

# Extendible cardinals, and Laver-generic large cardinal axioms for extendibility

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## Abstract

We introduce (super- $C^{(\infty)}$ -)Laver-generic large cardinal axioms for extendibility ((super- $C^{(\infty)}$ -)LgLCAs for extendible, for short), and show that most of the previously known consequences of the (super- $C^{(\infty)}$ -)LgLCAs for ultrahuge, in particular, general forms of Resurrection Principles, Maximality Principles, and Absoluteness Theorems, already follow from (super- $C^{(\infty)}$ -)LgLCAs for extendible.

The consistency of LgLCAs for extendible (for  $\omega_1$ -preserving, relevant transfinitely iterable  $\Sigma_2$ -definable classes of posets) follows from an extendible cardinal while the consistency of super- $C^{(\infty)}$ -LgLCAs for extendible follows from a model with a strongly super- $C^{(\infty)}$ -extendible cardinal. If  $\mu$  is an almost-huge cardinal, there are cofinally many  $\kappa < \mu$  such that  $V_\mu \models$  “ $\kappa$  is strongly super- $C^{(\infty)}$  extendible”.

Most of the known reflection properties follow already from some of the LgLCAs for supercompact. We give a survey on the related results.

We also show the separation between some of the LgLCAs as well as between LgLCAs and their consequences.

## Contents

1. Introduction .....	2
2. Extendible and super- $C^{(n)}$ -extendible cardinals .....	5
3. Models with super- $C^{(\infty)}$ -extendible cardinals .....	9
4. LgLCAs and super- $C^{(\infty)}$ -LgLCAs for extendibility imply (almost) everything .	12
5. Consistency of LgLCAs and super- $C^{(\infty)}$ -LgLCAs for extendibility .....	21
References .....	37

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# 1 Introduction

The present note is a short version of the more extensive [9] in preparation.

In Section 2, we begin with reviewing known characterizations of extendible cardinals (Proposition 2.4). We then look into the super- $C^{(n)}$  and super- $C^{(\infty)}$  large cardinal versions of extendibility, and give their characterizations (Proposition 2.5, Theorem 2.6).

In Section 3, we evaluate the consistency strength of super  $C^{(\infty)}$ -extendible cardinal: It is classical that if  $\mu$  is almost-huge, then  $V_\mu$  satisfies the Second-order Vopěnka Principle (Lemma 3.1). We show that the Second-order Vopěnka Principle implies that there are cofinally many  $\kappa < \mu$  such that  $V_\mu \models$  “ $\kappa$  is strongly super  $C^{(\infty)}$ -extendible” (Proposition 3.2).

In Section 4, we introduce Laver-generic large cardinal versions of these notions of large cardinals, and the axioms asserting the existence of a/the Laver-generic large cardinal — the (super- $C^{(\infty)}$ -)  $\mathcal{P}$ -Laver-generic large cardinal axioms for extendibility ((super- $C^{(\infty)}$ -) $\mathcal{P}$ -LgLCAs for extendible, for short) for various classes  $\mathcal{P}$  of posets, and show that most of the previously known consequences of the (super- $C^{(\infty)}$ -) $\mathcal{P}$ =LgLCA for ultrahugeness. In particular, the strong and general forms of Resurrection Principles, Maximality Principles, and Absoluteness Theorems, already follow from the (super- $C^{(\infty)}$ -)LgLCAs for extendible.

The consistency of the LgLCAs for extendible (and for a relevant transfinately iterable  $\Sigma_2$ -definable classes of posets — see below for the definition of these notions) follows from an extendible cardinal while a  $V_\mu$  with strongly super- $C^{(\infty)}$  extendible cardinal can be generically extended to a model with the super- $C^{(\infty)}$ -  $\mathcal{P}$ -LgLCAs for extendible, and for a relevant transfinately iterable classes  $\mathcal{P}$  of posets (which may be defined by formulas more complex than  $\Sigma_2$ , see Theorem 5.2).

In contrast, it is known that (super- $C^{(\infty)}$ -)LgLCAs for hyperhugeness for a relevant transfinately iterable class  $\mathcal{P}$  of posets, axioms apparently stronger than the corresponding axioms for ultrahugeness, are equiconsistent with the existence of a genuine (super- $C^{(\infty)}$ -)hyperhuge ([17]).

Most of the known reflection properties are consequences of some of the LgLCAs

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for supercompact. In Section 6, we give a survey on this topic.

In Section 7, we prove the separation between some of the **LgLCAs** as well as between **LgLCAs** and their consequences.

(super- $C^{(\infty)}$ -) $\mathcal{P}$ -**LgLCAs** are generic large cardinal axioms claiming for each  $\mathbb{P} \in \mathcal{P}$ , the existence of  $\mathbb{P}$ -name  $\mathbb{Q}$  and  $\mathbb{P} * \mathbb{Q}$ -generic elementary embedding with the critical point  $\kappa_{\text{refl}} := \max\{\aleph_2, 2^{\aleph_0}\}$  with certain closedness properties of the target model which depends on the notion of large cardinal used there (and certain further conditions in addition in case of “super- $C^{(\infty)}$ -”) — see the definition in Section 4. (super- $C^{(\infty)}$ -)Laver generic large cardinal axioms for *all posets* in terms of generic elementary embeddings with the critical point  $2^{\aleph_0} = \aleph_1$  is also possible. Such axioms has been already discussed in [17] and [8].

In Section 8, we abbreviate this type of axiom for the notion of extendibility as “the **LgLCAA** for extendible”, and examine their consistency and their consequences.

Our notation is standard, and mostly compatible with that of [26], [27], and/or [29], but with the following slight deviations: “ $j : M \xrightarrow{\sim}_{\kappa} V$ ” expresses the situation that  $M$  and  $N$  are transitive (sets or classes),  $j$  is an elementary embedding of  $M$  into  $N$  and  $\kappa$  is the critical point of  $j$ . We use letters with under-tilde to denote  $\mathbb{P}$ -names for a poset  $\mathbb{P}$ . Underline added to a symbol like  $\underline{\alpha}$  emphasizes that the symbol is used to denote a variable in a language, mostly the language of **ZFC** which is denoted by  $\mathcal{L}_{\in}$ . A letter with under-bracket like  $\underline{c}$  emphasizes that the letter denotes a (new) constant symbol added to the language.

In the following, we always denote with  $\mathcal{P}$ , a class of posets. We usually assume that the class  $\mathcal{P}$  of posets satisfies the following properties which we call *iterability*. A class  $\mathcal{P}$  of posets is said to be (two-step) *iterable* if

- (1.1)  $\mathcal{P}$  is closed with respect to forcing equivalence, and  $\{\mathbb{1}\} \in \mathcal{P}$ ;
- (1.2)  $\mathcal{P}$  is closed with respect to restriction. That is, for  $\mathbb{P} \in \mathcal{P}$  and  $\mathbb{p} \in \mathbb{P}$ , we always have  $\mathbb{P} \upharpoonright \mathbb{p} \in \mathcal{P}$ ; and
- (1.3) For any  $\mathbb{P} \in \mathcal{P}$ , and any  $\mathbb{P}$ -name  $\mathbb{Q}$  of a poset with  $\Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{P}\text{”}$ , we have  $\mathbb{P} * \mathbb{Q} \in \mathcal{P}$ .

$\mathcal{P}$  is *transfinitely iterable* if it is iterable and endowed with an iteration of arbitrary length for an appropriate notion of support, and the iteration satisfies the corresponding Iteration and Factor Lemmas, and all of these facts should be provably so. Thus if  $\mathcal{P}$  is transfinitely iterable then  $M \models \text{“}\mathcal{P} \text{ is transfinitely iterable”}$  for all models  $M$  of **ZFC**.

We shall call a transfinitely iterable class  $\mathcal{P}$  of posets *relevant* if it provably contains a poset collapsing of  $\omega_2$  or an adding of a new real (or both).

For a property  $P$  of posets, we shall say “ $\mathcal{P}$  is  $P$ ” (e.g. “ $\mathbb{P}$  is c.c.c.”) to indicate that all elements of  $\mathcal{P}$  have the property  $P$ . In contrast, if we say  $\mathcal{P}$  is the class of posets with the property  $P$ , we mean  $\mathcal{P} = \{\mathbb{P} : \mathbb{P} \models P\}$ .

We adopt the notation of [1] and denote  $C^{(n)} := \{\alpha \in \text{On} : V_\alpha \prec_{\Sigma_n} \mathbb{V}\}$  for  $n \in \mathbb{N}$ . Intuitively we put  $C^{(\infty)} := \{\alpha \in \text{On} : V_\alpha \prec \mathbb{V}\}$ , though this is not a definable class in the language of ZFC, due to undefinability of the truth. For each transitive set model  $M$ , however,  $C := \{\alpha \in \text{On} \cap M : V_\alpha^M \prec M\}$  is a(n existent) set. Note that the first order logic in this context on which the elementary submodel relation  $\prec$  relies, is not the meta-mathematical one but rather the logic in the set theory whose formulas are the corresponding subset of  $\omega$  (consisting of Gödel numbers) in ZFC. Also,  $C$  is not a first-order definable subset of  $M$ , again because of the undefinability of the truth in  $M$ .

The following (almost trivial) lemma is often used without mention:

**Lemma 1.1** (see e.g. Section 1 in Bagaria [1]) *For an uncountable cardinal  $\alpha$ ,  $\mathcal{H}(\alpha) = V_\alpha$  if and only if  $V_\alpha \prec_{\Sigma_1} \mathbb{V}$ .*

*If  $V_\alpha \prec_{\Sigma_1} \mathbb{V}$  then  $\alpha$  is an uncountable strong limit cardinal.*

*Thus, we have*

$$C^{(1)} = \{\alpha : \alpha \text{ is an uncountable limit cardinal with } V_\alpha = \mathcal{H}(\alpha)\}. \quad \square$$

Note that there is a (first-order) sentence  $\varphi$  such that  $V_\alpha \models \varphi$  if and only if  $V_\alpha = \mathcal{H}(\alpha)$  for a cardinal  $\alpha$ .

The following notes are results of an examination of what was suggested by Gabriel Goldberg in a discussion we had during his visit to Kobe after the RIMS Set Theory Workshop 2024. Toshimichi Usuba pointed out some elementary flaws in early sketches of the note. I learned some known arguments used below in conversation with Hiroshi Sakai. I am grateful for their comments and advices. Also I would like to thank Andreas Leitz for giving me a permission to present an exposition of his proof of Theorem 2.6 in the extended version of the present article.

Back in the summer of 2015, I enjoyed a pleasant walk around the port of Yokohama with Joel Hamkins when we were together on the way to Kyoto starting from Tokyo and made a short stop in Yokohama. On the walk, Joel told me about his then recent researches and research projects, and one of them was about the Resurrection Axioms.

Now that his Resurrection Axioms are shown to be restricted versions of the LgLCAs (see Theorem 4.2), I notice that what I learned from him on that walk might have influenced me subliminally when I introduced the LgLCAs in the late

2010s. In that case, I have to thank Joel again sincerely, also for the nice conversation we had in Yokohama.

## 2 Extendible and super- $C^{(n)}$ -extendible cardinals

In this section, we summarize some well-known and some other less well-known facts about extendible cardinals and introduce the notion of super- $C^{(n)}$ -extendible cardinals.

It appears that the notion of super- $C^{(n)}$ -extendible cardinals is equivalent to some other already known strong variants of extendibility, see Theorem 2.6. At the moment, it is yet unknown if similar equivalence is also available for super- $C^{(n)}$ -ultrahuge cardinals, or super- $C^{(n)}$ -hyperhuge cardinals.

It is easy to see that the definition of an extendible cardinal in Kanamori [27] is equivalent to its slight modification: a cardinal  $\kappa$  is *extendible* if (2.1): for any  $\alpha > \kappa$  there are  $\beta \in \text{On}$  and  $j : V_\alpha \xrightarrow{\simeq}_\kappa V_\beta$  such that (2.2):  $j(\kappa) > \alpha$ .

An extendible cardinal is supercompact (see e.g. Proposition 23.6 in [27]). The following is easy to prove:

**Lemma 2.1** *If  $\kappa$  is extendible then there are class many measurable cardinals.  $\square$*

The “measurable” in the Lemma above cannot be replaced by “supercompact”. See Exercise 23.9, (b) in [27].

Since existence of a supercompact cardinal does not imply existence of any large cardinal above it (see Exercise 22.8 in [27]), Lemma 2.1 explains the transcendence of extendible cardinals above supercompact.

In Jech [26], extendibility is defined by (2.1) without (2.2). We say in the following that  $\kappa$  is *Jech-extendible* if it satisfies (2.1) but not necessarily (2.2). The two definitions of extendibility are equivalent. In Proposition 2.4 below, we show the equivalence of these two together with some other characterizations of extendibility.

The key fact to Proposition 2.4 is that the elementary embedding in (2.1) can be often lifted to an elementary embedding of the whole universe  $V$ .

We call a mapping  $f : M \rightarrow N$  *cofinal* (in  $N$ ) if, for all  $b \in N$ , there is  $a \in M$  such that  $b \in f(a)$ .

**Lemma 2.2** (A special case of Lemma 6 in Fuchino and Sakai [15]) *Suppose that  $\theta$  is a cardinal and  $j_0 : \mathcal{H}(\theta) \xrightarrow{\simeq} N$  for a transitive set  $N$ . Let  $N_0 := \bigcup j_0''\mathcal{H}(\theta)$ . Then  $j_0 : \mathcal{H}(\theta) \xrightarrow{\simeq} N_0$  and  $j_0$  is cofinal in  $N_0$ .  $\square$*

**Lemma 2.3** (A special case of Lemma 7 in [15]) *For any regular cardinal  $\theta$  and any cofinal  $j_0 : \mathcal{H}(\theta) \xrightarrow{\sim} N$ , there are  $j, M \subseteq \mathbb{V}$  such that  $j : \mathbb{V} \xrightarrow{\sim} M$ ,  $N \subseteq M$  and  $j_0 \subseteq j$ .  $\square$*

The following equivalence should be a folklore. The author learned some arguments in the detail of the following proof from an answer by Farmer S in [33].

**Proposition 2.4** *For a cardinal  $\kappa$  the following are equivalent:*

- (a)  $\kappa$  is extendible.
- (b)  $\kappa$  is Jech-extendible.
- (a') For all  $\lambda > \kappa$ , there are  $j, M \subseteq \mathbb{V}$  such that  $j : \mathbb{V} \xrightarrow{\sim}_{\kappa} M$ ,  $j(\kappa) > \lambda$  and  $V_{j(\lambda)} \in M$ .
- (b') For all  $\lambda > \kappa$ , there are  $j, M \subseteq \mathbb{V}$  such that  $j : \mathbb{V} \xrightarrow{\sim}_{\kappa} M$ , and  $V_{j(\lambda)} \in M$ .

**Proof.** (a)  $\Rightarrow$  (b): is clear by definition.

(b)  $\Rightarrow$  (a): This can be proved by an argument similar to that of the proof of (b)  $\Rightarrow$  (a) of Proposition 2.5 below.

(a)  $\Rightarrow$  (a'): follows from Lemmas 2.2 and 2.3.

(a')  $\Rightarrow$  (b'): is trivial.

(b')  $\Rightarrow$  (b): For  $j$  as in (b'),  $j \upharpoonright V_{\lambda} : V_{\lambda} \xrightarrow{\sim}_{\kappa} V_{j(\lambda)}$  is as desired in (b).  $\square$  (Proposition 2.4)

The notion of super- $C^{(n)}$ -large cardinal was introduced in Fuchino and Usuba [17]. Proposition 2.4 in mind, we define the super- $C^{(n)}$ -extendibility as follows: For a natural number  $n$ , we call a cardinal  $\kappa$  *super- $C^{(n)}$ -extendible* if for any  $\lambda_0 > \kappa$  there are  $\lambda \geq \lambda_0$  with  $V_{\lambda} \prec_{\Sigma_n} \mathbb{V}$ , and  $j, M \subseteq \mathbb{V}$  such that  $j : \mathbb{V} \xrightarrow{\sim}_{\kappa} M$ ,  $j(\kappa) > \lambda$ ,  $V_{j(\lambda)} \in M$  and  $V_{j(\lambda)} \prec_{\Sigma_n} \mathbb{V}$ .

