Resurrection and Maximality in light of Laver-generic large cardinal

渕野 昌

Sakaé Fuchino*

Abstract

Set-theoretic axioms formulated in terms of existence of a Laver-generic large cardinal were introduced in [17] and studied further in [18], [19], [21]. These axioms, let us call them Laver-genericity axioms, claim the existence of a $\mathcal{P}$-Laver generic large cardinal for various classes $\mathcal{P}$ of proper or semi-proper posets, and they still vary depending on the notions of large cardinal involved, and a modification (tightness) of the definition of Laver-genericity.

Laver-genericity axioms we consider here are divided into three groups depending on whether they imply that the Laver generic large cardinal κ is $\aleph_2 = (2^{\aleph_0})^+$, or it is $\aleph_2 = 2^{\aleph_0}$, or else it is very large and $= 2^{\aleph_0}$ (see the Trichotomy Theorem (Theorem 3.5)).

Many set-theoretic axioms and principles considered in the recent development of set theory follow from a Laver-genericity axiom in one of these three groups, and by this, all of them can be placed in a global picture (see Figure 3).

In spite of this very strong unifying feature of the Laver genericity axioms, we show that Maximality Principle (MP) without parameters is independent over ZFC with any of the Laver-genericity axioms we consider in our present context (Theorem 4.8, Theorem 5.11). Similar independence is also shown for parameterized versions of Maximality Principles (Theorem 6.1, Theorem 6.5).

In contrast to these independence results, we can show that local versions of Maximality Principle as well as versions of Resurrection Axioms including

* Graduate School of System Informatics, Kobe University
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan
fuchino@diamond.kobe-u.ac.jp

the Unbounded Resurrection Axioms of Tsaprounis follow from the existence of a tightly Laver-generic large cardinal for a strong enough notion of large cardinal (Theorem 6.6, Theorem 7.1, Theorem 7.2).

Contents

1 Introduction

Set-theoretic axioms formulated in terms of existence of a Laver-generic large cardinal (see Section 3 for definition) were introduced in [17] and studied further in [18], [19], [21]. intro

More precisely, these axioms — let us call them here *Laver-genericity axioms* — claim the existence of a $\mathcal{P}$-Laver generic large cardinal for various classes $\mathcal{P}$ of proper or semi-proper posets, and they still vary depending on the notions of large cardinal involved, and a modification (tightness) of Laver-genericity.

Date: May 8, 2023 *Last update:* May 19, 2025 (21:04 JST)

MSC2020 Mathematical Subject Classification: 03E35, 03E50, 03E55, 03E37

Keywords: generic large cardinal, Laver-generic large cardinal, Maximality Principles

The research is supported by Kakenhi Grant-in-Aid for Scientific Research (C) 20K03717. The author would like to thank Kaethe Minden for making him aware of [30]. He also would like to thank Gunter Fuchs, Takehiko Gappo, Paul Larson, Hiroshi Sakai, and Kostas Tsaprounis for many helpful remarks and comments.

This is a note for the paper in preparation with the the same title [14]. An earlier version of a part of this note appeared as [11].

The most up-to-date pdf-file of this note is downloadable as:
https://fuchino.ddo.jp/papers/RIMS2022-RA-MP-x.pdf

We restrict ourselves here to classes $\mathcal{P}$ of posets which are proper or semi-proper since we want to have axioms which imply (or at least compatible with) various reflection principles, see Section 2 and Section 3.

Laver-genericity axioms are divided into three groups depending on whether they imply that the Laver generic large cardinal κ whose existence is claimed by the axioms is $\aleph_2 = (2^{\aleph_0})^+$, or it is $\aleph_2 = 2^{\aleph_0}$, or else it is very large and $= 2^{\aleph_0}$ (see the Trichotomy Theorem (Theorem 3.5)).

By this trichotomy, the Laver-generic large cardinal we consider here is proved to be unique under the respective Laver-genericity axiom.

Many set-theoretic axioms and principles considered in the recent development of set theory follow from a Laver-genericity axiom in one of these three groups, and by this, they are placed uniformly in a global context (see Figure 3).

In sections 2, 3 of the present note, we give an improved and streamlined presentation of the Laver-genericity axioms. Most of the materials presented in these sections are already stated in [16] or [17] but there are also a couple of improvements and new results. The extended version of the paper you are reading now also contains detailed proofs of the results mentioned in these sections.

In spite of the very strong unifying feature of the Laver genericity axioms, we can show that Maximality Principle (MP) without parameters is independent over ZFC with any of the Laver-genericity axioms we consider in our present context (Theorem 4.8, Theorem 5.11). Similar independence is also shown for parameterized versions of Maximality Principles (Theorem 6.1, Theorem 6.5).

In contrast to these independence results, the local versions of Maximality Principle as well as versions of Resurrection Axioms including the Unbounded Resurrection Axioms of Tsaprounis are consequences of the Laver-genericity axioms for a strong enough notion of large cardinal (Theorem 6.6, Theorem 7.1, Theorem 7.2).

In the following we are working in the framework of ZFC. All classes are definable by some $\mathcal{L}_\in$-formula where $\mathcal{L}_\in$ is the language of ZFC consisting solely of the $\in$-symbol. Sometimes the language is extended with a constant symbol or a unary relation symbol. This is in particular the case when we are talking about "$V_\delta \prec \mathsf{V}$" or that "there are stationarily many δ with certain large cardinal property and such that $V_\delta \prec \mathsf{V}$". Even in such cases ZFC is meant the axiom system in the original language $\mathcal{L}_\in$.

Regardless of this convention, we sometimes choose a narrative that may sound we are working in some higher order set theory. This happens in particular when we are talking about the notions of Laver-generic large cardinal: we may do this since it is proved in [21] that the notions of Laver generic cardinals are actually characterizable by a property which is formalizable the language of ZFC.

2 Generic large cardinals

Let us begin with recalling the definition of supercompact cardinal: A cardinal κ is *supercompact* if, for any $\lambda > \kappa$, there are classes j, M such that ① $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$, ② $j(\kappa) > \lambda$ and ③ ${}^{\lambda}M \subseteq M$. gen

Here, "$j : N \xrightarrow{\prec}_{\kappa} M$" denotes the set of conditions that N and M are transitive (sets or classes); j is a non-trivial elementary embedding of the structure $(N, \in)$ into the structure $(M, \in)$; $\kappa \in N$, and $crit(j) = \kappa$.

Note that a supercompact cardinal is a large large cardinal which is a normal measure one limit of measurable cardinals (see e.g. [28] Proposition 22.1), and more. This is not the case with the generic large cardinal version of the notion of supercompactness (e.g. see Examples 2.2, 2.3 below).

For a class $\mathcal{P}$ of posets, a cardinal κ is *$\mathcal{P}$-generically supercompact* (*$\mathcal{P}$-gen. supercompact*, for short) if, for every $\lambda > \kappa$, there is $\mathbb{P} \in \mathcal{P}$ such that, for $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$, there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that ① $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$, ② $j(\kappa) > \lambda$, and ③' $j''\lambda \in M$.

In case of genuine supercompactness, the condition $j''\lambda \in M$ is equivalent to ${}^{\lambda}M \subseteq M$ for M obtained as the ultrapower of V by an ω_1-complete ultrafilter (see Kanamori [28], Proposition 22.4). In general we do not have this equivalence for generic supercompactness. However this condition still implies certain closedness of M:

Lemma 2.1 (Lemma 2.5 in [17]) *Suppose that $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter for a poset $\mathbb{P} \in \mathsf{V}$, and $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M \subseteq \mathsf{V}[\mathbb{G}]$ for a cardinal κ is such that, for a cardinal in V λ with $\kappa \leq \lambda$, we have $j''\lambda \in M$.* L-lt-conti-0

(1) *For any set $A \in \mathsf{V}$ with $\mathsf{V} \models |A| \leq \lambda$, we have $j''A \in M$.*

(2) *$j \restriction \lambda$, $j \restriction \lambda^2 \in M$.*

(3) *For any $A \in \mathsf{V}$ with $A \subseteq \lambda$ or $A \subseteq \lambda^2$ we have $A \in M$.*

(4) *$(\lambda^+)^M \geq (\lambda^+)^{\mathsf{V}}$, Thus, if $(\lambda^+)^{\mathsf{V}} = (\lambda^+)^{\mathsf{V}[\mathbb{G}]}$, then $(\lambda^+)^M = (\lambda^+)^{\mathsf{V}}$.*

(5) *$\mathcal{H}(\lambda^+)^{\mathsf{V}} \subseteq M$.*

(6) *$j \restriction A \in M$ for all $A \in \mathcal{H}(\lambda^+)^{\mathsf{V}}$.*

Proof. (1): In V, let $f : \lambda \to A$ be a surjection.

For each $a \in A$ with $a = f(\alpha)$, we have

$$j(a) = j(f(\alpha)) = j(f)(j(\alpha)) \tag{2.1}$$ pr-L-lt-conti-0

by elementarity. Thus $j''A = j(f)''(j''\lambda)$. Since $j(f)$, $j''\lambda \in M$, it follows that $j''A \in M$.

(2): Since $j''\lambda \in M$ and $(j \restriction \lambda)(\xi)$ for $\xi \in \lambda$ is the ξth element of $j''\lambda$, $j \restriction \lambda$ is definable subset of $\lambda \times j''\lambda$ in M and hence is an element of M. Similarly, $j \restriction \lambda^2 \in M$.

(3): Suppose that $A \in \mathsf{V}$ and $A \subseteq \lambda$ (the case of $A \subseteq \lambda^2$ can be treated similarly). Then $j''A \in M$ by (1). Thus, by (2), $A = (j \restriction \lambda)^{-1}{}''(j''A) \in M$.

(4): Suppose that $\mu < (\lambda^+)^{\mathsf{V}}$. Then there is $A \in \mathsf{V}$ with $A \subseteq \lambda^2$ such that A codes the order type of μ. $A \in M$ by (3). Thus $M \models$ "$|\,\mu\,| \leq \lambda$".

If $(\lambda^+)^{\mathsf{V}} = (\lambda^+)^{\mathsf{V}[\mathbb{G}]}$, we have

$$(\lambda^+)^{\mathsf{V}} = (\lambda^+)^{\mathsf{V}[\mathbb{G}]} \geq (\lambda^+)^M \geq (\lambda^+)^{\mathsf{V}}. \tag{2.2}$$ pr-L-lt-conti-1

(5): For $A \in \mathcal{H}(\lambda^+)^{\mathsf{V}}$, let $U \in \mathsf{V}$ be such that $trcl(A) \subseteq U$ and $\mathsf{V} \models$ "$|\,U\,| = \lambda$". Let $c_A \subseteq \lambda^2$ and d_A, $e_A \subseteq \lambda$ be such that c_A, d_A, $e_A \in \mathsf{V}$ and

$$\langle \lambda, c_A, d_A, e_A \rangle \cong \langle U, \in\restriction U^2, trcl(A), A \rangle. \tag{2.3}$$ pr-L-lt-conti-2

By (3), c_A, d_A, $e_A \in M$ and hence $\langle \lambda, c_A, d_A, e_A \rangle \in M$. Since $trcl(A)$ and then A can be recovered from this quadruplet in M, it follows that $A \in M$.

(6): Suppose that $A \in \mathcal{H}(\lambda^+)^{\mathsf{V}}$. Since $A \in M$ by (5), it is enough to show that $j \restriction trcl(A) \in M$.

We have $trcl(A) \in \mathcal{H}(\lambda^+)^{\mathsf{V}}$ and hence $trcl(A) \in M$ by (5). Thus $j''\,trcl(A)$, $j''(\in\restriction trcl(A)) \in M$ by (1). But then the mapping $(j \restriction trcl(A))^{-1}$ is the Mostowski collapse of $j''\,trcl(A)$. Thus $j \restriction trcl(A) \in M$. ❑ (Lemma 2.1)

Example 2.2 *Suppose that κ is a supercompact cardinal and $\mathcal{P} = \mathrm{Col}(\aleph_1, \kappa)$ (the standard collapsing of all cardinals strictly between $\aleph_1$ and κ by countable conditions). Then for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, we have $\kappa = (\aleph_2)^{\mathsf{V}[\mathbb{G}]}$ and $\mathsf{V}[\mathbb{G}] \models$ "κ is σ-*closed-gen. supercompact*"*.* ❑ ex-gen-1

Example 2.3 *If* MA *is forced starting from an supercompact cardinal κ with an ccc-iteration of length κ in finite support along with a supercompact Laver function, then we obtain a model in which κ is the continuum (though still quite large, e.g. hyper-hyper etc. weakly Mahlo, and more) and it is ccc-gen. supercompact in the generic extension.* ❑ ex-gen-2

These examples will be revisited in Theorem 3.3 below. The situation created in Example 2.2 can be also seen as a strong reflection property.

Theorem 2.4 (B. König [29]) *The following are equivalent:* p-gen-0

(a) *Game Reflection Principle (*GRP*) holds.*

(b) $\aleph_2$ *is* σ*-closed-gen. supercompact.* ❑

As in [16], what we call the Game Reflection Principle (GRP) is the principle called GRP^+ in [29]. As its name suggests, GRP is actually a reflection statement about the non-existence of winning strategy of certain games of length ω_1 down to sub-games of size $< \aleph_2$.

We will not go into the details of the definition of GRP but just note that GRP implies the Continuum Hypothesis (CH) and it implies practically all reflection principles with reflection down to $< \aleph_2$ available under CH:

(2.4) GRP implies Rado's Conjecture (RC) (König, [29]). x-gen-0

(2.5) GRP implies strong downward Löwenheim-Skolem Theorem of $\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}$ down to $< \aleph_2$ ($\mathsf{SDLS}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}, < \aleph_2)$ in the notation of [16]). x-gen-1

(2.6) Each of RC and $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, < \aleph_2)$ (a weakening of $\mathsf{SDLS}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}, < \aleph_2)$) implies Fodor-type Reflection Principle (FRP), see [6] and [16]. x-gen-2

(2.7) RC implies a strong form of Chang's Conjecture (Todorčević, [31]) x-gen-2-0

(2.8) FRP is known to be equivalent to many "mathematical" reflection principles with reflection down to $< \aleph_2$, see [4], [5], [8], [15], [23]. x-gen-3

(2.9) GRP implies a "generic" solution to the Hamburger's problem (see Corollary 2.6 below, for the original Hamburger's Problem see [18] and reference given there). x-gen-4

Some of these and some other implications are put together in the following diagram:

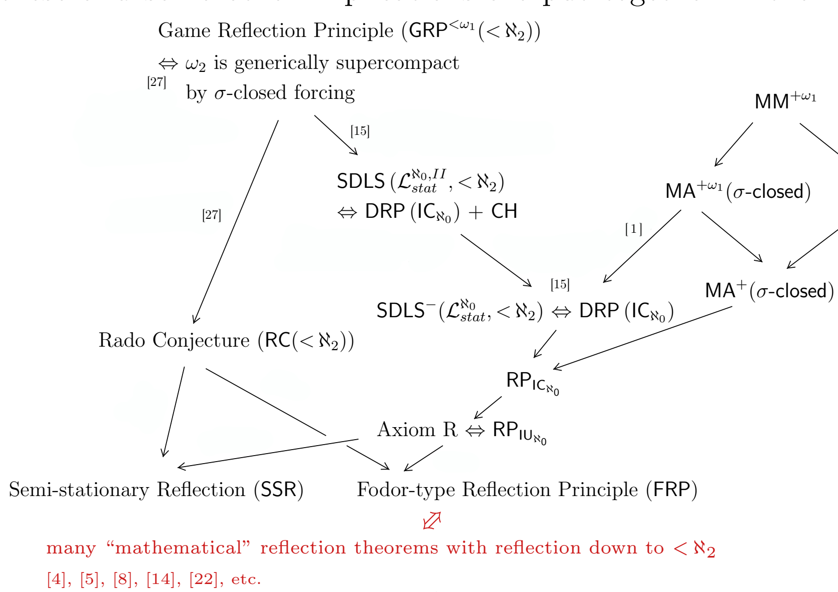

Figure 1. figure1

In the following we will elaborate on (2.9) above.

