Extendible cardinals, and Laver-generic large cardinal axioms for extendibility

Sakaé Fuchino* (渕野 昌)

Abstract

We introduce (super-$C^{(\infty)}$-)Laver-generic large cardinal axioms for extendibility ((super-$C^{(\infty)}$-)LgLCAs for extendible, for short), and show that most of the previously known consequences of the (super-$C^{(\infty)}$-)LgLCAs for ultrahuge, in particular, the strong and general forms of Resurrection Principles, Maximality Principles, and Absoluteness Theorems, already follow from (super-$C^{(\infty)}$-)LgLCAs for extendible.

The consistency of the LgLCAs for extendible (for transfinitely iterable Σ_2-definable classes of posets) follows from an extendible cardinal while the consistency of super-$C^{(\infty)}$-LgLCAs for extendible follows from a model with a super-$C^{(\infty)}$-extendible cardinal. If κ is an almost-huge cardinal, there are cofinally many $\kappa_0 < \kappa$ such that $V_\kappa \models$ "κ_0 is super-$C^{(\infty)}$ extendible".

Most of the known reflection properties follow already from some of the LgLCAs for supercompact. We give a survey on the related results.

We also show the separation between some of the LgLCAs as well as between LgLCAs and their consequences.

Contents

* Graduate School of System Informatics, Kobe University
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan
fuchino@diamond.kobe-u.ac.jp

1 Introduction

The present note is a short version of the more extensive [9] in preparation.

In Section 2, we begin with reviewing known characterizations of extendible cardinals (Proposition 2.4). We then look into the super-$C^{(n)}$ and super-$C^{(\infty)}$ large cardinal versions of extendibility, and give their characterizations (Proposition 2.6, Theorem 2.7).

In Section 3, we evaluate the consistency strength of super $C^{(\infty)}$-extendible cardinal: It is classical that if κ is almost huge, then V_κ satisfies the Second-order Vopěnka Principle (Lemma 3.1). We show that the Second-order Vopěnka Principle implies that there are cofinally many $\kappa_0 < \kappa$ such that $V_\kappa \models$ "κ_0 is super-$C^{(\infty)}$ extendible" (Proposition 3.2).

In Section 4, we introduce Laver-generic large cardinal versions of these large cardinals, and the axioms asserting the existence of a/the Laver-generic large cardinals — the (super-$C^{(\infty)}$-) $\mathcal{P}$-Laver-generic large cardinal axioms for extendibility ((super-$C^{(\infty)}$-)LgLCAs for extendible, for short) for various classes $\mathcal{P}$ of posets, and show that most of the previously known consequences of the (super-$C^{(\infty)}$-)LgLCAs for ultrahugeness, in particular, the strong and general forms of Resurrection Principles, Maximality Principles, and Absoluteness Theorems, already follow from (super-$C^{(\infty)}$-)LgLCAs for extendible.

The consistency of the LgLCAs for extendible (for transfinitely iterable Σ_2-definable classes of posets) follows from an extendible cardinal while the consistency of super-$C^{(\infty)}$- $\mathcal{P}$-LgLCAs for extendible for transfinitely iterable classes $\mathcal{P}$ of posets (which may be defined by formulas more complex than 2) follow from a model with super-$C^{(\infty)}$ extendible cardinal (see Theorem 5.2).

In contrast, it is known that (super-$C^{(\infty)}$-)LgLCAs for hyperhugeness for transfinitely iterable class $\mathcal{P}$ of posets, axioms apparently stronger than the corresponding axioms for ultrahugeness, are equiconsistent with the existence of a genuine (super-$C^{(\infty)}$-)hyperhuge ([16]).

Most of the known reflection properties are consequences of some of the LgLCAs for supercompact. In Section **??**, we give a survey on this topic.

Date: January 23, 2025 *Main update:* January 31, 2025 (23:59 JST)

MSC2020 Mathematical Subject Classification: 03E45, 03E50, 03E55, 03E57, 03E65

Keywords: generic large cardinal, Laver-generic large cardinal, Maximality Principles, Continuum Hypothesis, Ground Axiom

The most recent and extended version of this paper possibly icluding more details and proofs than the present one is downloadable as:
`https://fuchino.ddo.jp/papers/RIMS2024-extendible-x.pdf`

In Section **??**, we show the separation between some of the LgLCAs as well as between LgLCAs and their consequences.

Our notation is standard, and mostly compatible with that of [25], [26], and/or [28], but with the following slight deviations: "$j : M \overset{\prec}{\rightarrow}_\kappa V$" expresses the situation that M and N are transitive (sets or classes), j is an elementary embedding of M into N and κ is the critical point of j. We use letters with under-tilde to denote $\mathbb{P}$-names for a poset $\mathbb{P}$. Underline added to a symbol like $\underline{\alpha}$ emphasizes that the symbol is used to denote a variable in a language, mostly the language of ZFC which is denoted by $\mathcal{L}_\in$. A letter with under-bracket like $\underbracket{c}$ emphasizes that the letter denotes a (new) constant symbol added to the language.

In the following, we always denote with $\mathcal{P}$, a class of posets. We usually assume that the class $\mathcal{P}$ of posets satisfies some natural properties. A class $\mathcal{P}$ of posets is (two-step) *iterable* if

(1.1) $\mathcal{P}$ is closed with respect to forcing equivalence, and $\{\mathbb{1}\} \in \mathcal{P}$;

(1.2) $\mathcal{P}$ is closed with respect to restriction. That is, for $\mathbb{P} \in \mathcal{P}$ and $\mathbb{p} \in \mathbb{P}$, we always have $\mathbb{P} \restriction \mathbb{p} \in \mathcal{P}$; and

(1.3) For any $\mathbb{P} \in \mathcal{P}$, and any $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of a poset with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", we have $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathcal{P}$.

$\mathcal{P}$ is *transfinitely iterable* if it is iterable and it permits iteration of arbitrary length for an appropriate notion of support with reasonable iteration lemmas.

For a property P of posets, we shall say "$\mathcal{P}$ is P" to indicate that all elements of $\mathcal{P}$ have the property P. In contrast, if we say $\mathcal{P}$ is the class of posets with the property P, we mean $\mathcal{P} = \{\mathbb{P} : \mathbb{P} \models P\}$.

We adopt the notation of [1] and denote $C^{(n)} := \{\alpha \in \mathrm{On} : V_\alpha \prec_{\Sigma_n} \mathsf{V}\}$ for $n \in \mathbb{N}$. Intuitively we put $C^{(\infty)} := \{\alpha \in \mathrm{On} : V_\alpha \prec \mathsf{V}\}$, though this is not a definable class in the language of ZFC, due to undefinability of the truth. For each transitive set model M, however, $\{\alpha \in \mathrm{On} \cap M : {V_\alpha}^M \prec M\}$ is a(n existent) set except that the first order logic in this context which the elementary submodel relation $\prec$ relies on, is not the meta-mathematical one but rather the logic in the set theory whose formulas are the corresponding subset of ω (consisting of Gödel numbers) in ZFC. Also, this set is not a first-order definable subset of M, again because of the undefinability of the truth.

The following (almost trivial) lemma is often used without mention:

Lemma 1.1 (see e.g. Section 1 in Bagaria [1]) *For an uncountable cardinal α, $\mathcal{H}(\alpha) = V_\alpha$ if and only if $V_\alpha \prec_{\Sigma_1} \mathsf{V}$.*

If $V_\alpha \prec_{\Sigma_1} \mathsf{V}$ then α is an uncountable strong limit cardinal.

Thus, we have

$$C^{(1)} = \{\alpha : \alpha \textit{ is an uncountable limit cardinal with } V_\alpha = \mathcal{H}(\alpha)\}. \qquad \square$$

Note that there is a (first-order) sentence φ such that $V_\alpha \models \varphi$ if and only if $V_\alpha = \mathcal{H}(\alpha)$ for a cardinal α.

The following notes are results of an examination of what was suggested by Gabriel Goldberg in a discussion we had during his visit to Kobe after the RIMS Set Theory Workshop 2024. Toshimichi Usuba pointed out some elementary flows in early sketches of the note. I learned some known arguments used below in conversation with Hiroshi Sakai. I am grateful for their comments and advices. Also I would like to thank Andreas Leitz for giving me a permission to present an exposition of his proof of Theorem 2.7 in the extended version of the present article.

Back in the summer of 2015, I enjoyed a pleasant walk around the port of Yokohama with Joel Hamkins when we were together on the way to Kyoto starting from Tokyo and made a short stop in Yokohama. On the walk, Joel told me about his then recent researches and research projects, and one of them was about the Resurrection Axioms.

Now that his Resurrection Axioms are shown to be restricted versions of the LgLCAs (see Theorem 4.2), I notice that what I learned from him on that walk might have influenced me subliminally when I introduced the LgLCAs in the late 2010s. In that case, I have to to thank Joel again sincerely, also for the nice conversation we had in Yokohama.