We call a cardinal  $\kappa$  *super- $C^{(\infty)}$ -extendible* if  $\kappa$  is super- $C^{(n)}$ -extendible for all  $n \in \omega$ . In general, we cannot formulate the assertion “ $\kappa$  is super- $C^{(\infty)}$ -extendible” in the language of ZF since we would need an infinitary logic to do this unless we are allowed to introduce a new constant symbol to refer the cardinal across the infinitely many formulas expressing the  $C^{(n)}$ -extendibility of the cardinal for each  $n \in \mathbb{N}$ . However, there are certain situations in which we can say that a cardinal is super- $C^{(\infty)}$ -extendible. One of them is when we are talking about a cardinal in a set model. In this case, being “super- $C^{(\infty)}$ -extendible” in the model is an  $\mathcal{L}_{\omega_1, \omega}$  sentence which is satisfied by the cardinal in the model. Another situation is when we are talking about a cardinal in an inner model and the cardinal is definable in  $\mathbb{V}$  (e.g. as  $2^{\aleph_0}$  in the outer model  $\mathbb{V}$ ). Note that in the latter case, we can formulate the super- $C^{(\infty)}$ -extendibility of the cardinal in infinitely many formulas, and hence  $n$  in this case ranges only over metamathematical natural numbers.

Similarly to Proposition 2.4, we have the following equivalence:

**Proposition 2.5** *For a cardinal  $\kappa$  and  $n \geq 1$ , the following are equivalent:*

(a) *For any  $\lambda_0 > \kappa$  there are  $\lambda > \lambda_0$  with  $V_\lambda \prec_{\Sigma_n} \mathbf{V}$ ,  $j_0$ , and  $\mu$  such that  $j_0 : V_\lambda \xrightarrow{\prec_\kappa} V_\mu$ ,  $j(\kappa) > \lambda$ , and  $V_\mu \prec_{\Sigma_n} \mathbf{V}$ .*

(b) *For any  $\lambda_0 > \kappa$  there are  $\lambda > \lambda_0$  with  $V_\lambda \prec_{\Sigma_n} \mathbf{V}$ ,  $j_0$ , and  $\mu$  such that  $j_0 : V_\lambda \xrightarrow{\prec_\kappa} V_\mu$ , and  $V_\mu \prec_{\Sigma_n} \mathbf{V}$  (without the condition “ $j(\kappa) > \lambda$ ”).*

(a')  *$\kappa$  is super- $C^{(n)}$ -extendible.*

(b') *for any  $\lambda_0 > \kappa$  there are  $\lambda \geq \lambda_0$  with  $V_\lambda \prec_{\Sigma_n} \mathbf{V}$ , and  $j, M \subseteq \mathbf{V}$  such that  $j : \mathbf{V} \xrightarrow{\prec_\kappa} M$ ,  $V_{j(\lambda)} \in M$ , and  $V_{j(\lambda)} \prec_{\Sigma_n} \mathbf{V}$  (without the condition “ $j(\kappa) > \lambda$ ”).  $\square$*

**Proof.** The proof is similar to that of Lemma 2.4. We only show (b)  $\Rightarrow$  (a). The following proof is a modification of the proof of Lemma 2.4, (b)  $\Rightarrow$  (a) given by Farmer S in [33].

Assume, toward a contradiction, that  $\kappa$  satisfies (b) but not (a). Then there is a  $\gamma$  such that

(2.3) for all sufficiently large  $\lambda > \kappa$ , if (2.4):  $V_\lambda \prec_{\Sigma_n} \mathbf{V}$ , and  $\mu, j$  are such that  
(2.5):  $j : V_\lambda \xrightarrow{\prec_\kappa} V_\mu$  and (2.6):  $V_\mu \prec_{\Sigma_n} \mathbf{V}$ ,  
then  $j(\kappa) < \gamma$ .

In the following, let  $\gamma$  be the minimal among such  $\gamma$ 's satisfying (2.3).

**Claim 2.5.1**  *$\gamma$  is a limit ordinal. For all sufficiently large  $\lambda$  with (2.4) and for all  $\xi < \gamma$ , there are  $\mu, j$  with (2.5), (2.6) such that  $j(\kappa) > \xi$ .*

┆ Suppose  $\gamma$  is not a limit ordinal, say  $\gamma = \xi + 1$ . Then there are cofinally many  $\lambda \in \text{On}$  such that  $V_\lambda \prec_{\Sigma_n} \mathbf{V}$  (actually  $\lambda \in \text{Card}$ , see Lemma 1.1), and there are  $j$  and  $\mu$  with (2.5), (2.6) and  $j(\kappa) = \xi$ . Restrictions of  $j$ 's in the previous sentence witness that, for all  $\lambda > \xi$  with  $V_\lambda \prec_{\Sigma_n} \mathbf{V}$ , there are  $j$  and  $\mu$  such that (2.5), (2.6) and  $j(\kappa) = \xi$  hold.

Let  $\lambda^*$  with  $V_{\lambda^*} \prec_{\Sigma_n} \mathbf{V}$  be a sufficiently large such  $\lambda$  where “sufficiently large” is meant in terms of (2.3). Let  $j^*$  and  $\mu^*$  be such that  $j^* : V_{\lambda^*} \xrightarrow{\prec_\kappa} V_{\mu^*}$ ,  $j^*(\kappa) = \xi$ , and  $V_{\mu^*} \prec_{\Sigma_n} \mathbf{V}$ .

Since  $\lambda^* \leq \mu^*$ , there is also  $k : V_{\mu^*} \xrightarrow{\prec_\kappa} V_{\nu^*}$  such that  $V_{\nu^*} \prec_{\Sigma_n} \mathbf{V}$  and  $k(\kappa) = \xi$ . But then we have  $k \circ j^* : V_{\lambda^*} \xrightarrow{\prec_\kappa} V_{\nu^*}$  and  $k \circ j^*(\kappa) = k(\xi) > k(\kappa) = \xi$ . This is a contradiction to (2.3).

The second assertion of the claim follows from this and by the minimality of  $\gamma$ .

┆ (Claim 2.5.1)

**Claim 2.5.2** For all sufficiently large  $\mu > \kappa$  with  $V_\mu \prec_{\Sigma_n} \mathbf{V}$ , and  $k, \nu$  with  $V_\nu \prec_{\Sigma_n} \mathbf{V}$  and  $k : V_\mu \xrightarrow{\prec}_\kappa V_\nu$ , we have  $k''\gamma \subseteq \gamma$ .

⊢ Suppose otherwise. Then we find  $\xi < \gamma$  such that, for cofinally many  $\mu > \kappa$  with  $V_\mu \prec_{\Sigma_n} \mathbf{V}$ , there are  $\nu, k$  such that  $k : V_\mu \xrightarrow{\prec}_\kappa V_\nu$ ,  $V_\nu \prec_{\Sigma_n} \mathbf{V}$  and  $k(\xi) \geq \gamma$ . By considering restrictions of  $k$ 's as above, we conclude that for all  $\mu > \xi$  with  $V_\mu \prec_{\Sigma_n} \mathbf{V}$ , there are  $\nu$  and  $k$  as above.

Let  $\lambda > \xi$  and  $j$  (together with rechosen  $\mu$  and  $k$  for this  $\lambda$ ) be such that  $V_\lambda \prec_{\Sigma_n} \mathbf{V}$ ,  $j : V_\lambda \xrightarrow{\prec}_\kappa V_\mu$  and  $j(\kappa) > \xi$  (possible by the second half of Claim 2.5.1). Then we have  $k \circ j : V_\lambda \xrightarrow{\prec}_\kappa V_\nu$  and  $k \circ j(\kappa) > k(\xi) \geq \gamma$ .

Since  $\lambda, \mu, \nu, j, k$  can be chosen such that  $\lambda$  is sufficiently large (in terms of (2.3)), this is a contradiction. ⊣ (Claim 2.5.2)

Now, let  $\lambda > \kappa$  be sufficiently large with  $\lambda \geq \gamma + 2$ ,  $V_\lambda \prec_{\Sigma_n} \mathbf{V}$ , and  $j : V_\lambda \xrightarrow{\prec}_\kappa V_\mu$  with  $V_\mu \prec_{\Sigma_n} \mathbf{V}$ . By Claim 2.5.2, we have  $j''\gamma \subseteq \gamma$ .

**Case 1.**  $cf(\gamma) = \omega$ . Then  $j(\gamma) = \gamma$  and hence  $j \upharpoonright V_{\gamma+2} : V_{\gamma+2} \xrightarrow{\prec}_\kappa V_{\gamma+2}$ . This is a contradiction to Kunen's proof (see e.g. Kanamori [27] Corollary 23.14).

**Case 2.**  $cf(\gamma) > \omega$ . then, letting  $\kappa_0 := \kappa$ ,  $\kappa_{n+1} := j(\kappa_n)$  for  $n \in \omega$  and  $\kappa_\omega := \sup_{n \in \omega} \kappa_n$ , we have  $\kappa_\omega < \gamma$ , and  $j \upharpoonright V_{\kappa_\omega+2} : V_{\kappa_\omega+2} \xrightarrow{\prec}_\kappa V_{\kappa_\omega+2}$ . This is again a contradiction to Kunen's proof. □ (Proposition 2.5)

Super- $C^{(n)}$ -extendibility is actually equivalent to  $C^{(n)}$ -extendibility of Bagaria [1]. Konstantinos Tsaprounis proved the equivalence of  $C^{(n)}$ -extendibility with a variant of super- $C^{(n)}$ -extendibility which he called  $C^{(n)+}$ -extendibility in [38].

A cardinal  $\kappa$  is  $C^{(n)}$ -extendible if, for any  $\alpha > \kappa$ , there is  $\beta$  and  $j : V_\alpha \xrightarrow{\prec}_\kappa V_\beta$  such that  $j(\kappa) > \alpha$  and  $V_{j(\kappa)} \prec_{\Sigma_n} \mathbf{V}$ .

The following notion is introduced by Benjamin Goodman [20].

A cardinal  $\kappa$  is *supercompact for  $C^{(n)}$*  if, for any  $\lambda > \kappa$  there is  $j : \mathbf{V} \xrightarrow{\prec}_\kappa M$  such that  ${}^\lambda M \subseteq M$  and  $C^{(n)} \cap \lambda = (C^{(n)})^M \cap \lambda$ .

The equivalence of (a) and (b') in the following Theorem 2.6 was established in Tsaprounis [38]. Andreas Lietz recently found a short proof of the Theorem 2.6. Goodman possibly proved the equivalence of (a) and (c) in the theorem, but mentioned only the case of  $n = 1$  in his [20]. His proof is given in the extended version of the present article.

**Theorem 2.6** (Andreas Lietz) For a cardinal  $\kappa$  and for all  $n \geq 1$  the following are equivalent: (a)  $\kappa$  is  $C^{(n)}$ -extendible.

(b)  $\kappa$  is super- $C^{(n)}$ -extendible.

(b')  $\kappa$  is  $C^{(n)+}$ -extendible.

(c)  $\kappa$  is supercompact for  $C^{(n+1)}$ . □

**Problem 2.7** (a) *Does a similar equivalence hold for super- $C^{(n)}$ -hyperhuge cardinals?*

(b) *Do similar equivalence theorems hold for generic and/or Laver-generic versions of large cardinals?*

The answer to (b) in case of Laver-generic version is negative: in many natural cases Laver generic  $\kappa$  is  $\aleph_2$  or continuum. However, we have  $V_\kappa \not\prec_{\Sigma_1} V$ .

### 3 Models with super- $C^{(\infty)}$ -extendible cardinals

We prove that there are unboundedly many super- $C^{(\infty)}$ -extendible cardinals in  $V_\kappa$  below an almost-huge cardinal  $\kappa$  (Corollary 3.3).

For a cardinal  $\kappa$ , we say that  $V_\kappa$  satisfies the *Second-order Vopěnka's Principle* if for any set  $C \subseteq V_\kappa$  of structures of the same signature with  $C \not\subseteq V_\kappa$  (which is not necessarily a definable subset of  $V_\kappa$ ), there are non-isomorphic  $\mathfrak{A}, \mathfrak{B} \in C$  such that we have  $i : \mathfrak{A} \xrightarrow{\sim} \mathfrak{B}$  for an elementary embedding  $i$ .

The following is well-known (see e.g. Jech [26], Lemma 20.27), and attributed to William C. Powell.

**Lemma 3.1** (W.C. Powell [31]) *If  $\kappa$  is an almost-huge cardinal then  $V_\kappa$  satisfies the Second-order Vopěnka's Principle.*

**Proof.** Suppose that  $C \subseteq V_\kappa$  where  $C$  is a set of structures of the same signature, and  $C \not\subseteq V_\kappa$ . Without loss of generality, we may assume that (3.1):  $C$  is closed with respect to isomorphism. Then it is enough to show that there are non-isomorphic  $\mathfrak{A}, \mathfrak{B} \in C$  such that  $\mathfrak{B} \prec \mathfrak{A}$ . Note that  $rank(C) = \kappa$ .

Let  $j : V \xrightarrow{\sim}_\kappa M$  be an almost-huge elementary embedding (i.e.  $M$  satisfies (3.2):  $j^{(\kappa)} M \subseteq M$ ). Let  $\mathfrak{A}^* \in j(C) \setminus C$  — note that  $j(C) \setminus C \neq \emptyset$  since  $M \models$  “ $rank(j(C)) = j(\kappa) > \kappa$ ”, and hence the same inequality also holds in  $V$ . Let  $A$  be the underlying set of the structure  $\mathfrak{A}^*$ .

We have (3.3):  $M \models j(\mathfrak{A}^*) \not\cong \mathfrak{A}^*$  — otherwise  $M \models j(\mathfrak{A}^*) \cong \mathfrak{A}^* \in j(C)$ , and hence  $V \models \mathfrak{A}^* \in C$  by elementarity. This is a contradiction to the choice of  $\mathfrak{A}^*$ .

Let  $\mathfrak{A}' := j(\mathfrak{A}^*) \upharpoonright j''A$ .

**Claim 3.1.1** (1)  $M \models \mathfrak{A}' \in j(C)$ .

(2)  $M \models \mathfrak{A}' \prec j(\mathfrak{A}^*)$ .

$\vdash$  (1):  $\mathfrak{A}' \in M$ , and  $M \models \mathfrak{A}' \cong \mathfrak{A}^*$  by (3.2). Since  $\mathfrak{A}^* \in j(C)$  by the choice of  $\mathfrak{A}^*$ , the assumption (3.1) and the elementarity of  $j$  imply  $M \models \mathfrak{A}' \in j(C)$ .

(2): Working in  $M$ , we check that  $\mathfrak{A}' \subseteq j(\mathfrak{A}^*)$  satisfy Vaught's criterion.

Suppose  $a_0, \dots, a_{n-1} \in j''A$ ,  $a \in j(A)$  and  $j(\mathfrak{A}^*) \models \varphi(a, a_0, \dots, a_{n-1})$ . Let  $a'_0, \dots, a'_{n-1} \in A$  be such that  $a_0 = j(a'_0), \dots, a_{n-1} = j(a'_{n-1})$ .

