isomorphic to $\mathbb{C}_{\aleph_1}$ can be constructed already in ZFC.

Theorem 4.1 *Let κ be a regular cardinal and $S \subseteq \{\, \delta < \kappa : \text{cof}\,(\delta) = \omega \,\}$ be a stationary non-reflecting subset of κ (i.e. for every limit $\gamma < \kappa$, $S \cap \gamma$ is not stationary in γ.). Then there exists a continuously increasing sequence $(A_\alpha)_{\alpha<\kappa}$ of Boolean algebras such that*

(0) $|\, A_\alpha \,| < \kappa$ *and* $A_\alpha \cong \text{Fr}\,(|\, A_\alpha + \omega \,|)$ *for every* $\alpha < \kappa$*;*

(1) $A_\alpha \mid A_\beta$ *for every* $\alpha < \beta < \kappa$ *such that* $\alpha, \beta \in \kappa \setminus S$*;*

(2) A_α *is not a regular subalgebra of* $A_{\alpha+1}$ *for every* $\alpha \in S$.

Theorem 4.1 is proved after Lemma 4.4. The proof is a modification of a construction by S. Koppelberg.

Corollary 4.2 (a) *There exists an $\mathcal{L}_{\infty\aleph_1}$-free Boolean algebra A of cardinality $\aleph_1$ such that A satisfies the ccc and $\overline{A}$ is not isomorphic to $\mathbb{C}_{\aleph_1}$.*

(b) *$(V = L)$ Let κ be a non weakly compact regular cardinal. There exists an $\mathcal{L}_{\infty\kappa}$-free Boolean algebra A of cardinality κ such that $\overline{A}$ is not isomorphic to $\mathbb{C}_\kappa$.*

Note that, by Theorem 2.4, the completion of A in (a) (in (b) resp.) is an $\mathcal{L}_{\infty\aleph_1}$-Cohen algebra (an $\mathcal{L}_{\infty\kappa}$-Cohen algebra resp.). Also note that for $\kappa > \aleph_1$ every $\mathcal{L}_{\infty\kappa}$-free Boolean algebra satisfies the ccc.

Proof (a): Let $S \subseteq \{\, \alpha < \omega_1 : \alpha \text{ is a limit} \,\}$ be a stationary co-stationary set. S is non-reflecting. Hence there exists a continuously increasing sequence of countable Boolean algebras $(A_\alpha)_{\alpha<\omega_1}$ satisfying (0), (1) and (2) in Theorem 4.1. $A = \bigcup_{\alpha<\omega_1} A_\alpha$ is as desired: By (1) and Theorem 1.5, A is an $\mathcal{L}_{\infty\aleph_1}$-free Boolean algebra. Since (1) of Theorem 4.1 holds for stationary many $\alpha, \beta < \kappa$, A satisfies the ccc (see [24]). If $\overline{A}$ were isomorphic to $\mathbb{C}_{\omega_1}$ then there would be club many $\alpha < \kappa$ such that $\overline{A_\alpha} \parallel \overline{A}$. But if $\alpha \in S$ then $\overline{A_\alpha}$ is not a regular subalgebra of $\overline{A}$.

(b): Under $V = L$ there exists a non-reflecting stationary $S \subseteq \{\, \delta < \kappa : \text{cof}\,(\delta) = \omega \,\}$ for every non weakly compact regular κ (see e.g. [13]). Hence by the same argument as in (a) we obtain an A with the desired property. ▮ (Corollary 4.2)

For the proof of Theorem 4.1 we need the following lemmas:

Lemma 4.3 *Let $(B_\alpha)_{\alpha<\kappa}$ be a continuously increasing chain of Boolean algebras such that $B_\alpha \leq_{\text{proj}} B_{\alpha+1}$. Let $B = \bigcup_{\alpha<\kappa} B_\alpha$. Let $A \geq B$ and $x \in A$ be such that $A = B(x)$. If $B_\alpha \leq_{\text{rc}} B(x)$ for every $\alpha < \kappa$ then it holds that $B_\alpha \leq_{\text{proj}} B(x)$ for every $\alpha < \kappa$.*

Proof Let $\alpha < \kappa$. By Theorem 1.7 there exists a continuously increasing chain $(C_\eta)_{\eta<\delta}$ of subalgebras of B such that $C_0 = B_\alpha$; $C_\eta \leq_{\omega\text{rc}} C_{\eta+1}$ for all $\eta < \delta$ which is a refinement of the chain $(B_\beta)_{\alpha\leq\beta<\kappa}$. By the assumption we have $C_\eta \leq_{\text{rc}} B(x)$ for every $\eta < \delta$.