2 Extendible and super-$C^{(n)}$-extendible cardinals

In this section, we summarize some well-known and some other less well-known facts about extendible cardinals and introduce the notion of super-$C^{(n)}$-extendible cardinals.

It appears that the notion of super-$C^{(n)}$-extendible cardinals is equivalent to some other already known strong variants of extendibility, see Theorem 2.7. At the moment, it is yet unknown if similar equivalence is also available for super-$C^{(n)}$-ultrahuge cardinals, or super-$C^{(n)}$-hyperhuge cardinals.

It is easy to see that the definition of an extendible cardinal in Kanamori [26] is equivalent to its slight modification: a cardinal κ is *extendible* if (2.1) : for any $\alpha > \kappa$ there are $\beta \in \mathrm{On}$ and $j : V_\alpha \xrightarrow{\prec}_\kappa V_\beta$ such that (2.2) : $j(\kappa) > \alpha$.

An extendible cardinal is supercompact (see e.g. Proposition 23.6 in [26]). The following is easy to prove:

Lemma 2.1 *If κ is extendible then there are class many measurable cardinals.* ❑

Since existence of a supercompact cardinal does not imply existence of any large cardinal above it (see Exercise 22.8 in [26]), Lemma 2.1 explains the transcendence of extendible cardinals above supercompact.

In Jech [25], extendibility is defined by (2.1) without (2.2). We say in the following that κ is *Jech-extendible* if it satisfies (2.1) but not necessarily (2.2). The two definitions of extendibility are equivalent. In Proposition 2.4 below, we show the equivalence of these two together with some other characterizations of extendibility.

The key fact to Proposition 2.4 is that the elementary embedding in (2.1) can be often lifted to an elementary embedding of the whole universe V.

We call a mapping $f : M \to N$ *cofinal* (in N) if, for all $b \in N$, there is $a \in M$ such that $b \in f(a)$.

Lemma 2.2 (A special case of Lemma 6 in Fuchino and Sakai [14]) *Suppose that θ is a cardinal and $j_0 : \mathcal{H}(\theta) \overset{\preceq}{\to} N$ for a transitive set N. Let $N_0 := \bigcup j_0{}''\mathcal{H}(\theta)$. Then $j_0 : \mathcal{H}(\theta) \overset{\preceq}{\to} N_0$ and j_0 is cofinal in N_0.* ❑

Lemma 2.3 (A special case of Lemma 7 in [14]) *For any regular cardinal θ and any cofinal $j_0 : \mathcal{H}(\theta) \overset{\preceq}{\to} N$, there are j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \overset{\preceq}{\to} M$, $N \subseteq M$ and $j_0 \subseteq j$.* ❑

Proposition 2.4 *For a cardinal κ the following are equivalent:*

(a) *κ is extendible.*

(b) *κ is Jech-extendible.*

(a$'$) *For all $\lambda > \kappa$, there are j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \overset{\preceq}{\to}_\kappa M$, $j(\kappa) > \lambda$ and $V_{j(\lambda)} \in M$.*

(b$'$) *For all $\lambda > \kappa$, there are j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \overset{\preceq}{\to}_\kappa M$, and $V_{j(\lambda)} \in M$.*

Proof. (a) $\Rightarrow$ (b): is clear by definition.

(b) $\Rightarrow$ (a): This can be proved by an argument similar to that of the proof of (b) $\Rightarrow$ (a) of Proposition 2.6 below.

(a) $\Rightarrow$ (a$'$): follows from Lemmas 2.2 and 2.3.

(a$'$) $\Rightarrow$ (b$'$): is trivial.

(b$'$) $\Rightarrow$ (b): is obtained by restricting elementary embeddings on V to V_λ's.

❑ (Proposition 2.4)

The notion of super-$C^{(n)}$-large cardinal was introduced in Fuchino and Usuba [16]. Proposition 2.4 in mind, we define the super-$C^{(n)}$-extendibility as follows: For a natural number n, we call a cardinal κ *super-$C^{(n)}$-extendible* if for any $\lambda_0 > \kappa$ there are $\lambda \geq \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, and j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$, $j(\kappa) > \lambda$, $V_{j(\lambda)} \in M$ and $V_{j(\lambda)} \prec_{\Sigma_n} \mathsf{V}$.

It is easy to see that the definition of the super-$C^{(n)}$-extendibility is equivalent to the following variation:

Lemma 2.5 *κ is super-$C^{(n)}$-extendible if and only if the following holds:*

(*) *for any $\lambda \geq \kappa$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, there are j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$, $j(\kappa) > \lambda$, $V_{j(\lambda)} \in M$ and $V_{j(\lambda)} \prec_{\Sigma_n} \mathsf{V}$.* ❏

We call a cardinal κ *super-$C^{(\infty)}$-extendible* if κ is super-$C^{(n)}$-extendible for all $n \in \omega$. In general, we cannot formulate the assertion "κ is super-$C^{(\infty)}$-extendible" in the language of ZF since we would need an infinitary logic to do this unless we are allowed to introduce a new constant symbol for the large cardinal to refer it across infinitely many formulas. However, there are certain situations where we can say that a cardinal is super-$C^{(\infty)}$-extendible. One of them is when we are talking about a cardinal in a set model. In this case, being "super-$C^{(\infty)}$-extendible" in the model is an $\mathcal{L}_{\omega_1,\omega}$ sentence which is satisfied by the cardinal in the model. Another situation is when we are talking about a cardinal in an inner model and the cardinal is definable in V (e.g. as $2^{\aleph_0}$ in the outer model). Note that in the latter case, we can formulate the super-$C^{(\infty)}$-extendibility of the cardinal in infinitely many formulas, and hence n in this case ranges only over metamathematical natural numbers.

Similarly to Proposition 2.4, we have the following equivalence:

Proposition 2.6 *For a cardinal κ and $n \geq 1$, the following are equivalent:*

(a) *For any $\lambda_0 > \kappa$ there are $\lambda > \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, j_0, and μ such that $j_0 : V_\lambda \xrightarrow{\prec}_\kappa V_\mu$, $j(\kappa) > \lambda$, and $V_\mu \prec_{\Sigma_n} \mathsf{V}$.*

(b) *For any $\lambda_0 > \kappa$ there are $\lambda > \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, j_0, and μ such that $j_0 : V_\lambda \xrightarrow{\prec}_\kappa V_\mu$, and $V_\mu \prec_{\Sigma_n} \mathsf{V}$ (without the condition "$j(\kappa) > \lambda$").*

(a′) *κ is super-$C^{(n)}$-extendible.*

(b′) *for any $\lambda_0 > \kappa$ there are $\lambda \geq \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, and j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$, $V_{j(\lambda)} \in M$, and $V_{j(\lambda)} \prec_{\Sigma_n} \mathsf{V}$.* ❏

Proof. The proof is similar to that of Lemma 2.4. We only show (b) $\Rightarrow$ (a). The following proof is a modification of the proof of Lemma 2.4, (b) $\Rightarrow$ (a) given by Farmer S in [31].

Assume, toward a contradiction, that κ satisfies (b) but not (a). Then there is a γ such that

(2.3) for all sufficiently large $\lambda > \kappa$, if $(2.4)\colon\ V_\lambda \prec_{\Sigma_n} \mathsf{V}$, and μ, j are such that $(2.5)\colon\ j : V_\lambda \overset{\prec}{\to}_\kappa V_\mu$ and $(2.6)\colon\ V_\mu \prec_{\Sigma_n} \mathsf{V}$,
then $j(\kappa) < \gamma$.

In the following, let γ be the least such γ.

Claim 2.6.1 *γ is a limit ordinal. For all sufficiently large λ with* (2.4) *and for all $\xi < \gamma$, there are μ, j with* (2.5), (2.6) *such that $j(\kappa) > \xi$.*

$\vdash$ Suppose γ is not a limit ordinal, say $\gamma = \xi + 1$. Then there are cofinally many $\lambda \in \mathrm{On}$ such that $V_\lambda \prec_{\Sigma_n} \mathsf{V}$ (actually $\lambda \in \mathit{Card}$, see Lemma 1.1), and there are j and μ with (2.5), (2.6) and $j(\kappa) = \xi$. By restricting of j's as right above, it follows that, for all $\lambda > \xi$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, there are j and μ as above.

Let λ^* be a sufficiently large such λ where "sufficiently large" is meant in terms of (2.3). Let j^* and μ^* be such that $j^* : V_{\lambda^*} \overset{\prec}{\to}_\kappa V_{\mu^*}$, $j^*(\kappa) = \xi$, and $V_{\mu^*} \prec_{\Sigma_n} \mathsf{V}$.

Since $\lambda^* \leq \mu^*$, there is also $k : V_{\mu^*} \overset{\prec}{\to}_\kappa V_{\nu^*}$ such that $V_{\nu^*} \prec_{\Sigma_n} \mathsf{V}$ and $k(\kappa) = \xi$. But then we have $k \circ j^* : V_{\lambda^*} \overset{\prec}{\to}_\kappa V_{\nu^*}$ and $k \circ j^*(\kappa) = k(\xi) > k(\kappa) = \xi$. This is a contradiction to (2.3).