Since  $M \models \text{“}\exists \underline{a} \in j(A) (j(\mathfrak{A}^*) \models \varphi(\underline{a}, j(a'_0), \dots, j(a'_{n-1})))\text{”}$ , it follows that  $\mathbf{V} \models \text{“}\exists \underline{a} \in A (\mathfrak{A}^* \models \varphi(\underline{a}, a'_0, \dots, a'_{n-1}))\text{”}$ . Let  $a' \in A$  be such that  $\mathbf{V} \models \mathfrak{A}^* \models \varphi(a', a'_0, \dots, a'_{n-1})$ . Then  $j(a') \in j''A$ , and  $M \models j(\mathfrak{A}^*) \models \varphi(j(a'), j(a'_0), \dots, j(a'_{n-1}))$  by elementarity. ⊣ (Claim 3.1.1)

Now by (3.3) and Claim 3.1.1, (2), we have

$$M \models \text{“there is a } \underline{\mathfrak{B}} \in j(C) \text{ such that } \underline{\mathfrak{B}} \not\cong j(\mathfrak{A}^*) \wedge \underline{\mathfrak{B}} \prec j(\mathfrak{A}^*)\text{”}.$$

By elementarity of  $j$  it follows that

$$\mathbf{V} \models \text{“there is a } \underline{\mathfrak{B}} \in C \text{ such that } \underline{\mathfrak{B}} \not\cong \mathfrak{A}^* \wedge \underline{\mathfrak{B}} \prec \mathfrak{A}^*\text{”}.$$

Let  $\mathfrak{B} \in C$  be such that

$$\mathbf{V} \models \text{“}\mathfrak{B} \in C \wedge \mathfrak{B} \not\cong \mathfrak{A}^* \wedge \mathfrak{B} \prec \mathfrak{A}^*\text{”}.$$

By (3.1), we may assume that (the underlying set of)  $\mathfrak{B}$  is a subset of  $\kappa$ . Since  $|\mathfrak{B}| < \kappa$ , we have (3.4):  $j(\mathfrak{B}) = \mathfrak{B}$ . Thus we have

$$\mathbf{V} \models \text{“}\mathfrak{B} \in C \wedge j(\mathfrak{B}) \not\cong \mathfrak{A}^* \wedge j(\mathfrak{B}) \prec \mathfrak{A}^*\text{”}.$$

By absoluteness of “ $\not\cong$ ” and “ $\prec$ ” between  $\mathbf{V}$  and  $M$  for structures as above, it follows that

$$M \models \text{“}j(\mathfrak{B}) \in j(C) \wedge j(\mathfrak{B}) \not\cong \mathfrak{A}^* \wedge j(\mathfrak{B}) \prec \mathfrak{A}^*\text{”}.$$

Thus  $M \models \text{“there is } \underline{\mathfrak{A}} \in j(C) \text{ such that } j(\mathfrak{B}) \not\cong \underline{\mathfrak{A}} \wedge j(\mathfrak{B}) \prec \underline{\mathfrak{A}}\text{”}$ . By elementarity it follows that

$$(3.5) \quad \mathbf{V} \models \text{“there is } \underline{\mathfrak{A}} \in C \text{ such that } \mathfrak{B} \not\cong \underline{\mathfrak{A}} \wedge \mathfrak{B} \prec \underline{\mathfrak{A}}\text{”}.$$

A witness  $\underline{\mathfrak{A}} \in C$  of  $\underline{\mathfrak{A}}$  in (3.5) together with  $\mathfrak{B}$  as above is then as desired.

□ (Lemma 3.1)

At the present, we can only start from the existence of a cardinal with a property stronger than the super- $C^{(\infty)}$ -extendibility to prove the existence of a super- $C^{(\infty)}$ -extendible Laver function (Lemma 5.1). We call such a cardinal *strongly super- $C^{(\infty)}$  extendible*.

Formally, both the notion of a super- $C^{(\infty)}$  extendible cardinal and that of a strongly super- $C^{(\infty)}$  extendible cardinal are undefinable in the language of ZFC.

The reason of the undefinability is of course that the predicate “ $\underline{M} \prec \mathbf{V}$ ” (with the free variable  $\underline{M}$ ) is not definable due to Tarski’s undefinability of truth (we would need infinitely many formulas referring to the free variable  $\underline{M}$  which is impossible in the framework of first-order logic). This is also the reason why we are talking only about (strongly) super- $C^{(\infty)}$  extendible cardinals in a set structure  $V_\mu$  in the following.

For an inaccessible  $\mu$  and  $\kappa < \mu$ , we say that  $\kappa$  is *strongly super- $C^{(\infty)}$ -extendible* in  $V_\mu$ , and denote this as  $V_\mu \models$  “ $\kappa$  is strongly super- $C^{(\infty)}$ -extendible”, if for any  $\kappa < \lambda < \mu$ , there are  $\lambda < \lambda' < \lambda''$  and  $j$  such that  $V_{\lambda'} \prec V_{\lambda''} \prec V_\mu$ , and  $j : V_{\lambda'} \xrightarrow{\prec}_\kappa V_{\lambda''}$ .

Note that by Proposition 2.5,  $V_\mu \models$  “ $\kappa$  is strongly super- $C^{(\infty)}$ -extendible” implies  $V_\mu \models$  “ $\kappa$  is super- $C^{(\infty)}$ -extendible”.

**Proposition 3.2** *Suppose that  $\mu$  is an inaccessible cardinal, and (3.6):  $V_\mu$  satisfies the Second-order Vopěnka’s Principle. Then there are unboundedly many  $\kappa < \mu$  such that  $V_\mu \models$  “ $\kappa$  is strongly super- $C^{(\infty)}$ -extendible”.*

**Proof.** Suppose  $\beta^* < \mu$ . We want to show that there is  $\beta^* \leq \kappa < \mu$  such that  $V_\mu \models$  “ $\kappa$  is strongly super- $C^{(\infty)}$ -extendible”.

Let  $C := \{\alpha < \mu : V_\alpha \prec V_\mu\}$ , and  $\mathcal{V}_\mu := \langle V_\mu, \in, C \rangle$ . Let  $\mathcal{L}$  be the language of the structure  $\mathcal{V}_\mu$ .

Let

$$I := \{\alpha < \mu : \alpha > \beta^*, \alpha \text{ is an } \omega\text{-limit of ordinals } \eta \text{ such that } \mathcal{V}_\mu \upharpoonright V_\eta \prec \mathcal{V}_\mu\}.$$

Note that  $I$  is cofinal in  $\mu$  and we have  $\mathcal{V}_\mu \upharpoonright V_\alpha \prec \mathcal{V}_\mu$  for all  $\alpha \in I$ .

For each  $\alpha \in I$ , let  $C_\alpha \subseteq \alpha \setminus (\beta^* + 1)$  be a cofinal subset of  $\alpha$  of order-type  $\omega$  consisting of (increasing)  $\eta_n^\alpha$ ,  $n \in \omega$  with  $\mathcal{V}_\mu \upharpoonright V_{\eta_n^\alpha} \prec \mathcal{V}_\mu$ . In particular, we have  $C_\alpha \subseteq C$ . For each  $\alpha \in I$ , let  $\mathcal{W}_\alpha := \langle V_\alpha, \in, C_\alpha, \xi \rangle_{\xi < \beta^*}$  and

$$\mathcal{C} := \{\mathcal{W}_\alpha : \alpha \in I\}.$$

By (3.6), there are  $\alpha, \beta \in I$ ,  $\alpha < \beta$  such that there is an elementary embedding  $i : \mathcal{W}_\alpha \xrightarrow{\prec} \mathcal{W}_\beta$ . Since  $i''C_\alpha = C_\beta$  by elementarity,  $i''C_\alpha$  is cofinal in  $\beta$ , and hence  $i \upharpoonright \alpha$  can not be an identity mapping.

Let  $\kappa := \text{crit}(i)$ . Then  $\beta^* \leq \kappa < \alpha$  because of the constants  $\xi$ ,  $\xi < \beta^*$  in the structures.

For all  $k \in \omega$  such that  $\eta_k^\alpha > \kappa$ , we have  $i \upharpoonright V_{\eta_k^\alpha} : V_{\eta_k^\alpha} \xrightarrow{\prec}_\kappa V_{\eta_k^\beta}$ . Thus

$$\mathcal{V}_\mu \models “\exists \underline{\eta} \exists \underline{i} (\underline{\eta} \in \underline{C} \wedge \underline{i} : V_{\eta_k^\alpha} \xrightarrow{\prec}_\kappa V_{\eta_k^\beta})”.$$

By elementarity (in the extended language  $\mathcal{L}$ ),

$$\mathcal{V}_\mu \upharpoonright V_\alpha \models \text{“}\exists \underline{\eta} \exists \underline{i} (\underline{\eta} \in \underline{\mathcal{C}} \wedge \underline{i} : V_{\eta_k^\alpha} \xrightarrow{\sim}_\kappa V_{\underline{\eta}})\text{”}$$

for all  $k \in \omega$  such that  $\eta_k^\alpha > \kappa$ .

It follows that

$$\mathcal{V}_\mu \upharpoonright V_\alpha \models \forall \underline{\nu} \exists \underline{\eta}_0 \exists \underline{\eta} \exists \underline{i} (\underline{\eta}_0 > \underline{\nu}, \kappa \wedge \underline{\eta}_0, \underline{\eta} \in \underline{\mathcal{C}} \wedge \underline{i} : V_{\underline{\eta}_0} \xrightarrow{\sim}_\kappa V_{\underline{\eta}}).$$

Thus by elementarity (in the extended language  $\mathcal{L}$ ), we have:

$$\mathcal{V}_\mu \models \forall \underline{\nu} \exists \underline{\eta}_0 \exists \underline{\eta} \exists \underline{i} (\underline{\eta}_0 > \underline{\nu}, \kappa \wedge \underline{\eta}_0, \underline{\eta} \in \underline{\mathcal{C}} \wedge \underline{i} : V_{\underline{\eta}_0} \xrightarrow{\sim}_\kappa V_{\underline{\eta}}).$$

This simply means that  $V_\mu \models \text{“}\kappa \text{ is strongly super-}C^{(\infty)}\text{-extendible”}$ .  $\square$  (Proposition 3.2)

**Corollary 3.3** *Suppose that  $\mu$  is almost-huge. Then there are unboundedly many  $\kappa < \mu$  such that  $V_\mu \models \text{“}\kappa \text{ is strongly super-}C^{(\infty)}\text{-extendible”}$ .*

**Proof.** By Lemma 3.1 and Proposition 3.2.  $\square$  (Theorem 3.3)

The following can be proved by the same argument as that of Proposition 2.5.

**Lemma 3.4** *Suppose that  $\mu$  is an inaccessible cardinal,  $\kappa < \mu$ . Then  $V_\mu \models \text{“}\kappa \text{ is strongly super-}C^{(\infty)}\text{-extendible”}$  is equivalent to the condition that, for any  $\kappa < \lambda < \mu$  there are  $\lambda < \lambda' < \lambda''$ ,  $j$  and  $M$  such that  $\lambda'' = j(\lambda')$ ,  $V_{\lambda'} \prec V_{\lambda''} \prec V_\mu$ ,  $j : V_\mu \xrightarrow{\sim}_\kappa M$ ,  $M \subseteq V_\mu$ ,  $j(\kappa) > \lambda'$  and  $V_{\lambda''} \in M$ .  $\square$*

## 4 LgLCAs and super- $C^{(\infty)}$ -LgLCAs for extendibility imply (almost) everything

In this section, we prove that Laver-generic Large Cardinal Axioms (LgLCAs) and super- $C^{(\infty)}$ -LgLCAs for extendibility imply the strongest forms of resurrection axioms, maximality principles, and absoluteness known to be consistent under some large cardinal consistency. This was previously known to hold under LgLCAs for hugeness and super- $C^{(\infty)}$ -LgLCAs for hyperhugeness.

We begin with a short summary of definitions and known results around LgLCAs and super- $C^{(\infty)}$ -LgLCAs.

Laver-generic large cardinals were introduced in [12]. For a class  $\mathcal{P}$  of posets and a notion  $LC$  of large cardinal, a cardinal  $\kappa$  is said to be  $\mathcal{P}$ -Laver generic  $LC$  if the statement about the existence of elementary embedding  $j : \mathbb{V} \xrightarrow{\sim}_\kappa M$  for  $j$ ,  $M \subseteq \mathbb{V}$  with the closedness condition  $C_{LC}$  of  $M$  in the definition of the notion  $LC$  of large cardinal are replaced with the statement:

(4.1) for any  $\mathbb{P} \in \mathcal{P}$ , there is a  $\mathbb{P}$ -name  $\mathbb{Q}$  such that  $\Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{P}\text{”}$ , and for  $\mathbb{P} * \mathbb{Q}$ -generic  $\mathbb{H}$  there are  $j, M \subseteq V[\mathbb{H}]$  such that  $\mathbb{P}, \mathbb{P} * \mathbb{Q}, \mathbb{H} \in M, j : V \xrightarrow{\check{}}_{\kappa} M$ , and  $M$  satisfies  $C'_{LC}$  which is the generic large cardinal variant of the closedness property  $C_{LC}$  associated with the notion  $LC$  of large cardinal.

For supercompactness, the instance of (4.1) for an iterable  $\mathcal{P}$  is as follows: a cardinal  $\kappa$  is  $\mathcal{P}$ -Laver-generically supercompact ( $\mathcal{P}$ -Lg supercompact for short) if,

(4.2) for any  $\lambda > \kappa$ , and for any  $\mathbb{P} \in \mathcal{P}$ , there is a  $\mathbb{P}$ -name  $\mathbb{Q}$  such that  $\Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{P}\text{”}$ , and, for any  $(V, \mathbb{P} * \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $j, M \subseteq V[\mathbb{H}]$  such that  $j : V \xrightarrow{\check{}}_{\kappa} M, j(\kappa) > \lambda, \mathbb{P}, \mathbb{P} * \mathbb{Q}, \mathbb{H} \in M, j''\lambda \in M$ .

Note that, in (4.2), the closure property “ ${}^{\lambda}M \subseteq M$ ” in the usual definition of supercompactness is replaced with “ $j''\lambda \in M$ ”. For a genuine elementary embedding introduced by some ultrafilter, these two conditions are equivalent (see e.g. Kanamori [27], Proposition 22.4, (b)). This equivalence is no more valid in general for generic embeddings. Nevertheless, the condition “ $j''\lambda \in M$ ” can be still considered as a certain closure property (see Lemma 2.5 in Fuchino, Ottenbreit Maschio Rodrigues, and Sakai [12]).

We say that a  $\mathcal{P}$ -Lg supercompact cardinal  $\kappa$  is *tightly  $\mathcal{P}$ -Laver-generically supercompact* (*tightly  $\mathcal{P}$ -Lg supercompact*, for short) if additionally, we have  $|RO(\mathbb{P} * \mathbb{Q})| \leq j(\kappa)$ .

The present definition of tightness is slightly stronger than the one in some of the previous papers (e.g. see the corresponding definition in Fuchino, and Usuba [17]).

A (tightly)  $\mathcal{P}$ -Lg supercompact cardinal is often decided uniquely as the cardinal  $\kappa_{\text{refl}} := \sup(\{2^{\aleph_0}, \aleph_2\})$ . This is the case, if  $\mathcal{P}$  is the class of all  $\sigma$ -closed posets. Then CH holds under the existence of a  $\mathcal{P}$ -Lg supercompact  $\kappa$  and  $\kappa = \aleph_2 (= \kappa_{\text{refl}})$ .

Similarly, if  $\mathcal{P}$  is either the class of all proper posets or the class of all semi-proper posets, the existence of a  $\mathcal{P}$ -generically supercompact  $\kappa$  implies  $2^{\aleph_0} = \aleph_2$  and again  $\kappa = \kappa_{\text{refl}}$ .

For the case that  $\mathcal{P}$  is the class of all ccc posets, it is open whether a  $\mathbb{P}$ -Lg supercompact cardinal is decided to be  $\kappa_{\text{refl}}$ . However a tightly  $\mathcal{P}$ -Lg supercompact cardinal under the present definition of tightness<sup>1)</sup> is the continuum ( $= \kappa_{\text{refl}}$ ) and, in this case, the continuum is extremely large. There is a more general theorem which suggests that for a “natural” class  $\mathcal{P}$  of posets, the existence of (tightly)

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<sup>1)</sup>In course of the development of the theory of Laver-genericity, we strengthened the definition of tightness. However, the modification is chosen so that it still holds in all the standard models of Laver genericity.

$\mathcal{P}$ -Lg supercompact cardinal implies that the continuum is either  $\aleph_1$  or  $\aleph_2$  or else extremely large (see [12], [7], [8]).

The naming “Laver-generic ...” based on the fact that the standard models with this type of generic large cardinal is created by starting from a large cardinal, and then iterating along with a Laver function for the large cardinal with the support appropriate for the class of posets in consideration. This is exactly the way to create models of PFA and MM. Actually, for  $\mathcal{P}$  being the class of all proper posets or the class of all semi-proper posets, the existence of a  $\mathcal{P}$ -Lg supercompact cardinal implies the double-plus version of the corresponding forcing axiom (see Theorem 6.4 below), and can be considered as an axiomatization of the standard models of such axioms.