Let $D_0 = C_0$ $(= B_\alpha)$ and $D_\eta = C_\eta(x)$ for $0 < \eta < \delta$. $(D_\eta)_{\eta<\delta}$ is a continuously increasing chain of subalgebras of $B(x)$ and $\bigcup_{\eta<\delta} D_\eta = B(x)$ holds.

For every $\eta < \delta$, $D_{\eta+1}$ is countably generated over D_η: Let X be a countable subset of $C_{\eta+1}$ such that $C_\eta[X] = C_{\eta+1}$. Then $D_{\eta+1} = D_\eta[X \cup \{ x \}]$.

For every $\eta < \delta$ it holds that $D_\eta \leq_{\text{rc}} D_{\eta+1}$: For $\eta = 0$ we have that $D_0 = B_\alpha \leq_{\text{rc}} B(x)$ by the assumption. Since $D_0 \leq D_1 \leq B(x)$ it follows that $D_0 \leq_{\text{rc}} D_1$. For $0 < \eta < \delta$ we have that $C_\eta \leq_{\text{rc}} B(x)$. By Lemma 1.6 it follows that $D_\eta = C_\eta(x) \leq_{\text{rc}} B(x)$. Hence $D_\eta \leq_{\text{rc}} D_{\eta+1}$.

By Theorem 1.7 it follows that $B_\alpha \leq_{\text{proj}} B(x)$. ▮ (Lemma 4.3)

Lemma 4.4 *Let $(B_\alpha)_{\alpha<\kappa}$ be an increasing chain of Boolean algebras such that $B_\alpha \leq_{\text{rc}} B_\beta$ for every $\alpha \leq \beta < \kappa$. Let $B = \bigcup_{\alpha<\kappa} B_\alpha$. Let $A \geq B$ and let $x \in A$ be such that $A = B(x)$. If there exist greatest elements p_α and q_α of $\{ b \in B_\alpha : b \leq x \}$ and $\{ b \in B_\alpha : b \leq -x \}$ respectively for every $\alpha < \kappa$ then $B_\alpha \leq_{\text{rc}} A$ holds for every $\alpha < \kappa$.*

Proof Let $y \in A$, say $y = b \cdot x + b' \cdot -x$ for some b, $b' \in B$. For $\alpha < \omega_1$ let $\beta < \omega_1$ be such that $\alpha \leq \beta$ and b, $b' \in B_\beta$. For any $a \in B_\alpha$ we have:

$$\begin{aligned} a \leq y \quad &\Leftrightarrow \quad a \cdot x \leq b \text{ and } a \cdot -x \leq b' \\ &\Leftrightarrow \quad a \cdot -b \leq -x \text{ and } a \cdot -b' \leq x \\ &\Leftrightarrow \quad a \cdot -b \leq q_\beta \text{ and } a \cdot -b' \leq p_\beta \\ &\Leftrightarrow \quad a \leq (b + q_\beta) \cdot (b' + p_\beta) \\ &\Leftrightarrow \quad a \leq \text{pr}^{B_\beta}_{B_\alpha}((b + q_\beta) \cdot (b' + p_\beta)). \end{aligned}$$

Hence $\text{pr}^{B_\beta}_{B_\alpha}((b+q_\beta)\cdot(b'+p_\beta))$ is the greatest element of $\{ a \in B_\alpha : a \leq y \}$. ▮ (Lemma 4.4)

Proof of Theorem 4.1: For $\alpha < \kappa$ let A_α be defined inductively so that, for every $\alpha < \kappa$,

$(*_\alpha)$ $A_\alpha \cong \text{Fr} \mid \alpha + \omega \mid$; for every $\beta < \alpha$ if $\beta \notin S$ then $A_\beta \mid A_\alpha$ holds and if $\beta \in S$ then $A_\beta \leq_{\neg\text{rc}} A_\alpha$

holds. Suppose for $\alpha < \kappa$, $(A_\beta)_{\beta<\alpha}$ has been already chosen so that $(*_\beta)$ for all $\beta < \alpha$ is satisfied.