Proposition 2.5 *Suppose that κ is $\mathcal{P}$-gen. supercompact and P is a property of topological spaces which is ⓐ preserved by homeomorphism and ⓑ downward absolute, meaning that if W is a universe of set theory and W_0 is an inner model in W, any topological space $X \in \mathsf{W}$ satisfying the property P in W also satisfies P in W_0. Then, for any topological space X of local character $< \kappa$, if $\Vdash_{\mathbb{P}}$ "X satisfies P" for any $\mathbb{P} \in \mathcal{P}$ then there is a subspace Y of X of cardinality $< \kappa$ which satisfies P.* p-gen-1

Proof. Suppose that X is a topological space of local character $< \kappa$ and

ⓒ $\Vdash_{\mathbb{P}}$ "X satisfies P" for any $\mathbb{P} \in \mathcal{P}$.

Without loss of generality, we may assume that the underlying set of X is a cardinal $\lambda \geq \kappa$ and the topology of X is given by the system $\langle \tau_\alpha : \alpha < \lambda \rangle$ where τ_α is an open nbhd basis of α of cardinality $< \kappa$.

Let $\mathbb{P} \in \mathcal{P}$ be such that, for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that ① $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, ② $j(\kappa) > \lambda$ and ③ $j''\lambda \in M$.

Let

$$\begin{aligned} X' &:= \langle j''X, \{\langle \{j''U : U \in \tau_\alpha\}, j(\alpha)\rangle : \alpha \in \lambda\}\rangle \\ &= \langle j''X, \{\langle \{V \cap j''X : V \in j(\tau_\alpha)\}, j(\alpha)\rangle : \alpha \in \lambda\}\rangle. \end{aligned}$$

Then $X' \in M$ by ③ (and ①), X' is a subspace of $j(X)$ and $X \cong X'$ (in $\mathsf{V}[\mathbb{G}]$). By ⓐ and ⓒ, $\mathsf{V}[\mathbb{G}] \models$ "X' satisfies P" and hence $M \models$ "X' satisfies P" by ⓑ. Thus, by ②, $M \models$ "there is a subspace of X of size $< j(\kappa)$ satisfying P" By elementarity, it follows that $V \models$ "there is subspace of X of size $< \kappa$ satisfying P". ❑ (Proposition 2.5)

Corollary 2.6 *Suppose that GRP holds. Then, for any topological space X of character $\leq \aleph_1$ such that $\Vdash_{\mathbb{P}}$ "X is not metrizable" for any σ-closed poset $\mathbb{P}$, there is a non-metrizable subspace of X of cardinality $\leq \aleph_1$.* p-gen-2

Proof. By Theorem 2.4 and Proposition 2.5 for the property P being non-metrizable. ❑ (Corollary 2.6)

3 Laver-generic large cardinals

The Laver-genericity axioms (i.e. the axioms claiming the existence of Laver-generic large cardinals defined below) for respective classes of posets complete the picture of reflection and absoluteness in terms of double plus versions of forcing axioms given in Figure 1 (see Theorem 3.9 and Figure 3 below). laver

A (definable) class $\mathcal{P}$ of posets is said to be *iterable* if (a) $\{\mathbb{1}\} \in \mathcal{P}$, (b) $\mathcal{P}$ is closed with respect to forcing equivalence (i.e. if $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P} \sim \mathbb{P}'$ then $\mathbb{P}' \in \mathcal{P}$), (c) closed with respect to restriction (i.e. if $\mathbb{P} \in \mathcal{P}$ then $\mathbb{P} \restriction \mathbb{p} \in \mathcal{P}$ for any $\mathbb{p} \in \mathbb{P}$), and (d) for any $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$, $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" implies $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathcal{P}$.

For an iterable class $\mathcal{P}$ of posets, a cardinal κ is said to be *$\mathcal{P}$-Laver-gen. supercompact* [1)] if, for any $\lambda \geq \kappa$ and $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" such that, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with ① $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$, ② $j(\kappa) > \lambda$, and ③' $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H}$, $j''\lambda \in M$.

Recall that a cardinal κ is *superhuge* (*super-almost-huge*, resp.) if, for any $\lambda > \kappa$, there are classes j, M such that ① $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$, ② $j(\kappa) > \lambda$ and ③ ${}^{j(\kappa)}M \subseteq M$ (${}^{j(\kappa)>}M \subseteq M$, resp.).

These notions of large cardinals can be straightforwardly translated into their Laver-generic versions: For an iterable class $\mathcal{P}$ of posets, κ is *$\mathcal{P}$-Laver-gen. superhuge* (*$\mathcal{P}$-Laver-gen. super-almost-huge*, resp.) if, for any $\lambda \geq \kappa$, $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" such that, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with ① $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$, ② $j(\kappa) > \lambda$, and ③' $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, and $j''j(\kappa) \in M$ ($j''\mu \in M$ for all $\mu < j(\kappa)$, resp.).

Sometimes it is more convenient to consider the following additional property which we called the tightness of Laver-genericity: For an iterable $\mathcal{P}$, a $\mathcal{P}$-Laver-gen. supercompact cardinal ($\mathcal{P}$-Laver-gen. huge cardinal, etc., resp.) is *tightly $\mathcal{P}$-Laver-gen. supercompact* (*tightly $\mathcal{P}$-Laver-gen. huge*, etc., resp.) if the condition

④ $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a poset of cardinality $\leq j(\kappa)$.

additionally holds for the elementary embedding j in the definition.

The strongest notion of large cardinal we consider in this paper in connection with its Laver-generic version is that of ultrahuge cardinal introduced by Tsaprounis [33]. A cardinal κ is *ultrahuge* if for any $\lambda > \kappa$ there is $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$ such that $j(\kappa) > \lambda$ and ${}^{j(\kappa)}M, V_{j(\lambda)} \subseteq M$. In terms of consistency strength ultrahuge cardinal is placed between superhuge and 2-almost-huge (Theorem 3.4 in [33]).

For an iterable class $\mathcal{P}$ of posets, a cardinal κ is (*tightly*) *$\mathcal{P}$-Laver-generically ultrahuge*, if, for any $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", such that for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are $j, M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{P}, \mathbb{H}, (V_{j(\lambda)})^{\mathsf{V}[\mathbb{H}]} \in M$ (and $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a poset of size $j(\kappa)$).

[1)] The definition of Laver-generic large cardinals given here is slightly stronger than the one given in [17]. The Laver-generic large cardinals in the sense of present subsection is called *strongly* Laver-generic large cardinals in [17].

The following theorem is used to construct models with a Laver-generically ultrahuge cardinal:

Theorem A 3.1 (Tsaprounis [33]) *If κ is an ultrahuge cardinal, then κ carries a Laver function $f : \kappa \to V_\kappa$, i.e. a function f with the property:* p-gen-3

(3.1) *for every cardinal $\lambda \geq \kappa$ and any $x \in \mathcal{H}(\lambda^+)$ there is an $j : \mathsf{V} \overset{\preceq}{\rightarrow}_\kappa M$ with $j(\kappa) > \lambda$, and ${}^{j(\kappa)}M, V_{j(\lambda)} \subseteq M$ such that $x = j(f)(\kappa)$.* x-gen-5 ❑

By definition, it is obvious that we have the following implications:

$$
\begin{array}{ccccccccc}
\text{tightly } \mathcal{P}\text{-Laver-gen. ultrahuge} & \Rightarrow & \text{tightly } \mathcal{P}\text{-Laver-gen. superhuge} & \Rightarrow & \text{tightly } \mathcal{P}\text{-Laver-gen. super-almost-huge} & \Rightarrow & \text{tightly } \mathcal{P}\text{-Laver-gen. supercompact} & \Rightarrow & \text{tightly } \mathcal{P}\text{-Laver-gen. measurable} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\mathcal{P}\text{-Laver-gen. ultrahuge} & \Rightarrow & \mathcal{P}\text{-Laver-gen. superhuge} & \Rightarrow & \mathcal{P}\text{-Laver-gen. super-almost-huge} & \Rightarrow & \mathcal{P}\text{-Laver-gen. supercompact} & \Rightarrow & \mathcal{P}\text{-Laver-gen. measurable} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\mathcal{P}\text{-gen. ultrahuge} & \Rightarrow & \mathcal{P}\text{-gen. superhuge} & \Rightarrow & \mathcal{P}\text{-gen. super-almost-huge} & \Rightarrow & \mathcal{P}\text{-gen. supercompact} & \Rightarrow & \mathcal{P}\text{-gen. measurable}
\end{array}
$$

`Figure 2.` figure2

Some of the horizontal implications should be irreversible. At the moment however we can only prove the irreversibility of the implication from (tightly) $\mathcal{P}$-(Laver)-gen. ultrahugeness to (tightly) $\mathcal{P}$-(Laver)-gen. supercompactness.

Proposition 3.1 (Proposition 4 in [13]) *Suppose that $\mathbb{P}$ is a class of posets such that there is a construction of a model with a tightly $\mathcal{P}$-Laver generically supercompact cardinal starting from an arbitrary model with an supercompact cardinal κ by a poset of cardinality κ.*[2)] *Then tightly $\mathcal{P}$-Laver generic supercompactness of κ does not necessarily imply the $\mathcal{P}$-gen. super-almost-hugeness.* p-laver-a

For the proof of Proposition 3.1 we use the following observation:

Lemma 3.2 *Suppose that κ is $\mathcal{P}$-gen. ultrahuge for an arbitrary class $\mathcal{P}$ of posets. If there is an inaccessible $\lambda_0 > \kappa$ then there are cofinally many inaccessible in V.* p-laver-a-0

Proof. Let $\lambda > \lambda_0$ be an arbitrary cardinal. Then there is $\mathbb{P} \in \mathcal{P}$ such that, for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are $j, M \subseteq \mathsf{V}[\mathbb{G}]$ such that

(3.2) $j : \mathsf{V} \overset{\preceq}{\rightarrow}_\kappa M$, (3.3) : $j(\kappa) > \lambda$, and x-laver-1 x-laver-2

(3.4) $(V_{j(\lambda)})^{\mathsf{V}[\mathbb{G}]} \in M$. x-laver-3

2) By the following Theorem 3.3, the class of all σ-closed posets and the class of all ccc posets satisfy this condition.

By (3.3) and elementarity (3.2), we have $j(\lambda_0) > \lambda$. By elementarity (3.2), $M \models$ "$j(\lambda_0)$ is inaccessible". By (3.4), $\mathsf{V}[\mathbb{G}] \models$ "$j(\lambda_0)$ is inaccessible", and hence $\mathsf{V} \models$ "$j(\lambda_0)$ is inaccessible". ❏ (Lemma 3.2)

Proof of Proposition 3.1: Suppose that κ is a supercompact cardinal and $\lambda_0 > \kappa$ is an inaccessible cardinal.

We may assume that λ_0 is the largest inaccessible cardinal: if there is inaccessible cardinal larger than λ_0, then let λ_1 be the least such inaccessible cardinal. In V_{λ_1}, λ_0 is the largest inaccessible cardinal and κ is supercompact (see e.g. Exercise 22.8, (a) in [28] : $V_{\lambda_1} \models$ "κ is supercompact" can be seen using the characterization of supercompactness in terms of ultrafilters.).

Let $\mathbb{P}$ be a poset of size κ such that, for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, we have $\mathsf{V}[\mathbb{G}] \models$ "κ is tightly $\mathbb{P}$-Laver generically supercompact". Note that $\mathsf{V}[\mathbb{G}] \models$ "λ_0 is the largest inaccessible cardinal". Thus, by Lemma 3.2, it follows that $\mathsf{V}[\mathbb{G}] \models$ "κ is not $\mathcal{P}$-gen. ultrahuge". ❏ (Proposition 3.1)

(Tightly) Laver-generic large cardinal is actually first-order definable (i.e. it has a characterization formalizable in the language of ZFC), cf. [21]. Thus "Forcing Theorems" are available for arguments with Laver-genericity. Because of this and because an iterable class $\mathcal{P}$ is closed under restriction to a condition, by definition, we may be lazy about the quantification on generic filters like in the context of "for a/any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$ ..."

The Examples 2.2, 2.3 are actually examples of the construction of models with a Laver-generic large cardinal.

Theorem 3.3 (Theorem 5.2, [17]) (1) *Suppose that κ is supercompact (superhuge, etc., resp.) and $\mathbb{P} = \mathrm{Col}(\aleph_1, \kappa)$. Then, in $\mathsf{V}[\mathbb{G}]$, for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{\mathsf{V}[\mathbb{G}]}$ $(= \kappa)$ is tightly σ-closed-Laver-gen. supercompact (superhuge, etc., resp.) and* CH *holds.* p-laver-0

(2) *Suppose that κ is super-almost-huge (superhuge, etc., resp.) with a Laver function $f : \kappa \to V_\kappa$ for super-almost-hugeness (superhugeness, etc., resp.), and $\mathbb{P}$ is the CS-iteration for forcing* PFA *along with f. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{\mathsf{V}[\mathbb{G}]}$ $(= \kappa)$ is tightly proper-Laver-gen. super-almost-huge (superhuge, etc., resp.) and $2^{\aleph_0} = \aleph_2$ holds.*[3)]

(2′) *Suppose that κ is super-almost-huge (superhuge, resp.) with a Laver function $f : \kappa \to V_\kappa$ for super-almost-hugeness (superhugeness, etc., resp.), and $\mathbb{P}$ is the RCS-iteration for forcing* MM *along with f. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{\mathsf{V}[\mathbb{G}]}$ $(= \kappa)$ is tightly semi-proper-Laver-gen. super-almost-huge (superhuge, etc., resp.) and $2^{\aleph_0} = \aleph_2$ holds.*[3)]

(3) *Suppose that κ is supercompact (superhuge, etc., resp.) with a Laver function $f : \kappa \to V_\kappa$ for supercompactness (superhugeness, etc., resp.), and $\mathbb{P}$ is a FS-iteration for forcing* MA *along with f. Then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $2^{\aleph_0}$ $(= \kappa)$ is tightly ccc-Laver-gen. supercompact (superhuge, etc., resp.). $\kappa = 2^{\aleph_0}$, and κ is very large.*

In the following we give a proof of the case (2) of Theorem 3.3 for ultrahugeness and its Laver-generic version. For this case, we need the next lemma. Note that κ is almost huge if it is ultrahuge.