In the following, we call the axiom asserting that the cardinal  $\kappa_{\text{refl}}$  is a/the tightly  $\mathcal{P}$ -Laver generic  $LC$ , the  *$\mathcal{P}$ -Laver-generic large cardinal axiom for the notion of large cardinal  $LC$*  (the  $\mathcal{P}$ -LgLCA for  $LC$ , for short).

The instances of  $\mathcal{P}$ -LgLCA for other notions of large cardinal considered in [7], [8], [10], [12], [13], [17], etc. are summarized in the following chart.

The $\mathcal{P}$ -LgLCA for	The condition “ $j''\lambda \in M$ ” in the definition of “the $\mathcal{P}$ -LgLCA for super-compact” is replaced with:
hyperhuge	$j''j(\lambda) \in M$
ultrahuge	$j''j(\kappa) \in M$ and $V_{j(\lambda)}^{V^{\text{H}}} \in M$
superhuge	$j''j(\kappa) \in M$
super-almost-huge	$j''\mu \in M$ for all $\mu < j(\kappa)$

It has been proved that LgLCA for sufficiently strong notions of large cardinal imply strongest forms of resurrection, maximality and absoluteness among the known consistent variants of resurrection, maximality and absoluteness:

- (4.3) In [7], it is proved that a boldface variant of Resurrection Axiom by Hamkins and Johnstone ([22], [23]) for  $\mathcal{P}$  and parameters from  $\mathcal{H}(\kappa_{\text{refl}})$  follows from the  $\mathcal{P}$ -LgLCA for ultrahuge.
- (4.4) In [17] or [8] (see [10] for an improved version, or see also Theorem 4.3 below), it is proved that the  $\mathcal{P}$ -LgLCA for ultrahuge implies a restricted form of Maximality Principle for  $\mathcal{P}$  and  $\mathcal{H}(\kappa_{\text{refl}})$ . More specifically, It is proved that under the  $\mathcal{P}$ -LgLCA for ultrahuge implies  $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_2}$ -RcA<sup>+</sup> holds — see below for the definition of this principle (Theorem 21 in [8]).

It can be shown that **LgLCA** type axiom formulated in a single formula is incapable of covering the full Maximal Principle ([7]). The notion of super- $C^{(\infty)}$  **LgLCAs** is introduced in [17] to fill this gap.

For a notion  $LC$  of large cardinal let  $C'_{LC}$  be the closedness property of the target model of the generic large cardinal corresponding to  $LC$ . We call a cardinal  $\kappa$  *tightly super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver generically  $LC$*  if, for any  $n \in \mathbb{N}$ ,  $\lambda_0 > \kappa$ , and  $\mathbb{P} \in \mathcal{P}$ , there are  $\lambda \geq \lambda_0$  and a  $\mathbb{P}$ -name  $\tilde{Q}$  such that  $V_\lambda \prec_{\Sigma_n} V$ ,  $\Vdash_{\mathbb{P}} \tilde{Q} \in \mathcal{P}$  and for any  $(V, \mathbb{P} * \tilde{Q})$ -generic  $\mathbb{H}$ , there are  $j$ ,  $M \subseteq V[\mathbb{H}]$  such that  $V_{j(\lambda)}^{V[\mathbb{H}]} \prec_{\Sigma_n} V[\mathbb{H}]$ ,  $j : V \xrightarrow{\check{}}_{\kappa} M$ ,  $j(\kappa) > \lambda$ ,  $C'_{LC}$ , and  $|RO(\mathbb{P} * \tilde{Q})| \leq j(\kappa)$ .

The *super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver-generic large cardinal axiom for the notion  $LC$  of large cardinal (the super- $C^{(\infty)}$ - $\mathcal{P}$ -**LgLCA** for  $LC$ , for short)* is the assertion that  $\kappa_{\text{refl}}$  is a/the tightly super  $C^{(\infty)}$ - $\mathcal{P}$ -Laver-generic extendible cardinal.

Note that tightly super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver generically  $LC$  is not formalizable in general but the super- $C^{(\infty)}$ - $\mathcal{P}$ -**LgLCA** for  $LC$  is as an axiom scheme, since the generic large cardinal in the axiom is named as  $\kappa_{\text{refl}}$ .

It is shown that for a transfinitely iterable  $\mathcal{P}$ , the consistency of the super- $C^{(\infty)}$ - $\mathcal{P}$ -**LgLCA** for hyperhuge follows from a 2-huge cardinal ([17], Lemma 2.6 and Theorem 2.8).

(4.5) The super- $C^{(\infty)}$ - $\mathcal{P}$ -**LgLCA** for ultrahuge implies the full Maximality Principle for  $\mathcal{P}$  and  $\mathcal{H}(\kappa_{\text{refl}})$  ([17], Theorem 4.10).

(4.6) In [10], it is proved that under  $\text{BFA}_{< \kappa_{\text{refl}}}(\mathcal{P})$  and the  $\mathcal{P}$ -**LgLCA** for huge, a generalization of Viale's Absoluteness Theorem in Viale [42] holds (see Theorem 5.7 in [10]).

For extendibility and super- $C^{(\infty)}$ -extendibility, the natural Laver-generic versions of these notions of large cardinals should be the following: a cardinal  $\kappa$  is *tightly  $\mathcal{P}$ -Laver generically extendible (tightly  $\mathcal{P}$ -Lg extendible, for short)* if, for any  $\lambda > \kappa$ , and for any  $\mathbb{P} \in \mathcal{P}$ , there is a  $\mathbb{P}$ -name  $\tilde{Q}$  such that  $\Vdash_{\mathbb{P}} \tilde{Q} \in \mathcal{P}$  and for any  $(V, \mathbb{P} * \tilde{Q})$ -generic  $\mathbb{H}$ , there are  $j$ ,  $M \subseteq V[\mathbb{H}]$  such that  $j : V \xrightarrow{\check{}}_{\kappa} M$ ,  $j(\kappa) > \lambda$ ,  $V_{j(\lambda)}^{V[\mathbb{H}]} \in M$ , and  $|RO(\mathbb{P} * \tilde{Q})| \leq j(\kappa)$ .

The  *$\mathcal{P}$ -Laver-generic large cardinal axiom for the notion of extendibility (the  $\mathcal{P}$ -**LgLCA** for extendible, for short)* is the assertion that  $\kappa_{\text{refl}}$  is a/the tightly  $\mathcal{P}$ -Lg extendible cardinal.

A cardinal  $\kappa$  is *tightly super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver generically extendible (tightly super- $C^{(\infty)}$ -Lg extendible, for short)* if, for any  $n \in \mathbb{N}$ ,  $\lambda_0 > \kappa$ , and  $\mathbb{P} \in \mathcal{P}$ , there are  $\lambda \geq \lambda_0$  and a  $\mathbb{P}$ -name  $\tilde{Q}$  such that  $V_\lambda \prec_{\Sigma_n} V$ ,  $\Vdash_{\mathbb{P}} \tilde{Q} \in \mathcal{P}$  and for any  $(V, \mathbb{P} * \tilde{Q})$ -generic

$\mathbb{H}$ , there are  $j$ ,  $M \subseteq \mathbf{V}[\mathbb{H}]$  such that  $V_{j(\lambda)}^{\mathbf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathbf{V}[\mathbb{H}]$ ,  $j : \mathbf{V} \xrightarrow{\sim}_{\kappa} M$ ,  $j(\kappa) > \lambda$ ,  $V_{j(\lambda)}^{\mathbf{V}[\mathbb{H}]} \in M$ , and  $|RO(\mathbb{P} * \mathbb{Q})| \leq j(\kappa)$ .

The *super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver-generic large cardinal axiom* for the notion of extendibility (the *super- $C^{(\infty)}$ - $\mathcal{P}$ -LgLCA for extendible*, for short) is the assertion that  $\kappa_{\text{refl}}$  is a/the tightly super  $C^{(\infty)}$ - $\mathcal{P}$ -Lg extendible cardinal.

The $\mathcal{P}$ -LgLCA for	The condition “ $j''\lambda \in M$ ” in the definition of “the $\mathcal{P}$ -LgLCA for super-compact” is replaced with:
hyperhuge	$j''j(\lambda) \in M$
ultrahuge	$j''j(\kappa) \in M$ and $V_{j(\lambda)}^{\mathbf{V}[\mathbb{H}]} \in M$
superhuge	$j''j(\kappa) \in M$
super-almost-huge	$j''\mu \in M$ for all $\mu < j(\kappa)$
extendible	$V_{j(\lambda)}^{\mathbf{V}[\mathbb{H}]} \in M$

For genuine extendible cardinal the super- $C^{(n)}$ -ness is equivalent to  $C^{(n)}$ -ness of Joan Baraglia (see Lietz’s Theorem 2.6). However, similar equivalence is not available for Laver-gen. extendible cardinals, since these cardinals can be the continuum or  $\aleph_2$  and both of them cannot be  $C^{(1)}$  ( $\kappa \in C^{(1)}$  implies that  $\kappa$  is a strong limit).

As we have shown an almost-huge cardinal produces a transitive model with cofinally many super- $C^{(\infty)}$ -extendible cardinals. In the next section, we show that we can generic extend such models to a model of the super- $C^{(\infty)}$   $\mathcal{P}$ -LgLCA for extendible for each reasonable (i.e. transfinitely iterable) class  $\mathcal{P}$  of posets. Thus the super- $C^{(\infty)}$   $\mathcal{P}$ -LgLCA for extendible and the  $\mathcal{P}$ -LgLCA for extendible are of relatively low consistency strength.

These LgLCAs are placed at the expected places in the web of implications of LgLCAs (see also the chart on page 25):

**Lemma 4.1** *Suppose that  $\mathcal{P}$  is an arbitrary class of posets. (1) The  $\mathcal{P}$ -LgLCA for hyperhuge implies the  $\mathcal{P}$ -LgLCA for extendible. The super- $C^{(\infty)}$ - $\mathcal{P}$ -LgLCA for hyperhuge implies the super- $C^{(\infty)}$ - $\mathcal{P}$ -LgLCA for extendible.*

(2) *The  $\mathcal{P}$ -LgLCA for extendible implies the  $\mathcal{P}$ -LgLCA for supercompact.*

**Proof.** (1): By definition.

(2): Assume that the  $\mathcal{P}$ -LgLCA for extendible holds. Let  $\kappa := \kappa_{\text{refl}}$ . Suppose that  $\lambda > \kappa$  is regular, and  $\mathbb{P} \in \mathcal{P}$ . Then there are  $\mathbb{Q}$ ,  $\mathbb{H}$ ,  $j$ ,  $M \subseteq \mathbf{V}[\mathbb{H}]$  such that  $\Vdash_{\mathbb{P}} “\mathbb{Q} \in \mathcal{P}”$ ,  $\mathbb{H}$  is  $(\mathbf{V}, \mathbb{P} * \mathbb{Q})$ -generic,  $j$ ,  $M \subseteq \mathbf{V}[\mathbb{H}]$ , (4.7):  $j : \mathbf{V} \xrightarrow{\sim}_{\kappa} M$ , (4.8):  $j(\kappa) > \lambda$ , (4.9):  $|RO(\mathbb{P} * \mathbb{Q})| \leq j(\kappa)$ , and (4.10):  $V_{j(\lambda)}^{\mathbf{V}[\mathbb{H}]} \in M$ .

$M \models “j(\lambda) \text{ is regular}”$  by elementarity (4.7). Thus  $\mathbf{V} \models “j(\lambda) \text{ is regular}”$ . By  $j(\kappa) < j(\lambda)$  and (4.9), it follows that  $\mathbf{V}[\mathbb{H}] \models “j(\lambda) \text{ is regular}”$ . Thus  $\mathbf{V}[\mathbb{H}] \models$

$cf(j''\lambda) \leq \lambda < j(\lambda) = cf(j(\lambda))$ , and  $\mathbb{V}^{[\mathbb{H}]} \models j''\lambda \in V_{j(\lambda)}^{\mathbb{V}^{[\mathbb{H}]}} \in M$ . Hence  $j''\lambda \in M$  by transitivity of  $M$ .

This shows that  $j$ 's for all regular  $\lambda > \kappa$  and  $\mathbb{P} \in \mathcal{P}$  witness that  $\kappa$  is tightly  $\mathcal{P}$ -Lg. supercompact. □ (Lemma 4.1)

All of the results mentioned in (4.3)  $\sim$  (4.6) can be proved under the assumption of the **LgLCA** for extendible instead of the stronger assumptions in the original results. Below we shall state these results. In the extended version of the present paper, we shall include all the details of the proofs (though the proofs are practically identical with the original ones) for the convenience of the reader.

The following boldface version of the Resurrection Axioms was studied by Hamkins and Johnstone in [23]: For a class  $\mathcal{P}$  of posets and a definition  $\mu^\bullet$  of a cardinal (e.g. as  $\aleph_1$ ,  $\aleph_2$ ,  $2^{\aleph_0}$ ,  $(2^{\aleph_0})^+$ . etc.) the *Resurrection Axiom in Boldface for  $\mathcal{P}$  and  $\mathcal{H}(\mu^\bullet)$*  is defined by:

$\mathbb{R}\mathcal{A}_{\mathcal{H}(\mu^\bullet)}^{\mathcal{P}}$  : For any  $A \subseteq \mathcal{H}(\mu^\bullet)$  and any  $\mathbb{P} \in \mathcal{P}$ , there is a  $\mathbb{P}$ -name  $\mathbb{Q}$  of poset such that  $\Vdash_{\mathbb{P}} \mathbb{Q} \in \mathcal{P}$  and, for any  $(\mathbb{V}, \mathbb{P} * \mathbb{Q})$ -generic  $\mathbb{H}$ , there is  $A^* \subseteq \mathcal{H}(\mu^\bullet)^{\mathbb{V}^{[\mathbb{H}]}}$  such that  $(\mathcal{H}(\mu^\bullet)^{\mathbb{V}}, A, \in) \prec (\mathcal{H}(\mu^\bullet)^{\mathbb{V}^{[\mathbb{H}]}} , A^*, \in)$ .

**Theorem 4.2** (Theorem 7.1 in [7] reformulated under the **LgLCA** for extendible) *For an iterable class  $\mathcal{P}$  of posets, assume that the  $\mathcal{P}$ -LgLCA for extendible holds. Then  $\mathbb{R}\mathcal{A}_{\mathcal{H}(\kappa_{\text{refl}})}^{\mathcal{P}}$  holds.* □

Recurrence Axioms are introduced in Fuchino and Usuba [17].

For an iterable class  $\mathcal{P}$  of posets, a set  $A$  (of parameters), and a set  $\Gamma$  of  $\mathcal{L}_{\in}$ -formulas,  *$\mathcal{P}$ -Recurrence Axiom<sup>+</sup> for formulas in  $\Gamma$  with parameters from  $A$*  ( $((\mathcal{P}, A)_{\Gamma}\text{-RcA}^+$ , for short) is the following assertion expressed as an axiom scheme formulated in  $\mathcal{L}_{\in}$ :

$(\mathcal{P}, A)_{\Gamma}\text{-RcA}^+$  : For any  $\varphi(\bar{x}) \in \Gamma$  and  $\bar{a} \in A$ , if  $\Vdash_{\mathbb{P}} \varphi(\bar{a})$ , then there is a  $\mathcal{P}$ -ground  $\mathbb{W}$  of  $\mathbb{V}$  such that  $\bar{a} \in \mathbb{W}$  and  $\mathbb{W} \models \varphi(\bar{a})$ .

Here, an inner model  $M$  of  $\mathbb{V}$  is said to be a  *$\mathcal{P}$ -ground of  $\mathbb{V}$* , if there are  $\mathbb{P} \in M$  and  $\mathbb{G} \in \mathbb{V}$  such that  $M \models \mathbb{P} \in \mathcal{P}$ ,  $\mathbb{G}$  is an  $(M, \mathbb{P})$ -generic filter, and  $\mathbb{V} = M[\mathbb{G}]$ . If  $M$  is a  $\mathcal{P}$ -ground of  $\mathbb{V}$  for the class  $\mathcal{P}$  of all posets, we shall say that  $M$  is a *ground of  $\mathbb{V}$* . The Recurrence Axiom  $(\mathcal{P}, A)_{\Gamma}\text{-RcA}$  without  $+$  is obtained when “ $\mathcal{P}$ -ground” in the definition of  $(\mathcal{P}, A)_{\Gamma}\text{-RcA}^+$  is replaced with “ground”.

