Remarks on the coloring number of graphs †

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Abstract

We give two characterizations of graphs with coloring number $\leq \kappa$ in terms of elementary submodels; one under ZFC and another under SSH and the version of very weak square principle of [8].

These characterizations suggest that the graphs with coloring number $\leq \kappa$ behave very much like the Boolean algebras with κ-Freese-Nation property (see [5], [8]).

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† This is an extended version of the note with the same title. Some details and proofs omitted in the version appeared in RIMS Kôkyûroku No.1754 (2011) as well as a new result (see Theorem A2.3) are added in typewriter font. The most up-to-date version of this note is downloadable as:

`http://fuchino.ddo.jp/papers/RIMS10-graph-square-x.pdf`

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1 Introduction

A graph $G = \langle G, K \rangle$ ($K \subseteq [G]^2$) has *coloring number* $\leq \kappa$ (notation: $col(G) \leq \kappa$) if there is a well ordering $\Subset$ on G such that $K^a_\Subset = \{b \in G : b \Subset a$ and $\{a, b\} \in K\}$ has cardinality $< \kappa$ for all $a \in G$ ([3]). The *coloring number* $col(G)$ of G is then defined as the minimum of such κ's. It is easy to see that the chromatic number $\chi(G)$ of G is less or equal to $col(G)$.

The purpose of this note is to show that the graphs with coloring number $\leq \kappa$ behave quite similarly to the Boolean algebras with κ-Freese-Nation property (see e.g. [5], [8]).

In Section 2 we give a characterization of graphs with coloring number $\leq \kappa$ in terms of elementary submodels (Theorem 2.4). As an application of the characterization, we present in Section 3 a short proof of the countability of the coloring number of the plane.

In Section 4, we show that the characterization of Section 2 can be yet sharpened under SSH and the version of the very weak square principle introduced in [8] (Theorem 4.2).

Both Theorems 2.4 and 4.2 find their parallels in the theory of Boolean algebras with κ-Freese-Nation property (see PROPOSITION 3 and THEOREM 10 in [8]).

The following theorem also underlines the analogy between the Boolean algebras with the κ-Freese-Nation property and the graphs with coloring number $\leq \kappa$ in the case of $\kappa = \aleph_0$. Note that Boolean algebras with $\aleph_0$-Freese Nation property are also called *openly generated*.

If $G = \langle G, K \rangle$ is graph then we identify any subset H of G with the graph $G \restriction H = \langle H, K \cap [H]^2 \rangle$.

Theorem 1.1 ([6] and [7]). *The following assertions are equivalent over* ZFC*:*

(α) *For any Boolean algebra B if there are club many subalgebras of B of cardinality $\aleph_1$ which are openly generated then B is openly generated.*

(β) *For any graph G if $col(H) \leq \aleph_0$ for every $H \in [G]^{\aleph_1}$ then $col(G) \leq \aleph_0$.*

❑

Theorem 1.1 in this formulation is a sort of bluff since we actually proved that each of the assertions (α) and (β) is equivalent to the set-theoretic principle FRP introduced in [4].

2 A characterization of graphs with coloring number $\leq \kappa$

We use here the following notations. The first one was already used in the introduction:

For a linear ordering $\Subset$ on a graph $G = \langle G, K \rangle$ we denote

(2.1) $\quad K^a_{\Subset} = \{b \in G \,:\, b \Subset a \text{ and } \{a, b\} \in K\}.$

If $H \subseteq G$ then we write

(2.2) $\quad K^{H,a}_{\Subset} = \{b \in H \,:\, b \Subset a \text{ and } \{a, b\} \in K\}$

For a graph $G = \langle G, K \rangle$, $H \subseteq G$ and $a \in G$, let

(2.3) $\quad K^a_H = \{b \in H \,:\, \{a, b\} \in K\}.$

We write $H \sqsubseteq_\kappa G$ if $|\, K^a_H \,| < \kappa$ for all $a \in G \setminus H$.

A mapping $f : G \to [G]^{<\kappa}$ is a *κ-coloring mapping on G* if for any $a, b \in G$ with $\{a, b\} \in K$, at least one of $a \in f(b)$ or $b \in f(a)$ holds.

Lemma 2.1 (Erdős and Hajnal [3])**.** *For any graph G and any infinite cardinal κ, the following are equivalent:*

(a) *$col(G) \leq \kappa$.*

(b) *There is a κ-coloring mapping on G.*

Proof. (a) $\Rightarrow$ (b): Suppose that $col(G) \leq \kappa$ and let $\Subset$ be a well-ordering on G such that $|\, K^a_{\Subset} \,| < \kappa$ for all $a \in G$. Then $f : G \to [G]^{<\kappa}$ defined by $f(a) = K^a_{\Subset}$ for $a \in G$ is a κ-coloring mapping.