Lemma 3.4 (1) *If κ is almost huge and $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$ is an almost huge elementary embedding, then the target of j (i.e. $j(\kappa)$) is inaccessible.* p-laver-0-0

(2) *Suppose that κ is super almost huge. Then there are cofinally many inaccessible cardinals.*

Proof. (1): Suppose that $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$ is an almost huge elementary embedding. Thus we have in particular (3.5): ${}^{j(\kappa)>}M \subseteq M$. Since κ is inaccessible, $M \models$ "$j(\kappa)$ is inaccessible" by elementarity. By (3.5), it follows that $j(\kappa)$ is really inaccessible. x-laver-4

(2): follows from (1) since, if κ is super almost huge, the targets of almost huge elementary embeddings are unbounded. ❑ (Lemma 3.4)

Proof of Theorem 3.3: We prove (2) for ultrahugeness and its Laver-generic version. Other cases can be proved similarly.

Suppose that κ is ultrahuge and $f : \kappa \to V_\kappa$ is an ultrahuge Laver function (see Theorem A 3.1).

Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be a CS-iteration with

$$\underset{\sim}{\mathbb{Q}}_\beta := \begin{cases} f(\beta), & \text{if } \Vdash_{\mathbb{P}_\beta} \text{“} f(\beta) \text{ is a proper poset; ”} \\ \mathbb{1}, & \text{otherwise.} \end{cases}$$

We show that $\mathbb{P}_\kappa$ forces that κ is tightly proper-Laver generically ultrahuge.

Let $\mathbb{G}_\kappa$ be a $(\mathsf{V}, \mathbb{P}_\kappa)$-generic filter. Suppose $\lambda > \kappa$ and $\mathbb{P}$ be a proper poset in $\mathsf{V}[\mathbb{G}_\kappa]$. Let $\underset{\sim}{\mathbb{P}}$ be a $\mathbb{P}_\kappa$-name of $\mathbb{P}$.

By Lemma 3.4, (2), we may assume that λ is inaccessible.

Let $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$ be such that $j(f)(\kappa) = \underset{\sim}{\mathbb{P}}$, $j(\kappa) > \lambda$, and

$$\text{(3.6)} \qquad {}^{j(\kappa)}M, V_{j(\lambda+1)} \subseteq M.\text{[4]}$$ x-laver-5

---

[3] It seems that the construction does not work with supercompact κ here.

By elementarity, we have

$$M \models \text{“ } j(\mathbb{P}_\kappa) \text{ is a CS-iteration } \langle \mathbb{P}^*_\alpha, \underset{\sim}{\mathbb{Q}}^*_\beta : \alpha \leq j(\kappa), \beta < j(\kappa) \rangle \text{ of proper posets with the book-keeping } j(f) \text{ and } |\,\mathbb{P}^*_\alpha\,| < j(\kappa) \text{ for all } \alpha < \kappa\text{”}.$$

Note that $\mathbb{P}^*_\alpha = \mathbb{P}_\alpha$ for all $\alpha \leq \kappa$, $\mathbb{P}_\kappa \in M$, and $\underset{\sim}{\mathbb{Q}}^*_\kappa = \underset{\sim}{\mathbb{P}}$. Thus, by the Factor Lemma

$$M[\mathbb{G}_\kappa] \models \text{“ } j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa \text{ is (forcing equivalent to) a CS-iteration of proper posets of length } j(\kappa) \text{ and its 0th iterand is } \mathbb{P}\text{ ”}.$$

By the κ-cc of $\mathbb{P}_\kappa$ and by (3.6),we have ${}^\lambda(M[\mathbb{G}_\kappa]) \subseteq M[\mathbb{G}_\kappa]$.

$$\mathsf{V}[\mathbb{G}_\kappa] \models \text{“ } j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa \text{ is (forcing equivalent to) a CS-iteration of proper posets of length } j(\kappa) \text{ and its 0th iterand is } \mathbb{P}\text{ ”}.$$

It follows that, in $\mathsf{V}[\mathbb{G}_\kappa]$, we have $j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa \sim \mathbb{P} * \underset{\sim}{\mathbb{Q}}^*$ where

$$\mathsf{V}[\mathbb{G}_\kappa] \models \;\Vdash_{\mathbb{P}} \text{“ } \underset{\sim}{\mathbb{Q}}^* \text{ is proper ”}.$$

Let $\mathbb{H}$ be a $(\mathsf{V}[\mathbb{G}_\kappa], j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa)$-generic filter: Note that $\mathbb{H}$ corresponds to a $(\mathsf{V}[\mathbb{G}], \mathbb{P} * \underset{\sim}{\mathbb{Q}}^*)$-generic filter, and $\mathbb{G}_\kappa * \mathbb{H}$ corresponds to a $(\mathsf{V}, j(\mathbb{P}_\kappa))$-generic filter extending $\mathbb{G}_\kappa$. I shall denote the latter also with $\mathbb{G} * \mathbb{H}$.

Let $\tilde{j}$ be the “lifting” of j defined by

$$\tilde{j} : \mathsf{V}[\mathbb{G}_\kappa] \to M[\mathbb{G}_\kappa * \mathbb{H}]; \quad \underset{\sim}{a}[\mathbb{G}_\kappa] \;\mapsto\; j(\underset{\sim}{a})[\mathbb{G}_\kappa * \mathbb{H}] \quad \text{for all } \mathbb{P}_\kappa\text{-name } \underset{\sim}{a}.$$

Then we have $j \subseteq \tilde{j}$, $\tilde{j} : \mathsf{V}[\mathbb{G}_\kappa] \xrightarrow{\preceq}_\kappa M[\mathbb{G}_\kappa * \mathbb{H}]$, $\tilde{j}\,''\lambda = j\,''\lambda \in M \subseteq M[\mathbb{G}_\kappa * \mathbb{H}]$, $|\, j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa \,|^{\mathsf{V}[\mathbb{G}_\kappa]} \leq |\, j(\mathbb{P}_\kappa)\,|^M = j(\kappa)$.

$\mathbb{G}_\kappa * \mathbb{H}$ seen as a $(\mathsf{V}, j(\mathbb{P}_\kappa))$-gen. filter has cardinality $j(\kappa) < j(\lambda)$ and it is $\in V_{j(\lambda)}$.

Thus, there is a $j(\mathbb{P}_\kappa * \underset{\sim}{\mathbb{Q}})$-name $\underset{\sim}{V}$ of $(V_{j(\lambda)})^{\mathsf{V}[\mathbb{G}_\kappa * \mathbb{H}]}$ in $V_{j(\lambda)+1} = V_{j(\lambda+1)}$.

It follows that

$$(V_{j(\lambda)})^{\mathsf{V}[\mathbb{G}_\kappa * \mathbb{H}]} = \underset{\sim}{V}[\mathbb{G}_\kappa * \mathbb{H}] \in M[\mathbb{G}_\kappa * \mathbb{H}].$$

This shows that $\mathsf{V}[\mathbb{G}_\kappa] \models$ “κ is tightly proper-Laver-gen. ultrahuge”.

❑ (Theorem 3.3)

The circumstance that the three possibilities of the cardinality of the continuum: $\aleph_1$, $\aleph_2$, or very large, are highlighted in Theorem 3.3, has also an explanation in terms of Laver-genericity:

---

[4] It is intentional that we choose M here with slightly stronger closure property by saying $V_{j(\lambda+1)} \subseteq M$ instead of $V_{j(\lambda)} \subseteq M$.

Theorem 3.5 (The Trichotomy Theorem [17], see also [10]) (A) *If κ is $\mathcal{P}$-Laver-gen. supercompact for an iterable class $\mathcal{P}$ of posets such that* (a) *all $\mathbb{P} \in \mathcal{P}$ are ω_1 preserving,* (b) *all $\mathbb{P} \in \mathcal{P}$ do not add reals, and* (c) *there is a $\mathbb{P}_1 \in \mathcal{P}$ which collapses ω_2, then $\kappa = \aleph_2$ and* CH *holds.* p-laver-1

(B) *If κ is $\mathcal{P}$-Laver-gen. supercompact for an iterable class $\mathcal{P}$ of posets such that* (a) *all $\mathbb{P} \in \mathcal{P}$ are ω_1-preserving,* (b′) *there is a $\mathbb{P}_0 \in \mathcal{P}$ which add a real, and* (c) *there is a $\mathbb{P}_1$ which collapses ω_2, then $\kappa = \aleph_2 \leq 2^{\aleph_0}$. If $\mathcal{P}$ contains enough many proper posets then $\kappa = \aleph_2 = 2^{\aleph_0}$ (For the last assertion see Theorem 3.9 below).*

(Γ) *If κ is $\mathcal{P}$-Laver-gen. supercompact for an iterable class $\mathcal{P}$ of posets such that* (a′) *all $\mathbb{P} \in \mathcal{P}$ preserve cardinals, and* (b′) *there is a $\mathbb{P}_0 \in \mathcal{P}$ which adds a real, then κ is "very large" and $\kappa \leq 2^{\aleph_0}$. If κ is tightly $\mathcal{P}$-Laver-gen. superhuge then $\kappa = 2^{\aleph_0}$.*

Theorem 3.5 follows from the next Lemma 3.6, Lemma 3.7, and Theorem 3.8.

Lemma 3.6 *Suppose that $\mathcal{P}$ is a class of posets and κ is $\mathcal{P}$-generically measurable. Then:* p-Lg-RA-1-1

(1) *κ is regular.*

(2) *If all elements of $\mathcal{P}$ preserve κ, then κ is a weakly inaccessible cardinal.*

(3) *If all elements of $\mathcal{P}$ also preserve regularity of κ, then κ is a weakly inaccessible cardinal which is a stationary limit of weakly inaccessible, limit of limits of weakly inaccessible cardinals, etc.*

(4) *If all elements of $\mathcal{P}$ preserve stationarity of subsets of all regular $\lambda \leq \kappa$, then κ is a weakly inaccessible which is a stationary limit of weakly inaccessible, stationary limit of stationary limits of weakly inaccessible cardinals, etc.*

Proof. Let $\mathbb{P} \in \mathcal{P}$ be such that, for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that $j : \mathsf{V} \overset{\preceq}{\rightarrow}_\kappa M$.

(1): If κ were not regular, there would be $\mu < \kappa$ and cofinal $f : \mu \rightarrow \kappa$. $j(f) = f$ by $crit(j) = \kappa$. Hence, by elementarity, $M \models j(\kappa) = \sup_{\alpha<\mu} f(\alpha) = \kappa$. This is a contradiction.

(2): Suppose that $\mathbb{P}$ preserves κ. κ is a limit cardinal: If κ were a successor cardinal, there would be $\mu < \kappa$ with a sequence $\langle f_\alpha : \alpha < \kappa \rangle$ such that each f_α is a surjection form μ to α (μ to $\alpha + 1$ for finite α). Let $\langle f^*_\alpha : \alpha < j(\kappa) \rangle := j(\langle f_\alpha : \alpha < \kappa \rangle)$. Then f^*_κ is a surjection from μ to κ (in M, and hence in $\mathsf{V}[\mathbb{G}]$). This is a contradiction to $\mathbb{P} \in \mathcal{P}$.

Together with (1), this implies that κ is a weakly inaccessible cardinal.

(3): Let $\mathbb{P} \in \mathcal{P}$ be such that for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ there are j, $N \subseteq \mathsf{V}[\mathbb{G}]$ such that $j : \mathsf{V} \overset{\preceq}{\rightarrow}_\kappa M$. κ is weakly inaccessible by (2).

Suppose that $D \subseteq \kappa$ is a club (in V). We want to show that D contains a weakly inaccessible cardinal. Since $j(D)$ is closed by elementarity and since $j(D) \cap \kappa = D$, $\kappa \in j(D)$. Since $\mathbb{P}$ preserves cofinality, $\mathsf{V}[\mathbb{G}] \models$ "κ is weakly inaccessible". Hence $M \models$ "κ is weakly inaccessible". Thus $M \models$ "$j(D)$ contains a weakly inaccessible cardinal". By elementarity, it follows that $\mathsf{V} \models$ "D contains a weakly inaccessible cardinal".

(4): Let $\mathbb{P}$, $\mathbb{G}$, j, M be as in (3) where $\mathbb{P}$ now preserves stationary subsets of κ.

Suppose that $D \subseteq \kappa$ is a club (in V). We want to show that D contains a weakly inaccessible cardinal which is a stationary limit of weakly inaccessible cardinals.

By (3), $S = \{\alpha < \kappa : \alpha \text{ is weakly inaccessible}\}$ is stationary. It follows that S remains stationary subset of κ in $\mathsf{V}[\mathbb{G}]$. Since $S = j(S) \cap \kappa$, We have $S \in M$. As in (3), we also have $\kappa \in j(D)$.

Thus $M \models$ "there is a weakly inaccessible $\delta \in j(D)$ which is a stationary limit of inaccessible cardinals". By elementarity it follows that $\mathsf{V} \models$ "there is a weakly inaccessible$\delta \in D$ which is a stationary limit of inaccessible cardinals". ❑ (Lemma 3.6)

Lemma 3.7 (Proposition 4, in [10]) (1) *If κ is $\mathcal{P}$-gen. measurable for an ω_1 preserving $\mathcal{P}$, then $\omega_1 < \kappa$.* p-Lg-RA-1-2-0

(2) *If κ is $\mathcal{P}$-Laver-gen. supercompact for an iterable ω_1-preserving $\mathcal{P}$ with* $\mathrm{Col}(\omega_1, \{\omega_2\}) \in \mathcal{P}$ *then $\kappa = \omega_2$.*

(3) *If κ is $\mathcal{P}$-Laver-gen. supercompact for an iterable $\mathcal{P}$ which contains a poset adding a new real, then $\kappa \leq 2^{\aleph_0}$.*

(4) *If κ is $\mathcal{P}$-gen. supercompact for a $\mathcal{P}$ such that all posets in $\mathcal{P}$ do not add any reals then $2^{\aleph_0} < \kappa$.*

Proof. (1): Suppose $\kappa \leq \omega_1$. Since $\kappa = \omega$ is impossible, we have $\kappa = \omega_1$. Let $\mathbb{P} \in \mathcal{P}$ and $\mathbb{G}$ be $(\mathsf{V}, \mathbb{P})$-generic such that $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M \subseteq \mathsf{V}[\mathbb{G}]$. Then we have $M \models$ "$j(\kappa) = \omega_1$" by elementarity. Since $\kappa < j(\kappa)$, $M \models$ "κ is countable". Thus $\mathsf{V}[\mathbb{G}] \models$ "κ is countable". This is a contradiction to the assumption that elements of $\mathcal{P}$ are ω_1 preserving.

(2): Suppose that $\kappa \neq \omega_2$. By (1), we then have

$$\kappa > \omega_2. \tag{3.7}$$ x-Lg-0

Let $\mathbb{P} := \mathrm{Col}(\omega_1, \{\omega_2\})$, and let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" such that there is a $(V, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$ with j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$, and $\mathbb{H} \in M$. By (3.7), $j({\omega_2}^{\mathsf{V}}) = {\omega_2}^{\mathsf{V}}$ and $M \models$ "$j({\omega_2}^{\mathsf{V}}) = \omega_2$" by elementarity. On the other hand, $\mathbb{H}$ codes a collapsing of ${\omega_2}^{\mathsf{V}}$. Thus $M \models$ "$|\,{\omega_2}^{\mathsf{V}}\,| = \aleph_1$". This is a contradiction.

(3): Suppose that $\mu < \kappa$ and $\langle a_\alpha : \alpha < \mu \rangle$ is a sequence of reals. We show that $\langle a_\alpha : \alpha < \mu \rangle$ is not an enumeration of $\mathcal{P}(\omega)$.