If  $\Gamma$  is the set of all  $\mathcal{L}_{\in}$ -formulas, we drop the subscript  $\Gamma$  and say simply  $(\mathcal{P}, A)\text{-RcA}^+$  or  $(\mathcal{P}, A)\text{-RcA}$ .

As it is noticed in [17],  $(\mathcal{P}, A)\text{-RcA}^+$  is equivalent to the Maximality Principle  $\text{MP}(\mathcal{P}, A)$  (see Proposition 2.2, (1) in [17]).

When [10] was written, we didn't consider the notion of **LgLCA** for extendible among the possible **LgLCAs**. This is why the following theorem was stated there under the assumption of the  $\mathcal{P}$ -**LgLCA** for ultrahuge. However the proof given in [10] works perfectly under the  $\mathcal{P}$ -**LgLCA** for extendible without any change.

**Theorem 4.3** (Theorem 6.1 in Fuchino, Gappo and Parente [10] reformulated under **LgLCAs** for extendible) *Assume the  $\mathcal{P}$ -**LgLCA** for extendible. Then  $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Gamma}$ -**RcA**<sup>+</sup> holds where  $\Gamma$  is the set of all formulas which are conjunctions of a  $\Sigma_2$ -formula and a  $\Pi_2$ -formula.  $\square$*

In the proof of Theorem 4.3 as well as in the proof of Theorem 4.7, we use the following lemma which should be a well-known fact.

**Lemma 4.4** *If  $\alpha$  is a limit ordinal and  $V_\alpha$  satisfies a sufficiently large finite fragment of ZFC, then for any  $\mathbb{P} \in V_\alpha$  and  $(V, \mathbb{P})$ -generic  $\mathbb{G}$ , we have  $V_\alpha[\mathbb{G}] = V_\alpha^{V[\mathbb{G}]}$ .*

Theorem 4.3 has an important application (Theorem 4.6). For this theorem, we need the following facts about Recurrence Axioms.

**Lemma 4.5** (Fuchino and Usuba [17], see also Lemma 20 in the extended version of [8]) *Assume that  $\mathcal{P}$  is an iterable class of posets. (1) If  $\mathcal{P}$  contains a poset which adds a real (over the universe), then  $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_1}$ -**RcA** implies  $\neg\text{CH}$ .*

(2) *Suppose that  $\mathcal{P}$  contains a poset which forces  $\aleph_2^V$  to be equinumerous with  $\aleph_1^V$ . Then  $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$ -**RcA** implies  $2^{\aleph_0} \leq \aleph_2$ .*

(2') *If  $\mathcal{P}$  contains a posets which forces  $\aleph_2^V$  to be equinumerous with  $\aleph_1^V$ , then  $(\mathcal{P}, \mathcal{H}((\aleph_2)^+))_{\Sigma_1}$ -**RcA** does not hold.*

(3) *If  $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_1}$ -**RcA** holds then all  $\mathbb{P} \in \mathcal{P}$  preserve  $\aleph_1$  and they are also stationary preserving.*

(4) *If  $\mathcal{P}$  contains a poset which adds a real as well as a poset which collapses  $\aleph_2^V$ , then  $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_1}$ -**RcA** implies  $2^{\aleph_0} = \aleph_2$ .*

(5) *If  $\mathcal{P}$  contains a poset which collapses  $\aleph_1^V$ , then  $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$ -**RcA** implies **CH**.*

(5') *If  $\mathcal{P}$  contains a poset which collapses  $\aleph_1^V$  then  $(\mathcal{P}, \mathcal{H}((2^{\aleph_0})^+))_{\Sigma_1}$ -**RcA** does not hold.*

(6) *Suppose that all  $\mathbb{P} \in \mathcal{P}$  preserve cardinals and  $\mathcal{P}$  contains posets adding at least  $\kappa$  many reals for each  $\kappa \in \text{Card}$ . Then  $(\mathcal{P}, \emptyset)_{\Sigma_2}$ -**RcA**<sup>+</sup> implies that  $2^{\aleph_0}$  is very large.*

(6') *Suppose that  $\mathcal{P}$  is as in (6). Then  $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_2}$ -**RcA**<sup>+</sup> implies that  $2^{\aleph_0}$  is a limit cardinal. Thus if  $2^{\aleph_0}$  is regular in addition, then  $2^{\aleph_0}$  is weakly inaccessible.  $\square$*

**Theorem 4.6** *Suppose that  $\mathcal{P}$ -LgLCA for extendible holds. Then we have:*

- (1) *Elements of  $\mathcal{P}$  are stationary preserving.*
- (2) *For all classes  $\mathcal{P}$  of posets covered by Lemma 4.5,  $\mathcal{P}$ -LgLCA for extendible implies that the continuum is either  $\aleph_1$  or  $\aleph_2$  or very large.*

**Proof.** (1): By Theorem 4.3 and Lemma 4.5, (3).

(2): By Theorem 4.3 and the rest of Lemma 4.5.  $\square$  (Theorem 4.6)

We consider the following principle  $(\mathcal{P}, A, \lambda)$ -RcA<sup>+</sup> for a class  $\mathcal{P}$  of posets, a set  $A$  and cardinal  $\lambda$ . An inner model  $M$  of  $\mathbf{V}$  is said to be  $(\mathcal{P}, \lambda)$ -ground of  $\mathbf{V}$  if there is  $\mathbb{P} \in M$  with  $M \models \text{“}\mathbb{P} \in \mathcal{P} \text{ and } |\mathbb{P}| < \lambda\text{”}$  and a  $(M, \mathbb{P})$ -generic  $\mathbb{G} \in \mathbf{V}$  such that  $\mathbf{V} = M[\mathbb{G}]$ .

$(\mathcal{P}, A, \lambda)$ -RcA<sup>+</sup>: For any formula  $\varphi = \varphi(\bar{x}) \in \mathcal{L}_\epsilon$  and  $\bar{a} \in A$ , if  $\Vdash_{\mathbb{P}} \text{“}\varphi(\bar{a})\text{”}$  for a  $\mathbb{P} \in \mathcal{P}$ , then there is a  $(\mathcal{P}, \lambda)$ -ground  $M$  such that  $\bar{a} \in M$  and  $M \models \varphi(\bar{a})$ .

Clearly this principle implies  $(\mathcal{P}, A)$ -RcA<sup>+</sup>. As we already noticed, the latter principle is equivalent to the Maximality Principle  $\text{MP}(\mathcal{P}, A)$  (see Proposition 2.2, (1) in [17]).

The proof of the following theorem is almost identical with the corresponding proof in [17]. Nevertheless I would like to write down the proof here since this proof shows how the notion of super- $C^{(\infty)}$  version of the LgLCA is incorporated into the whole picture.

**Theorem 4.7** (A slight improvement of Fuchino and Usuba [17], Theorem 4.10)

- (1) *Suppose that  $\mathcal{P}$  is an iterable class of posets. If super- $C^{(\infty)}$ - $\mathcal{P}$ -LgLCA for extendible holds, then  $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}), \kappa_{\text{refl}}^+)$ -RcA<sup>+</sup> holds.*
- (2) *If super- $C^{(\infty)}$ -LgLCAA for extendible holds, then  $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}), (2^{\aleph_0})^+)$ -RcA<sup>+</sup> holds.*

**Proof.** We prove (1). (2) can be proved similarly.

Suppose that  $\kappa := \kappa_{\text{refl}}$  is tightly super- $C^{(\infty)}$ - $\mathcal{P}$ -Lg extendible,  $\mathbb{P} \in \mathcal{P}$ , and  $\Vdash_{\mathbb{P}} \text{“}\varphi(\bar{a})\text{”}$  for an  $\mathcal{L}_\epsilon$ -formula  $\varphi$  and  $\bar{a} \in \mathcal{H}(\kappa)$ . We want to show that  $\varphi(\bar{a})$  holds in some  $(\mathcal{P}, \kappa^+)$ -ground of  $\mathbf{V}$ .

Let  $n$  be a sufficiently large natural number  $\geq 2$  such that the details of the following argument work out. In particular, we assume that  $V_\alpha^{\mathbf{V}} \prec_{\Sigma_n} \mathbf{V}$  implies that  $\text{“}\varphi(\bar{x})\text{”}$  and  $\Vdash \text{“}\varphi(\bar{x})\text{”}$  are absolute between  $V_\alpha^{\mathbf{V}}$  and  $\mathbf{V}$ , and  $V_\alpha^{\mathbf{V}} \prec_{\Sigma_n} \mathbf{V}$  also implies that a sufficiently large fragment of ZFC holds in  $V_\alpha$ .

Let  $\mathbb{Q}$  be a  $\mathbb{P}$ -name such that  $\Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{P}\text{”}$  and, for  $(\mathbf{V}, \mathbb{P} * \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $\lambda > \kappa$  with

$$(4.11) \quad V_\lambda \prec_{\Sigma_n} \mathbf{V},$$

and  $j, M \subseteq \mathbf{V}[\mathbb{H}]$  such that

$$(4.12) \quad \begin{aligned} & \text{(a) } j : \mathbf{V} \xrightarrow{\prec_\kappa} M, \quad \text{(b) } j(\kappa) > \lambda, \quad \text{(c) } \mathbb{P}, \mathbb{H}, V_{j(\lambda)}^{\mathbf{V}[\mathbb{H}]} \in M, \\ & \text{(d) } |RO(\mathbb{P} * \mathbb{Q})| \leq j(\kappa) (< j(\lambda)), \text{ and} \quad \text{(e) } V_{j(\lambda)}^{\mathbf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathbf{V}[\mathbb{H}]. \end{aligned}$$

By (4.12), (c) and (d), we may assume that  $\mathbb{P} * \mathbb{Q} \in V_{j(\lambda)}^{\mathbf{V}}$  by replacing  $\mathbb{P} * \mathbb{Q}$  by an isomorphic poset (and replacing  $\mathbb{H}$  by corresponding filter) if necessary.

By the choice of  $n$ , we have  $V_\lambda \models \Vdash_{\mathbb{P}} \text{“}\varphi(\bar{a})\text{”}$ .  $j(V_\lambda^{\mathbf{V}}) = V_{j(\lambda)}^M \prec_{\Sigma_n} M$  by elementarity of  $j$ , and

$$(4.13) \quad V_{j(\lambda)}^M = V_{j(\lambda)}^{\mathbf{V}[\mathbb{H}]}$$

by the closedness of  $M$ . Since  $V_\lambda \prec_{\Sigma_n} \mathbf{V}$ , we have  $V_\lambda[\mathbb{H}] \prec_{\Sigma_{n_0}} \mathbf{V}[\mathbb{H}]$  for a still large enough  $n_0 \leq n$ . By (4.12), (e) above, it follows that  $V_\lambda^{\mathbf{V}[\mathbb{H}]} \prec_{\Sigma_{n_0}} V_{j(\lambda)}^{\mathbf{V}[\mathbb{H}]}$ . Thus

$$(4.14) \quad V_\lambda^{\mathbf{V}} \prec_{\Sigma_{n_1}} V_{j(\lambda)}^{\mathbf{V}}$$

for a still large enough  $n_1 \leq n_0$ .

In particular, we have  $V_{j(\lambda)}^{\mathbf{V}} \models \Vdash_{\mathbb{P}} \text{“}\varphi(\bar{a})\text{”}$ , and hence  $V_{j(\lambda)}[\mathbb{G}] \models \varphi(\bar{a})$  where  $\mathbb{G}$  is the  $\mathbb{P}$ -part of  $\mathbb{H}$ . Note that by (4.11) and (4.14),  $V_{j(\lambda)}$  satisfies a sufficiently large fragment of ZFC.

Thus we have  $V_{j(\lambda)}[\mathbb{H}] \models \text{“there is a } (\mathcal{P}, j(\kappa)^+)\text{-ground satisfying } \varphi(\bar{a})\text{”}$ , and hence

$$V_{j(\lambda)}^{\mathbf{V}[\mathbb{H}]} \models \text{“there is a } (\mathcal{P}, j(\kappa)^+)\text{-ground satisfying } \varphi(\bar{a})\text{”}$$

by Lemma 4.4. By (4.13) and elementarity, it follows that

$$V_\lambda \models \text{“there is a } (\mathcal{P}, \kappa^+)\text{-ground satisfying } \varphi(\bar{a})\text{”}.$$

Finally, this implies  $\mathbf{V} \models \text{“there is a } (\mathcal{P}, \kappa^+)\text{-ground satisfying } \varphi(\bar{a})\text{”}$  by (4.11).

□ (Theorem 4.7)

For an ordinal  $\alpha$ , let  $\alpha^{(+)} := \sup(\{|\beta|^+ : \beta < \alpha\})$ . Note that  $\alpha^{(+)} = \alpha$  if  $\alpha$  is a cardinal. Otherwise, we have  $\alpha^{(+)} = |\alpha|^+$ .

**Theorem 4.8** (Theorem 5.7 in Fuchino, Gappo and Parente [10] restated under LgLCA for extendible) *For an iterable class  $\mathcal{P}$  of posets, assume that  $\mathcal{P}$ -LgLCA for extendible holds. Then, for  $\kappa := \kappa_{\text{refl}}$*

$$(4.15) \quad \text{for any } \mathbb{P} \in \mathcal{P} \text{ such that } \Vdash_{\mathbb{P}} \text{“BFA}_{< \kappa}(\mathcal{P})\text{”}, \mathcal{H}(\mu^+)^{\mathbf{V}} \prec_{\Sigma_2} \mathcal{H}(\mu^+)^{\mathbf{V}[\mathbb{G}]} \text{ holds for all } \mu < \kappa \text{ and for } (\mathbf{V}, \mathbb{P})\text{-generic } \mathbb{G}.\text{^2)}$$

Thus, we have  $\mathcal{H}(\kappa)^{\mathbf{V}} \prec_{\Sigma_2} \mathcal{H}((\kappa^{(+)})^{\mathbf{V}[\mathbb{G}]})^{\mathbf{V}[\mathbb{G}]}$  for  $\mathbb{G}$  as above. □

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<sup>2)</sup>  $\mu^+$  in “ $\mathcal{H}(\mu^+)^{\mathbf{V}[\mathbb{G}]}$ ” is actually  $(\mu^+)^{\mathbf{V}[\mathbb{G}]}$ .

## 5 Consistency of LgLCAs and super- $C^{(n)}$ -LgLCAs for extendibility

We examine first that extendible and super- $C^{(\infty)}$ -extendible cardinals are endowed with Laver functions.

A function  $f : \kappa \rightarrow V_\kappa$  for an extendible cardinal  $\kappa$  is said to be a *Laver function for extendibility* (of  $\kappa$ ) if for any  $\lambda > \kappa$  and any set  $a$ , there are  $j$ ,  $M \subseteq V$  such that  $j : V \xrightarrow{\lambda} M$ ,  $j(\kappa) > \lambda$ ,  $V_{j(\kappa)} \in M$ , and  $j(f)(\kappa) = a$ .

Suppose that  $\mu$  is an inaccessible cardinal and  $V_\mu \models$  “ $\kappa$  is a super- $C^{(\infty)}$ -extendible”. Then  $f : \kappa \rightarrow V_\kappa$  is said to be a *super- $C^{(\infty)}$ -extendible Laver function for  $\kappa$  in  $V_\mu$*  if, for any  $x \in V_\mu$ ,  $n \in \omega$  and for any  $\kappa < \lambda < \mu$  with  $\lambda \in C^{(n)}$ , there are  $j$ ,  $M \subseteq V_\mu$  such that  $j : V_\mu \xrightarrow{\lambda} M$ ,  $j(\kappa) > \lambda$ ,  $j(\lambda) \in C^{(n)}$ ,  $V_{j(\lambda)} \in M$  and  $j(f)(\kappa) = x$ .