Case I α is a limit: Let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Since $\alpha \cap S$ is not stationary in α there exists a club $X \subseteq \alpha$ such that for every $\beta, \beta' \in X$ such that $\beta < \beta'$ we have $A_\beta | A_{\beta'}$. Since $A_\alpha = \bigcup_{\beta \in X} A_\beta$ it follows that $A_\alpha \cong \mathrm{Fr}\,(|\ \alpha + \omega\ |)$. Clearly $(*_\alpha)$ is satisfied.

Case II $\alpha = \gamma + 1$ for some $\gamma \in \kappa \setminus S$: Let $A_\alpha = A_\gamma \oplus \mathrm{Fr}\,(|\ \gamma + \omega\ |)$.

Case III $\alpha = \gamma + 1$ for some $\gamma \in S$: In this case we have that $\mathrm{cof}\,(\gamma) = \omega$. Let $(\beta_n)_{n\in\omega}$ be a strictly increasing sequence of ordinals $< \gamma$ such that $\beta_n \notin S$ for all $n \in \omega$ and $\bigcup_{n\in\omega} \beta_n = \gamma$. Hence $A_\gamma = \bigcup_{n\in\omega} A_{\beta_n}$. Since $(*_\beta)$ holds for all $\beta \leq \gamma$, we have that $A_{\beta_n} | A_{\beta_{n+1}}$ and $A_{\beta_n} \cong \mathrm{Fr}\,(|\ \beta_n + \omega\ |)$. Let $(a_n)_{n\in\omega}$ be a sequence of elements of A_γ such that $a_n \in A_{\beta_n}$, $\mathrm{pr}^{A_{\beta_{n+1}}}_{A_{\beta_n}}(a_{n+1}) = a_n$ and $\sum^{A_\gamma}_{n\in\omega} a_n = 1$. Let $I = \{\, b \in A_\gamma \,:\, b \leq a_n$ for some $n \in \omega \,\}$. Let x (in some Boolean algebra extending A_γ) be such that $\{\, b \in A_\gamma \,:\, b \leq x \,\} = I$ and $\{\, b \in A_\gamma \,:\, b \leq -x \,\} = \{\, 0 \,\}$ hold. By Lemma 4.4 it holds that $A_{\beta_n} \leq_{\mathrm{rc}} A_\gamma(x)$ for every $n \in \omega$. By Lemma 4.3 it follows that $A_{\beta_n} \leq_{\mathrm{proj}} A_\gamma(x)$ for every $n \in \omega$. Since $\sum^{A_\gamma(x)}_{n\in\omega} a_n \leq x < 1$, A_γ is not relatively complete in $A_\gamma(x)$. Let $A_\alpha = A_\gamma(x) \oplus \mathrm{Fr}\,(|\ \gamma\ |)$. Then $A_{\beta_n} \mid A_\alpha$ and A_γ is not a regular subalgebra of A_α. For $\beta < \gamma$ such that $\beta \notin S$ let $n \in \omega$ be such that $\beta \leq \beta_n$. By $(*_{\beta_n})$ we have $A_\beta \mid A_{\beta_n}$. It follows that $A_\beta \mid A_\alpha$.
▮ (Theorem 4.1)

By L. B. Shapiro [22] there exist Boolean algebras S^0, S^1 with the following properties[1]:

$S^0 \cong S^1 \cong \mathrm{Fr}\,\aleph_1$;

$S^0 \leq_{\mathrm{rc}} S^1$;

S^1 is π-homogeneous over S^0 and $\pi(S^1/S^0) = \aleph_1$;

$\overline{S^0} \not\Vdash \overline{S^1}$;

there exists a continuously increasing chain $(S^0_\alpha)_{\alpha<\omega_1}$ of subalgebras of S^0 such that

- $S^0_\alpha \cong \mathrm{Fr}\,\aleph_0$ for every $\alpha < \omega_1$;
- $S^0 = \bigcup_{\alpha<\omega_1} S^0_\alpha$;
- $S^0_\alpha \mid S^0_{\alpha+1}$ for every $\alpha < \omega_1$ and
- $S^0_\alpha \mid S^1$ for every $\alpha < \omega_1$.