(b) $\Rightarrow$ (a): Suppose that $f : G \to [G]^{<\kappa}$ is a κ-coloring mapping on G. Let $\Subset$ be a well-ordering on G such that all initial segments of G of order-type of the form $\kappa \cdot \alpha$ with respect to $\Subset$ are closed with respect to f. Then $\Subset$ is as desired:

Claim 2.1.1. $|\, K^a_{\Subset} \,| < \kappa$ *for all* $a \in G$.

$\vdash$ Suppose that $a \in G$ is the $\kappa \cdot \alpha + \beta$'th element with respect to $\Subset$ where $\beta < \kappa$. Then the first $\kappa \cdot \alpha$ elements of G are closed with respect to f and hence if b is among them and $\{a, b\} \in K$ then we have $b \in f(a)$. Thus

$$K^a_{\Subset} \subseteq \{b \in G \,:\, b \text{ is the } \gamma\text{'th element for some } \kappa \cdot \alpha \leq \gamma < \kappa \cdot \alpha + \beta\} \\ \cup f(a).$$

The right side of the inclusion has size $< \kappa$ (note that we need here the infinity of κ). Hence $|\, K^a_{\Subset} \,| < \kappa$. ⊣ (Claim 2.1.1)

❑ (Lemma 2.1)

Lemma 2.2. *Suppose that $\langle G_\alpha : \alpha < \delta \rangle$ is a filtration of a graph $G = \langle G, K \rangle$ and κ is an infinite cardinal. If $G_\alpha \sqsubseteq_\kappa G$ and $col(G_{\alpha+1}) \leq \kappa$ for all $\alpha < \delta$, then we have $col(G) \leq \kappa$.*

Proof. For $a \in G$ let $o(a) = \min\{\alpha < \delta : a \in G_{\alpha+1}\}$. For $\alpha < \delta$, let $\Subset_{\alpha+1}$ be a well-ordering of $G_{\alpha+1}$ witnessing $col(G_{\alpha+1}) \leq \kappa$. Let $\Subset$ be the ordering on G defined by:

$$(2.4) \quad a \Subset b \quad \Leftrightarrow \quad o(a) < o(b) \text{ or } \Big(o(a) = o(b) \text{ and } a \Subset_{o(a)+1} b \Big).$$

Then $\Subset$ is a well ordering on G. The following claim shows that $\Subset$ witnesses that G has coloring number $< \kappa$.

Claim 2.2.1. $|\, K^a_{\Subset} \,| < \kappa$ *for all $a \in G$.*

⊢ For $a \in G$, we have $K^a_{\Subset} \subseteq K^a_{G_{o(a)}} \cup K^{G_{o(a)+1},\, a}_{\Subset_{o(a)+1}}$. Since the right side of the inclusion is of cardinality $< \kappa$, it follows that $|\, K^a_{\Subset} \,| < \kappa$. ⊣ (Claim 2.2.1)

❑ (Lemma 2.2)

Lemma 2.3. *Suppose that H_0 and H_1 are subsets of G with $H_0 \sqsubseteq_\kappa G$ and $H_1 \sqsubseteq_\kappa G$. Then we have $H_0 \cap H_1 \sqsubseteq_\kappa G$.*

Proof. Suppose that $a \in G \setminus (H_0 \cap H_1)$. Then we have $a \in G \setminus H_0$ or $a \in G \setminus H_1$. If $a \in G \setminus H_0$, then $K^a_{H_0 \cap H_1} \subseteq K^a_{H_0}$. And hence $|\, K^a_{H_0 \cap H_1} \,| < \kappa$. If $a \in G \setminus H_1$, then $K^a_{H_0 \cap H_1} \subseteq K^a_{H_1}$. And hence again we have $|\, K^a_{H_0 \cap H_1} \,| < \kappa$.