The second assertion of the claim follows from this and the minimality of γ.

$\dashv$ (Claim 2.6.1)

Claim 2.6.2 *For all sufficiently large $\mu > \kappa$ with $V_\mu \prec_{\Sigma_n} \mathsf{V}$, and k, ν with $V_\nu \prec_{\Sigma_n} \mathsf{V}$ and $k : V_\mu \overset{\prec}{\to}_\kappa V_\nu$, we have $k''\gamma \subseteq \gamma$.*

$\vdash$ Suppose otherwise. Then we find $\xi < \gamma$ such that, for cofinally many $\mu > \kappa$ with $V_\mu \prec_{\Sigma_n} \mathsf{V}$, there are ν, k such that $k : V_\mu \overset{\prec}{\to}_\kappa V_\nu$, $V_\nu \prec_{\Sigma_n} \mathsf{V}$ and $k(\xi) \geq \gamma$. By considering restrictions of k's as above, we conclude that for all $\mu > \xi$ with $V_\mu \prec_{\Sigma_n} \mathsf{V}$, there are ν and k as above.

Let $\lambda > \xi$ and j (together with rechosen μ and k for this λ) be such that $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, $j : V_\lambda \overset{\prec}{\to}_\kappa V_\mu$ and $j(\kappa) > \xi$ (possible by the second half of Claim 2.6.1). Then we have $k \circ j : V_\lambda \overset{\prec}{\to}_\kappa V_\nu$ and $k \circ j(\kappa) > k(\xi) \geq \gamma$.

Since λ, μ, ν, j, k can be chosen such that λ is sufficiently large (in terms of (2.3)), this is a contradiction. $\dashv$ (Claim 2.6.2)

Now, let $\lambda > \kappa$ be sufficiently large with $\lambda \geq \gamma + 2$, $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, and $j : V_\lambda \overset{\prec}{\to}_\kappa V_\mu$ with $V_\mu \prec_{\Sigma_n} \mathsf{V}$. By Claim 2.6.2, we have $j''\gamma \subseteq \gamma$.

Case 1. $cf(\gamma) = \omega$. Then $j(\gamma) = \gamma$ and hence $j \restriction V_{\gamma+2} : V_{\gamma+2} \overset{\prec}{\to}_\kappa V_{\gamma+2}$. This is a contradiction to Kunen's proof (see e.g. Kanamori [26], Corollary 23.14).

Case 2. $cf(\gamma) > \omega$. then, letting $\kappa_0 := \kappa$, $\kappa_{n+1} := j(\kappa_n)$ for $n \in \omega$ and $\kappa_\omega := \sup_{n\in\omega} \kappa_n$, we have $\kappa_\omega < \gamma$, and $j \restriction V_{\kappa_\omega+2} : V_{\kappa_\omega+2} \overset{\prec}{\to}_\kappa V_{\kappa_\omega+2}$. This is again a contradiction to Kunen's proof. ❑ (Proposition 2.6)

Note that variants of (a), (b) and (b') in Proposition 2.6 similar to (*) of Lemma 2.5 can also be proved to be equivalent to the super-$C^{(n)}$-extendibility of κ.

Super-$C^{(n)}$-extendibility is actually equivalent to $C^{(n)}$-extendibility of Bagaria [1]. Konstantinos Tsaprounis proved the equivalence for a variant of super-$C^{(n)}$-extendibility which he called $C^{(n)+}$-extendibility in [32].

A cardinal κ is *$C^{(n)}$-extendible* if, for any $\alpha > \kappa$, there is β and $j : V_\alpha \xrightarrow{\preceq}_\kappa V_\beta$ such that $j(\kappa) > \alpha$ and $V_{j(\kappa)} \prec_{\Sigma_n} \mathsf{V}$.

The following notion is introduced by Benjamin Goodman [19].

A cardinal κ is *supercompact for $C^{(n)}$* if, for any $\lambda > \kappa$ there is $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$ such that ${}^\lambda M \subseteq M$ and $C^{(n)} \cap \lambda = (C^{(n)})^M \cap \lambda$.

Andreas Lietz recently found a short proof of the following Theorem 2.7. Goodman apparently proved the equivalence of (a) and (c) in the theorem, but mentioned only the case of $n = 1$ in his [19]. His proof is given in the extended version of the present article.

Theorem 2.7 (Andreas Lietz) *For a cardinal κ and for all $n \geq 1$ the following are equivalent:* (a) *κ is $C^{(n)}$-extendible.*

(b) *κ is super-$C^{(n)}$-extendible.*

(b') *κ is $C^{(n)+}$-extendible.*

(c) *κ is supercompact for $C^{(n+1)}$.* ❑

3 Models with super-$C^{(n)}$-extendible cardinals

We prove that there are unboundedly many super-$C^{(\infty)}$-extendible cardinals in V_κ below an almost huge cardinal κ (Theorem 3.3).

For a cardinal κ, we say that V_κ satisfies the *second-order Vopěnka's principle* if for any set $C \subseteq V_\kappa$ of structures of the same signature with $C \notin V_\kappa$ (which is not necessarily a definable subset of V_κ), there are non-isomorphic $\mathfrak{A}, \mathfrak{B} \in C$ such that we have $i : \mathfrak{A} \xrightarrow{\preceq} \mathfrak{B}$ for an elementary embedding i.

The following is well-known (see e.g. Jech [25], Lemma 20.27), and attributed to William C. Powell.

Lemma 3.1 (W.C. Powell [29]) *If κ is an almost-huge cardinal then V_κ satisfies the second-order Vopěnka's principle.*

Proof. Suppose that $C \subseteq V_\kappa$ where C is a set of structures of the same signature, and $C \notin V_\kappa$. Without loss of generality, we may assume that (3.1) : C is closed with respect to isomorphism. Then it is enough to show that there are non-isomorphic $\mathfrak{A}$, $\mathfrak{B} \in C$ such that $\mathfrak{A} \prec \mathfrak{B}$.

Let $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$ be an almost-huge elementary embedding (i.e. M satisfies (3.2): ${}^{j(\kappa)>}M \subseteq M$). Let $\mathfrak{A} \in j(C) \setminus C$ — note that $j(C) \setminus C \neq \emptyset$ since $M \models$ "$rank(j(C)) = j(\kappa) > \kappa$". Let A be the underlying set of the structure $\mathfrak{A}$.

We have (3.3): $M \models j(\mathfrak{A}) \not\cong \mathfrak{A}$ — otherwise $M \models j(\mathfrak{A}) \cong \mathfrak{A} \in j(C)$, and hence $\mathsf{V} \models \mathfrak{A} \in C$ by elementarity. This is a contradiction to the choice of $\mathfrak{A}$.

Let $\mathfrak{A}' := j(\mathfrak{A}) \restriction j''A$.

Claim 3.1.1 (1) $M \models \mathfrak{A}' \in j(C)$.
(2) $M \models \mathfrak{A}' \prec j(\mathfrak{A})$.

$\vdash$ (1): $\mathfrak{A}' \in M$ by (3.2). Since $\mathfrak{A}' \cong \mathfrak{A} \in j(C)$, (3.1), and the elementarity of j imply $M \models \mathfrak{A}' \in j(C)$.

(2): Working in M, we check Vaught's criterion.

Suppose $a_0, ..., a_{n-1} \in j''A$, $a \in j(A)$ and $j(\mathfrak{A}) \models \varphi(a, a_0, ..., a_{n-1})$. Let a_0', ..., $a_{n-1}' \in A$ be such that $a_0 = j(a_0'), ..., a_{n-1} = j(a_{n-1}')$. Since $M \models \exists \underline{a} \in j(A)\ j(\mathfrak{A}) \models \varphi(\underline{a}, j(a_0'), ..., j(a_{n-1}'))$, it follows that $\mathsf{V} \models \exists \underline{a} \in A\ \mathfrak{A} \models \varphi(\underline{a}, a_0', ..., a_{n-1}')$. Let $a' \in A$ be such that $\mathsf{V} \models \mathfrak{A} \models \varphi(a', a_0', ..., a_{n-1}')$. Then $j(a') \in j''A$, and $M \models j(\mathfrak{A}) \models \varphi(j(a'), j(a_0'), ..., j(a_{n-1}'))$ by elementarity, as desired. $\dashv$ (Claim 3.1.1)

Now by (3.3) and Claim 3.1.1, (2), $M \models$ "there are non-isomorphic $\mathfrak{A}, \mathfrak{B} \in j(C)$ such that $\mathfrak{A} \prec \mathfrak{B}$". By elementarity it follows that $\mathsf{V} \models$ "there are non-isomorphic $\mathfrak{A}, \mathfrak{B} \in C$ such that $\mathfrak{A} \prec \mathfrak{B}$". ❑ (Lemma 3.1)

Proposition 3.2 *Suppose that κ is a Mahlo cardinal, and (3.4): V_κ satisfies the second-order Vopěnka's principle. Then there are unboundedly many $\kappa_0 < \kappa$ such that $V_\kappa \models$ "κ_0 is super-$C^{(\infty)}$-extendible".*

Proof. Suppose $\beta^* < \kappa$. We want to show that there is $\beta^* \leq \kappa_0 < \kappa$ such that $V_\kappa \models$ "κ_0 is super-$C^{(\infty)}$-extendible".