S^0,S^1 as above show that Lemma 1.3 with $\aleph_2$ in place of $\aleph_1$ does not hold.

Using S^0, S^1 as above we shall construct (in ZFC) an $\mathcal{L}_{\infty\aleph_2}$-free Boolean algebra of cardinality $\aleph_2$ whose completion is not isomorphic to $\mathfrak{C}_{\aleph_2}$.

[1] I would like to thank Ingo Bandlow who made me familiar with this result.

Lemma 4.5 *Let A, B be complete Boolean algebras such that $A \leq_{\mathrm{reg}} B$, B is π-homogeneous over A, $\pi(B/A) = \aleph_1$ and $A \not\parallel B$.*

(a) *For any complete Boolean algebra C such that $B \leq_{\mathrm{reg}} C$ we have that $A \not\parallel C$.*

(b) *For any ccc complete Boolean algebra C we have that $\overline{A \oplus C} \not\parallel \overline{B \oplus C}$.*

Proof (a): If there were a complete Boolean algebra C such that $B \leq_{\mathrm{reg}} C$ and $A \parallel C$, we may assume that $C \cong_A \overline{A \oplus \mathfrak{C}_{\aleph_1}}$ by $\pi(B/A) = \aleph_1$. Then we would have

$$\Vdash_A \text{“}(B : A) \leq_{\mathrm{reg}} (C : A) \cong \mathfrak{C}_{\aleph_1}\text{”}.$$

By the assumption we have

$$\Vdash_A \text{“}(B : A) \text{ is } \pi\text{-homogeneous and } \pi((B : A)) = \aleph_1\text{”}.$$

Hence, by a theorem in [14], it follows that $\Vdash_A$ “$(B : A) \cong \mathfrak{C}_{\aleph_1}$”. By Lemma 1.4 it follows that $B \cong_A \overline{A \oplus \mathfrak{C}_{\aleph_1}}$. This is a contradiction to the assumption: $A \not\parallel B$.

(b): Assume that $\overline{A \oplus C} \parallel \overline{B \oplus C}$ holds for some complete Boolean algebra C. Then it follows that $A \parallel \overline{B \oplus C}$. By a this is a contradiction. ▮ (Lemma 4.5)

Lemma 4.6 *Let $(A_\alpha)_{\alpha<\omega_1}$ be a continuously increasing sequence of free Boolean algebras such that $A_{\alpha+1} \cong_{A_\alpha} A_\alpha \oplus \mathrm{Fr}\,\kappa_\alpha$ for some infinite cardinal κ_α and for $\alpha < \beta < \omega_1$. Let $A = \bigcup_{\alpha<\omega_1} A_\alpha$. Then there exists a free Boolean algebra B such that; $A \leq_{\mathrm{rc}} B$, B is π-homogeneous over A and $\pi(B/A) = \aleph_1$, $\overline{A} \not\parallel \overline{B}$ and $\overline{A}_\alpha \parallel \overline{B}$ for every $\alpha < \omega_1$.*

Proof Let S^0, S^1 be Boolean algebras as above. Let $\kappa = \sup_{\alpha<\omega_1} \kappa_\alpha$ and let $(X_\alpha)_{\alpha<\kappa}$ be a partition of κ such that $|\, X_\alpha \,| = \kappa_\alpha$ for all $\alpha < \omega_1$. Let $T^0 = S^0 \oplus \mathrm{Fr}\,\kappa$ and $T^1 = S^1 \oplus \mathrm{Fr}\,\kappa$. Then T^0 and T^1 are free Boolean algebras and $T^0 \leq_{\mathrm{rc}} T^1$. By Lemma 4.5 ($b$) we have $\overline{T^0} \not\parallel \overline{T^1}$. For $\alpha < \kappa$ let $T^0_\alpha = S^0_\alpha \oplus \mathrm{Fr}\,(\bigcup_{\beta<\alpha} X_\beta)$. Then $T^0 = \bigcup_{\alpha<\omega_1} T^0_\alpha$ and $(A_\alpha)_{\alpha<\omega_1}$ is isomorphic to $(T^0_\alpha)_{\alpha<\omega_1}$. Hence by identifying T^0_α with A_α, the Boolean algebra $B = T^1$ is as desired. ▮ (Lemma 4.6)