**Lemma 5.1** (1) *Suppose that  $\kappa$  is extendible. Then there is a Laver function  $f : \kappa \rightarrow V_\kappa$  for extendibility.*

(2) *Suppose that, for an inaccessible cardinal  $\mu$  and a cardinal  $\kappa < \mu$ , we have  $V_\mu \models$  “ $\kappa$  is strongly super- $C^{(\infty)}$ -extendible”. Then there is a super- $C^{(\infty)}$ -extendible Laver function  $f : \kappa \rightarrow V_\kappa$  for  $\kappa$  in  $V_\mu$ .*

**Proof.** (1): is known previously (see e.g. Corazza [2]). The proof of (2) below can be easily modified and slimmed down to a proof of (1).

(2): Suppose that  $\mu$  is an inaccessible cardinal and  $\kappa < \mu$  is such that  $V_\mu \models$  “ $\kappa$  is strongly super- $C^{(\infty)}$ -extendible”. Assume, towards a contradiction, that

$$(5.1) \quad \text{there is no super-}C^{(\infty)}\text{-extendible Laver function } f : \kappa \rightarrow V_\kappa \text{ for } \kappa \text{ in } V_\mu.$$

Let  $C := \{\alpha < \mu : V_\alpha \prec V_\mu\}$ , and let  $n^* \in \omega$  be sufficiently large for the following argument.

For  $n \in \omega \setminus n^*$ , let  $\varphi_n(\underline{f}, \underline{x})$  be the formula

$$(5.2) \quad \forall \underline{\delta} \forall \underline{\delta}' \forall \underline{j} \forall \alpha \varphi_n^*(\underline{f}, \underline{\alpha}, \underline{x}, \underline{\delta}, \underline{\delta}', \underline{j})$$

where  $\varphi_n^*(\underline{f}, \underline{\alpha}, \underline{x}, \underline{\delta}, \underline{\delta}', \underline{j})$  is the formula:

$$\begin{aligned} ((\underline{f} : \underline{\alpha} \rightarrow V_{\underline{\alpha}} \wedge \underline{\alpha} < \underline{\delta} \wedge V_{\underline{\delta}} \prec_{\Sigma_n} V \wedge V_{\underline{\delta}'} \prec_{\Sigma_n} V \wedge \underline{j} : V_{\underline{\delta}} \xrightarrow{\underline{\alpha}} V_{\underline{\delta}'} \wedge \underline{j}(\underline{\alpha}) > \underline{\delta}) \\ \rightarrow \underline{j}(\underline{f})(\underline{\alpha}) \neq \underline{x}). \end{aligned}$$

If  $V_\mu \models \exists \underline{x} \varphi_n(\underline{f}, \underline{x})$  for some  $n \in \omega \setminus n^*$  and  $f \in V_\mu$ , then let  $x_{n,f} \in V_\mu$  be a witness for  $\underline{x}$  in  $\varphi_n(\underline{f}, \underline{x})$ , and let  $\mu_{n,f} := \text{rank}(x_{n,f})$ . If  $V_\mu \models \exists \underline{x} \varphi_n(\underline{f}, \underline{x})$  does not hold, we let  $x_{n,f} := \emptyset$  and  $\mu_{n,f} := 0$ .

$x_{n,f}$  might not be determined uniquely. However, we can choose  $x_{n,f}$  such that  $\mu_{n,f}$  is minimal and, in this way,  $\mu_{n,f}$  is determined uniquely to each  $n$  and  $f$ . Let  $\vec{x} := \langle x_{n,f} : n \in \omega, f : \alpha \rightarrow V_\alpha \text{ for some } \alpha \leq \kappa \rangle$ .

Since  $\kappa$  is strongly super- $C^{(\infty)}$ -extendible in  $V_\mu$ , we can find  $\lambda < \mu$  and  $j^*$ ,  $M^* \subseteq V_\mu$  by Lemma 3.4 such that

$$(5.3) \quad \lambda \geq \lambda_0 := \max\{\mu_{n,f} : n \in \omega, f : \alpha \rightarrow V_\alpha \text{ for an } \alpha \leq \kappa\}, \kappa,$$

$$(5.4) \quad \lambda \in \text{Lim}(C),$$

$$(5.5) \quad j^* : V_\mu \xrightarrow{\sim}_\kappa M^* \text{ with (a) } j^*(\kappa) > \lambda, \quad (\text{b}) \quad j^*(\lambda) \in C, \text{ and} \\ (\text{c}) \quad V_{j^*(\lambda)} \in M^*.$$

Note that we have (5.6):  $\vec{x} \in V_\lambda$ .

Let

$$A := \{ \alpha < \kappa : \text{for all } f : \alpha \rightarrow V_\alpha, \text{ there is } n \in \omega \\ \text{such that } V_\mu \models \exists \underline{x} \varphi_n(f, \underline{x}) \}.$$

**Claim 5.1.1**  $M^* \models \kappa \in j^*(A)$ .

$\vdash$  For each  $f : \kappa \rightarrow V_\kappa$ , there is  $n \in \omega$  such that  $V_\mu \models \exists \underline{x} \varphi_n(f, \underline{x})$  by the definition (5.2) of  $\varphi_n$  and by the assumption (5.1). It follows that  $V_{j^*(\lambda)} \models \exists \underline{x} \varphi_n(f, \underline{x})$  by (5.5), (b). By (5.4) and by the elementarity of  $j^*$ ,  $V_{j^*(\lambda)} \prec_{\Sigma_m} M^*$  holds for all  $m \in \omega$ . Thus  $M^* \models \exists \underline{x} \varphi_n(f, \underline{x})$ . This implies  $M^* \models \kappa \in j^*(A)$ .  $\dashv$  (Claim 5.1.1)

Remembering (5.6), let  $f^* : \kappa \rightarrow V_\kappa$  be defined by

$$f^*(\alpha) := \begin{cases} x_{n,f^* \upharpoonright \alpha}, & \text{if } \alpha \in A, f^* \upharpoonright \alpha : \alpha \rightarrow V_\alpha \\ & \text{and } n \text{ is minimal with } x_{n,f^*} \neq \emptyset; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let (5.7):  $x^* := j^*(f^*)(\kappa)$ . By definition of  $f^*$ , by Claim 5.1.1, and elementarity, we have a witness of  $\exists \underline{x} \varphi_n(f^*, \underline{x})$  for some  $n \in \omega$  in  $M^*$ . Letting  $n_0$  be the minimal such  $n$ , we have

$$(5.8) \quad x^* = j^*(\vec{x})(n_0, f^*) \neq \emptyset$$

by definition of  $f^*$  and since  $j^*(f^*) \upharpoonright \kappa = f^*$ .

Thus, we have  $M^* \models \varphi_{n_0}(f^*, x^*)$ . It follows  $V_{j^*(\lambda)}^{M^*} \models \varphi_{n_0}(f^*, x^*)$  by (5.3), and (5.4). By (5.5), (c), it follows that  $V_{j^*(\lambda)}^V \models \varphi_{n_0}(f^*, x^*)$ . Hence

$$(5.9) \quad V \models \varphi_{n_0}(f^*, x^*)$$

by (5.5), (b).

by definition (5.8) of  $x^*$

Thus, we have  $x^* \neq \underbrace{j^*(f^*)(\kappa)}_{\text{by (5.9)}} = x^*$ . This is a contradiction. □ (Lemma 5.1)

**Theorem 5.2** *Suppose that  $\mathcal{P}$  is  $\omega_1$  preserving, relevant and transfinitely iterable class of posets.*

(1) (This improves Theorem 5.2, (2) in [12]) *If  $\kappa$  is extendible, and  $\mathcal{P}$  is  $\Sigma_2$ -definable, then there is a poset  $\mathbb{P}_\kappa \in \mathcal{P}$  such that  $\Vdash_{\mathbb{P}_\kappa} \kappa = \kappa_{\text{refl}}$  and  $\mathcal{P}$ -LgLCA for extendible holds”.*

(2) *Suppose that  $\kappa$  is super- $C^{(n^*)}$ -extendible for  $n \in \mathbb{N}$  and sufficiently large  $n^* > n$ . If super- $C^{(n^*)}$ -extendible Laver function for  $\kappa$  exists, and  $\mathcal{P}$  is  $\Sigma_n$ -definable, then there is a poset  $\mathbb{P}_\kappa \in \mathcal{P}$  such that  $\Vdash_{\mathbb{P}_\kappa} \kappa = \kappa_{\text{refl}}$  and super- $C^{(n)}$ - $\mathcal{P}$ -LgLCA for extendible holds”.*

(3) *Suppose  $V_\mu \models \text{“}\kappa \text{ is strongly super-}C^{(\infty)} \text{ extendible”}$  for an inaccessible cardinal  $\mu$  and  $\kappa < \mu$ . Then there is  $\mathbb{P}_\kappa \in \mathcal{P}^{V_\mu}$  such that  $V_\mu \models \Vdash_{\mathbb{P}_\kappa} \kappa = \kappa_{\text{refl}}$  and super- $C^{(\infty)}$ - $\mathcal{P}$ -LgLCA for extendible holds”.*

Note that many natural classes of posets including the classes of all ccc posets, all  $\sigma$ -closed posets, all proper posets, all semi-proper posets, etc., are  $\Sigma_2$ -definable,  $\omega_1$  preserving and transfinitely iterable.

**Proof of Theorem 5.2:** (1): We show the assertion for the case that  $\mathcal{P}$  is the class of all proper posets. The proof for the general case can be done by replacing the CS-iteration in the following proof by the type of iteration for which the class  $\mathcal{P}$  is transfinitely iterable. Let  $f$  be a Laver function for extendibility of  $\kappa$  ( $f$  exists by Lemma 5.1, (1)).

Let  $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$  be an CS-iteration of elements of  $\mathcal{P}$  such that

$$\mathbb{Q}_\beta := \begin{cases} f(\beta), & \text{if } f(\beta) \text{ is a } \mathbb{P}_\beta\text{-name} \\ & \text{and } \Vdash_{\mathbb{P}_\beta} \text{“} f(\beta) \in \mathcal{P} \text{”}; \\ \mathbb{P}_\beta\text{-name of the trivial forcing,} & \text{otherwise.} \end{cases}$$

We show that  $\Vdash_{\mathbb{P}_\kappa} \kappa$  is tightly  $\mathcal{P}$ -Lg extendible”. Note that once this has been established, it follows that  $V[\mathbb{G}_\kappa] \models \kappa = \kappa_{\text{refl}}$  by Section 5 of [12] (see also [7], Theorem 3.5), and hence  $V[\mathbb{G}_\kappa] \models \mathcal{P}$ -LgLCA for extendible.

Let  $\mathbb{G}_\kappa$  be a  $(V, \mathbb{P}_\kappa)$ -generic filter. In  $V[\mathbb{G}_\kappa]$ , suppose that  $\mathbb{P} \in \mathcal{P}$  and let  $\mathbb{P}$  be a  $\mathbb{P}_\kappa$ -name for  $\mathbb{P}$ .

Suppose that  $\lambda > \kappa$ . Let  $\lambda^* > \lambda$  be a cardinal such that  $\mathcal{H}(\lambda^*) = V_{\lambda^*}$ . Note that we have  $V_{\lambda^*} \prec_{\Sigma_1} V$ .

Let  $j : \mathbb{V} \xrightarrow{\lambda} M$  be such that (5.10):  $j(\kappa) > \lambda^*$ , (5.11):  $V_{j(\lambda^*)} \in M$ , and (5.12):  $j(f)(\kappa) = \mathbb{P}$ . The last condition is possible since  $f$  is a Laver function for the extendible  $\kappa$ . By (5.11), we have  $\mathcal{H}(j(\lambda^*))^M = V_{j(\lambda^*)}^M = V_{j(\lambda^*)}^{\mathbb{V}} = \mathcal{H}(j(\lambda^*))^{\mathbb{V}}$ , and hence (5.13):  $V_{j(\lambda^*)} \prec_{\Sigma_1} M$  and  $V_{j(\lambda^*)} \prec_{\Sigma_1} \mathbb{V}$ .

In  $M$ , there is a  $\mathbb{P}_\kappa * \mathbb{P}$ -name  $\mathbb{Q}$  such that

$$M \models \Vdash_{\mathbb{P}_\kappa * \mathbb{P}} \text{“} \mathbb{Q} \in \mathcal{P} \text{ and } \mathbb{Q} \text{ is the direct limit of CS-iteration of small posets in } \mathcal{P} \text{ of length } j(\kappa)\text{”},$$

$$\text{and } \mathbb{P}_\kappa * \mathbb{P} * \mathbb{Q} \sim j(\mathbb{P}_\kappa).$$

By (5.13), and since “ $\mathbb{P} \in \mathcal{P}$ ” is  $\Sigma_2$ , the same statement holds in  $\mathbb{V}$  with the same  $\mathbb{Q}$  (see the proof of (2) for more details).

We have  $j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa \sim \mathbb{P} * \mathbb{Q}$  where we identify  $\mathbb{Q}$  with a corresponding  $\mathbb{P}$ -name.

Let  $\mathbb{H}$  be  $(\mathbb{V}, j(\mathbb{P}_\kappa))$ -generic filter with  $\mathbb{G}_\kappa \subseteq \mathbb{H}$ . The lifting  $\tilde{j} : \mathbb{V}[\mathbb{G}_\kappa] \xrightarrow{\lambda} M[\mathbb{H}]; \underline{a}[\mathbb{G}_\kappa] \mapsto j(\underline{a})[\mathbb{H}]$  witnesses that  $\kappa$  is tightly  $\mathcal{P}$ -Laver generic extendible in  $\mathbb{V}[\mathbb{G}_\kappa]$ .  $|RO(j(\mathbb{P}_\kappa/\mathbb{G}_\kappa))| \leq j(\kappa)$  follows from  $\langle j(\kappa)$ -c.c. of  $j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa$ ,  $M \models j(\kappa)^{< j(\kappa)} = j(\kappa)$ , and (5.11). So it is enough to show:

**Claim 5.2.1**  $V_\alpha^{\mathbb{V}[\mathbb{H}]} \in M[\mathbb{H}]$  for all  $\alpha \leq j(\lambda)$ .

— By induction on  $\alpha \leq j(\lambda)$ . The successor step from  $\alpha < j(\lambda)$  to  $\alpha + 1$  can be proved by showing that  $\mathbb{P}_\kappa$ -names of subsets of  $V_\alpha^{\mathbb{V}[\mathbb{H}]}$  can be chosen as elements of  $M$ . This is the case because of (5.11) and  $\kappa$ -c.c. of  $\mathbb{P}_\kappa$ . — (Claim 5.2.1)

(2): Let  $f$  be a Laver function for super- $C^{(n^*)}$  extendible  $\kappa$ .

Let  $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$  be an iteration of elements of  $\mathcal{P}$  with the support appropriate for  $\mathcal{P}$  such that

$$\mathbb{Q}_\beta := \begin{cases} f(\beta), & \text{if } f(\beta) \text{ is a } \mathbb{P}_\beta\text{-name} \\ & \text{and } \Vdash_{\mathbb{P}_\beta} \text{“} f(\beta) \in \mathcal{P}\text{”}; \\ \mathbb{P}_\beta\text{-name of the trivial forcing,} & \text{otherwise.} \end{cases}$$

We show that  $\Vdash_{\mathbb{P}_\kappa}$  “ $\kappa$  is tightly super- $C^{(n)}$ - $\mathcal{P}$ -Lg extendible”. As in the proof of (1), this is enough to show.

Let  $\mathbb{G}_\kappa$  be a  $(\mathbb{V}, \mathbb{P}_\kappa)$ -generic filter. In  $\mathbb{V}[\mathbb{G}_\kappa]$ , suppose that  $\mathbb{P} \in \mathcal{P}$  and let  $\mathbb{P}$  be a  $\mathbb{P}_\kappa$ -name for  $\mathbb{P}$ .

Suppose that  $\lambda > \kappa$ . Then, there is  $\lambda^* > \lambda$ ,  $\lambda^* \in C^{(n^*)}$  with  $j : \mathbb{V} \xrightarrow{\lambda} M$  such that (5.14):  $j(\kappa) > \lambda^*$ , (5.15):  $V_{j(\lambda^*)} \prec_{\Sigma_{n^*}} \mathbb{V}$ , (5.16):  $V_{j(\lambda^*)} \in M$ , and (5.17):  $j(f)(\kappa) = \mathbb{P}$ . The last condition is possible since  $f$  is a Laver function for extendibility of  $\kappa$ .