This shows $H_0 \cap H_1 \sqsubseteq_\kappa G$. ❑ (Lemma 2.3)

Theorem 2.4. *For any graph $G = \langle G, K \rangle$ and an infinite cardinal κ, the following are equivalent:*

(a) *$col(G) \leq \kappa$.*

(a') *There is a well-ordering $\Subset$ of G of order-type $|\, G \,|$ such that $|\, K^a_{\Subset} \,| < \kappa$ for all $a \in G$.*

(b) *G has a κ-coloring mapping.*

(c) *For a/all sufficiently large regular χ and for all $M \prec \mathcal{H}(\chi)$ such that $\langle G, K \rangle \in M$ and $\kappa + 1 \subseteq M$ we have $G \cap M \sqsubseteq_\kappa G$.*

Proof. (a) $\Rightarrow$ (b) was already proved in Lemma 2.1. (a') $\Rightarrow$ (a) is trivial. The proof of (b) $\Rightarrow$ (a) in Lemma 2.1 actually proves (b) $\Rightarrow$ (a').

For (a) $\Rightarrow$ (c), suppose that $G = \langle G, K\rangle$ has coloring number $\leq \kappa$. Let χ be a sufficiently large regular cardinal and $M \prec \mathcal{H}(\chi)$ be such that $G \in M$ and $\kappa + 1 \subseteq M$. By elementarity and (a) $\Leftrightarrow$ (b), there is $f \in M$ such that f is a κ-coloring mapping on G. Note that by $\kappa+1 \subseteq M$ and by elementarity, $G \cap M$ is closed with respect to f. For $a \in G \setminus M$ and $b \in K^a_{G\cap M}$, since $a \notin f(b) \subseteq M$, we have $b \in f(a)$. Thus $K^a_{G\cap M} \subseteq f(a)$ and hence $|\,K^a_{G\cap M}\,| < \kappa$. This shows that $G \cap M \sqsubseteq_\kappa G$.

Now we prove (c) $\Rightarrow$ (a) by induction on $|\,G\,|$.

If $|\,G\,| \leq \kappa$, then (c) $\Rightarrow$ (a) holds since G then has coloring number $\leq \kappa$ anyway — any well-ordering of G of order-type $|\,G\,|$ will witness this.

Suppose that $|\,G\,| > \kappa$ and we have shown the implication (c) $\Rightarrow$ (a) for all graphs of cardinality $< |\,G\,|$. Let $\lambda = |\,G\,|$, $\lambda^* = \mathrm{cf}(\lambda)$ and $\langle M_\alpha : \alpha < \lambda^*\rangle$ a continuously increasing chain of elementary submodels of $\mathcal{H}(\chi)$ such that

(2.5) $G \in M_0$; $\kappa + 1 \subseteq M_0$;

(2.6) $|\,M_\alpha\,| < \lambda$ for all $\alpha < \lambda^*$; and

(2.7) $G \subseteq \bigcup_{\alpha<\lambda^*} M_\alpha$.

For $\alpha < \lambda^*$, let $G_\alpha = G \cap M_\alpha$. Then $\langle G_\alpha : \alpha < \lambda^*\rangle$ is a filtration of G by (2.6) and (2.7). $G_\alpha \sqsubseteq_\kappa G$ for all $\alpha < \kappa$ by (2.5) and by the assumption of (c). By Lemma 2.3, G_α also satisfies (c) for $\alpha < \lambda^*$. Since $|\,G_\alpha\,| < \lambda$, it follows that *col*$(G_\alpha) \leq \kappa$ for all $\alpha < \lambda^*$ by the induction hypothesis. Hence we have *col*$(G) \leq \kappa$ by Lemma 2.2. ❑ (Theorem 2.4)

Lemma A.2.1. *Suppose that $\mathcal{C} \subseteq [\lambda]^\kappa$ is a club in $[\lambda]^\kappa$. If $A \in [\mathcal{C}]^{\leq\kappa}$ is upward directed then $\bigcup A \in \mathcal{C}$.*

Proof. We prove this by induction on the size of $|\,A\,|$.

If A is finite then the assertion is trivial since $\bigcup A = a$ for the maximal element a of A.

Suppose now that $\mu = |\,A\,| \leq \kappa$ and we have the claim of the Lemma for all upward directed subsets of $\mathcal{C}$ of cardinality $< \mu$.