Theorem 4.7 *Let κ be a regular cardinal and $S \subseteq \{\, \delta < \kappa \,:\, \mathrm{cof}\,(\delta) = \omega_1 \,\}$ be a non-reflecting stationary subset of κ. Then there exists a continuously increasing sequence $(A_\alpha)_{\alpha<\kappa}$ of Boolean algebras such that*

(*0*) *$|\, A_\alpha \,| < \kappa$ and A_α is free for every $\alpha < \kappa$. $A_\alpha \leq_{\mathrm{rc}} A_{\alpha+1}$ for every $\alpha < \kappa$;*

(*1*) *$A_\alpha \mid A_\beta$ for every $\alpha < \beta < \kappa$ such that $\alpha, \beta \in \kappa \setminus S$;*

(*2*) *$A_{\alpha+1}$ is π-homogeneous over A_α but $\overline{A_\alpha} \not\parallel \overline{A_{\alpha+1}}$ for every $\alpha \in S$.*

Proof Similarly to the proof of Theorem 4.1 we can take A_α, $\alpha < \kappa$ inductively. At the $\alpha + 1$st step for $\alpha \in S$ use Lemma 4.6. ▮ (Theorem 4.7)

Corollary 4.8 (a) *There exists an $\mathcal{L}_{\infty\aleph_2}$-free Boolean algebra A of cardinality $\aleph_2$ such that $\overline{A}$ is not isomorphic to $\mathfrak{C}_{\aleph_2}$.*

(b) *Let κ be a regular cardinal. If there exists a non-reflecting stationary $S \subseteq \{\,\delta < \kappa : \mathrm{cof}\,(\delta) = \omega_1\,\}$ then there exists an $\mathcal{L}_{\infty\kappa}$-free Boolean algebra A of cardinality κ such that $\overline{A}$ is not isomorphic to $\mathfrak{C}_\kappa$.*

Proof (a): Let $S = \{\,\alpha < \omega_2 : \mathrm{cof}\,(\alpha) = \omega_1\,\}$. Then S is a non-reflecting stationary subset of ω_2. Let $(A_\alpha)_{\alpha<\omega_2}$ be as in Theorem 4.7 for this S. Let $A = \bigcup_{\alpha<\omega_2} A_\alpha$. By (1) in Theorem 4.7 and Theorem 1.5 A is an $\mathcal{L}_{\infty\aleph_2}$-free Boolean algebra. $\overline{A}$ is not isomorphic to $\mathfrak{C}_{\aleph_2}$: Otherwise there would exist an $\alpha \in S$ such that $\overline{A_\alpha} \parallel \overline{A}$. But by Lemma 4.5 ($a$) this is a contradiction.

(b): Similar. ▮ (Corollary 4.8)

Note that the Boolean algebra A in Corollary 4.8 is represented as a union of a continuously increasing chain of relatively complete free subalgebras. A Boolean algebra A is said to be *openly generated* if the set $\{\,C : C \leq_{\mathrm{rc}} B$, and C is countable$\,\}$ contains a club subset of $[B]^{\aleph_0}$. By Ščepin [19] union of a continuously increasing chain of openly generated relatively complete subalgebras is again openly generated. It follows that the Boolean algebra A in Corollary 4.8 is openly generated.

Since we need a non-reflecting stationary subset of $\{\,\alpha < \omega_2 : \mathrm{cof}\,(\alpha) = \omega\,\}$ for the construction of $\mathcal{L}_{\infty\aleph_2}$-free Boolean algebra in Theorem 4.1 and this is the only known construction of non openly generated $\mathcal{L}_{\infty\aleph_2}$-free Boolean algebras, one may ask if the assertion: "every $\mathcal{L}_{\infty\aleph_2}$-free Boolean algebra is openly generated" is consistent with ZFC. Ideed it is proved in [8] that this assertion follows from Martin's Maximum.

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