Let $\mathbb{P} \in \mathcal{P}$ be a poset which adds a real, and let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name of a poset such that, for a $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$ there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$ and $\mathbb{H} \in M$. By elementarity and since $\mu < \kappa$, we have $j(\langle a_\alpha : \alpha < \mu \rangle) = \langle a_\alpha : \alpha < \mu \rangle$. Since there is a new reals coded in $\mathbb{H}$,

$$M \models \text{“}\langle a_\alpha : \alpha < \mu \rangle \text{ is not an enumeration of } \mathcal{P}(\omega)\text{”}.$$

By elementarity it follows that

$$\mathsf{V} \models \text{“}\langle a_\alpha : \alpha < \mu \rangle \text{ is not an enumeration of } \mathcal{P}(\omega)\text{”}.$$

(4) : Suppose, toward a contradiction, that $\kappa \leq 2^{\aleph_0}$. Let $\lambda > 2^{\aleph_0}$ and let $\mathbb{P} \in \mathcal{P}$ be such that, for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, and $j(\kappa) > \lambda$. We have $M \models 2^{\aleph_0} \geq j(\kappa)$ by elementarity. Thus $(2^{\aleph_0})^M \geq j(\kappa) > \lambda > (2^{\aleph_0})^{\mathsf{V}}$. Since $(2^{\aleph_0})^{\mathsf{V}} = (2^{\aleph_0})^M$ by $\mathbb{P} \in \mathcal{P}$, this is a contradiction. ❑ (Lemma 3.7)

Theorem 3.8 (Theorem 5.8 in [17]) *Suppose that each element of an iterable class $\mathcal{P}$ of posets is μ-cc for some $\mu < \kappa$ and $\mathcal{P}$ contains a poset $\mathbb{P}$ which adds a real. If κ is tightly $\mathcal{P}$-Laver-gen. superhuge then $\kappa = 2^{\aleph_0}$.* p-MP-2-0

Proof. Suppose that κ is tightly $\mathcal{P}$-Laver-gen. superhuge for the class of posets $\mathcal{P}$ as above. Then $\kappa \leq 2^{\aleph_0}$ by Lemma 3.7, (3).

To prove $2^{\aleph_0} \leq \kappa$, let $\lambda \geq \kappa, 2^{\aleph_0}$ be large enough. By assumption there is a μ-cc poset $\mathbb{Q}$ such that there are $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$ and $j : \mathsf{V} \xrightarrow{\preceq} M \subseteq \mathsf{V}[\mathbb{H}]$ with (a) $crit(j) = \kappa$,
(b) $|\mathbb{Q}| \leq j(\kappa) > \lambda$, (c) $\mathbb{H} \in M$ and (d) $j''j(\kappa) \in M$.

Since κ is regular (Lemma 3.6, (1)), and by elementarity, we have $M \models$ “$j(\kappa)$ is regular”. By the closedness (d) of M, it follows that $j(\kappa)$ is regular in $\mathsf{V}[\mathbb{H}]$. Hence it is also regular in V.

Thus, we have $\mathsf{V} \models$ “$j(\kappa)^\mu = j(\kappa)$”, since SCH holds above κ by Proposition 2.8, (1) in [17]. Since $\mathbb{Q}$ has the μ-cc and μ, $|\mathbb{Q}| \leq j(\kappa)$, it follows that $\mathsf{V}[\mathbb{H}] \models$ “$2^{\aleph_0} \leq j(\kappa)$”. Again by (d) (see Lemma 2.1 (4)), we have $(j(\kappa)^+)^M = (j(\kappa)^+)^{\mathsf{V}} = (j(\kappa)^+)^{\mathsf{V}[\mathbb{H}]}$. Thus $M \models$ “$2^{\aleph_0} \leq j(\kappa)$”.

By elementarity, it follows that $\mathsf{V} \models$ “$2^{\aleph_0} \leq \kappa$”. ❑ (Theorem 3.8)

Proof of Theorem 3.5: (A): By Lemma 3.7, (2) and (4).

(B): By Lemma 3.7, (2) and (3). The last claim follows since $\mathsf{MA}(\mathcal{P})$ (and actually much more) holds by Theorem 3.9 below.

(Γ): κ is "very large" by Lemma 3.6. $\kappa \leq 2^{\aleph_0}$ follows from Lemma 3.7, (3). The last statement follows from Theorem 3.8. ❑ (Theorem 3.5)

Laver-generic supercompactness also implies double plus versions of forcing axioms. For a class $\mathcal{P}$ of posets and cardinals κ, μ, let us denote with $\mathsf{MA}^{+\mu}(\mathcal{P}, <\kappa)$ and $\mathsf{MA}^{++<\mu}(\mathcal{P}, <\kappa)$ the following versions of Martin's Axiom:

$\mathsf{MA}^{+\mu}(\mathcal{P}, <\kappa)$: For any $\mathbb{P} \in \mathcal{P}$, any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ and any family $\mathcal{S}$ of $\mathbb{P}$-names such that $|\mathcal{S}| \leq \mu$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{S}$ is a stationary subset of ω_1" for all $\underset{\sim}{S} \in \mathcal{S}$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ over $\mathbb{P}$ such that $\underset{\sim}{S}[\mathbb{G}]$ is a stationary subset of ω_1 for all $\underset{\sim}{S} \in \mathcal{S}$.

$\mathsf{MA}^{++\leq\mu}(\mathcal{P}, <\kappa)$: For any $\mathbb{P} \in \mathcal{P}$, any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ and any family $\mathcal{S}$ of $\mathbb{P}$-names such that $|\mathcal{S}| \leq \mu$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{S}$ is a stationary subset of $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$" for some $\omega < \eta_{\underset{\sim}{S}} \leq \theta_{\underset{\sim}{S}} \leq \mu$ with $\eta_{\underset{\sim}{S}}$ regular, for all $\underset{\sim}{S} \in \mathcal{S}$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ over $\mathbb{P}$ such that $\underset{\sim}{S}[\mathbb{G}]$ is stationary in $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$ for all $\underset{\sim}{S} \in \mathcal{S}$.

Clearly $\mathsf{MA}^{++\leq\omega_1}(\mathcal{P}, <\kappa)$ is equivalent to $\mathsf{MA}^{+\omega_1}(\mathcal{P}, <\kappa)$.
MM^{++} is $\mathsf{MA}^{+\omega_1}$(stationary preserving posets, $<\aleph_2$).

Theorem 3.9 *Suppose $\mathcal{P}$ is an iterable class of posets such that* p-laver-2

(3.8) *the elements of $\mathcal{P}$ preserve stationarity of subsets of $\mathcal{P}_\mu(\theta)$ for all $\mu \leq \theta < \kappa$.* x-laver-6

If $\kappa > \aleph_1$ is $\mathcal{P}$-Laver generically supercompact then $\mathsf{MA}^{++\leq\mu}(\mathcal{P}, <\kappa)$ holds for all $\mu < \kappa$.

Proof. Suppose that $\mathcal{P}$ is an iterable class of posets, $\kappa > \aleph_1$ is $\mathcal{P}$-Laver generically supercompact, and $\mu < \kappa$. Let $\mathcal{D}$ and $\mathcal{S}$ be as in the definition of $\mathsf{MA}^{++\leq\mu}(\mathcal{P}, <\kappa)$. Without loss of generality, we may assume that the underlying set of $\mathbb{P}$ is some cardinal λ_0 and elements of $\mathcal{S}$ are nice $\mathbb{P}$-names.

Let $\lambda > \lambda_0$ be sufficiently large, and let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for a $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic filter $\mathbb{H}$, there are transitive $M \subseteq \mathsf{V}[\mathbb{H}]$ and $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$ with

(3.9) $j(\kappa) > \lambda$, laver-0

(3.10) $\mathbb{P}$, $\mathbb{H} \in M$ and laver-0-0

(3.11) $j''\lambda \in M$. laver-1

By the choice of λ, (3.11) and Lemma 2.1, (5), we have $\mathbb{P}$, $\mathcal{D}$, $\mathcal{S} \in M$. Let $\mathbb{G} = \mathbb{H} \cap \mathbb{P}$. Then $\mathbb{G} \in M$ by (3.10). By (3.8), $\mathbb{G}$ witnesses

(3.12) $M \models$ "there is a $\mathcal{D}$-generic filter G over $\mathbb{P}$ such that $\underset{\sim}{S}[G]$ is a stationary subset of $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$ for all $\underset{\sim}{S} \in \mathcal{S}$". laver-1-3

Since $j(\mathcal{D}) = \{j(D) : D \in \mathcal{D}\}$ and $j(\mathcal{S}) = \{j(S) : S \in \mathcal{S}\}$ by $|\mathcal{D}|, |\mathcal{S}| < \kappa$, $j(D) \supseteq j''D$ for all $D \in \mathcal{D}$, $j(S) \supseteq j''S$ for all $S \in \mathcal{S}$ and $j''\mathbb{G} \in M$ by Lemma 2.1, (6), $j''\mathbb{G}$ witnesses the following:

(3.13) $M \models$ "there is a $j(\mathcal{D})$-generic filter G over $j(\mathbb{P})$ such that $\underset{\sim}{S}(G)$ is a stationary subset of $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$ for all $\underset{\sim}{S} \in j(\mathcal{S})$". laver-1-4

By elementarity, it follows that

(3.14) $\mathsf{V} \models$ "there is a $\mathcal{D}$-generic filter G over $\mathbb{P}$ such that $\underset{\sim}{S}[G]$ is a stationary subset of $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$ for all $\underset{\sim}{S} \in \mathcal{S}$". laver-1-5

❑ (Theorem 3.9)

Proposition 3.10 *If* ZFC + *"there are two supercompact cardinals" is consistent, then* ZFC + FRP + *"there is a tightly ccc-Laver-generically supercompact cardinal" is consistent as well.* p-laver-3

Proof. Let κ_0 and κ_1 with $\kappa_0 < \kappa_1$ be two supercompact cardinals. We can use κ_0 to force $\mathsf{MA}^+(\sigma\text{-closed})$ by a poset of size κ_0. In the generic extension we have FRP and κ_2 is still supercompact. Now we use κ_1 to force that κ_1 is tightly ccc-Laver-gen. supercompact in the generic extension as described in Theorem 3.3, (3). FRP still holds in the second generic extension since FRP is preserved by ccc forcing (Theorem 3.4 in [15]). ❑ (Proposition 3.10)

These results together with some other implications proved [17] as well as some results that are going to be discussed bellow are integrated in Figure 1 to obtain the following extended diagram:

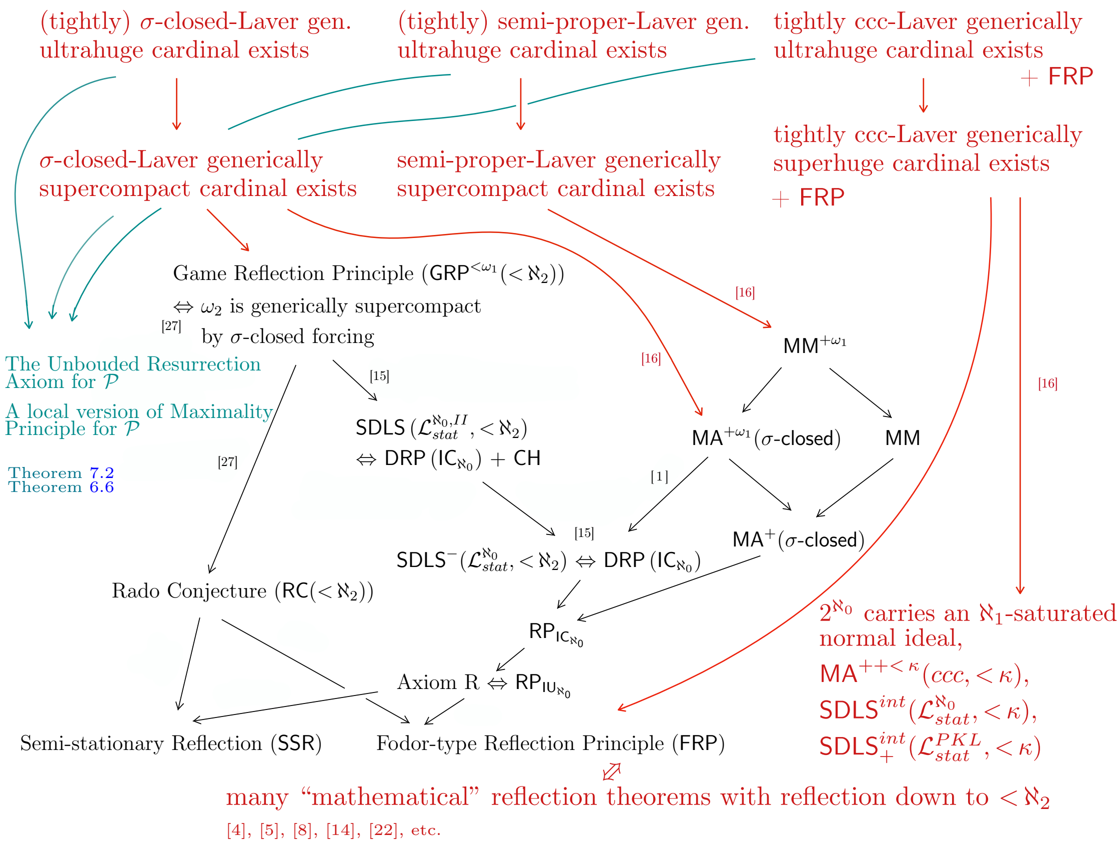

Figure 3. figure3

For some iterable classes $\mathcal{P}$ of posets, even though they look quite natural, we can prove that there is no $\mathcal{P}$-Laver generic large cardinal (see Propositions 3.11, 3.13, and Corollary 3.12 below). These results suggest that our Trichotomy Theorem 3.5 does cover a large range of possible classes $\mathcal{P}$ of posets in connection with Laver-genericity.

Proposition 3.11 *Suppose that $\mathcal{P}$ is an iterable class of posets such that all $\mathbb{P} \in \mathcal{P}$ are ω_1-preserving and $\mathcal{P}$ contains a poset $\mathbb{P}^*$ whose generic filter destroys a stationary subset of ω_1.*[5] *Then there is no $\mathcal{P}$-Laver-gen. measurable cardinal.* p-Lg-RA-2

Proof. Suppose, toward a contradiction, that $\mathcal{P}$ is as above and κ is $\mathcal{P}$-Laver generically measurable cardinal.

---

5) "$\mathbb{P}^*$ destroys a stationary subset of ω_1" means here that a $\mathbb{P}^*$-generic set codes a club subset of $\omega_1 \setminus S$ in some absolute way.

Note that, for stationary and co-stationary subset S of ω_1, various posets are known which preserve ω_1 while shooting a club in $\omega_1 \setminus S$ (e.g. see [34]).