Let $\langle A_\alpha : \alpha < \mathrm{cf}(\mu)\rangle$ be a filtration of A such that each A_α for $\alpha < \mathrm{cf}(\mu)$ is upward directed. Then, by the induction hypothesis, $a_\alpha = \bigcup A_\alpha \in \mathcal{C}$ for $\alpha < \mathrm{cf}(\mu)$. $\langle a_\alpha : \alpha < \mathrm{cf}(\mu)\rangle$ is an increasing sequence with respect to $\subseteq$. Since $\mathcal{C}$ is a club, it follows that $\bigcup A = \bigcup_{\alpha<\mathrm{cf}(\mu)} a_\alpha \in \mathcal{C}$. ❑ (Lemma A.2.1)

Lemma A. 2.2. *Suppose that $\mathcal{C} \subseteq [\lambda]^\kappa$ is a club and let χ be a sufficiently large regular cardinal. Then, for any $M \prec \mathcal{H}(\lambda)$ with $|M| = \kappa$, $\mathcal{C} \in M$ and $\kappa + 1 \subseteq M$, we have $\lambda \cap M \in \mathcal{C}$.*

Proof. By $\kappa + 1 \subseteq M$, we have $a \subseteq M$ for all $a \in \mathcal{C} \cap M$. It follows that $\bigcup(\mathcal{C} \cap M) = \lambda \cap M$. $\mathcal{C} \cap M$ is upward directed. Thus $\lambda \cap M = \bigcup(\mathcal{C} \cap M) \in \mathcal{C}$ by Lemma A2.1. ❑ (Lemma A.2.2)

I learned the following characterization of coloring number being less than or equal to κ from Hiroshi Sakai. The characterization will be used extensively in a fortcomming paper of mine jointly with Hiroshi Sakai and André Ottenbereit Maschio Rodrigues.

Theorem A.2.3. *For a graph $G = \langle G, K \rangle$ and a regular cardinal κ the following are equivalent:*

(a) *$col(G) \leq \kappa$;*

(b) *$\{H \in [G]^\kappa : H \sqsubseteq_\kappa G\}$ contains a club set $\subseteq [G]^\kappa$.*

Proof. (a) $\Rightarrow$ (b) follows from Theorem 2.4, (a) $\Leftrightarrow$ (c).

For (b) $\Rightarrow$ (a), assume that (b) holds for G. Let χ be sufficiently large and let $M \prec \mathcal{H}(\chi)$ be such that $G \in M$ and $\kappa + 1 \subseteq M$. Again by Theorem 2.4, (a) $\Leftrightarrow$ (c), it is enough to show that $G \cap M \sqsubseteq_\kappa G$ holds.

Suppose for contradiction that this were not the case. Then there is a $g \in G \setminus (G \cap M)$ such that $|G^g_{G\cap M}| \geq \kappa$. Let $S \in [G^g_{G\cap M}]^\kappa$ and let $N \prec M$ be such that $|N| = \kappa$, S, $\kappa + 1 \subseteq N$. By elementarity there is a club $\mathcal{C} \subseteq \{H \in [G]^\kappa : H \sqsubseteq_\kappa G\}$ with $\mathcal{C} \in N$. By Lemma A2.2, it follows that $G \cap N \in \mathcal{C} \subseteq \{H \in [G]^\kappa : H \sqsubseteq_\kappa G\}$. Thus

$G \cap N \sqsubseteq_\kappa G$. This is a contradiction to $S \subseteq G \cap N$.

❑ (Theorem A.2.3)

3 Coloring number of the plane

The *plane*, or *the unit distance graph of the plane*, is the graph $G^1(\mathbb{R}^2)$ defined by $G^1(\mathbb{R}^2) = \langle \mathbb{R}^2, K^1_{\mathbb{R}^2} \rangle$ where $K^1 = \{\{x, y\} \in [\mathbb{R}^2]^2 : d(x, y) = 1\}$. Applying Theorem 2.4, we can show easily that the coloring number of the plane is equal to $\aleph_0$.

Theorem 3.1. *$col(G^1(\mathbb{R}^2)) = \aleph_0$.*

Proof. In [2] it is noted that the list-chromatic number $list(G^1(\mathbb{R}^2))$ of $G^1(\mathbb{R}^2)$ is infinite since finite regular graph of arbitrarily large degree d can be embedded in $G^1(\mathbb{R}^2)$ (e.g., throwing down of n-dimensional cube onto the plane) and the list-chromatic number of such finite graph is d (see [1]). Thus we have $\aleph_0 \leq list(G^1(\mathbb{R}^2)) \leq col(G^1(\mathbb{R}^2))$.

To prove the inequality $col(G^1(\mathbb{R}^2)) \leq \aleph_0$, let χ be sufficiently large and $N \prec \mathcal{H}(\chi)$. Note that we have $G^1(\mathbb{R}^2) \in N$ since the plane is definable. Suppose $x \in \mathbb{R}^2 \setminus N$. Let us write simply K for $K^1_{\mathbb{R}^2}$ By Theorem 2.4, it is enough to show that $K^x_{\mathbb{R}^2\cap N}$ is finite. Actually, we can show that $|\,K^x_{\mathbb{R}^2\cap N}\,| \leq 1$:

Toward a contradiction, suppose that $|\,K^x_{\mathbb{R}^2\cap N}\,| > 1$. Then there are two distinct y, $z \in G \cap N$ such that $d(x,y) = d(x,z) = 1$. But then $X = \{u \in \mathbb{R}^2 : d(u,y) = d(u,z) = 1\}$ is a two element set definable with parameters from N. It follows that $x \in X \subseteq N$. This is a contradiction to the choice of x.