In  $M$ , there is a  $\mathbb{P}_\kappa * \mathbb{P}$ -name  $\mathbb{Q}$  such that



## 6 Simultaneous and diagonal reflections, and total failure of square principles under LgLCAs

In this section, we return to the narration that LgLCAs are strong form of reflection principles. Since square principles imply the existence of structures with strong non-reflection properties, a natural guess is that LgLCAs imply the total failure of square principles, at least above a certain cardinal. Corollary 6.2 below confirms this intuition.

Let us start with the following variant of the Reflection Property RP in Jech [26]:

(RP\*): For any uncountable  $\lambda$  and  $S_\xi \subseteq [\lambda]^{\aleph_0}$ ,  $\xi < \eta$  for some  $\eta < \kappa_{\text{refl}}$  such that  $S_\xi$  is stationary in  $[\lambda]^{\aleph_0}$  for all  $\xi < \eta$ , there is  $X \in [\lambda]^{<\kappa_{\text{refl}}}$  such that  $S_\xi \cap [X]^{\aleph_0}$  is stationary in  $[X]^{\aleph_0}$  for all  $\xi < \eta$ .

**Theorem 6.1** (1) *Suppose that  $\mathcal{P}$  is proper. Then  $\mathcal{P}$ -LgLCA for supercompact implies RP\*.*

(2) *Suppose that  $\mathcal{P}$  is stationary preserving (i.e. preserving stationarity of stationary subsets of  $\omega_1$ ), and  $\mathcal{P}$  contains all  $\sigma$ -closed posets. Then  $\mathcal{P}$ -LgLCA for supercompact implies RP\*.*

**Proof.** (1): Assume that  $\mathcal{P}$  is proper, and  $\mathcal{P}$ -LgLCA for supercompact holds. Let  $\mathcal{S} = \langle S_\xi : \xi < \eta \rangle$  for some  $\eta < \kappa_{\text{refl}}$  be such that each  $S_\xi$ ,  $\xi < \eta$  is a stationary subset of  $[\lambda]^{\aleph_0}$ . Without loss of generality, we may assume  $\lambda \geq \kappa$ ,

Let  $\kappa := \kappa_{\text{refl}}$ .  $\mathbb{P} \in \mathcal{P}$  be arbitrary, and let  $\mathbb{P} * \mathbb{Q}$ ,  $\mathbb{H}$ ,  $j$ ,  $M$  be such that  $\Vdash_{\mathbb{P}} \mathbb{Q} \in \mathcal{P}$ ,  $\mathbb{H}$  is a  $(\mathbb{V}, \mathbb{P} * \mathbb{Q})$ -generic set,  $j : \mathbb{V} \xrightarrow{\sim} M$ ,  $j(\kappa) > \lambda$ ,  $\mathbb{P}$ ,  $\mathbb{P} * \mathbb{Q}$ ,  $\mathbb{H}$ ,  $j''\lambda \in M$ , and  $|RO(\mathbb{P} * \mathbb{Q})| \leq j(\kappa)$ .

Then  $j''S_\xi = j(S_\xi) \cap [j''\lambda]^{\aleph_0}$  and  $\mathbb{V}[\mathbb{H}] \models \text{“}j''S \text{ is stationary in } [j''\lambda]^{\aleph_0}\text{”}$  for all  $\xi < \eta$ , the latter holds since  $\mathbb{P} * \mathbb{Q}$  is proper. Thus  $M \models \text{“}j''S \text{ is stationary in } [j''\lambda]^{\aleph_0}\text{”}$  for all  $\xi < \eta$ .

Since  $j(\mathcal{S}) = \langle j(S_\xi) : \xi < \eta \rangle$  by  $\eta < \kappa$ , we have

$$M \models \text{“there is } X \in [j(\lambda)]^{<j(\kappa)} \text{ such that for all components } S \text{ of } j(\mathcal{S}) \\ S \cap [X]^{\aleph_0} \text{ is stationary in } [X]^{\aleph_0}\text{”}.$$

By elementarity, it follows that

$$\mathbb{V} \models \text{“there is } X \in [\lambda]^{<\kappa} \text{ such that for all components } S \text{ of } \mathcal{S} \\ S \cap [X]^{\aleph_0} \text{ is stationary in } [X]^{\aleph_0}\text{”}.$$

(2): Suppose that  $\mathcal{P}$  is a class consisting of stationary preserving posets such that all  $\sigma$ -closed posets are elements of  $\mathcal{P}$ .

Suppose that  $\lambda$  and  $\mathcal{S} = \langle S_\xi : \xi < \eta \rangle$  are as above. Let  $\mathbb{P} := \text{Col}(\aleph_1, \lambda^+)$  and  $\mathbb{Q}$  be a  $\mathcal{P}$ -name with  $\mathbb{H}, j, M$  such that  $\Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{P}\text{”}$ ,  $\mathbb{H}$  is a  $(\mathbb{V}, \mathbb{P} * \mathbb{Q})$ -generic set,  $j : \mathbb{V} \xrightarrow{\sim} M$ ,  $j(\kappa) > \lambda$ ,  $\mathbb{P}, \mathbb{P} * \mathbb{Q}, \mathbb{H}, j''\lambda \in M$ , and  $|RO(\mathbb{P} * \mathbb{Q})| \leq j(\kappa)$ .

Let  $\mathbb{G}$  be the  $\mathbb{P}$  part of  $\mathbb{H}$ . Since  $\mathbb{P}$  is proper, we have  $\mathbb{V}[\mathbb{G}] \models \text{“}S_\xi \text{ is stationary in } [\lambda]^{\aleph_0}\text{”}$  where we have  $\mathbb{V}[\mathbb{G}] \models |\lambda| = \aleph_1$ . Thus  $\mathbb{V}[\mathbb{H}] \models \text{“}j''S \text{ is stationary in } [j''\lambda]^{\aleph_0}\text{”}$ , since  $\mathbb{Q}[\mathbb{G}]$  is stationary preserving in  $\mathbb{V}[\mathbb{G}]$ .

Now the final part of the proof of (1) can be repeated here to obtain the same conclusion as in (1). □ (Theorem 6.1)

Theorem 6.1, (2) will be surpassed by Theorem 6.4: the assumptions of Theorem 6.1, (2) imply  $\text{MA}^{++}(\sigma\text{-closed})$  by Theorem 6.4. Cox [3] proved that  $\text{MA}^{++}(\sigma\text{-closed})$  implies the strongest form of Diagonal Reflection Principle and our  $\text{RC}^*$  follows from this.

Actually, with almost the same proof, we can also prove Theorem 6.1 with  $\text{RP}^*$  replaced by the following Diagonal Reflection Principle which implies  $\text{RP}^*$ :

( $\text{RP}^{**}$ ): For any uncountable  $\lambda \geq \kappa$ ,  $\eta < \kappa_{\text{refl}}$  and  $S_\xi \subseteq [\lambda]^{\aleph_0}$ ,  $\xi < \lambda$  such that  $S_\xi$  is stationary in  $[\lambda]^{\aleph_0}$  for all  $\xi < \lambda$ , there are stationarily many  $X \in [\lambda]^{< \kappa_{\text{refl}}}$  such that  $\eta \subseteq X$ , and  $S_\xi \cap [X]^{\aleph_0}$  is stationary in  $[X]^{\aleph_0}$  for all  $\xi \in X$ .

**Corollary 6.2** *Assume that  $\mathcal{P}$  is a class of posets which is either proper, or (stationary preserving and containing all  $\sigma$ -closed posets). Then  $\mathcal{P}$ -LgLCA for supercompact implies  $\neg \square_\mu$  for all  $\mu \geq \kappa_{\text{refl}}$ . Further, if  $\kappa_{\text{refl}} \leq \aleph_2$ , then  $\mathcal{P}$ -LgLCA for supercompact implies the total failure of square principles.*

**Proof.**  $\text{RP}^*$  implies that all stationary  $S \subseteq E_{\omega_0}^{\mu^+} := \{\alpha < \mu^+ : cf(\alpha) = \omega_0\}$  for all  $\mu$  with  $\mu^+ \geq \kappa_{\text{refl}}$  reflects. This implies  $\neg \square_\mu$ . □ (Corollary 6.2)

“ $\mu \geq \kappa_{\text{refl}}$ ” in the Corollary above is optimal.

**Lemma 6.3** *Suppose that  $\mathcal{P}$  is transfinitely iterable class of posets such that all elements of  $\mathcal{P}$  are cardinality preserving. Then  $\mathcal{P}$ -LgLCA for supercompact is compatible with the statement that there are cofinally many  $\mu < \kappa_{\text{refl}}$  with  $\square_\mu$ .*

**Proof.** Let  $\kappa$  be a supercompact cardinal. By Easton support iteration of length  $\kappa$  we can force that cofinally many  $\mu < \kappa$  satisfy  $\square_\mu$  while  $\kappa$  is kept to be supercompact. If we force  $\mathcal{P}$ -LgLCA by the standard construction starting from this model,  $\square_\mu$ -sequences of the cofinally many  $\mu < \kappa$  remain  $\square_\mu$ -sequences in the generic extension since  $\mu^+$  in the model at the start remains  $\mu^+$  in the generic extension.

□ (Lemma 6.3)

If  $\mathcal{P}$  is either the class of all proper posets, or the class of all semi-proper posets, the reason of the reflection properties available under  $\mathcal{P}$ -LgLCA for supercompact is simply that  $\mathcal{P}$ -LgLCA implies  $\text{PFA}^{++}$  and  $\text{MM}^{++}$  respectively. More generally, we have the following.

We consider the following axiom for a class  $\mathcal{P}$  of posets:

( $\text{MA}^{**}(\mathcal{P})$ ): For any  $\mathbb{P} \in \mathcal{P}$  and  $\langle D_\alpha : \alpha < \mu \rangle, \langle \mathcal{S}_\beta : \beta < \nu \rangle$  where  $\mu, \nu < \kappa_{\text{refl}}$ ,  $D_\alpha \subseteq \mathbb{P}$  is dense subset of  $\mathbb{P}$  for all  $\alpha < \mu$  and  $\mathcal{S}_\beta$  is a  $\mathbb{P}$ -name of a stationary subset of  $[\lambda_\beta]^{\aleph_0}$  for some uncountable  $\lambda_\beta < \kappa_{\text{refl}}$ , for all  $\beta < \nu$ , there is a filter  $\mathbb{G} \subseteq \mathbb{P}$  such that  $\mathbb{G} \cap D_\alpha \neq \emptyset$  for all  $\alpha < \mu$  and  $\mathcal{S}_\beta[\mathbb{G}]$  is a stationary subset of  $[\lambda_\beta]^{\aleph_0}$  for all  $\beta < \nu$ .

Note that, if  $\kappa_{\text{refl}} = \aleph_2$ ,  $\text{MA}^{**}(\mathcal{P})$  is simply equivalent to the usual  $\text{MA}^{++}(\mathcal{P})$ .

**Theorem 6.4** (Theorem 5.7 in [12] restated under LgLCA for extendible) *Suppose that an iterable class of posets  $\mathcal{P}$  is either proper, or (semi-proper and contains all  $\sigma$ -closed posets). Then  $\mathcal{P}$ -LgLCA for supercompact implies  $\text{MA}^{**}(\mathcal{P})$ .*

**Proof.** We show the proof for the case that  $\mathcal{P}$  is proper. The proof for the other case is similar using the fact that  $\mathcal{P}$ -LgLCA implies  $\kappa_{\text{refl}} = \aleph_2$  for  $\omega_1$  preserving  $\mathbb{P}$  containing all  $\sigma$ -closed posets (see [12]).

Thus, let us assume that  $\mathcal{P}$  is proper, and let  $\kappa := \kappa_{\text{refl}}$ .

Let  $\mathbb{P} \in \mathcal{P}$  be arbitrary, and  $\mathcal{D} := \langle D_\alpha : \alpha < \mu \rangle, \mathcal{S} := \langle \mathcal{S}_\beta : \beta < \nu \rangle, \langle \lambda_\beta : \beta < \nu \rangle$  be as in the definition of  $\text{MA}^{**}(\mathcal{P})$ . We show that there is a  $\mathbb{G} \subseteq \mathbb{P}$  as in the definition of  $\text{MA}^{**}(\mathcal{P})$ . for these sequences.

Without loss of generality, we may assume that the underlying set of  $\mathbb{P}$  is a cardinal  $\lambda > \kappa$ , and  $\mathcal{S}_\beta$  are nice  $\mathbb{P}$ -names (in the sense of Kunen [29]).

Let  $\mathbb{Q}$  be a  $\mathbb{P}$ -name such that

$$(6.1) \quad \Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{P}\text{”}$$

and that, for  $(\mathbb{V}, \mathbb{P} * \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $j, M \subseteq \mathbb{V}[\mathbb{H}]$  such that  $j : \mathbb{V} \xrightarrow{\lambda} M$ ,  $j(\kappa) > \lambda$ ,  $\mathbb{P}, \mathbb{P} * \mathbb{Q}, \mathbb{H}, j''\lambda \in M$ , and  $|RO(\mathbb{P} * \mathbb{Q})| \leq j(\kappa)$ .

Let  $\mathbb{H}, j, M$  be as above and let  $\mathbb{G}$  be the  $\mathbb{P}$  part of  $\mathbb{H}$ . Note that  $j(\mathcal{D}) = \langle j(D_\alpha) : \alpha < \mu \rangle$ , and  $j(\mathcal{S}) = \{j(\mathcal{S}_\beta) : \beta < \nu\}$ .

By the closure properties of  $M$ , we have  $j''\mathbb{G} \in M, j''\mathbb{G} \subseteq j(\mathbb{P})$ .  $j''\mathbb{G}$  is a subset of  $j(\mathbb{P})$  with finite intersection property by elementarity of  $j$ . Let  $\mathbb{G}^*$  be the filter  $\subseteq j(\mathbb{P})$  generated by  $j''\mathbb{G}$ . Then  $\mathbb{G}^* \in M$ , and  $\mathbb{G}^* \cap j(D_\alpha) \supseteq \mathbb{G}^* \cap j''D_\alpha \neq \emptyset$  for all  $\alpha < \mu$ .

We also have  $\mathbb{V}[\mathbb{H}] \models \text{“}j(\mathcal{S}_\beta)[\mathbb{G}^*] = \mathcal{S}_\beta[\mathbb{G}] \text{ is a stationary subset of } [\lambda_\beta]^{\aleph_0}\text{”}$  for  $\beta < \nu$  by the choice of  $\mathcal{S}$  and (6.1). It follows that

$M \models “j(\mathcal{S}_\beta)[\mathbb{G}^*] = \mathcal{S}_\beta[\mathbb{G}] \text{ is a stationary subset of } [\lambda_\beta]^{\aleph_0}”$ .

Thus,

$M \models “\exists \underline{G} (\underline{G} \text{ is a } j(\mathcal{D})\text{-generic filter on } j(\mathbb{P}), \text{ and for each component } \mathcal{S} \text{ of } j(\mathcal{S}) \text{ with the corresponding } \lambda, \mathcal{S}[\underline{G}] \text{ is a stationary subset of } [\lambda]^{\aleph_0})”$ .

By elementarity of  $j$ , it follows that

$V \models “\exists \underline{G} (\underline{G} \text{ is a } \mathcal{D}\text{-generic filter on } \mathbb{P}, \text{ and for each component } \mathcal{S} \text{ of } \mathcal{S} \text{ with the corresponding } \lambda, \mathcal{S}[\underline{G}] \text{ is a stationary subset of } [\lambda]^{\aleph_0})”$

as desired. □ (Theorem 6.4)

Fodor-type Reflection Principle (FRP) is a reflection principle with reflection point  $< \aleph_2$ . FRP has many “mathematical” characterizations. The following is one of such characterizations in terms of non-metrizability of topological spaces.

**Proposition 6.5** (see [16], Theorem 2.8) *FRP is equivalent to the following statement:*

(6.2) *For any locally countably compact topological space  $X$  if  $X$  is not metrizable then there is a subspace  $Y$  of  $X$  of cardinality  $< \aleph_2$  which is not metrizable.*

In the following, we shall not go into the combinatorial definition of FRP. Instead, we use the following basic facts about FRP to establish Theorem 6.7 showing the connection of FRP to LgLCA.