Let $S \subseteq \omega_1$ be stationary (and co-stationary), and let $\mathbb{P}^* \in \mathcal{P}$ be a poset shooting a club in $\omega_1 \setminus S$. By assumption, there is a $\mathbb{P}^*$-name $\underset{\sim}{\mathbb{Q}}$ of a poset such that $\Vdash_{\mathbb{P}^*}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for $(\mathsf{V}, \mathbb{P}^* * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that

$$j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M \text{ and} \tag{3.15}$$ x-Lg-RA-0

$$\mathbb{P}, \mathbb{H} \in M. \tag{3.16}$$ x-Lg-RA-1

By the choice of $\mathbb{P}^*$ ($\lessdot \mathbb{P}^* * \underset{\sim}{\mathbb{Q}}$) and (3.16), $M \models$ "S is a non-stationary subset of ω_1". Since $crit(j) = \kappa > \omega_1$ by Lemma 3.7, (1), we have $S = j(S)$. By $\mathsf{V} \models$ "S is stationary subset of ω_1", this is a contradiction to the elementarity (3.15). ❑ (Proposition 3.11)

Corollary 3.12 *Suppose that $\mathcal{P} = \{\mathbb{P} : \mathbb{P}$ is a poset preserving cardinals$\}$ or $\mathcal{P} = \{\mathbb{P} : \mathbb{P}$ is an ω_1-preserving poset$\}$. Then there is no $\mathcal{P}$-Laver-gen. measurable cardinal.* ❑ p-Lg-RA-2-0

Proposition 3.13 *Suppose that $\mathcal{P}$ is an iterable class of posets such that, for a regular cardinal $\delta > \aleph_1$,* p-Lg-RA-3

$$\textit{all } \mathbb{P} \in \mathcal{P} \textit{ are } \delta\textit{-cc, and} \tag{3.17}$$ x-Lg-RA-2

$$\mathrm{Fn}(\omega, \omega_1, {<}\aleph_0) \in \mathcal{P}.\text{[6]} \tag{3.18}$$ x-Lg-RA-3

Then there is no $\mathcal{P}$-Laver-gen. supercompact cardinal.

Proof. Suppose, toward a contradiction, that κ is a $\mathcal{P}$-Laver-gen. supercompact cardinal for $\mathcal{P}$ as above.

Claim 3.13.1 $\kappa = \omega_1$. p-cl-Lg-RA-0

$\vdash$ Suppose $\kappa \neq \omega_1$. Then, since $\kappa = \omega$ is impossible, we have $\kappa > \omega_1$. Let $\mathbb{P} := \mathrm{Fn}(\omega, \omega_1, {<}\aleph_0)$. Since $\mathbb{P} \in \mathcal{P}$ by (3.18), there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of a poset such that, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $\mathbb{H} \in M$ and $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$. But then, by $\mathbb{H} \in M$, we have $M \models$ "$j(\omega_1) = \omega_1$ is countable". This is a contradiction to the elementarity of j. $\dashv$ (Claim 3.13.1)

Let $\mathbb{P} \in \mathcal{P}$ be such that, for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that $j : \mathsf{V} \xrightarrow{\preceq}_{\omega_1} M$, and $j(\omega_1) > \delta$. Since $\mathbb{P}$ is δ-cc by (3.17), $\mathsf{V}[\mathbb{G}] \models$ "δ is a cardinal". It follows that $M \models$ "δ is a cardinal and $\omega < \delta < j(\omega_1)$". This is a contradiction to the elementarity of j. ❑ (Proposition 3.13)

---

[6] Note that $\mathrm{Fn}(\omega, \omega_1, {<}\aleph_0)$ has ω_2-cc (since its size is $\aleph_1$). In particular It has the δ-cc.

4 Maximality Principle

Maximality Principle (MP) in its non parameterized version as given in Joel Hamkins' max [26] was first formulated by Paul Larson following the ideas suggested by Christophe Chalons.

In the language $\mathcal{L}_\in$ of ZFC, the Maximality Principle can only be formulated in an infinite set of $\mathcal{L}_\in$-sentences asserting for each $\mathcal{L}_\in$-sentence φ that if it is a button then it is already pushed. That is, for all $\mathcal{L}_\in$-sentences φ, if, there is a poset $\mathbb{P}$ such that

$$\Vdash_{\mathbb{Q}} \text{“}\varphi\text{” holds for all poset } \mathbb{Q} \text{ with } \mathbb{P} \leqslant \mathbb{Q}, \tag{4.1}$$ x-max-0

then φ holds.

If (4.1) holds, then we shall say that φ is a *button with the push* $\mathbb{P}$.[7)]

One of the easy consequences of MP is the following:

Proposition 4.1 (Hamkins [26]) MP *implies* $\mathsf{V} \neq \mathsf{L}$. ❑ p-max-0

Note that the statement “$\mathsf{V} \neq \mathsf{L}$” is apparently a button. For another consequence of MP, see Lemma 5.8 below.

For an $\mathcal{L}_\in$-sentence φ let mp_φ be the $\mathcal{L}_\in$-sentence:

$$\exists P\,(P \text{ is a poset} \land \forall Q(P \leqslant Q \rightarrow \Vdash_Q \text{“}\varphi\text{”})) \rightarrow \varphi. \tag{4.2}$$ x-max-1

Formally we define MP to be the collection of all $\mathcal{L}_\in$-sentence of the form mp_φ for $\mathcal{L}_\in$-sentence φ.

For an $\mathcal{L}_\in$-sentence φ let mp^+_φ be the $\mathcal{L}_\in$-sentence:

$$\begin{aligned} &\exists P\,(P \text{ is a poset} \land \forall Q(P \leqslant Q \rightarrow \Vdash_Q \text{“}\varphi\text{”})) \\ &\quad \rightarrow \forall R(R \text{ is a poset } \rightarrow \Vdash_R \text{“}\varphi\text{”}). \end{aligned} \tag{4.3}$$ x-max-4

Let MP^+ be the collection of $\mathcal{L}_\in$-sentences of the form mp^+_φ for all $\mathcal{L}_\in$-sentences φ.

Proposition 4.2 (Hamkins [26]) MP *and* MP^+ *are equivalent over* ZFC. p-max-4-1

Proof. It is clear that MP^+ implies MP.

To see that MP implies MP^+, let φ be an arbitrary $\mathcal{L}_\in$-sentence. Let us write $\Box\varphi$ for $\forall R(R$ is a poset $\rightarrow \Vdash_R$ “φ”$)$.

It is easy to see that we have $\Box\varphi \leftrightarrow \Box\Box\varphi$. Thus mp^+_φ is equivalent to $mp_{\Box\varphi}$. The latter sentence is a member of MP. ❑ (Lemma 4.2)

7) This paper was written just when it was intensively discussed whether a dictator was going to push the red button.

Lemma 4.3 *Suppose that mp^+_φ holds for an $\mathcal{L}_\in$-sentence φ. Then for any poset $\mathbb{P}$, $\Vdash_\mathbb{P}$ "mp^+_φ" holds.* p-max-4-1-0

Proof. This follows from the fact that the premise of mp^+_φ is forcing absolute while the conclusion is forcing upward absolute. ❑ (Lemma 4.3)

Lemma 4.4 *Suppose that $\varphi_0,\ldots,\varphi_{n-1}$ are $\mathcal{L}_\in$-sentences. If ZFC is consistent, then so is $\mathsf{ZFC} + mp^+_{\varphi_0} + \cdots + mp^+_{\varphi_{n-1}}$.* p-max-1

Proof. Suppose otherwise. Then for some $\mathcal{L}_\in$-sentences $\varphi_0,\ldots,\varphi_n$, $\mathsf{ZFC} + mp^+_{\varphi_0} + \cdots + mp^+_{\varphi_n}$ is inconsistent. We can take $\varphi_0,\ldots,\varphi_n$ such that n is minimal possible for such set of sentences. Then we have

(4.4) $\mathsf{ZFC} + mp^+_{\varphi_0} + \cdots + mp^+_{\varphi_{n-1}}$ is consistent and p-max-4-2

(4.5) $\mathsf{ZFC} + mp^+_{\varphi_0} + \cdots + mp^+_{\varphi_{n-1}} \vdash \neg mp^+_{\varphi_n}$. x-max-2

Note that

$$\begin{aligned}(4.6)\quad \neg mp^+_{\varphi_n} \ \leftrightarrow\ & \exists P\,(P \text{ is a poset} \wedge \forall Q(P \leqslant Q \rightarrow \Vdash_Q \text{“}\varphi_n\text{”}))\\ & \wedge \exists R\,(R \text{ is a poset} \wedge \nVdash_R \text{“}\varphi_n\text{”}).\end{aligned}$$ x-max-3

In $\mathsf{ZFC} + mp^+_{\varphi_0} + \cdots + mp^+_{\varphi_{n-1}}$, let $\mathbb{P}$ be a poset whose existence is guaranteed by the first half of the right side of (4.6).

Since all formulas of $\mathsf{ZFC} + mp^+_{\varphi_0} + \cdots + mp^+_{\varphi_{n-1}}$ are forced by $\mathbb{P}$ by Lemma 4.3, we have $\Vdash_\mathbb{P}$ "$\neg mp^+_{\varphi_n}$". Hence $\Vdash_\mathbb{P}$ "$\exists R\,(\nVdash_R$ "φ_n")" by the second half of the right side of (4.6).

On the other hand, by the choice of $\mathbb{P}$ we have $\Vdash_\mathbb{P}$ "$\forall R\,(\Vdash_R$ "φ_n")".

Thus we have obtained a proof of contradiction from $\mathsf{ZFC}+ mp^+_{\varphi_0} + \cdots + mp^+_{\varphi_{n-1}}$. This is a contradiction to the assumption (4.4). ❑ (Lemma 4.4)

Theorem 4.5 (Hamkins [26]) *If ZFC is consistent then so is $\mathsf{ZFC} + \mathsf{MP}$.* p-max-3

Proof. Assume that $\mathsf{ZFC} + \mathsf{MP}$ is inconsistent. Then there are $\mathcal{L}_\in$-sentences $\varphi_0,\ldots,\varphi_{n-1}$ such that $\mathsf{ZFC} + mp^+_{\varphi_0} + \cdots + mp^+_{\varphi_{n-1}}$ is inconsistent (see Proposition 4.2). This is a contradiction to Lemma 4.4. ❑ (Theorem 4.5)

By practically the same argument as above, we can prove also the following:

Theorem 4.6 *Suppose that "x-large cardinal" is a notion of a large cardinal formalizable in $\mathcal{L}_\in$ such that,* p-max-4

(4.7) *if κ is an x-large cardinal then the x-largeness of κ is preserved by any* x-max-4-0 *set-forcing of size $< \kappa$.*

If ZFC+ *"there are class many x-large cardinals" is consistent, then so is* ZFC + MP + *"there are class many x-large cardinals".*

Proof. Working in the theory ZFC + "there are class many x-large cardinals" we have that $\Vdash_{\mathbb{P}}$ "there are class many x-large cardinals" holds for any poset $\mathbb{P}$ since, by (4.7), only set many x-large cardinals are destroyed by $\mathbb{P}$. Thus Lemma 4.4 with ZFC replaced by ZFC + "there are class many x-large cardinals" can be shown by the same argument. ❑ (Theorem 4.6)

Theorem 4.7 (Hamkins [26]) MP *is preserved by any set-generic extension.* p-max-4-0

Proof. By Proposition 4.2 and Lemma 4.3. ❑ (Theorem 4.7)

A sort of inverse of Theorem 4.6 also holds:

Theorem 4.8 (Hamkins [26]) *Suppose that* MP *holds. If "x-large cardinal" is a notion of large cardinal formalizable in $\mathcal{L}_\in$ such that* p-max-5

(4.8) *If κ is an x-large cardinal, then κ is (weakly, resp.) inaccessible; and* x-max-5

(4.9) *no new x-large cardinal is created by set-forcing.* x-max-7

If there is an x-large cardinal, then there are cofinally many x-large cardinals in V.

Proof. Suppose otherwise. Let κ be an x-large cardinal, and $\lambda > \kappa$ be a cardinal above which there are no x-large cardinals.

Let $\mathbb{P}$ be a poset which collapses λ to be, say, of cardinality ω_1, and let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic filter. Then by (4.8) and (4.9), there is no x-large cardinal in $\mathsf{V}[\mathbb{G}]$. Also there is no x-large cardinal in any further generic extension by (4.9).

By MP it follows that there is no x-large cardinal in V. This is however a contradiction to the assumption of the theorem. ❑ (Theorem 4.8)

If "x-large cardinal" implies (strong) inaccessibility, we can also prove Theorem 4.8 by adding λ many reals instead of collapsing cardinals below λ. This remark is going to be relevant in the proof of Proposition 6.3..

In the following corollary, I call a cardinal κ *resurrectably x-large* for a notion "x-large" of large cardinal, if there is a poset $\mathbb{P}$ such that $\Vdash_{\mathbb{P}}$ "κ is x-large".

Corollary 4.9 *Suppose that* MP *holds.* *p-max-6*

(1) *If there is a (weakly, resp.) inaccessible cardinal then there are class many (weakly, resp.) inaccessible cardinals.*

(2) *If "x-large" is a large cardinal property satisfying* (4.8) *and there is a resurrectably x-large cardinal then there are class many resurrectably x-large cardinals.*

Proof. (1): The (weakly rep.) inaccessible cardinals satisfy (4.8) and (4.9).

(2): The notion of resurrectably x-large cardinal as a large cardinal property satisfies (4.8) and (4.9). □ (Corollary 4.9)

5 Independence of MP under a Laver-gen. large cardinal

In the following it is convenient to consider an abstract notion of large cardinal. indep As a generic name for a notion of large cardinal, we shall use the fancy words "x-large cardinal", "y-large cardinal" etc. which are in association with the German expression "x-beliebig" meaning "really arbitrary". This way of narration has been already used in the last section.

Suppose that "... is an x-large cardinal" is a notion of large cardinal. We say this notion of large cardinal is *normal* if the following hold:

(5.1) "κ is an x-large cardinal" is formalizable in an $\mathcal{L}_\in$-sentence over ZFC. x-max-7-0

(5.2) "κ is an x-large cardinal" implies that κ is inaccessible; x-max-8

(5.3) "κ is an x-large cardinal" cannot be destroyed by a forcing of size $<\kappa$; x-max-9

(5.4) No new x-large cardinal can be created by small forcing; and x-max-10

(5.5) ZFC + "there are stationarily many x-large cardinals" is consistent.[8] x-max-11

Note that most of the known notions of large cardinal are normal in the sense above under the assumption of the consistency of the existence of a sufficiently large cardinal.

Example 5.1 *The notion of ultrahuge cardinal is normal under the consistency of* ZFC + *"there is a 2-almost-huge cardinal".* *ex-indep-a-0*

---

[8] "there are stationarily many x-large cardinals" is the axiom scheme consisting of the statements "if $C = \{\alpha \in \mathrm{On} : \varphi(\alpha)\}$ is a club in On then there is an x-large cardinal κ such that $\varphi(\kappa)$" for all $\mathcal{L}_\in$-formulas $\varphi = \varphi(x)$.

Proof. By Theorem 3.4 in Tsaprounis [33], if κ is 2-almost-huge then there is a normal ultrafilter $\mathcal{U}$ over κ such that $\{\alpha < \kappa : V_\kappa \models \text{“}\alpha \text{ is ultrahuge”}\} \in \mathcal{U}$. Thus V_κ is a model of ZFC+ “there are stationarily many ultrahuge cardinals”.

□ (Example 5.1)

The notion of super $C^{(n)}$-hyperhuge cardinal is normal under 2-huge. (Lemma 3.4 in `recurrence-axioms.tex` label: `x-Lg-RcA-5`)

Example 5.2 *The notion of super almost-huge cardinal is normal under the consistency of* ZFC + *“there is a huge cardinal”.* ex-indep-a

The example above follows from the next theorem which should be a folklore:

Theorem 5.3 *Suppose that κ is huge. Then,* $\{\alpha < \kappa : V_\kappa \models \text{“}\alpha \text{ is super almost-huge”}\}$ *is a normal measure 1 subset of κ.* p-indep-0

Proof. □ (Theorem 5.3)

Note that Theorem 5.3 implies that V_κ for a huge cardinal κ models ZFC + there is a super almost-huge cardinal + there are stationarily many inaccessible cardinals (actually there are stationarily many super almost-huge cardinals). See Theorems 5.10, 5.11.