❑ (Theorem 3.1)

With the same proof we can also show:

$$col(G^{\mathrm{Odd}}(\mathbb{R}^2)) = col(G^{\mathbb{N}}(\mathbb{R}^2)) = col(G^{\mathbb{Q}}(\mathbb{R}^2)) = col(G^{\mathrm{algebraic}}(\mathbb{R}^2)) = \cdots = \aleph_0.$$

Theorem 3.1 may be already known. However I could not find any direct mention or proof of the theorem in the literature. Also, in [2] the authors prove $list(G^{\mathrm{Odd}}(\mathbb{R}^2)) \leq \aleph_0$ directly and it seems that idea of the proof cannot be extended to a proof of $col(G^{\mathrm{Odd}}(\mathbb{R}^2)) \leq \aleph_0$.

I first learned a proof of $col(G^1(\mathbb{R}^2)) \leq \aleph_0$ from Hiroshi Sakai in November 2009 who proved the inequality straightforwardly.

Theorem 2.4 is often quite useful to decide the coloring number of infinite graphs. For example, $col(K(\kappa,\kappa)) = \kappa$ and $col(K(\kappa,\lambda)) = \kappa^+$ for any $\aleph_0 \leq \kappa < \lambda$; $col(G^{\mathrm{Odd}}(\mathbb{R}^3)) = \aleph_1$ etc. can be seen immediately by this theorem.

We shall demonstrate the last equality. Recall $G^{\mathrm{Odd}}(\mathbb{R}^3) = \langle \mathbb{R}^3, K^{\mathrm{Odd}}_{\mathbb{R}^3}\rangle$ where $K^{\mathrm{Odd}}_{\mathbb{R}^3} = \{\langle \vec{x},\vec{y}\rangle \in [\mathbb{R}^3]^2 : d(\vec{x},\vec{y})$ is an odd (natural) number$\}$.

Theorem A.3.1. $col(G^{\mathrm{Odd}}(\mathbb{R}^3)) = \aleph_1$.

Proof. For notational simplicity, let $G = G^{\mathrm{Odd}}(\mathbb{R}^3) = \langle G, K\rangle$ with $G = \mathbb{R}^3$ and $K = K^{\mathrm{Odd}}_{\mathbb{R}^3}$. Suppose that χ is sufficiently large. By Theorem 2.4, it is enough to show that $G \cap M \sqsubseteq_{\aleph_1} G$ for all $M \prec \mathcal{H}(\chi)$ but $G \cap M \not\sqsubseteq_{\aleph_0} G$ for some $M \prec \mathcal{H}(\chi)$.

Suppose that $M \prec \mathcal{H}(\chi)$. If $\mathbb{R} \subseteq M$ then $G \subseteq M$ and we have $G \cap M \sqsubseteq_{\aleph_i} G$ vacuously.

Otherwise, letting $C = \{\langle x, y, 0\rangle \in \mathbb{R}^3 : d(\langle x, y, 0\rangle, \vec{0}) = 1\}$, we have $C \not\subseteq M$. Let $\vec{x} \in C \setminus M$. Then, for any odd $n \in \omega$, $\sqrt{n^2 - 1} \in M$ and $d(\vec{x}, \langle 0, 0, \sqrt{n^2-1}\rangle) = n$. Thus $\langle 0, 0, \sqrt{n^2-1}\rangle \in K^{\vec{x}}_{G\cap M}$. This shows that $G \cap M \not\sqsubseteq_{\aleph_0} G$.

To show $G \cap M \sqsubseteq_{\aleph_1} G$, assume for contradiction that there is $\vec{x} \in G \setminus M$ such that $K^{\vec{x}}_{G\cap M}$ is uncountable. Then there is an odd $n \in \omega$ such that $X = \{\vec{y} \in G \cap M : d(\vec{x}, \vec{y}) = n\}$ is uncountable. Let y_0, y_1, y_3 be three distinct elements of X. $Y = \{\vec{z} \in G : d(\vec{z}, \vec{y_0}) = d(\vec{z}, \vec{y_1}) = d(\vec{z}, \vec{y_2}) = n\}$ is a two-elements set definable with parameters form M. It follows that $\vec{x} \in Y \subseteq M$. This is a contradiction to the choice of $\vec{x}$. ❑ (Theorem A.3.1)

4 Coloring number under very weak square

The following version of the very weak square was introduced in [8].