Let

$$I := \{\alpha < \kappa : \alpha \text{ is an } \omega\text{-limit of inaccessible cardinals } \eta \text{ such that } V_\eta \prec \mathsf{V}_\kappa\}.$$

I is cofinal in κ since κ is a Mahlo cardinal.

For each $\alpha \in I$, let $C_\alpha \subseteq \alpha$ be a cofinal subset of α of order-type ω consisting of (increasing sequence of) inaccessible cardinals η_n^α, $n \in \omega$ with $V_{\eta_n^\alpha} \prec V_\kappa$.

Let

$$\mathcal{C} := \{(V_{\alpha+1}, \in, C_\alpha, \xi)_{\xi<\beta^*} : \alpha \in I\}.$$

By (3.4), there are $\mathcal{V}_{\alpha+1} = (V_{\alpha+1}, \in, C_\alpha, \xi)_{\xi<\beta^*}$, $\mathcal{V}_{\beta+1} = (V_{\beta+1}, \in, C_\beta, \xi)_{\xi<\beta^*} \in \mathcal{C}$ such that there is an elementary embedding $i : \mathcal{V}_{\alpha+1} \xrightarrow{\preceq} \mathcal{V}_{\beta+1}$. Let $n \in \omega$ be arbitrary and let $\kappa_0 := crit(i)$. Then $\alpha > \kappa_0 \geq \beta^*$. Let $k \in \omega$ be large enough so that we have $\eta^\alpha_k > \kappa_0$.

Since $i(\eta^\alpha_k) \in C_\beta$, we have:

$$V_\kappa \models \text{“}V_{i(\eta^\alpha_k)} \prec_{\Sigma_n} \mathsf{V}\text{”} \ \wedge \ i \restriction V_{\eta^\alpha_k} : V_{\eta^\alpha_k} \xrightarrow{\prec}_{\kappa_0} V_{i(\eta^\alpha_k)}.$$

Thus $V_\kappa \models \exists \underline{\nu}\, \exists \underline{i}\, (\text{“}V_{\underline{\nu}} \prec_{\Sigma_n} \mathsf{V}\text{”} \ \wedge \ \underline{i} : V_{\eta^\alpha_k} \xrightarrow{\prec}_{\kappa_0} V_{\underline{\nu}})$. By elementarity, it follows that

$$V_{\eta^\alpha_k} \models \exists \underline{\eta}\, \exists \underline{\nu}\, \exists \underline{i}\, (\text{“}V_{\underline{\eta}} \prec_{\Sigma_n} \mathsf{V}\text{”} \ \wedge \ \text{“}V_{\underline{\nu}} \prec_{\Sigma_n} \mathsf{V}\text{”} \ \wedge \ \underline{i} : V_{\eta^\alpha_k} \xrightarrow{\prec}_{\kappa_0} V_{\underline{\nu}})$$

for all $n \in \omega$.

By Lemma 2.6, this implies $V_{\eta^\alpha_k} \models$ “κ_0 is super-$C^{(n)}$-extendible”. By the elementarity $V_{\eta^\alpha_k} \prec V_\kappa$, it follows that $V_\kappa \models$ “κ_0 is super-$C^{(n)}$-extendible” for all $n \in \omega$. ❑ (Proposition 3.2)

Theorem 3.3 *Suppose that κ is almost-huge. Then there are unboundedly many $\kappa_0 < \kappa$ such that $V_\kappa \models$ “κ_0 is super-$C^{(\infty)}$-extendible”.*

Proof. By Lemma 3.1 and Proposition 3.2. ❑ (Theorem 3.3)

4 LgLCAs and super-$C^{(n)}$-LgLCAs for extendibility imply (almost) everything

In this section, we prove that Laver-generic Large Cardinal Axioms (LgLCAs) and super- $C^{(\infty)}$-LgLCAs for extendibility imply maximal amount of resurrection axioms, maximality principles, and absoluteness. I.e., at least maximal, amount out of variations of these axioms and principles which is known to be consistent. This was previously known to hold under LgLCAs for hugeness and super-$C^{(\infty)}$-LgLCAs for hyperhugeness.

We begin with a short summary of definitions and known results around LgLCAs and super- $C^{(\infty)}$-LgLCAs.

Laver-generic large cardinals were introduced in [12]. For a class $\mathcal{P}$ of posets and a notion LC of large cardinal, a cardinal κ is said to be $\mathcal{P}$-Laver generic LC if the statement about the existence of elementary embedding $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$ for j, $M \subseteq \mathsf{V}$ with the closedness condition C_{LC} of M in the definition of the notion LC of large cardinal are replaced with the statement:

(4.1) for any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", and for $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$-generic $\mathbb{H}$ there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$, and M satisfies C'_{LC} which is the generic large cardinal variant of the closedness property C_{LC} associated with the notion LC of large cardinal.

For supercompactness, the instance of (4.1) for an iterable $\mathcal{P}$ is as follows: a cardinal κ is *$\mathcal{P}$-Laver-generically supercompact* if,

(4.2) for any $\lambda > \kappa$, and for any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ such that $\Vdash_{\mathbb{P}}$ "$\mathbb{Q} \in \mathcal{P}$", and, for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, $j''\lambda \in M$.

Note that, in (4.2), the closure property "${}^{\lambda}M \subseteq M$" in the usual definition of supercompactness is replaced with "$j''\lambda \in M$". For a genuine elementary embedding introduced by some ultrafilter, these two conditions are equivalent (see e.g. Kanamori [26], Proposition 22.4, (b)). This equivalence is no more valid in general for generic embeddings. Nevertheless, the condition "$j''\lambda \in M$" can be still considered as a certain closure property (see Lemma 3.5 in Fuchino-Rodrigues-Sakai [12]).

We say that a $\mathcal{P}$-Laver-generically supercompact cardinal κ is *tightly $\mathcal{P}$-Laver-generically supercompact* if additionally, we have $|\,\mathsf{RO}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| \leq j(\kappa)$.

A (tightly) $\mathcal{P}$-Laver-generically supercompact cardinal is often decided uniquely as the cardinal $\kappa_{\mathfrak{refl}} := \sup(\{2^{\aleph_0}, \aleph_2\})$. This is the case, if $\mathcal{P}$ is the class of all σ-closed posets. Then CH holds under the existence of a $\mathcal{P}$-Laver-generically supercompact κ and $\kappa = \aleph_2$ $(= \kappa_{\mathfrak{refl}})$.

Similarly, if $\mathcal{P}$ is either the class of all proper posets or the class of all semi-proper posets, the existence of a $\mathcal{P}$-generically supercompact κ implies $2^{\aleph_0} = \aleph_2$ and again $\kappa = \kappa_{\mathfrak{refl}}$.

For the case that $\mathcal{P}$ is the class of all ccc posets, it is open whether a $\mathbb{P}$-Laver-generically supercompact cardinal is decided to be $\kappa_{\mathfrak{refl}}$. However a tightly $\mathcal{P}$-Laver-generically supercompact cardinal under the present definition of tightness[1)] is the continuum $(= \kappa_{\mathfrak{refl}})$ and, in this case, the continuum is extremely large. There is a more general theorem which suggests that for a "natural" class $\mathcal{P}$ of posets, the existence of (tightly) $\mathcal{P}$-Laver generically supercompact cardinal implies that the continuum is either $\aleph_1$ or $\aleph_2$ or else extremely large (see [12], [7], [8]).

[1)] In course of the development of the theory of Laver-genericity, we strengthened the definition of tightness. However, the modification is chosen so that it still holds in all the standard models of Laver genericity.

The naming "Laver-generic ..." based on the fact that the standard models with this type of generic large cardinal is created by starting from a large cardinal, and then iterating along with a Laver function for the large cardinal with the support appropriate for the class of posets in consideration. This is exactly the way to create models of PFA and MM. Actually, for $\mathcal{P}$ being the class of all proper posets or the class of all semi-proper posets, the existence of a $\mathcal{P}$-Laver generically supercompact cardinal implies the double-plus version of the corresponding forcing axiom (see Theorem **??** below), and can be considered as an axiomatization of the standard models of such axioms.

In the following, we call the axiom claiming the existence of a/the tightly $\mathcal{P}$-Laver generic LC, the *$\mathcal{P}$-Laver-generic large cardinal axiom for the notion of large cardinal LC* (*$\mathcal{P}$-LgLCA for LC*, for short).

The instances of $\mathcal{P}$-LgLCAs for other notions of large cardinal considered in [7], [8], [10], [12], [?], [16], etc. are summarized in the following chart.