Theorem 5.3 also tells that the existence of huge cardinal implies the consistency of the theory ZFC + δ is super almost-huge + $V_\delta \prec \mathsf{V}$, c.f. Theorem 6.1.

We use the following Theorem 5.4 and Lemma 5.5 for the proof of Theorem 5.3.

For cardinals $\kappa \le \lambda$ and a sequence $\vec{\mathcal{U}} = \langle \mathcal{U}_\gamma : \kappa \le \gamma < \lambda \rangle$ such that $\mathcal{U}_\gamma$ is a normal ultrafilter over $\mathcal{P}_\kappa(\gamma)$ for all $\kappa \le \gamma < \lambda$, we say that $\vec{\mathcal{U}}$ is coherent if $\mathcal{U}_\gamma = \mathcal{U}_\delta | \gamma := \{\{a \cap \gamma : a \in A\} : A \in \mathcal{U}_\delta\}$ for all $\kappa \le \gamma \le \delta < \lambda$.

For a coherent sequence of normal ultrafilters $\vec{\mathcal{U}} = \langle \mathcal{U}_\gamma : \kappa \le \gamma < \lambda \rangle$, We let $j_\gamma : \mathsf{V} \xrightarrow{\prec} M_\gamma \cong Ult(\mathsf{V}, U_\gamma)\kappa$ be the standard embedding, and, for $\kappa \le \gamma \le \delta < \lambda$, we define $k_{\gamma,\delta} : M_\gamma \xrightarrow{\prec} M_\delta$ by $k_{\gamma,\delta}([f]_{\mathcal{U}_\gamma}) := [\langle f(x \cap \gamma) : x \in \mathcal{P}_\kappa(\delta) \rangle]_{\mathcal{U}_\delta}$.

Then we have $j_\delta = k_{\gamma\delta} \circ j_\gamma$.

The following Theorem 5.4 is a slight modification of Theorem 24.11 in [28].

Theorem 5.4 *For a cardinal κ and inaccessible $\lambda > \kappa$ the following are equivalent:* p-indep-0-0

(a) *κ is an almost-huge cardinal with almost-huge elementary embedding j with the target $j(\kappa) = \lambda$.*

(b) *There is a coherent sequence $\langle \mathcal{U}_\gamma : \kappa \le \gamma < \lambda \rangle$ of normal ultrafilters such that*

(5.6) *for all $\kappa \le \gamma < \lambda$ and α with $\gamma \le \alpha < j_\gamma(\kappa)$, there is $\gamma \le \delta < \lambda$ such that $k_{\gamma,\delta}(\alpha) = \delta$.* x-indep-a-0

□

Lemma 5.5 *If κ is an (almost) huge cardinal and* p-indep-0-1

(5.7) $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$ *is a(n almost) huge elementary embedding.* x-indep-a-1

Thus, in particular,

(5.8) ${}^{j(\kappa)>}M \subseteq M$. x-indep-a-2

Then (1) *$j(\kappa)$ is inaccessible.*

(2) *$\{\alpha < \kappa : \alpha$ is measurable$\}$ is normal measure 1 subset of κ.*

(3) *$M \models$ "$\{\alpha < j(\kappa) : \alpha$ is measurable$\}$ is stationary in $j(\kappa)$".*

(4) *$\{\alpha < j(\kappa) : \alpha$ is measurable$\}$ is cofinal in $j(\kappa)$.*

Proof. (1): Since κ is inaccessible. $M \models$ "$j(\kappa)$ is inaccessible" by elementarity (5.7). By (5.8), it follows that $j(\kappa)$ is really inaccessible.

(2): κ is measurable and an ultrafilter witnessing this is an element of M by (5.8) and (1). Thus $M \models$ "κ is measurable". $\mathcal{U} := \{A \subseteq \kappa : \kappa \in j(A)\}$ is a normal ultrafilter over κ and $\{\alpha < \kappa : \alpha$ is measurable$\} \in \mathcal{U}$.

(3): By (2), $\{\alpha < \kappa : \alpha$ is measurable$\}$ is a stationary subset of κ. By elementarity (5.7), it follows that $M \models$ "$\{\alpha < j(\kappa) : \alpha$ is measurable$\}$ a is a stationary subset of $j(\kappa)$".

(4): follows from (3) and (5.8). $\square$ (Lemma 5.5)

Proof of Theorem 5.3: Let $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$ be a huge elementary embedding, so that we have

(5.9) ${}^{j(\kappa)}M \subseteq M$. x-indep-a-3

For $\kappa \leq \gamma < j(\kappa)$, let $\mathcal{U}_\gamma := \{A \subseteq \mathcal{P}_\kappa(\gamma) : j''\gamma \in j(A)\}$. Then $\vec{\mathcal{U}} := \langle \mathcal{U}_\gamma : \kappa \leq \gamma < j(\kappa)\rangle \in M$ by (5.9), and $\vec{\mathcal{U}} \models$(5.6) (see the proof of [28], Theorem 24.11).

Since (5.6) is a closure property, M knows that there are club many $\alpha < j(\kappa)$ such that $\langle \mathcal{U}_\gamma : \kappa \leq \gamma < \alpha\rangle \models$ (5.6) .

By Lemma 5.5, (2), M thinks that there are stationarily many $\alpha < \kappa$ which are inaccessible (actually even measurable!). Thus $M \models$ "there are stationarily many inaccessible $\alpha < j(\kappa)$ such that $\langle \mathcal{U}_\gamma : \kappa \leq \gamma < \alpha\rangle \models$ (5.6)"

By Theorem 5.4,

(5.10) $M \models$ "$V_{j(\kappa)} \models \kappa$ is super almost-huge". x-indep-a-4

$\mathcal{U} := \{A \subseteq \kappa : \kappa \in j(A)\}$ is a normal ultrafilter over κ. By (5.10) $\{\alpha < \kappa : V_\kappa \models$ "α is super almost-huge"$\} \in \mathcal{U}$. $\square$ (Theorem 5.3)

For an $\mathcal{L}_\in$-formula $\psi = \psi(\overline{x})$, we shall call a large cardinal κ *ψ-absolute* if the formula ψ is absolute between V_κ and V (i.e. if for any $\overline{a} \in V_\kappa$, we have $V_\kappa \models \psi(\overline{a})$ $\Leftrightarrow \mathsf{V} \models \psi(\overline{a})$, or more formally, if the $\mathcal{L}_\in$-formula $(\forall \overline{x} \in V_y)(\psi^{V_y}(\overline{x}) \leftrightarrow \psi(\overline{x}))$ holds for $y = \kappa$).

Lemma 5.6 *For any concretely given $n \in \mathbb{N}$, there is an $\mathcal{L}_\in$-formula ψ^*_n such that for any inaccessible κ, κ is ψ^*_n-absolute if and only if* p-indep-0-2

(5.11) *for any $\mathcal{M} \subseteq \mathsf{V}$ such that $\mathcal{M}$ is a set forcing ground of V with $\mathsf{V} = \mathcal{M}[\mathbb{G}]$ where $\mathbb{G}$ is an $(\mathcal{M}, \mathbb{P})$-generic filter for some poset $\mathbb{P} \in (V_\kappa)^{\mathcal{M}}$ (including the case of $\mathbb{P} = \{\mathbb{1}_{\mathbb{P}}\}$ and $\mathcal{M} = \mathsf{V}$), we have that all Σ^{ZFC}_n-formulas are absolute between $(V_\kappa)^{\mathcal{M}}$ and $\mathcal{M}$.* x-indep-a-5

Proof. By the analysis of set forcing ground in connection with Laver-Woodin theorem on definability of grounds (see e.g. [37]). ❑ (Lemma 5.6)

Lemma 5.7 *Suppose that ψ^*_n is as in Lemma 5.6. Then ψ^*_n-absolute inaccessible cardinals are not resurrectable. I.e., if $\mathbb{P}$ is a posets and $\Vdash_{\mathbb{P}}$ "$\check{\lambda}$ is ψ^*_n-absolute inaccessible cardinal", then λ is really ψ^*_n-absolute inaccessible cardinal.* p-indep-0-3

Proof. This is clear by the choice (5.11) of ψ^*_n. ❑ (Lemma 5.7)

Lemma 5.8 *Assume that* MP *holds. Suppose that ψ^*_n for some $n \in \mathbb{N}$ is as in Lemma 5.6. If there is a ψ^*_n-absolute inaccessible cardinal, then there are unboundedly may ψ^*_n-absolute inaccessible cardinal.* p-indep-0-4

Proof. "ψ^*_n-absolute inaccessible cardinal" as an abstract notion of large cardinal satisfies (4.8) and (4.9). Thus the Lemma follows from Theorem 4.8. ❑ (Lemma 5.8)

Lemma 5.9 *Suppose that ψ is an arbitrary $\mathcal{L}_\in$-formula. If there are stationarily many inaccessible cardinals, then there are cofinally many ψ-absolute inaccessible cardinals.* p-indep-0-5

Proof. Let λ be an arbitrary cardinal. By Montague-Lévy Reflection Lemma

(5.12) $\mathcal{C} := \{\kappa \in \mathit{Card} \,:\, \kappa > \lambda,\ \psi \text{ is absolute between } V_\kappa \text{ and } \mathsf{V}\}$ x-indep-a-6

contains a (definable) club subclass of *Card*. By assumption there is an inaccessible $\kappa \in \mathcal{C}$. Then $\kappa > \lambda$ is ψ-absolute. ❑ (Lemma 5.9)

The following theorem says that there is no reasonable notion of large cardinal such that existence of that large cardinal implies MP.

Theorem 5.10 *Suppose that "x-large cardinal" is a normal notion of large cardinal. Then* ZFC + *"there is an x-large cardinal"* + ¬MP *is consistent.* p-indep-0-6

Proof. Let $n \in \mathbb{N}$ be such that "κ is an x-large cardinal" is Σ_n^{ZFC}.

We work in ZFC + "there is an x-large cardinals" + "there are stationarily many inaccessible cardinals". Note that this theory is consistent by the normality of the x-largeness.

Let κ be an x-large cardinal, and let κ_0 and κ_1 be the first two ψ_n^*-absolute inaccessible cardinals above κ (they exist by Lemma 5.9).

By ψ_n^*-absoluteness of κ_1 and the choice of n, we have $V_{\kappa_1} \models$ "κ is an x-large cardinal". Since κ_0 is the unique ψ_n^*-absolute inaccessible cardinal in V_{κ_1}, we have $V_{\kappa_1} \not\models$ MP by Lemma 5.8. ❑ (Theorem 5.10)

Similar theorem also holds for Laver-generic versions of normal notions of large cardinal.

Theorem 5.11 *Suppose that "x-large cardinal" is a normal notion of large cardinal with Laver function and that its tight Laver-gen. version can be forced similarly to Theorem 3.3 for an iterable class $\mathcal{P}$ of posets given Theorem 3.3. Then* ZFC + *"there is a tightly Laver generically x-large cardinal for $\mathcal{P}$"* + ¬MP *is consistent.* p-indep-0-7

Proof. Let $n \in \mathbb{N}$ be such that the statement "κ is an x-large cardinal" is Σ_n^{ZFC}.

Let κ be an x-large cardinal, and let κ_0 and κ_1 be the first two ψ_n^*-absolute inaccessible cardinals above the x-large cardinal κ (as before they exist by Lemma 5.9).

By the choice of n, $V_{\kappa_1} \models$ "κ is an x-large cardinal". Thus by the assumption on the property "x-large cardinal", there is $\mathbb{P} \in V_{\kappa_1}$ such that, for $(V_{\kappa_1}, \mathbb{P})$-generic $\mathbb{G}$, we have

(5.13) $\quad V_{\kappa_1}[\mathbb{G}] \models$ "κ is tightly $\mathcal{P}$-Laver-gen. x-large cardinal".

In $V_{\kappa_1}[\mathbb{G}]$, κ_0 is still the unique ψ_n^*-absolute inaccessible cardinal above κ by Lemma 5.7. Thus $V_{\kappa_1}[\mathbb{G}] \not\models$ MP by Lemma 5.9. ❑ (Theorem 5.11)

6 Boldface Maximality Principle for an iterable class $\mathcal{P}$ of posets and Laver-genericity

For an iterable class $\mathcal{P}$ of posets and (a definition of) a set Σ, the *Maximality Principle for $\mathcal{P}$ with parameters from Σ* ($\mathsf{MP}(\mathcal{P}, \Sigma)$) is the following principle: para-max

$\mathsf{MP}(\mathcal{P}, \Sigma)$: For any $\mathcal{L}_\in$-formula $\varphi = \varphi(x_0, ...)$ and $a_0, ... \in \Sigma$, if there is $\mathbb{P} \in \mathcal{P}$ such that for any $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of a poset with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", we have $\Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}}$ "$\varphi(\check{a}_0, ...)$", then we actually have $\Vdash_{\mathbb{R}}$ "$\varphi(\check{a}_0, ...)$" for all $\mathbb{R} \in \mathcal{P}$.[9)]

Similarly to (4.1), we shall call $\varphi(a_0, \ldots)$ as above a $\mathcal{P}$-button, and $\mathbb{P}$ a push of the $\mathcal{P}$-button.

Theorem 6.1 *Suppose that "x-large cardinal" is a normal notion of large cardinal with a Laver function such that the tight Laver-gen. version of x-large cardinal can be forced similarly to Theorem 3.3 for one of the iterable classes $\mathcal{P}$ of posets given in Theorem 3.3. Working in $\mathsf{V} \models$ "ZFC $+$ κ is an x-large cardinal $+$ $V_\kappa \prec \mathsf{V}$", there is a poset $\mathbb{P}$ such that for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, we have* p-para-max-0

$$\mathsf{V}[\mathbb{G}] \models \text{"}\mathsf{ZFC} \ + \ \kappa \text{ is } \mathcal{P}\text{-Laver gen. x-large cardinal} \ + \ \mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa))\text{"}.$$

The following rather trivial lemma is used in the proof of Theorem 6.1.

Lemma 6.2 *If $f : \kappa \to V_\kappa$ is a Laver function for an x-large cardinal κ, then it is a book-keeping of elements of V_κ. I,e., for any $a \in V_\kappa$ and $\alpha < \kappa$, there is $\beta \in \kappa \setminus \alpha$ such that $f(\beta) = a$.* p-para-max-1

Proof. Suppose that $a \in V_\kappa$ and $\alpha < \kappa$. Since f is a Laver function for x-largeness of κ, there is $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$ with the closure property of M corresponding to the x-largeness of κ such that $j(f)(\kappa) = a = j(a)$. Note that we have $j(\alpha) = \alpha < \kappa < j(\kappa)$. By elementarity of j, it follows that there is $\beta \in \kappa \setminus \alpha$ such that $f(\beta) = a$.

$\square$ (Lemma 6.2)

Proof of Theorem 6.1: We shall only consider the case that $\mathcal{P}$ is the class of all proper posets. The other cases can be treated similarly.