For a regular cardinal κ and $\mu > \kappa$, let $\square^{***}_{\kappa,\mu}$ be the following assertion: there exists a sequence $\langle C_\alpha : \alpha < \mu^+\rangle$ and a club set $D \subseteq \mu^+$ such that, for all $\alpha \in D$ with $\mathrm{cf}(\alpha) \geq \kappa$, we have

(4.1) $C_\alpha \subseteq \alpha$, C_α is unbounded in α; and

(4.2) $[\alpha]^{<\kappa} \cap \{C_{\alpha'} : \alpha' < \alpha\}$ dominates $[C_\alpha]^{<\kappa}$ (with respect to $\subseteq$).

For a (sufficiently large regular) cardinal χ and $M \prec \mathcal{H}(\chi)$, M is *κ-internally cofinal* if $[M]^{<\kappa} \cap M$ is cofinal in $[M]^{<\kappa}$ with respect to $\subseteq$. For $\mathcal{D} \subseteq [\mathcal{H}(\chi)]^{<\kappa}$, M is *$\mathcal{D}$-internally cofinal* if $\mathcal{D} \cap M$ is cofinal in $[M]^{<\kappa}$ with respect to $\subseteq$.

Suppose now that κ is a regular cardinal and $\mu > \kappa$ is such that $\mathrm{cf}(\mu) < \kappa$. Let $\mu^* = \mathrm{cf}(\mu)$. For a sufficiently large χ and $x \in \mathcal{H}(\chi)$, let us call a sequence $\langle M_{\alpha,\beta} : \alpha < \mu^+, \beta < \mu^*\rangle$ a (κ, μ)*-dominating matrix* (*of elementary submodels of* $\mathcal{H}(\chi)$) *over* x if the following conditions (4.3) – (4.6) hold:

(4.3) $M_{\alpha,\beta} \prec \mathcal{H}(\chi)$, $x \in M_{\alpha,\beta}$, $\kappa + 1 \subseteq M_{\alpha,\beta}$ and $|\, M_{\alpha,\beta}\,| < \mu$ for all $\alpha < \mu^+$ and $\beta < \mu^*$;

(4.4) $\langle M_{\alpha,\beta} : \beta < \mu^*\rangle$ is an increasing sequence for each fixed $\alpha < \mu^+$;

(4.5) if $\alpha < \mu^+$ is such that $\mathrm{cf}(\alpha) \geq \kappa$, then there is $\beta^* < \mu^*$ such that, for every $\beta^* \leq \beta < \mu^*$, $M_{\alpha,\beta}$ is κ-internally cofinal.

For $\alpha < \mu^+$, let $M_\alpha = \bigcup_{\beta<\mu^*} M_{\alpha,\beta}$. By (4.3) and (4.4), we have $M_\alpha \prec \mathcal{H}(\chi)$.

(4.6) $\langle M_\alpha : \alpha < \mu^+\rangle$ is continuously increasing and $\mu^+ \subseteq \bigcup_{\alpha<\mu^+} M_\alpha$.

Theorem 4.1 (Theorem 7 in [8])**.** *Suppose that κ is a regular cardinal and $\mu > \kappa$ is such that $\mathrm{cf}(\mu) < \kappa$. If we have $\mathrm{cf}([\lambda]^{<\kappa}, \subseteq) = \lambda$ for cofinally many $\lambda < \mu$ and $\square^{***}_{\kappa,\mu}$ holds, then, for any sufficiently large χ and $x \in \mathcal{H}(\chi)$, there is a (κ, μ)-dominating matrix over x.*

Theorem 4.2. *Assume* SSH *and $\square^{***}_{\kappa,\mu}$ for a regular uncountable κ and all singular cardinal μ with $\mathrm{cf}(\mu) < \kappa < \mu$.*

Then, for any graph $G = \langle G, K\rangle$ the following are equivalent:

(a) *$col(G) \le \kappa$.*

(d) *For a/all sufficiently large regular χ and κ-internally cofinal $M \prec \mathcal{H}(\chi)$ with $G \in M$ we have $G \cap M \sqsubseteq_\kappa G$.*

(e) *For a/all sufficiently large regular χ there is $\mathcal{D} \subseteq [\mathcal{H}(\chi)]^{<\kappa}$ such that $\mathcal{D}$ is cofinal in $[\mathcal{H}(\chi)]^{<\kappa}$ and, for any $\mathcal{D}$-internally cofinal $M \prec \mathcal{H}(\chi)$, we have $G \cap M \sqsubseteq_\kappa G$.*

Proof. (a) $\Rightarrow$ (d) follows from Theorem 2.4, (d) $\Rightarrow$ (e) is trivial (just put $\mathcal{D} = [\mathcal{H}(\chi)]^{<\kappa}$).