$\mathcal{P}$-LgLCA for	The condition "$j''\lambda \in M$" in the definition of "$\mathcal{P}$-LgLCA for super-compact" is replaced with:
hyperhuge	$j''j(\lambda) \in M$
ultrahuge	$j''j(\kappa) \in M$ and ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$
superhuge	$j''j(\kappa) \in M$
super-almost-huge	$j''\mu \in M$ for all $\mu < j(\kappa)$

It has been proved that LgLCAs for sufficiently strong notions of large cardinal imply strongest forms of resurrection, maximality and absoluteness among the known consistent variants of resurrection, maximality and absoluteness:

(4.3) In Fuchino [7], it is proved that a boldface variant of Resurrection Axiom by Hamkins and Johnstone, [21], [22] for $\mathcal{P}$ and parameters from $\mathcal{H}(\kappa_{\mathfrak{refl}})$ follows $\mathcal{P}$-LgLCA for ultrahuge.

(4.4) In [16] or [8] (see [10] for an improved version, see also Theorem 4.3 below), it is proved that $\mathcal{P}$-LgLCA for ultrahuge implies a restricted form of Maximality Principle for $\mathcal{P}$ and $\mathcal{H}(\kappa_{\mathfrak{refl}})$. More specifically, It is proved that under $\mathcal{P}$-LgLCA for ultrahuge implies $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_2}$-$\mathsf{RcA}^+$ holds — see below for the definition of this priniple (Theorem 21 in [8]).

It can be shown that LgLCA type axiom formulated in a single formula is incapable of covering the full Maximal Principle ([7]). The notion of super-$C^{(\infty)}$ LgLCAs is introduced in [16] to fill this gap.

For a notion LC of large cardinal let C'_{LC} be the closedness property of the target model of the generic large cardinal corresponding to LC. We call a cardinal κ *tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver generically LC* if, for any $n \in \mathbb{N}$, $\lambda_0 > \kappa$, and $\mathbb{P} \in \mathcal{P}$, there are $\lambda \geq \lambda_0$ and a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ such that $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$, $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, $j(\kappa) > \lambda$, C'_{LC}, and $|\,\mathsf{RO}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| \leq j(\kappa)$.

The *super-$C^{(\infty)}$-$\mathcal{P}$-Laver-generic large cardinal axiom for the notion LC of large cardinal* (*super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for LC*, for short) is the assertion that $\kappa_{\mathfrak{refl}}$ is a/the tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver-generic extendible cardinal.

Note that tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver generically LC is not formalizable in general but super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for LC is as an axiom scheme, since the generic large cardinal in the axiom is named as $\kappa_{\mathfrak{refl}}$.

It is shown that for a transfinitely iterable $\mathcal{P}$, the consistency of super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for hyperhuge follows from a 2-huge cardinal ([16], Lemma 2.6 and Theorem 2.8).

(4.5) Super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for ultrahuge implies the full Maximality Principle for $\mathcal{P}$ and $\mathcal{H}(\kappa_{\mathfrak{refl}})$ ([16], Theorem 4.10).

(4.6) In [10], it is proved that under $\mathsf{BFA}_{<\kappa_{\mathfrak{refl}}}(\mathcal{P})$ and $\mathcal{P}$-LgLCA for huge, a generalization of Viale's Absoluteness Theorem in Viale [34] holds (see Theorem 5.7 in [10]).

For extendibility and super-$C^{(\infty)}$-extendibility, the natural Laver-generic versions of these notions of large cardinals should be the following: a cardinal κ is *tightly $\mathcal{P}$-Laver generically extendible* if, for any $\lambda > \kappa$, and for any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, $j(\kappa) > \lambda$, ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$, and $|\,\mathsf{RO}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| \leq j(\kappa)$.

The *$\mathcal{P}$-Laver-generic large cardinal axiom* for the notion of extendibility (*$\mathcal{P}$-LgLCA for extendible*, for short) is the assertion that $\kappa_{\mathfrak{refl}}$ is a/the tightly $\mathcal{P}$-Laver-generic extendible cardinal.

A cardinal κ is *tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver generically extendible* if, for any $n \in \mathbb{N}$, $\lambda_0 > \kappa$, and $\mathbb{P} \in \mathcal{P}$, there are $\lambda \geq \lambda_0$ and a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ such that $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$, $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, $j(\kappa) > \lambda$, ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$, and $|\,\mathsf{RO}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| \leq j(\kappa)$.

The *super-$C^{(\infty)}$-$\mathcal{P}$-Laver-generic large cardinal axiom* for the notion of extendibility (*super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for extendible*, for short) is the assertion that $\kappa_{\mathfrak{refl}}$ is a/the tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver-generic extendible cardinal.

$\mathcal{P}$-LgLCA for	The condition "$j''\lambda \in M$" in the definition of "$\mathcal{P}$-LgLCA for super-compact" is replaced with:
hyperhuge	$j''j(\lambda) \in M$
ultrahuge	$j''j(\kappa) \in M$ and $V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \in M$
superhuge	$j''j(\kappa) \in M$
super-almost-huge	$j''\mu \in M$ for all $\mu < j(\kappa)$
extendible	$V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \in M$

As we have shown an almost-huge cardinal produces a transitive model with cofinally many super-$C^{(\infty)}$-extendible cardinals. In the next section, we show that we can generic extend such models to a model of super-$C^{(\infty)}$ $\mathcal{P}$-LgLCA for extendible for each reasonable (i.e. transfinitely iterable) class $\mathcal{P}$ of posets. Thus super-$C^{(\infty)}$ $\mathcal{P}$-LgLCA for extendible and $\mathcal{P}$-LgLCA for extendible are of relatively low consistency strength.

These LgLCAs are placed at the expected places in the web of implications of LgLCAs (see also the chart on page 20):

Lemma 4.1 *Suppose that $\mathcal{P}$ is an arbitrary class of posets.* (1) *$\mathcal{P}$-LgLCA for hyperhuge implies $\mathcal{P}$-LgLCA for extendible. super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for hyperhuge implies super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for extendible.*

(2) *$\mathcal{P}$-LgLCA for extendible implies $\mathcal{P}$-LgLCA for supercompact.*

Proof. (1): By definition.

(2): This follows since if $V_{j(\lambda)}^{\mathsf{V}[\mathbb{H}]} \in M$ for a regular $\lambda > \kappa$ with $j(\kappa) > \lambda$ then since $\lambda < j(\kappa) < j(\lambda)$, and $cf(j''\lambda) \leq \lambda < j(\lambda)$, we have $j''\lambda \in V_{j(\lambda)}^{\mathsf{V}[\mathbb{H}]} \in M$. Since M is transitive it follows that $j''\lambda \in M$. □ (Lemma 4.1)

All of the results mentioned in (4.3) $\sim$ (4.6) can be proved under the assumption of LgLCA for extendible instead of the stronger assumptions in the original results. Below we shall state these results. In the extended version of the present paper, we shall include all the details of the proofs (though the proofs are practically identical with the original ones) for the convenience of the reader.

The following boldface version of the Resurrection Axioms was studied by Hamkins and Johnstone in [22]: For a class $\mathcal{P}$ of posets and a definition $\mu^\bullet$ of a cardinal (e.g. as $\aleph_1$, $\aleph_2$, $2^{\aleph_0}$, $(2^{\aleph_0})^+$. etc.) the *Resurrection Axiom in Boldface for $\mathcal{P}$ and $\mathcal{H}(\mu^\bullet)$* is defined by:

$\mathbb{RA}^{\mathcal{P}}_{\mathcal{H}(\mu^\bullet)}$: For any $A \subseteq \mathcal{H}(\mu^\bullet)$ and any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of poset such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there is $A^* \subseteq \mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}$ such that $(\mathcal{H}(\mu^\bullet)^{\mathsf{V}}, A, \in) \prec (\mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}, A^*, \in)$.

Theorem 4.2 (Theorem 7.1 in [7] rewriten under LgLCA for extendible) *For an iterable class $\mathcal{P}$ of posets, assume that $\mathcal{P}$-LgLCA for extendible holds. Then $\mathbb{RA}^{\mathcal{P}}_{\mathcal{H}(\kappa_{\mathfrak{refl}})}$ holds.* ❑

Recurrence Axioms are introduced in Fuchino and Usuba [16].

For an iterable class $\mathcal{P}$ of posets, a set A (of parameters), and a set Γ of $\mathcal{L}_{\in}$- formulas, *$\mathcal{P}$-Recurrence Axiom$^+$ for formulas in Γ with parameters from A* ($(\mathcal{P}, A)_\Gamma$-RcA^+, for short) is the following assertion expressed as an axiom scheme formulated in $\mathcal{L}_{\in}$:

$(\mathcal{P}, A)_\Gamma$-RcA^+ *:* For any $\varphi(\overline{x}) \in \Gamma$ and $\overline{a} \in A$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$", then there is a $\mathcal{P}$-ground W of V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$.