Let $f : \kappa \to V_\kappa$ be a Laver function for x-largeness of κ. Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be the CS-iteration of proper posets defined by

$$\text{(6.1)} \qquad \underset{\sim}{\mathbb{Q}}_\beta = \begin{cases} f(\beta), & \text{if } f(\beta) \text{ is a } \mathbb{P}_\beta\text{-name and } \Vdash_{\mathbb{P}_\beta} \text{"} f(\beta) \text{ is a proper poset"}; \quad (*) \\ \text{a } \mathbb{P}_\beta\text{-name of a push of the } \mathcal{P}\text{-button } \varphi(\underset{\sim}{a}_0, \ldots) \text{ in } V_\kappa, \\ \qquad \text{if } f(\beta) \text{ is the } \mathcal{L}_\in\text{-formula with } \mathbb{P}_\beta\text{-names } \underset{\sim}{a}_0, \ldots \in V_\kappa, \text{ and} \\ \qquad V_\kappa \models \text{"} \Vdash_{\mathbb{P}_\beta} \text{"}\varphi(\underset{\sim}{a}_0, \ldots) \text{ is a } \mathcal{P}\text{-button""}; \quad (**) \\ \mathbb{P}_\beta\text{-name of the trivial poset}, \quad \text{otherwise} \end{cases}$$ x-para-max-0

for $\beta < \kappa$.

Let $\mathbb{P} = \mathbb{P}_\kappa$. Note that $\mathbb{P} \in \mathcal{P}$. Let $\mathbb{G}$ be $(\mathsf{V}, \mathbb{P})$-generic. Then $\mathsf{V}[\mathbb{G}] \models$ "κ is $\mathcal{P}$-Laver gen. x-large" by (*) in (6.1) (see the proof of Theorem 3.3).

To show that $\mathsf{V}[\mathbb{G}]$ satisfies $\mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa))$, let $a_0, \ldots \in \mathcal{H}(\kappa)^{\mathsf{V}[\mathbb{G}]}$ and $\mathcal{L}_\in$-formula $\varphi = \varphi(x_0, \ldots)$ be such that

---

9) In particular, since $\{\mathbb{1}\} \in \mathcal{P}$, $\varphi(a_0, \ldots)$ holds.

$$\mathsf{V}[\mathbb{G}] \models \varphi(a_0, \ldots) \text{ is a } \mathcal{P}\text{-button.}$$

Without loss of generality, we may assume that $\underset{\sim}{a}_0, \ldots$ are $\mathbb{P}$-names of $a_0, \ldots$ respectively and

(6.2) $\quad \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \ldots)$ is a $\mathcal{P}$-button".[10)] x-para-max-1

Let $\alpha < \kappa$ be such that $\underset{\sim}{a}_0, \ldots$ are $\mathbb{P}_\alpha$ names.

For all $\alpha \leq \beta < \kappa$, since $\mathbb{P}_\kappa \sim \mathbb{P}_\beta * \underset{\sim}{\mathbb{R}}$ where $\Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\mathbb{R}}$ is proper", (6.2) implies that we have

(6.3) $\quad \Vdash_{\mathbb{P}_\beta}$ "$\varphi(\underset{\sim}{a}_0, \ldots)$ is a $\mathcal{P}$-button". x-para-max-2

By Lemma 6.2, there is $\alpha \leq \beta^* < \kappa$ such that $f(\beta^*) = \varphi(\underset{\sim}{a}_0, \ldots)$. By (6.3), and since $V_\kappa \prec \mathsf{V}$, there is a push of the $\mathcal{P}$-button $\varphi(\underset{\sim}{a}_0, \ldots)$ in V_κ. Thus, by $(**)$ in (6.1), $\underset{\sim}{\mathbb{Q}}_\beta$ is such a push. Since $\mathbb{P} \sim \mathbb{P}_{\beta^*} * \underset{\sim}{\mathbb{Q}}_{\beta^*} * \underset{\sim}{\mathbb{R}}$ where $\Vdash_{\mathbb{P}_{\beta^*} * \underset{\sim}{\mathbb{Q}}_{\beta^*}}$ "$\underset{\sim}{\mathbb{R}}$ is proper", it follows that $\Vdash_{\mathbb{P}}$ "$\forall Q\,(Q \in \mathcal{P} \ \rightarrow \ \Vdash_Q$ "$\varphi(\check{\underset{\sim}{a}}_0, \ldots)$")". Thus $\mathsf{V}[\mathbb{G}] \models \forall Q\,(Q \in \mathcal{P} \ \rightarrow$ $\Vdash_Q$ "$\varphi(a_0, \ldots)$"). ❑ (Theorem 6.1)

The following proposition is a variation of Theorem 4.8

Proposition 6.3 *Suppose that* $\mathsf{MP}(\mathcal{P}, \Sigma)$ *holds for an iterable class* $\mathcal{P}$ *of posets which contains either all posets of the form* $\mathrm{Col}(\omega_1, \lambda)$ *or posets adding arbitrary number of reals. Suppose further that "x-large cardinal" is a notion of large cardinals formalizable in* $\mathcal{L}_\in$ *such that* p-para-max-1-0

(6.4) $\quad$ *If* κ *is an x-large cardinal, then* κ *is (weakly, resp.) inaccessible; and* x-para-max-3

(6.5) $\quad$ *no new x-large cardinal is created by forcing by any* $\mathbb{P} \in \mathcal{P}$. x-para-max-4

If there is an x-large cardinal, then there are cofinally many x-large cardinals in V.

Proof. Similarly to the proof of Theorem 4.8. See also the remark after the proof of Theorem 4.8. ❑ (Proposition 6.3)

For $n \in \mathbb{N}$, let ψ_n^* be the $\mathcal{L}_\in$-formula introduced in Lemma 5.6. Since it is clear that "ψ_n^*-absolute inaccessible cardinal" a notion of large cardinal satisfying the conditions in Proposition 6.3, we obtain the following:

---

10) E.g., by replacing φ with $((\varphi(x_0, \ldots) \land y \equiv 0) \lor y \equiv 1)$, and $\underset{\sim}{a}_0, \ldots$ with $\underset{\sim}{a}_0, \ldots, \underset{\sim}{b}$. where $\underset{\sim}{b}$ is defined as follows: Let A be a maximal antichain $\subseteq \{\mathbb{p} \in \mathbb{P} : \mathbb{p} \parallel_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \ldots)$ is a button"$\}$ (here, φ is the original φ before the replacement), and $\underset{\sim}{b} := \{\langle \check{0}, \mathbb{p}\rangle : \mathbb{p} \in A,\ \mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \ldots)$ is a button"$\} \cup \{\langle \check{1}, \mathbb{p}\rangle : \mathbb{p} \in A,\ \mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{a}_0, \ldots)$ is not a button"$\}$.

Corollary 6.4 *Suppose that* $\mathsf{MP}(\mathcal{P},\Sigma)$ *holds for an iterable class* $\mathcal{P}$ *of posets which contains either all posets of the form* $\mathrm{Col}(\omega_1,\lambda)$ *or posets adding arbitrary number of reals. Then for any* $n\in\mathbb{N}$*, if there is a* ψ_n^**-absolute inaccessible cardinal. Then there are cofinally many* ψ_n^**-absolute inaccessible cardinals.* ❑ p-para-max-1-1

Theorem 6.5 *Suppose that "x-large cardinal" is a normal notion of large cardinal with Laver function such that the tight Laver-gen. version of x-large cardinal can be forced similarly to Theorem 3.3 for one of the iterable classes* $\mathcal{P}$ *of posets given in Theorem 3.3. Working in* $\mathsf{V}\models$ "$\mathsf{ZFC}\ +\ \kappa$ is an x-large cardinal $+\ V_\kappa\prec\mathsf{V}$ + there are ψ_n^*-absolute inaccessible cardinals above κ" *for sufficiently large* n*, and letting* κ_1 *the second* ψ_n^**-absolute inaccessible cardinal above* κ*, there is a poset* $\mathbb{P}$ *such that for* $(V_{\kappa_1},\mathbb{P})$*-generic* $\mathbb{G}$*, we have* p-para-max-2

$$\mathsf{V}_{\kappa_1}[\mathbb{G}]\models\text{“}\mathsf{ZFC}\ +\ \kappa\ \textit{is } \mathcal{P}\textit{-Laver gen. x-large cardinal}\\ +\ \neg\mathsf{MP}\ +\ \neg\mathsf{MP}(\mathcal{P},\mathcal{H}(\kappa))\text{”}.$$

Proof. The proof of Theorem 5.10 works also here by Corollary 6.4. ❑ (Theorem 6.5)

In spite of Theorem 6.5, the existence of a $\mathcal{P}$-Laver generically ultrahuge cardinal implies a local version of maximality principle.

To define the local version of maximality principle we are going to talk about below, let us call an $\mathcal{L}_\in$-formula $\varphi=\varphi(x,a)$ with a parameter a a *local property of cardinals* if, for any limit ordinal δ with $a\in V_\delta$ and a cardinal $\mu<\delta$, we have $\big(V_\delta\models\varphi(\mu,a)\big)\ \leftrightarrow\ \varphi(\mu,a)$ and that this fact is provable in ZFC (+ some formulas with the parameter a which depict features of the set a). Being an inaccessible cardinal is a local property of cardinals, as well as being a Mahlo cardinal or being a measurable cardinal. In contrast, being a supercompact cardinal is not necessarily a local property of cardinals.

A local property of cardinals $\varphi=\varphi(x,a)$ is a *local definition of a cardinal* if there is provably at most one cardinal which satisfies the formula.

"The first inaccessible cardinal above a given cardinal μ" is a local definition of a cardinal as well as "the first measurable above μ" but not "the least supercompact cardinal above μ".

If $\varphi(x,a)$ is a local definition of a cardinal, we denote the cardinal defined by $\varphi(x,a)$ with $\kappa^\bullet_{\varphi(x,a)}$, $\mu^\bullet_{\varphi(x,a)}$, etc. or just with $\kappa^\bullet$, $\mu^\bullet$, etc. if we want to drop the explicit mention of the formula $\varphi(x,a)$ which defines the term. In the latter notation we identify the term $\kappa^\bullet$ with its definition $\varphi(x,a)$ and say also that $\kappa^\bullet$ is a local definition of the cardinal.

$\beth_\alpha(\omega_\beta)$ for any concretely given finite or countable ordinal α, β is another example of a local definition of a cardinal.

Using this notation, we can show now that the existence of a $\mathcal{P}$-Laver gen. ultrahuge cardinal implies the following local version of Maximality Principle for $\mathcal{P}$. The following theorem is in line with the results in [30]:

Theorem 6.6 *Suppose that $\mathcal{P}$ is an iterable class of posets and κ is tightly $\mathcal{P}$-Laver gen. ultrahuge. Then, for any $\mathcal{L}_\in$-formula $\varphi(x_0, \ldots, x_{n-1})$, $a_0, \ldots, a_{n-1} \in \mathcal{H}(\kappa)$, and a local definition $\mu^\bullet$ of a cardinal, if there is $\mathbb{P} \in \mathcal{P}$ such that,* p-UR-4

(6.6) *for any $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", we have $\Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}}$ "$V_{\mu^\bullet} \models \varphi(\check{a}_0, \ldots, \check{a}_{n-1})$",* x-UR-6

then we have $(V_{\mu^\bullet})^{\mathsf{V}} \models \varphi(a_0, \ldots, a_{n-1})$.

Proof. Let κ, φ, $a_0, \ldots, a_{n-1}$, $\mu^\bullet$, $\mathbb{P}$ as above. Let $\lambda > (\mu^\bullet)^{\mathsf{V}}$ be a limit ordinal. Then there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" such that, for $(V, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that

(6.7) $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, x-UR-7

(6.8) $j(\kappa) > \lambda$, x-UR-8

(6.9) $\mathbb{P}$, $\mathbb{H}$, $(V_{j(\lambda)})^{\mathsf{V}[\mathbb{H}]} \in M$, and x-UR-9

(6.10) $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a poset of size $j(\kappa)$. x-UR-10

By the choice of λ and (6.7), we have $j(\lambda) > (\mu^\bullet)^M$. By (6.9) and (6.10), we have $(V_{j(\lambda)})^M = (V_{j(\lambda)})^{\mathsf{V}[\mathbb{H}]}$. Since $\mu^\bullet$ is a local definition, it follows that $(\mu^\bullet)^M = (\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}$, and $(V_{\mu^\bullet})^M = (V_{\mu^\bullet})^{\mathsf{V}[\mathbb{H}]}$. Thus, by the choice of $\mathbb{P}$, we have $M \models$ "$V_{\mu^\bullet} \models \varphi(a_0, \ldots, a_{n-1})$". Since $a_i = j(a_i)$ for $i < n$ by (6.7), it follows by the elementarity that $(V_{\mu^\bullet})^{\mathsf{V}} \models \varphi(a_0, \ldots, a_{n-1})$. ❑ (Theorem 6.6)

7 Resurrection Axioms

The following variants of Resurrection Axioms are introduced and studied by J. Hamkins and T. Johnstone ([35], [36]). resurr

For a class $\mathcal{P}$ of posets and a definition $\mu^\bullet$ of a cardinal (e.g. as $\aleph_1$, $\aleph_2$, $2^{\aleph_0}$, $(2^{\aleph_0})^+$. etc.) the *Resurrection Axiom for $\mathcal{P}$ and $\mathcal{H}(\mu^\bullet)$* is defined by:

$\mathsf{RA}^{\mathcal{P}}_{\mathcal{H}(\mu^\bullet)}$: For any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of poset such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, we have $\mathcal{H}(\mu^\bullet)^{\mathsf{V}} \prec \mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}$.

Here, $\mu^\bullet$'s in the left and right side of the last formula are actually meant $(\mu^\bullet)^{\mathsf{V}}$ and $(\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}$ respectively.

The following boldface version of the Resurrection Axioms is also considered in [36]: For a class $\mathcal{P}$ of posets and a definition $\mu^\bullet$ of a cardinal (e.g. as $\aleph_1$, $\aleph_2$, $2^{\aleph_0}$, $(2^{\aleph_0})^+$. etc.) the *Resurrection Axiom in Boldface for $\mathcal{P}$ and $\mathcal{H}(\mu^\bullet)$* is defined by:

$\mathbb{RA}^{\mathcal{P}}_{\mathcal{H}(\mu^\bullet)}$: For any $A \subseteq \mathcal{H}(\mu^\bullet)$ and any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of poset such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there is $A^* \subseteq \mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}$ such that $(\mathcal{H}(\mu^\bullet)^{\mathsf{V}}, A, \in) \prec (\mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}, A^*, \in)$.

Clearly $\mathbb{RA}^{\mathcal{P}}_{\mathcal{H}(\mu^\bullet)}$ implies $\mathsf{RA}^{\mathcal{P}}_{\mathcal{H}(\mu^\bullet)}$.

In the following we write $\kappa_{\mathfrak{refl}} := \max\{\aleph_2, 2^{\aleph_0}\}$. Note that this cardinal is the reflection point of the reflection properties we obtain in all scenarios of the trichotomy in Theorem 3.3 or Theorem 3.5.

Theorem 7.1 *For an iterable class of posets $\mathcal{P}$, if $\kappa_{\mathfrak{refl}}$ is tightly $\mathcal{P}$-Laver-gen. superhuge, then $\mathbb{RA}^{\mathcal{P}}_{\mathcal{H}(\kappa_{\mathfrak{refl}})}$ holds.* T-Lg-RA-0

Proof. The following proof is based on the idea suggested by Gunter Fuchs during a talk I gave at the New York Set Theory Seminar on October 7, 2022.