For (e) $\Rightarrow$ (a), we proceed with induction on $|G|$. If $|G| \le \kappa$ then the implication (e) $\Rightarrow$ (a) is trivial since $col(G) \le \kappa$ holds always for any graph of size $\le \kappa$. Suppose now that $|G| > \kappa$ and we have shown the implication (e) $\Rightarrow$ (a) for all graphs of cardinality $< |G|$.

Assume that G satisfies (e) with χ and $\mathcal{D}$. Let χ^* be sufficiently large above χ such that we have in particular $\mathcal{H}(\chi) \in \mathcal{H}(\chi^*)$.

Claim 4.2.1. *If M is a κ-internal cofinal elementary submodel of $\mathcal{H}(\chi^*)$ such that*

(4.7) *G, χ, $\mathcal{D} \in M$ and $\kappa + 1 \subseteq M$,*

then we have $G \cap M \sqsubseteq_\kappa G$.

$\vdash$ Suppose not. Then there is $a \in G \setminus M$ such that $|K^a_{G\cap M}| \ge \kappa$. Let $N = \mathcal{H}(\chi) \cap M$. By elementarity we have $N \prec \mathcal{H}(\chi)$. Let $\langle N_\alpha : \alpha < \kappa\rangle$ be an increasing sequence such that, for all $\alpha < \kappa$, we have

(4.8) $N_\alpha \in \mathcal{D} \cap M$;

(4.9) $N_\alpha \in N_{\alpha+1}$;

(4.10) there is $N^*_\alpha \in [N]^{<\kappa} \cap M$ such that $N^*_\alpha \prec N$ and $N_\alpha \subseteq N^*_\alpha \subseteq N_{\alpha+1}$; and

(4.11) $K^a_{G\cap M} \cap (N_{\alpha+1} \setminus N_\alpha) \neq \emptyset$.

The construction is possible by elementarity of M and since $\mathcal{D}$ is cofinal in $[\mathcal{H}(\chi)]^{<\kappa}$.

Let $N^* = \bigcup_{\alpha<\kappa} N_\alpha$. By (4.10) we have $N^* \prec N \prec \mathcal{H}(\chi)$. By (4.8) and (4.9) N^* is $\mathcal{D}$-internally cofinal. On the other hand, we have $|\, K^a_{G\cap N^*} \,| \geq \kappa$ by (4.11). This is a contradiction to the assumption of (e). ⊣ (Claim 4.2.1)

Claim 4.2.2. *If $H \sqsubseteq_\kappa G$ then for every $\mathcal{D}$-internally cofinal $M \prec \mathcal{H}(\chi)$ we have $H \cap M \sqsubseteq_\kappa H$. In particular, H also satisfies the condition* (e).

Proof. Suppose that $M \prec \mathcal{H}(\chi)$ is $\mathcal{D}$-internally approachable. For $a \in H \setminus (H \cap M)$, since $a \in G \setminus (G \cap M)$, we have $K^a_{H\cap M} \subseteq K^a_{G\cap M}$. The right side of the inclusion is of cardinality $< \kappa$ by the assumption of (e) on G. This shows that $H \cap M \sqsubseteq_\kappa H$. ⊣ (Claim 4.2.2)

Now we finish the induction step for the proof of (e) ⇒ (a) in two cases. Let $\nu = |\, G \,|$.

Case I. ν is a limit cardinal or $\nu = \delta^+$ with $\mathrm{cf}(\delta) \geq \kappa$.

Let $\nu^* = cf(\nu)$. Note that, in this case, we have that

(4.12) the cardinals $\lambda < \nu$ such that $cf([\lambda]^{<\kappa}) = \lambda$ are cofinal among cardinals below ν

by SSH.

Let $\langle M_\alpha : \alpha < \nu^* \rangle$ be an increasing sequence of elementary submodels of $\mathcal{H}(\chi^*)$ of cardinality $< \nu$ satisfying (4.7) and $G \subseteq \bigcup_{\alpha<\nu^*} M_\alpha$. We can find such a sequence by (4.12).