Here, an inner model M of V is said to be a *$\mathcal{P}$-ground of* V, if there are $\mathbb{P} \in M$ and $\mathbb{G} \in \mathsf{V}$ such that $M \models$ "$\mathbb{P} \in \mathcal{P}$", $\mathbb{G}$ is an $(M, \mathbb{P})$-generic filter, and $\mathsf{V} = M[\mathbb{G}]$. If M is a $\mathcal{P}$-ground of V for the class $\mathcal{P}$ of all posets, we shall say that M is a *ground of* V. The Recurrence Axiom $(\mathcal{P}, A)_\Gamma$-RcA without $+$ is obtained when "$\mathcal{P}$-ground" in the definition of $(\mathcal{P}, A)_\Gamma$-RcA^+ is replaced with "ground".

If Γ is the set of all $\mathcal{L}_{\in}$-formulas, we drop the subscript Γ and say simply $(\mathcal{P}, A)$-RcA^+ or $(\mathcal{P}, A)$-RcA.

As it is noticed in [16], $(\mathcal{P}, A)$-RcA^+ is equivalent to the Maximality Principle $\mathsf{MP}(\mathcal{P}, A)$ (see Proposition 2.2, (2) in [16]).

When [10] was written, we didn't consider the notion of LgLCA for extendible among the possible LgLCAs. This is why the following theorem was stated there under the assumption of $\mathcal{P}$-LgLCA for ultrahuge. However the proof given in [10] works perfectly under $\mathcal{P}$-LgLCA for extendible without any change.

Theorem 4.3 (Theorem 6.1 in Fuchino, Gappo and Parente [10] restated under LgLCA for extendible) *Assume $\mathcal{P}$-LgLCA for extendible. Then $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_\Gamma$-$\mathsf{RcA}^+$ holds where Γ is the set of all formulas which are conjunctions of a Σ_2-formula and a Π_2-formula.* ❑

Theorem 4.3 has an important application (Theorem 4.5). For this theorem, we need the following facts about Recurrence Axioms.

Lemma 4.4 (Fuchino and Usuba [16], see also Lemma 20 in the extended version of [8]) *Assume that $\mathcal{P}$ is an iterable class of posets.* (1) *If $\mathcal{P}$ contains a poset which adds a real (over the universe), then $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-$\mathsf{RcA}$ implies $\neg\mathsf{CH}$.*

(2) *Suppose that $\mathcal{P}$ contains a poset which forces ${\aleph_2}^{\mathsf{V}}$ to be equinumerous with ${\aleph_1}^{\mathsf{V}}$. Then $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-$\mathsf{RcA}$ implies $2^{\aleph_0} \leq \aleph_2$.*

(2') *If $\mathcal{P}$ contains a posets which forces ${\aleph_2}^{\mathsf{V}}$ to be equinumerous with ${\aleph_1}^{\mathsf{V}}$, then* $(\mathcal{P}, \mathcal{H}((\aleph_2)^+))_{\Sigma_1}$-$\mathsf{RcA}$ *does not hold.*

(3) *If* $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-$\mathsf{RcA}$ *holds then all $\mathbb{P} \in \mathcal{P}$ preserve $\aleph_1$ and they are also stationary preserving.*

(4) *If $\mathcal{P}$ contains a poset which adds a real as well as a poset which collapses ${\aleph_2}^{\mathsf{V}}$, then* $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-$\mathsf{RcA}$ *implies $2^{\aleph_0} = \aleph_2$.*

(5) *If $\mathcal{P}$ contains a poset which collapses ${\aleph_1}^{\mathsf{V}}$, then* $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-$\mathsf{RcA}$ *implies* CH.

(5') *If $\mathcal{P}$ contains a poset which collapses ${\aleph_1}^{\mathsf{V}}$ then* $(\mathcal{P}, \mathcal{H}((2^{\aleph_0})^+))_{\Sigma_1}$-$\mathsf{RcA}$ *does not hold.*

(6) *Suppose that all $\mathbb{P} \in \mathcal{P}$ preserve cardinals and $\mathcal{P}$ contains posets adding at least κ many reals for each $\kappa \in Card$. Then* $(\mathcal{P}, \emptyset)_{\Sigma_2}$-$\mathsf{RcA}^+$ *implies that $2^{\aleph_0}$ is very large.*

(6') *Suppose that $\mathcal{P}$ is as in (6). Then* $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_2}$-$\mathsf{RcA}^+$ *implies that $2^{\aleph_0}$ is a limit cardinal. Thus if $2^{\aleph_0}$ is regular in addition, then $2^{\aleph_0}$ is weakly incaccessible.* ❑

Theorem 4.5 *Suppose that $\mathcal{P}$-LgLCA for extendible holds. Then we have:*

(1) *Elements of $\mathcal{P}$ are stationary preserving.*

(2) *For all classes $\mathcal{P}$ of posets covered by Lemma 4.4, $\mathcal{P}$-LgLCA for extendible implies that the continuum is either $\aleph_1$ or $\aleph_2$ or very large.*

Proof. (1): By Theorem 4.3 and Lemma 4.4, (3).

(2): By Theorem 4.3 and the rest of Lemma 4.4. ❑ (Theorem 4.5)

Theorem 4.6 (Fuchino and Usuba [16], Theorem 4.10) *Suppose that $\mathcal{P}$ is an iterable class of posets and super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for extendible holds. Then* $\mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))$ *holds.* ❑

Theorem 4.7 (Theorem 5.7 in Fuchino, Gappo and Parente [10] restated under LgLCA for extendible) *For an iterable class $\mathcal{P}$ of posets, assume that $\mathcal{P}$-LgLCA for extendible holds. Then,*

(4.7) *for any $\mathbb{P} \in \mathcal{P}$ such that* $\Vdash_{\mathbb{P}}$ "$\mathsf{BFA}_{<\kappa}(\mathcal{P})$", $\mathcal{H}(\mu^+)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}(\mu^+)^{\mathsf{V}[\mathbb{G}]}$ *holds for all $\mu < \kappa_{\mathfrak{refl}}$ and for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$.*

Thus, we have $\mathcal{H}(\kappa_{\mathfrak{refl}})^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}((\kappa_{\mathfrak{refl}})^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]}$ *for $\mathbb{G}$ as above.* ❑

5 Consistency of LgLCAs and super-$C^{(n)}$-LgLCAs for extendibility

Lemma 5.1 (1) *Suppose that κ is extendible. Then there is a Laver function $f : \kappa \to V_\kappa$ for extendibility.*

(2) *Suppose that κ is super $C^{(n)}$-extendible for an $n \in \mathbb{N} \setminus 1$. Then there is a Laver function $f : \kappa \to V_\kappa$ for super $C^{(n)}$-extendibility.*

(3) *Suppose that, for an inaccessible cardinals κ^* and a cardinal $\kappa < \kappa^*$ we have $V_{\kappa^*} \models$ "κ is super-$C^{(\infty)}$-extendible". Then there is a Laver function $f : \kappa \to V_\kappa$ for super-$C^{(\infty)}$-extendible κ in V_{κ^*}.*

Proof. (1) has been known previously (see e.g. Corraza [2]). The proof of (2) below can be easily slimmed down to a proof of (1).

(2): Assume, toward a contradiction, that there is no Laver function $f : \kappa \to V_\kappa$ for super $C^{(n)}$-extendible κ.

Let $\varphi_n(f)$ be the formula

$$\exists \underline{\alpha} \exists \underline{x} \forall \underline{\delta} \forall \underline{\delta}' \forall \underline{j} \, ((f : \underline{\alpha} \to V_{\underline{\alpha}} \ \wedge \ \underline{\alpha} < \underline{\delta} \ \wedge \ V_{\underline{\delta}} \prec_{\Sigma_n} \mathsf{V} \ \wedge \ \underline{x} \in V_{\underline{\delta}} \ \wedge \ V_{\underline{\delta}'} \prec_{\Sigma_n} \mathsf{V} \\ \wedge \ \underline{j} : V_{\underline{\delta}} \overset{\prec}{\to}_{\underline{\alpha}} V_{\underline{\delta}'} \ \wedge \ \underline{j}(dom(f)) > \underline{\delta} \ \wedge \ \underline{j} \text{ is cofinal in } V_{\underline{\delta}'}) \ \to \ \underline{j}(f)(\underline{\alpha}) \neq \underline{x}).$$

If $\varphi_n(f)$ holds then the witness of $\underline{\alpha}$ in $\varphi_n(f)$ is uniquely determined. In this case, let x_f be a witnesses for $\underline{x}$ in $\varphi_n(x)$, and let $\mu_f := rank\,(x_f)$. x_f might not be determined uniquely. However, we will choose x_f such that μ_f is minimal and thus μ_f is determined uniquely to each f. If $\varphi_n(f)$ does not hold, we let $\mu_f := 0$.

By assumption, we have

(5.1) $\quad \varphi_n(f)$ for all $f : \kappa \to V_\kappa$.