Suppose $A \subseteq \mathcal{H}(\kappa_{\mathfrak{refl}})$ and $\mathbb{P} \in \mathcal{P}$. By the tightly $\mathcal{P}$-Laver-gen. superhugeness of $\kappa_{\mathfrak{refl}}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of a poset with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" such that, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with

$$j : \mathsf{V} \xrightarrow{\prec}_{\kappa_{\mathfrak{refl}}} M, \tag{7.1}$$ was-a

$$j(\kappa_{\mathfrak{refl}}) = |\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,|, \tag{7.2}$$ was-b

$$\mathbb{P},\ \mathbb{H} \in M, \text{ and} \tag{7.3}$$ was-c

$$j''j(\kappa_{\mathfrak{refl}}) \in M. \tag{7.4}$$ was-d

Without loss of generality, we may assume that the underlying set of $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is $j(\kappa_{\mathfrak{refl}})$.

Since $crit(j) = \kappa_{\mathfrak{refl}}$, $j(a) = a$ for all $a \in (\mathcal{H}(\kappa_{\mathfrak{refl}}))^{\mathsf{V}}$.

Claim 7.1.1 *$\mathcal{H}(j(\kappa_{\mathfrak{refl}}))^{\mathsf{V}[\mathbb{H}]} \subseteq M$ and hence $j(\mathcal{H}(\kappa_{\mathfrak{refl}})) = \mathcal{H}(j(\kappa_{\mathfrak{refl}}))^M = \mathcal{H}(j(\kappa_{\mathfrak{refl}}))^{\mathsf{V}[\mathbb{H}]}$.* Cl-Lg-RA-0

$\vdash$ Suppose that $b \in \mathcal{H}(j(\kappa_{\mathfrak{refl}}))^{\mathsf{V}[\mathbb{H}]}$ and let $c \subseteq j(\kappa_{\mathfrak{refl}})$ be a code of b. Let $\underset{\sim}{c}$ be a nice $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$-name of c. By (7.2), $|\,\underset{\sim}{c}\,| \leq j(\kappa_{\mathfrak{refl}})$. By (7.4), it follows that $\underset{\sim}{c} \in M$ (see Lemma 2.1, (5)). Thus $c \in M$ by (7.3) , and hence $b \in M$. $\dashv$ (Claim 7.1.1)

Thus, we have

$$id_{\mathcal{H}(\kappa_{\mathfrak{refl}})^{\mathsf{V}}} = j \restriction \mathcal{H}(\kappa_{\mathfrak{refl}})^{\mathsf{V}} : \ (\mathcal{H}(\kappa_{\mathfrak{refl}})^{\mathsf{V}}, A, \in) \ \overset{\preceq}{\rightarrow} \ (\mathcal{H}(j(\kappa_{\mathfrak{refl}}))^{\mathsf{V}[\mathbb{H}]}, j(A), \in).$$

□ (Theorem 7.1)

The following strengthening of the Resurrection Axiom is introduced by Tsaprounis [32]:

For an iterable class $\mathcal{P}$ of posets, the *Unbounded Resurrection Axiom for $\mathcal{P}$* is the following assertion.

$\mathsf{UR}(\mathcal{P})$: For any $\lambda > \kappa_{\mathfrak{refl}}$, and $\mathbb{P} \in \mathcal{P}$, there exists a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" such that, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-gen. $\mathbb{H}$, there are $\lambda^* \in \mathrm{On}$ and $j_0 \in \mathsf{V}[\mathbb{H}]$ such that $j_0 : \mathcal{H}(\lambda)^{\mathsf{V}} \overset{\preceq}{\rightarrow}_{\kappa_{\mathfrak{refl}}} \mathcal{H}(\lambda^*)^{\mathsf{V}[\mathbb{H}]}$, and $j_0(\kappa_{\mathfrak{refl}}) > \lambda$.

The "tight" version of the Unbounded Resurrection Axiom for $\mathcal{P}$ will be also considered.

$\mathsf{TUR}(\mathcal{P})$: For any $\lambda > \kappa_{\mathfrak{refl}}$, and $\mathbb{P} \in \mathcal{P}$, there exists a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" such that, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-gen. $\mathbb{H}$, there are $\lambda^* \in \mathrm{On}$, and $j_0 \in \mathsf{V}[\mathbb{H}]$ such that $j_0 : \mathcal{H}(\lambda)^{\mathsf{V}} \overset{\preceq}{\rightarrow}_{\kappa_{\mathfrak{refl}}} \mathcal{H}(\lambda^*)^{\mathsf{V}[\mathbb{H}]}$, $j_0(\kappa_{\mathfrak{refl}}) > \lambda$, and $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a poset of size $j_0(\kappa_{\mathfrak{refl}})$.

Both of the principles can be yet extended to boldface versions similarly to the boldface version $\mathbb{RA}^{\mathcal{P}}_{\mathcal{H}(\mu^\bullet)}$ of $\mathsf{RA}^{\mathbb{P}}_{\mathcal{H}(\mu^\bullet)}$. However, $\mathsf{UR}(\mathcal{P})$ and $\mathsf{TUR}(\mathcal{P})$ can be easily proved to be equivalent to their respective boldface (apparent) extensions.

Theorem 7.2 *For an iterable class $\mathcal{P}$, if $\kappa_{\mathfrak{refl}}$ is tightly $\mathcal{P}$-Laver gen. ultrahuge, then $\mathsf{TUR}(\mathcal{P})$ holds.* p-resurr-0

Proof. Suppose that $\kappa_{\mathfrak{refl}}$ is tightly $\mathcal{P}$-Laver gen. ultrahuge. Assume $\lambda > \kappa_{\mathfrak{refl}}$, and $\mathbb{P} \in \mathcal{P}$.

Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-gen. filter $\mathbb{H}$, there are $j, M \subseteq \mathsf{V}[\mathbb{H}]$ such that

(7.5) $\quad j : \mathsf{V} \overset{\preceq}{\rightarrow}_{\kappa_{\mathfrak{refl}}} M,$ x-resurr-0

(7.6) $\quad j(\kappa_{\mathfrak{refl}}) > \lambda,$ x-resurr-1

(7.7) $\quad \mathbb{P}, \mathbb{H}, V_{j(\lambda)} \in M$ and x-resurr-2

(7.8) $\quad \mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a poset of cardinality $j(\kappa_{\mathfrak{refl}})$. x-resurr-3

Without loss of generality, let us assume that

(7.9) $\quad |\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,| = j(\kappa_{\mathfrak{refl}})$, and $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \subseteq V_{j(\kappa_{\mathfrak{refl}})}$. x-resurr-4

Then $\mathcal{H}(j(\lambda))^{\mathsf{V}[\mathbb{H}]} \subseteq M$, since the code $\subseteq j(\lambda)$ of each element of $\mathcal{H}(j(\lambda))^{\mathsf{V}[\mathbb{H}]}$ has a $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$-name in $V_{j(\lambda)} \subseteq M$ by (7.8), (7.9) and (7.7). Thus the $\mathbb{H}$ interpretation of the $\mathbb{P}$-name of the code is in M by (7.7). Hence the coded element of $\mathcal{H}(j(\lambda))^{\mathsf{V}[\mathbb{H}]}$ is also in M.

It follows that $\mathcal{H}(j(\lambda))^M = \mathcal{H}(j(\lambda))^{\mathsf{V}[\mathbb{H}]}$.

Thus, letting $j_0 := j \restriction \mathcal{H}(\lambda)^{\mathsf{V}}$, and $\lambda^* := j(\lambda)$, we have

$$j_0 : \mathcal{H}(\lambda)^{\mathsf{V}} \xrightarrow{\preceq}_{\kappa_{\mathfrak{refl}}} \mathcal{H}(\lambda^*)^{\mathsf{V}[\mathbb{H}]} \quad \text{and} \quad j_0(\kappa_{\mathfrak{refl}}) = j(\kappa_{\mathfrak{refl}}) > \lambda \text{ by (7.6).}$$

This shows that $\mathsf{TUR}(\mathcal{P})$ holds. □ (Theorem 7.2)

References

ref

[1] Sean Cox, The diagonal reflection principle, Proceedings of the American Mathematical Society, Vol.140, No.8 (2012), 2893–2902.

[2] Alan Dow, Franklin D. Tall and W.A.R. Weiss, New proofs of the consistency of the normal Moore space conjecture I, II, Topology and its Applications, 37 (1990) 33–51, 115-129.

[3] Sakaé Fuchino, On potential embedding and versions of Martin's axiom, Notre Dame Journal of Logic, Vol.33, No.4, (1992), 481–492.

[4] ______________, Left-separated topological spaces under Fodor-type Reflection Principle, RIMS Kôkyûroku, No.1619, (2008), 32–42. 6

[5] ______________, Fodor-type Reflection Principle and Balogh's reflection theorems, RIMS Kôkyûroku, No.1686, (2010), 41–58. 6

[6] ______________, Rado' s Conjecture implies the Fodor-type Reflection Principle, Note, https://fuchino.ddo.jp/notes/RCimpliesFRP2.pdf 6

[7] ______________, On reflection numbers under large continuum, RIMS Kôkyûroku, No.1988, 1–16, (2016).

[8] ______________, Pre-Hilbert spaces without orthonormal bases, preprint. 6

[9] ______________, A reflection principle as a reverse-mathematical fixed point over the base theory ZFC, Annals of the Japan Association for the Philosophy of Science, Vol.25, (2017), 67–77.

[10] ____________, digital images of a presentation of a seminar talk given at Nagoya Set Theory Seminar on May 31, 2019,
https://fuchino.ddo.jp/talks/talk-nagoya-2019-05-31.pdf 13, 14

[11] ____________, Maximality Principle under a Laver-generic supercompact cardinal, RIMS Kôkûroku, No.2261, New Developments in Forcing and Cardinal Arithmetic (2022/10/25～2022/10/28), ed.: 酒井 拓史 (Hiroshi Sakai), (2023), 23–37. 2

[12] ____________, Resurrection and Maximality under a/the tightly Laver-generically ultrahuge cardinal, slides of a series zoom talks given at May 29, and June 5, 2023 at Kobe Set Theory Seminar.
https://fuchino.ddo.jp/slides/kobe2023-05-29-pf.pdf

[13] ____________, Resurrection and Maximality under a/the tightly Laver-generically ultrahuge cardinal — additional slides, additional slides of the zoom talks given at June 5, 2023 at Kobe Set Theory Seminar.
https://fuchino.ddo.jp/slides/kobe2023-06-05a-pf.pdf 9

[14] ____________, Resurrection and Maximality in light of Laver-generic large cardinal, in preparation. 2

[15] Sakaé Fuchino, István Juhász, Lajos Soukup, Zoltán Szentmiklóssy, and Toshimichi Usuba, Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness, Topology and its Applications, Vol.157, 8 (2010), 1415–1429. 6, 17

[16] Sakaé Fuchino, André Ottenbreit Maschio Rodrigues and Hiroshi Sakai, Strong Löwenheim-Skolem theorems for stationary logics, I, Archive for Mathematical Logic, Volume 60, issue 1-2, (2021), 17–47.
https://fuchino.ddo.jp/papers/SDLS-x.pdf 3, 6

[17] ________________________________, Strong Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum, Archive for Mathematical Logic, Volume 60, issue 3-4, (2021), 495–523.
https://fuchino.ddo.jp/papers/SDLS-II-x.pdf 1, 2, 3, 4, 8, 10, 13, 15, 17

[18] ________________________________, Strong downward Löwenheim-Skolem theorems for stationary logics, III — mixed support iteration, to appear in the Proceedings of the Asian Logic Conference 2019.
https://fuchino.ddo.jp/papers/SDLS-III-xx.pdf 1, 2, 6

[19] Sakaé Fuchino, and André Ottenbreit Maschio Rodrigues, Reflection principles, generic large cardinals, and the Continuum Problem, to appear in the Proceedings of the Symposium on Advances in Mathematical Logic 2018.
https://fuchino.ddo.jp/papers/refl_principles_gen_large_cardinals_continuum_problem-x.pdf 1, 2

[20] Sakaé Fuchino and Assaf Rinot, Openly generated Boolean algebras and the Fodor-type Reflection Principle, Fundamenta Mathematicae 212, (2011), 261-283.

[21] Fuchino, Sakaé, and Sakai, Hiroshi, The first-order definability of generic large cardinals, submitted. Extended version of the paper:
https://fuchino.ddo.jp/papers/definability-of-glc-x.pdf 1, 2, 3, 10

[22] Sakaé Fuchino, and Hiroshi Sakai, Generically and Laver-generically supercompact cardinals by forcing with chain conditions, RIMS Kôkûroku, No.2213, (2022), 94–111.
https://repository.kulib.kyoto-u.ac.jp/dspace/bitstream/2433/275453/1/2213-08.pdf

[23] Sakaé Fuchino, Hiroshi Sakai, Lajos Soukup and Toshimichi Usuba, More about Fodor-type Reflection Principle,
https://fuchino.ddo.jp/papers/moreFRP-x.pdf 6

[24] Sakaé Fuchino, Hiroshi Sakai, Victor Torres Perez and Toshimichi Usuba, Rado's Conjecture and the Fodor-type Reflection Principle, in preparation.

[25] Gunter Fuchs, Closed Maximality Principles: Implications, Separations and Combinations, The Journal of Symbolic Logic, Vol. 73, No. 1, (2008), 276–308.

[26] Joel Hamkins, A simple maximality principle, The Journal of Symbolic Logic Vol.68, no.7, (2003), 527–550. 20, 21, 22

[27] Thomas Jech, Set Theory, The Third Millennium Edition, Springer (2001/2006).

[28] Akihiro Kanamori, The Higher Infinite, Springer–Verlag (1994/2003). 4, 10, 24, 25

[29] Bernhard König, Generic compactness reformulated, Archive for Mathematical Logic 43, (2004), 311–326. 6

[30] Kaethe Minden, Combining resurrection and maximality, The Journal of Symbolic Logic, Vol. 86, No. 1, (2021), 397–414. 2, 31

[31] Stevo Todorčević, Conjectures of Rado and Chang and cardinal arithmetic, In: Sauer, N.W., Woodrow, R.E., Sands, B. (eds.) Finite and Infinite Combinatorics in Sets and Logic. NATO ASI Series, vol 411, (1993), 385 – 398. 6

[32] Konstantinos Tsaprounis, On resurrection axioms, The Journal of Symbolic Logic, Vol.80, No.2, (2015), 587–608. 33

[33] ______________________, Ultrahuge cardinals, Mathematical Logic Quarterly, Vol.62, No.1-2, (2016), 1–2. 8, 9, 24

Additional references cited in the extended version

[34] Uri Abraham, and Saharon Shelah, Forcing closed unbounded sets, Journal of Symbolic Logic, Vol.48(3), (1983), 643–657. 18

[35] Joel David Hamkins, and Thomas A. Johnstone, Resurrection axioms and uplifting cardinals, Archive for Mathematical Logic, Vol.53, Iss.3-4, (2014), 463–485. 31

[36] Joel David Hamkins, and Thomas A. Johnstone, Strongly uplifting cardinals and the boldface resurrection axioms, Archive for Mathematical Logic volume 56, (2017), 1115–1133. 31, 32

[37] Jonas Reitz, The Ground Axiom, The Journal of Symbolic Logic, Vol. 72, No. 4 (2007), 1299–1317. 26