Let

$$G_\alpha = \begin{cases} G \cap M_\alpha & \text{if } \alpha = 0 \text{ or } \alpha \text{ is a successor ordinal;} \\ G \cap \left(\bigcup_{\beta<\alpha} M_\beta\right) & \text{otherwise} \end{cases}$$

for $\alpha < \nu^*$. Then $\langle G_\alpha : \alpha < \nu^* \rangle$ is a filtration of G.

Claim 4.2.3. *$G_\alpha \sqsubseteq_\kappa G$ for all $\alpha < \nu^*$.*

Proof. If $\alpha < \nu^*$ is 0 or a successor ordinal, this follows from Claim 4.2.1.

If $\alpha < \nu^*$ is a limit and $\mathrm{cf}(\alpha) < \kappa$, Then G_α is a union of less than κ many G_β's where $\beta < \alpha$ may be chosen to be a successor ordinal and hence $G_\beta \sqsubseteq_\kappa G$. It follows that we have $G_\alpha \sqsubseteq_\kappa G$ also in this case.

If $\mathrm{cf}(\alpha) \geq \kappa$, then $\bigcup_{\beta<\alpha} M_\beta$ is κ-internally cofinal and hence we have $G_\beta \sqsubseteq_\kappa G$ again by Claim 4.2.1. $\dashv$ (Claim 4.2.3)

Now by Claim 4.2.2 and by the induction hypothesis, all of G_α, $\alpha < \nu^*$ are of coloring number $\leq \kappa$. By Lemma 2.2, it follows that G also has coloring number $\leq \kappa$.

Case II. $\nu = \mu^+$ with $\mathrm{cf}(\mu) < \kappa$. Let $\mu^* = \mathrm{cf}(\mu)$.

By Theorem 4.1, there is a (κ, μ)-dominating matrix $\langle M_{\alpha,\beta} : \alpha < \nu, \beta < \mu^* \rangle$ of submodels of $\mathcal{H}(\chi^*)$ over $x = \langle G, \mathcal{H}(\chi) \rangle$.

For $\alpha < \nu$ and $\beta < \mu^*$, let $G_{\alpha,\beta} = G \cap M_{\alpha,\beta}$ and $G_\alpha = \bigcup_{\beta<\mu^*} G_{\alpha,\beta} = G \cap \left(\bigcup_{\beta<\mu^*} M_{\alpha,\beta}\right)$. By (4.6), the sequence $\langle G_\alpha : \alpha < \nu \rangle$ is continuously increasing and $\bigcup_{\alpha<\nu} G_\alpha = G$. By (4.3), we have $|\, G_\alpha \,| \leq \mu < \nu$. Thus $\langle G_\alpha : \alpha < \nu \rangle$ is a filtration of G.

Let

$$(4.13) \quad C = \{\alpha < \nu : \mathrm{cf}(\alpha) \geq \kappa \text{ or } \{\alpha' < \alpha : \mathrm{cf}(\alpha') \geq \kappa\} \text{ is cofinal in } \alpha\}.$$

C is a club subset of ν.

Claim 4.2.4. *$G_\alpha \sqsubseteq_\kappa G$ for all $\alpha \in C$.*

$\vdash$ Suppose $\alpha \in C$. If $\mathrm{cf}(\alpha) \geq \kappa$, $M_{\alpha,\beta}$ is κ-internally cofinal for all sufficiently large $\beta < \mu^*$ by (4.5). Hence by Claim 4.2.1, we have $G_{\alpha,\beta} \sqsubseteq_\kappa G$ for all such β. Since $\mu^* < \kappa$, it follows that $G_\alpha \sqsubseteq_\kappa G$.

If $\mathrm{cf}(\alpha) < \kappa$, then let $X \subseteq \alpha$ be a cofinal subset of α with $|\, X \,| < \kappa$ such that all $\alpha' \in X$ have cofinality $\geq \kappa$. Since $G_\alpha = \bigcup_{\alpha' \in X} G_{\alpha'}$ and $G_{\alpha'} \sqsubseteq_\kappa G$ for all $\alpha' \in X$ by the first part of the proof, it follows that $G_\alpha \sqsubseteq_\kappa G$. $\dashv$ (Claim 4.2.4)

By Claim 4.2.2 and by the induction hypothesis, we have $col(G_\alpha) \leq \kappa$ for all $\alpha \in C$. Hence by Lemma 2.2 we can conclude that $col(G) \leq \kappa$.

❑ (Theorem 4.2)

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