Let ν, ν_0 be be cardinals such that $\nu \geq \nu_0 \geq \max\{\mu_f \ : \ f : \alpha \to V_\alpha$ for an inaccessible $\alpha \leq \kappa\}$, (5.2): $V_{\nu_0} \prec_{\Sigma_m} \mathsf{V}$ for sufficiently large $m \in \mathbb{N}$, $V_\nu \prec_{\Sigma_n} \mathsf{V}$, and there is $j^* : \mathsf{V} \overset{\prec}{\to}_\kappa M$ with (5.3): $j^*(\kappa) > \nu$, (5.4): $V_{j(\nu)} \prec_{\Sigma_n} \mathsf{V}$, and (5.5): $V_{j(\nu)} \in M$.

Let $A := \{\alpha < \kappa \ : \ \alpha$ is inaccessible, and $\forall \underline{f} \, ((\underline{f} : \alpha \to V_\alpha) \ \to \ \varphi_n(\underline{f}))\}$.

By assumption, $\mathsf{V} \models$ "$\forall \underline{f} \, ((\underline{f} : \kappa \to V_\kappa) \ \to \ \varphi_n(\underline{f}))$". By (5.2) and (5.5), it follows that $M \models$ "$\forall \underline{f} \, ((\underline{f} : \kappa \to V_\kappa) \ \to \ \varphi_n(\underline{f}))$".[2)] Thus we have $M \models j^*(A) \ni \kappa$.

Let $f^* : \kappa \to V_\kappa$ be defined by

[2)] Actually, by analyzing the statement of $\varphi_n(f)$ carefully we see that (5.2) and the details connected to it are redundant here.

$$f^*(\alpha) := \begin{cases} x_{f^*\restriction\alpha}, & \text{if } \alpha \in A; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let $x^* := j^*(f^*)(\kappa)$. By definition of f^*, by (5.1), and since $j(f^*) \restriction \kappa = f^*$, x^* witnesses $\varphi_n(f^*)$.
(μ_{f^*} is just as chosen before since it is uniquely determined. x^* may be different from x_{f^*} but this does not matter.)

In particular, $x^* \neq (j^* \restriction V_{\delta_{f^*}})(f^*)(\kappa) = j(f^*)(\kappa)$. This is a contradiction.

(3): Let $\varphi_n(f)$ be as in (1) where n runs now over ω, and let $\varphi(f) := \bigwedge\!\!\bigwedge_{n\in\omega}\varphi_n(f)$. As we already have noticed in Section 2, we cannot discuss about the validity of $\varphi(f)$ in V (at least not in the framework of ZFC) while $V_{\kappa^*} \models \varphi(f)$ is a well-defined notion. The conclusion of (2) is obtained by arguing analogously to (1) in V_{κ^*} with $\varphi_n(f)$ replaced by $\varphi(f)$. ❑ (Lemma 5.1)

Theorem 5.2 (1) *For an extendible κ, and for a Σ_2-definable transfinitely iterable class $\mathcal{P}$ of posets, there is a poset $\mathbb{P}_\kappa$ such that $\Vdash_{\mathbb{P}_\kappa}$ "$\kappa = \kappa_{\mathfrak{refl}}$ and κ is tightly $\mathcal{P}$-Laver generic extendible".*

(2) *Suppose that κ is super-$C^{(n^*)}$-extendible for $n \in \mathbb{N}$ and $n^* = \max\{n, 2\}$. Then for any Σ_{n+1}-definable transfinitely iterable class $\mathcal{P}$ of posets, there is a poset $\mathbb{P}_\kappa$ such that*
$\Vdash_{\mathbb{P}_\kappa}$ "$\kappa = \kappa_{\mathfrak{refl}}$ and κ is tightly super-$C^{(n)}$-$\mathcal{P}$-Laver generic extendible".

(3) *$V_\mu \models$ "κ is super-$C^{(\infty)}$ extendible" for an inaccessible μ. Then for any transfinitely iterable class $\mathcal{P}$ of posets there is a poset $\mathbb{P}_\kappa \in V_\mu$ such that $V_\mu \models$ " $\Vdash_{\mathbb{P}_\kappa}$ "$\kappa = \kappa_{\mathfrak{refl}}$ and κ is tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver generic extendible" ".*

Note that most of the natural classes of posets including the classes of all ccc posets, all σ-closed posets, all proper posets, all semi-proper posets, etc. are Σ_2.

Proof of Theorem 5.2: (1): We show the assertion for the case that $\mathcal{P}$ is the class of all proper posets. The proof for the general case can be done by replacing the CS-iteration in the following proof by the iteration for which the class $\mathcal{P}$ is transfinitely iterable. Let f be a Laver function for extendible κ (f exists by Lemma 5.1, (1)).

Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be an CS-iteration of elements of $\mathcal{P}$ such that

$$\underset{\sim}{\mathbb{Q}}_\beta := \begin{cases} f(\beta), & \text{if } f(\beta) \text{ is a } \mathbb{P}_\beta\text{-name} \\ & \text{and } \Vdash_{\mathbb{P}_\beta} \text{“} f(\beta) \in \mathcal{P} \text{”}; \\ \mathbb{P}_\beta\text{-name of the trivial forcing}, & \text{otherwise.} \end{cases}$$

We show that $\Vdash_{\mathbb{P}_\kappa}$ "$\mathcal{P}$-LgLCA for extendibility".

First, note that $\Vdash_{\mathbb{P}_\kappa}$ "$\kappa = 2^{\aleph_0} = \kappa_{\mathfrak{refl}}$" by definition of $\mathbb{P}_\kappa$.

Let $\mathbb{G}_\kappa$ be a $(\mathsf{V}, \mathbb{P}_\kappa)$-generic filter. In $\mathsf{V}[\mathbb{G}_\kappa]$, suppose that $\mathbb{P} \in \mathcal{P}$ and let $\underset{\sim}{\mathbb{P}}$ be a $\mathbb{P}_\kappa$-name for $\mathbb{P}$.

Suppose that $\lambda > \kappa$. Let $\lambda^* > \lambda$ be a cardinal such that $\mathcal{H}(\lambda^*) = V_{\lambda^*}$. Let $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$ be such that (5.6): $j(\kappa) > \lambda^*$, (5.7): $V_{j(\lambda^*)} \in M$, and (5.8): $j(f)(\kappa) = \underset{\sim}{\mathbb{P}}$. The last condition is possible since f is a Laver function for the extendible κ. By (5.7), we have $\mathcal{H}(j(\lambda^*))^M = {V_{j(\lambda^*)}}^M = {V_{j(\lambda^*)}}^{\mathsf{V}} = \mathcal{H}(j(\lambda^*))^{\mathsf{V}}$, and hence

(5.9): $V_{j(\lambda^*)} \prec_{\Sigma_1} M$ and $V_{j(\lambda^*)} \prec_{\Sigma_1} \mathsf{V}$.

In M, there is a $\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}}$-name $\underset{\sim}{\mathbb{Q}}$ such that

$$M \models \Vdash_{\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}}} \text{“}\underset{\sim}{\mathbb{Q}} \in \mathcal{P} \text{ and } \underset{\sim}{\mathbb{Q}} \text{ is the direct limit of CS-iteration of small posets in } \mathcal{P} \text{ of length } j(\kappa), \text{ and } \mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}} * \underset{\sim}{\mathbb{Q}} \sim j(\mathbb{P}_\kappa)\text{”}.$$

By (5.9), and since "$\underset{\sim}{\mathbb{P}} \in \mathcal{P}$" is Σ_2, the same statement holds in V with the same $\underset{\sim}{\mathbb{Q}}$.

We have $j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa \sim \mathbb{P} * \underset{\sim}{\mathbb{Q}}$ where we identify $\underset{\sim}{\mathbb{Q}}$ with a corresponding $\mathbb{P}$-name.

Let $\mathbb{H}$ be $(\mathsf{V}, j(\mathbb{P}_\kappa))$-generic filter with $\mathbb{G}_\kappa \subseteq \mathbb{H}$.

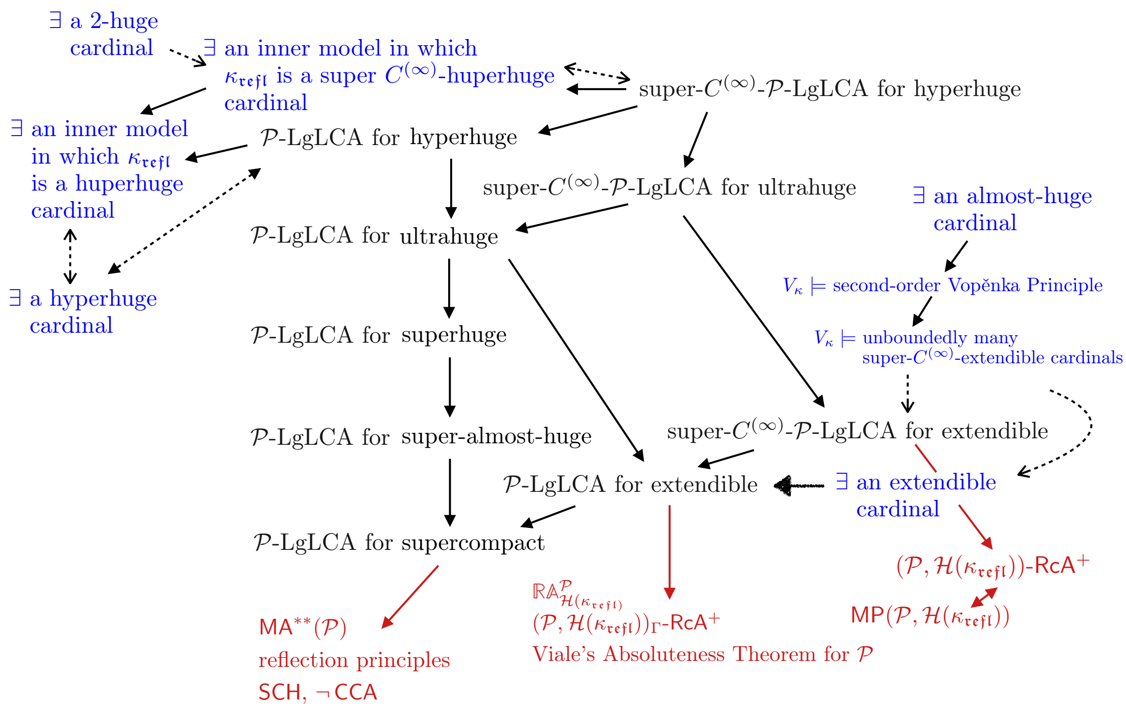
The lifting $\tilde{j} : \mathsf{V}[\mathbb{G}_\kappa] \overset{\prec}{\to}_\kappa M[\mathbb{H}]$; $\underset{\sim}{a}[\mathbb{G}_\kappa] \mapsto j(\underset{\sim}{a})[\mathbb{H}]$ witnesses that $\kappa = (\kappa_{\mathfrak{refl}})^{\mathsf{V}[\mathbb{G}_\kappa]}$ is tightly $\mathcal{P}$-Laver generic extendible. For this, it suffices to show:

Claim 5.2.1 *${V_\alpha}^{\mathsf{V}[\mathbb{H}]} \in M[\mathbb{H}]$ for all $\alpha \leq j(\lambda)$.*

$\vdash$ By induction on $\alpha \leq j(\lambda)$. The successor step from $\alpha < j(\lambda)$ to $\alpha + 1$ can be proved by showing that $\mathbb{P}_\kappa$-names of subsets of ${V_\alpha}^{\mathsf{V}[\mathbb{H}]}$ can be chosen as elements of M. This is possible because of (5.7). $\dashv$ (Claim 5.2.1)

(2) and (3) can be proved similarly. ❑ (Theorem 5.2)

The following chart summarizes our view of the landscape with LgLCAs.



B ← A : "the axiom A implies the axiom B"

B ↔ A : "the axioms A and B are equi-consistent."

B ← A : "the consistency of A implies the consistency of B but not the other way around."

B ← A : "the consistency of A implies the consistency of B but the equi-consistency is not (yet?) known."

References

[1] Joan Bagaria, $C^{(n)}$-cardinals, Archive for Mathematical Logic, Vol.51, (2012), 213–240. 3, 8

[2] Paul Corazza, Laver Sequences for Extendible and Super-Almost-Huge Cardinals, The Journal of Symbolic Logic Vol.64 (3), (1999), 963–983. 17

[3] Sean Cox, The diagonal reflection principle, Proceedings of the American Mathematical Society, Vol.140, No.8 (2012), 2893–2902.

[4] __________, Forcing axioms, approachability, and stationary set reflection, The Journal of Symbolic Logic Volume 86, Number 2, June 2021, 499–530.

[5] Alan Dow, Franklin D. Tall and W.A.R. Weiss, New proofs of the consistency of the normal Moore space conjecture I, II, Topology and its Applications, 37 (1990) 33–51, 115-129.

[6] Sakaé Fuchino, On potential embedding and versions of Martin's axiom, Notre Dame Journal of Logic, Vol.33, No.4, (1992), 481–492.

[7] Sakaé Fuchino, Maximality Principles and Resurrection Axioms in light of a Laver-generic large cardinal, Pre-preprint.
https://fuchino.ddo.jp/papers/RIMS2022-RA-MP-x.pdf 11, 12, 15

[8] ____________, Reflection and Recurrence, to appear in the festschrift on the occasion of the 75. birthday of Professor Janos Makowsky, Birkhäuser, (2024).
https://fuchino.ddo.jp/papers/reflection_and_recurrence-Janos-Festschrift-x.pdf 11, 12, 15

[9] ____________, Laver-generic large cardinal axioms for extendibility, extended version of the present note, in preparation. 2

[10] Sakaé Fuchino, Takehiko Gappo, and Francesco Parente, Generic Absoluteness revisited, submitted. Extended version of the paper:
https://fuchino.ddo.jp/papers/generic-absoluteness-revisited-x.pdf 12, 13, 15, 16

[11] Sakaé Fuchino, István Juhász, Lajos Soukup, Zoltán Szentmiklóssy, and Toshimichi Usuba, Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness, Topology and its Applications, Vol.157, 8 (2010), 1415–1429.

[12] Sakaé Fuchino, André Ottenbreit Maschio Rodrigues, and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum, Archive for Mathematical Logic, Volume 60, issue 3-4, (2021), 495–523.
https://fuchino.ddo.jp/papers/SDLS-II-x.pdf 10, 11, 12

[13] Sakaé Fuchino and Assaf Rinot, Openly generated Boolean algebras and the Fodor-type Reflection Principle, Fundamenta Mathematicae 212, (2011), 261-283.

[14] Fuchino, Sakaé, and Sakai, Hiroshi, The first-order definability of generic large cardinals, submitted. Extended version of the paper:
https://fuchino.ddo.jp/papers/definability-of-glc-x.pdf 5

[15] Sakaé Fuchino, Hiroshi Sakai, Lajos Soukup and Toshimichi Usuba, More about Fodor-type Reflection Principle,
https://fuchino.ddo.jp/papers/moreFRP-x.pdf

[16] Sakaé Fuchino, and Toshimichi Usuba, On recurrence axioms, submitted.
https://fuchino.ddo.jp/papers/recurrence-axioms-x.pdf 2, 5, 12, 13, 15, 16

[17] Gabriel Goldberg, Usuba's theorem is optimal, Journal of Mathematical Logic, (2024).

[18] Martin Goldstern, and Saharon Shelah, The Bounded Proper Forcing Axiom, The Journal of Symbolic Logic, Vol.60, No.1 (1995), 58–73.

[19] Benjamin P. Goodman, Sigma_n-correct Forcing Axioms, CUNY Academic Works (2024).
`https://academicworks.cuny.edu/gc_etds/5808/` 8

[20] Joel David Hamkins, A simple maximality principle, The Journal of Symbolic Logic Vol.68, No.7, (2003), 527–550.

[21] Joel David Hamkins, and Thomas A. Johnstone, Resurrection axioms and uplifting cardinals, Archive for Mathematical Logic, Vol.53, Iss.3-4, (2014), 463–485. 12

[22] Joel David Hamkins, and Thomas A. Johnstone, Strongly uplifting cardinals and the boldface resurrection axioms, Archive for Mathematical Logic Vol. 56, (2017), 1115–1133. 12, 14

[23] Yair Hayut, Menachem Magidor, and Ajejandro Poveda, Identity crisis between supercompactness and Vopěnka principle, JSL, Vol.87 (2), 2022, 626-648.

[24] Daisuke Ikegami and Nam Trang, On a class of maximality principles, Archive for Mathematical Logic, Vol. 57, (2018), 713–725.

[25] Thomas Jech, Set Theory, The Third Millennium Edition, Springer, (2003). 3, 5, 8

[26] Akihiro Kanamori, The Higher Infinite, Springer–Verlag (1994/2003). 3, 4, 5, 7, 11

[27] Bernhard König, Generic compactness reformulated, Archive for Mathematical Logic 43, (2004), 311–326.

[28] Kenneth Kunen, Set Theory, College Publications (2011). 3

[29] W.C. Powell, Almost huge cardinal and Vopenka's principle, Notices of Mamerican Mathematical Society, 19 (1972), A-616. 8

[30] Jonas Reitz, The Ground Axiom, The Journal of Symbolic Logic, Vol. 72, No. 4 (2007), 1299–1317.

[31] Farmer S, An answer to the question: Why are these two definitions of extendible cardinal equivalent?, Mathematics StackExchange (2022). 6

[32] Konstantinos Tsaprounis, On $C^{(n)}$-extendible cardinals, The Journal of Symbolic Logic, Vol.83, No.3, (2018), 1112–1131. 8

[33] ________________, Extendible cardinals and the mantle, Archive for Mathematical Logic, Vol.58, (2019), 71-75.

[34] Matteo Viale, Martin's maximum revisited, Archive for Mathematical Logic, Vol.55, (2016), 295–317. 13

[35] Hugh Woodin, The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal, De Gruyter, 2nd rev. ed. Edition (2010).