**Lemma 6.6** (1) ([11], Theorem 2.5) *RP\* implies FRP.*

(2) ([11], Proposition 1.5) *Non-reflecting stationary set  $S \subseteq E_{\omega_0}^\lambda$  for any uncountable  $\lambda$  creates a counter-example to (6.2).*

(3) ([11], Theorem 3.4) *FRP is preserved by c.c.c. forcing.* □

**Proposition 6.7** (1) *Suppose that  $\mathcal{P}$  is stationary preserving and contains all  $\sigma$ -closed posets. then  $\mathcal{P}$ -LgLCA for supercompact implies FRP.*

(2) *If  $\mathcal{P}$  is c.c.c. and it is an relevant and transfinitely iterable class of posets, then FRP is independent over  $\mathcal{P}$ -LgLCA for any (reasonable) notion of large cardinal.*

**Proof.** (1): By Theorem 6.1, (2) and Lemma 6.6, (1).

(2): Assume that  $\mathcal{P}$  is c.c.c. (remember that by the convention we introduced in Section 1, this means that all  $\mathbb{P} \in \mathcal{P}$  are c.c.c.), relevant and transfinitely iterable. We prove the independence over  $\mathcal{P}$ -LgLCA for supercompact. Proofs of independence for other notions of large cardinal can be done in the same way.

To establish the consistency of FRP over  $\mathcal{P}$ -LgLCA for supercompact, we start from a model with two supercompact cardinals. We use the first supercompact to force FRP such that the second supercompact survives. Using this remaining supercompact cardinal  $\kappa$  we force  $\mathcal{P}$ -LgLCA for supercompact over the first generic extension. This can be done by a forcing in  $\mathcal{P}$  (see Theorem 5.2, (3) in [12]) which is c.c.c. and thus FRP survives the generic extension by Lemma 6.6, (3).

To show the consistency of  $\neg$ FRP over  $\mathcal{P}$ -LgLCA for supercompact, we now start from a model with a supercompact cardinal  $\kappa$ . We force  $\square_\mu$  for some  $\mu < \kappa$  with a small forcing. Over this generic extension in which  $\kappa$  remains supercompact, we force  $\mathcal{P}$ -LgLCA for supercompact using  $\kappa$  by a forcing in  $\mathcal{P}$  (see, Theorem 5.2, (3) in [12]).  $\square_\mu$ -sequence survives the second generic extension since stationarity of subsets of  $\mu^+$  is preserved by the second generic extension. Now  $\square_\mu$  implies the existence of a non-reflecting stationary subset of  $E_{\omega_0}^{\mu^+}$ . Thus by Lemma 6.6, (2), FRP does not hold in the second generic extension. □ (Proposition 6.7)

LgLCAs behave similarly to Proposition 6.7 in connection with the (non-)existence of Kurepa trees.

**Proposition 6.8** (1) *Suppose that  $\mathcal{P}$  is stationary preserving and contains all  $\sigma$ -closed posets. then  $\mathcal{P}$ -LgLCA for supercompact implies the Strong Chang Conjecture  $\text{CC}^*$  in [35]. Hence  $\mathcal{P}$ -LgLCA for supercompact implies the non-existence of Kurepa trees.*

(2) *If  $\mathcal{P}$  is c.c.c. and it is an relevant and transfinitely iterable class of posets, then the existence of Kurepa tree, as well as  $\text{CC}$  is independent over  $\mathcal{P}$ -LgLCA for any (reasonable) notion of large cardinal.* □

Note that, in Proposition 6.8, (2), “ $\text{CC}$ ” cannot be replaced by “ $\text{CC}^*$ ” since  $\text{CC}^*$  implies  $2^{\aleph_0} \leq \aleph_2$  (See the remark after Lemma 6 in Todorćević [35]).

There are several other reflection of properties for which their preservation in the generic extensions is known only for special classes of posets, and thus the corresponding LgLCAs are also rather special.

*Game Reflection Principle* (GRP, which is called the global Game Reflection Principle and denoted by  $\text{GRP}^+$  in König [28]) is the statement that the non-existence of a winning strategy of Player II of the game of certain kind is reflected

down to a subgame over an underlying set of cardinality  $< \aleph_2$ . Since non-existence of a winning strategy is preserved by  $\sigma$ -closed posets it is easy to prove that  $\mathcal{P}$ -LgLCA for supercompact for the class  $\mathcal{P}$  of all  $\sigma$ -closed posets implies GRP by an argument similar to the proof of Theorem 6.1.

However, König proved in [28] that GRP is characterized by  $\aleph_2$  being generic supercompact by  $\sigma$ -closed forcing which is apparently a weakening of  $\mathcal{P}$ -LgLCA for  $\mathcal{P}$  as above. It is also proved in [28] that GRP implies Rado's Conjecture. Thus, we have

**Theorem 6.9** *Let  $\mathcal{P}$  be the class of all  $\sigma$ -closed posets. Then  $\mathcal{P}$ -LgLCA for supercompact implies GRP. In particular,  $\mathcal{P}$ -LgLCA for supercompact implies Rado's Conjecture.  $\square$*

GRP implies CH (Theorem 8 in König [28]), and Rado's Conjecture implies  $\neg\text{MA}_{\aleph_1}$  ([34]). These facts indicate that Theorem 6.9 can not be further generalized for other classes  $\mathcal{P}$  of posets.

Non-freeness of a structure is preserved by c.c.c. forcing (see e.g. [6], Theorem 2.1). Using this fact, an argument similar to the proof of Theorem 6.1 shows the following:

**Theorem 6.10** *Suppose that  $\mathcal{P}$  is c.c.c. Then  $\mathcal{P}$ -LgLCA for supercompact implies the following reflection theorem:*

*For any non-free algebra  $A$  (in a universal algebraic class of structures), there is a non-free subalgebra  $B$  of  $A$  of cardinality  $< 2^{\aleph_0}$ .  $\square$*

Let us call  $\mathcal{P}$  (and elements of  $\mathcal{P}$ ) Cohen, if

$$\mathcal{P} = \{\mathbb{P} : \mathbb{P} \text{ is forcing equivalent to } \text{Fn}(\kappa, 2) \text{ for some infinite } \kappa\}$$

where  $\text{Fn}(\kappa, 2)$  is the usual poset adding  $\kappa$  many Cohen reals by finite conditions.

Dow, Tall and Weiss [5] proved that Cohen posets preserve the non-metrizability of topological spaces. In a supercompact elementary embedding context with  $j : \mathbf{V} \xrightarrow{\kappa} M$ ,  $j''\mathcal{O}$  generates the subspace topology of  $j''X$ , in  $(j(X), j(\mathcal{O}))$  for a topological space  $X = (X, \mathcal{O})$  with character  $< \kappa$ .

Thus an argument similar to the proof of Theorem 6.1 establishes:

**Theorem 6.11** *Suppose that  $\mathcal{P}$  is Cohen. Then  $\mathcal{P}$ -LgLCA for supercompact implies the following reflection theorem:*

*If  $X$  is a non-metrizable topological space with character  $< 2^{\aleph_0}$  then there is a subspace  $Y$  of  $X$  of cardinality  $< 2^{\aleph_0}$  which is also non-metrizable.  $\square$*

In the theorem above, the reflection point “ $< 2^{\aleph_0}$ ” is optimal since

Singular Cardinal Hypothesis (SCH) can be seen as a reflection principle — see e.g. Fuchino, and Rinot [14], Theorem 2.4 in which it is shown that Shelah’s Strong Hypothesis (SSH), a generalization of SCH, is characterized as reflection of a topological property.

In [14], it is shown that FRP implies SSH. Thus Theorem 6.7 above implies that some instances of LgLCA for supercompact imply SSH. By virtue of the definition of LgLCA in the present paper, we can reformulate Proposition 2.8 in [12] as:

**Theorem 6.12** *Suppose that  $\mathcal{P}$  is a class of posets such that either all element of  $\mathcal{P}$  are  $\mu$ -cc for some  $\mu \leq 2^{\aleph_0}$ , or  $\mathcal{P}$  contains all proper posets. Then  $\mathcal{P}$ -LgLCA for supercompact implies SCH.*

**Proof.** Assume that  $\mathcal{P}$ -LgLCA for supercompact holds.

If elements of  $\mathcal{P}$  are  $\mu$ -cc for some  $\mu \leq 2^{\aleph_0}$ . Then the conclusion follows from Proposition 2.8 in [12]. If  $\mathcal{P}$  contains all proper posets, then we have PFA by Theorem 6.4. Thus the main theorem of Viale [41] implies that SCH holds.  $\square$  (Theorem 6.12)

We can remove the restriction on the class  $\mathcal{P}$  in Theorem 6.12, if we assume  $\mathcal{P}$ -LgLCA for hyperhuge (see, Corollary 5.4 in Fuchino and Usuba [17]).

## 7 Separation of some axioms under Continuum Coding Axiom

In this section, we show the separation of some of the axioms and principles we considered in the previous sections.

We start by showing that  $\text{MM}^{++}$ , as well as the property in (4.15) for all stationary preserving posets  $\mathcal{P}$ , does not imply LgLCA for supercompact. The main tools of the separation are the Ground Axiom ([32]) and some other related axioms.

Recall that a ground is an inner model  $M$  of the universe  $\mathbf{V}$  from which we can return to the universe by a set generic extension (for a poset in  $M$ ). *Ground Axiom* (GA) is the axiom asserting that there is no non-trivial ground. If a ground  $M$  satisfies the Ground Axiom then we call  $M$  the bedrock. *Bedrock Axiom* (BA) is the assertion that the bedrock exists.

*Continuum Coding Axiom* (CCA) is the axiom saying that each set is coded class many times by intervals of cardinal numbers in the pattern  $\{\mu^+ : \alpha < \mu^+ < \beta, 2^{\mu^+} = \mu^{++}\}$ . It is easy to see that CCA implies global choice and GA. On the other hand, CCA negates any form of  $\mathcal{P}$ -LgLCA if  $\mathcal{P}$  is non-trivial (see below).

**Theorem 7.1** (1) *CCA negates  $\mathcal{P}$ -LgLCA for any non-trivial class  $\mathcal{P}$  of posets, and for any notion of large cardinal.*

(2)  *$\text{MM}^{++}$  does not imply  $\mathcal{P}$ -LgLCA for any non-trivial class  $\mathcal{P}$  of posets, and for any notion of large cardinal.*

(3) *The conclusion of Viale's Absoluteness Theorem, namely the assertion (4.15) for all stationary preserving posets does not imply  $\mathcal{P}$ -LgLCA for any non-trivial class  $\mathcal{P}$  of posets, and for any notion of large cardinal.*

**Proof.** (1): Assume towards a contradiction, that the CCA and the  $\mathcal{P}$ -LgLCA for a non-trivial class  $\mathcal{P}$  of posets hold.

Let  $\mathbb{P} \in \mathcal{P}$  be a non-trivial poset. By  $\mathcal{P}$ -LgLCA, there is a  $\mathbb{P}$ -name  $\mathbb{Q}$  and  $(\mathbb{V}, \mathbb{P} * \mathbb{Q})$ -generic  $\mathbb{H}$  such that there are  $j, M \subseteq \mathbb{V}[\mathbb{H}]$  with  $j : \mathbb{V} \xrightarrow{\sim} M$  and  $\mathbb{H} \in M$ . Then the  $\mathbb{P}$  part of  $\mathbb{H}$  adds a new set while  $\mathbb{H}$  keeps the end segment of the continuum class function (i.e. the class function  $\kappa \mapsto 2^\kappa$ ) unchanged. Thus we have  $M \models \neg\text{CCA}$ . By elementarity of  $j$  it follows that  $\mathbb{V} \models \neg\text{CCA}$ . This is a contradiction.

(2): In [4], it is shown that  $\text{MM}^{++}$  is preserved by  $< \omega_2$ -directed closed forcing. Thus, starting from a model of  $\text{MM}^{++}$  with class many supercompact cardinals, we can class force with  $< \omega_2$ -directed closed class forcing as Laver did in [30] to obtain a model  $\mathbb{W}$  of  $\text{MM}^{++}$  with class many indestructible supercompact cardinals. By Lemma 7.2 below, we have  $\mathbb{W} \models \text{CCA}$ . Thus,  $\mathbb{W} \models \neg\mathcal{P}\text{-LgLCA}$  by (1).

(3): By Viale's Absoluteness Theorem in [42], (4.15) for  $\mathcal{P} =$  semi-proper posets holds in the model  $\mathbb{W}$  while  $\mathbb{W} \models \neg\mathcal{P}\text{-LgLCA}$  for any non-trivial  $\mathcal{P}$  as we showed above. □ (Theorem 7.1)

**Lemma 7.2** (G. Goldberg) *If there are class many (directed closed) indestructible supercompact cardinals, then CCA holds.*

**Proof.** Note that by AC, all sets are coded by ordinals and we can choose the coding translation invariant.

Let  $\gamma \in \text{On}$  be a limit ordinal and  $S \subseteq \gamma$  be an arbitrary set. For  $\alpha \in \text{On}$ , let  $S_\alpha := \{\alpha + \xi : \xi \in S\}$ .

Let  $\beta \in \text{On}$ . We show that there is  $\alpha^* > \beta$  such that  $S_{\alpha^*} = C_{\alpha^*, \alpha^* + \gamma}$  where  $C_{\alpha, \beta} := \{\xi : \alpha \leq \xi < \beta, 2^{\aleph_{\xi+1}} = \aleph_{\xi+2}\}$ .

Let  $\kappa > \beta, \gamma$  be indestructible supercompact cardinal. Then there is a  $< \kappa$ -directed closed  $\mathbb{P}$  such that, for  $(\mathbb{V}, \mathbb{P})$ -generic  $\mathbb{G}$ , we have  $\mathbb{V}[\mathbb{G}] \models S_\kappa = C_{\kappa, \kappa + \gamma}$ . Since  $\kappa$  is still supercompact in  $\mathbb{V}[\mathbb{G}]$  and hence  $\Sigma_2$ -correct, it follows that there is

$\beta < \delta < \kappa$ , such that  $V[G] \models S_\delta = C_{\delta, \delta+\gamma}$ . Since  $\mathbb{P}$  does not change the continuum function below  $\kappa$ , it follows that  $V \models S_\delta = C_{\delta, \delta+\gamma}$ .  $\square$  (Lemma 7.2)

We show that some **LgLCAs** are separated by their consistency strength. The following theorem has actually a generalization for tight  $\mathcal{P}$ -generic hyperhuge cardinals, and the generalization also implies Usuba’s Main Theorem in [39] asserting that a hyper-huge cardinal implies **BA**. Usuba proved that this theorem in [39] already holds under the existence of an extendible cardinal [40]. At the moment, it is open if corresponding improvement of the theorem below down to **LgLCA** for extendible is also possible.

**Theorem 7.3** (Fuchino and Usuba [17]) (1) *For any relevant and transfinately iterable  $\Sigma_2$  class  $\mathcal{P}$  of posets, then the following assertions are equiconsistent:*

- (a) **ZFC** +  $\mathcal{P}$ -**LgLCA** for hyperhuge;
- (b) **ZFC** + “there is a hyperhuge cardinal”;
- (c) **ZFC** + **BA** +  $\kappa_{\text{refl}}$  is hyperhuge in the bedrock.

(2) *For any relevant and transfinately iterable class  $\mathcal{P}$  of posets, the following assertions are equiconsistent:*

- (d) **ZFC** + super  $C^{(\infty)}$ - $\mathcal{P}$ -**LgLCA** for hyperhuge;
- (e) **ZFC** + **BA** +  $\kappa_{\text{refl}}$  is super- $C^{(\infty)}$  hyperhuge in the bedrock.  $\square$

**Theorem 7.4** (Tsaprounis [37], Proposition 3.2) *If  $\kappa$  is an ultrahuge cardinal then there exists a normal measure one many superhuge  $\alpha < \kappa$ .*  $\square$

**Theorem 7.5** (1) *Suppose that  $\mathcal{P}$  is a relevant and transfinately iterable  $\Sigma_2$ -definable class of posets and **ZFC** +  $\mathcal{P}$ -**LgLCA** for superhuge is consistent. Then **ZFC** +  $\mathcal{P}$ -**LgLCA** for superhuge does not imply  $\mathcal{P}$ -**LgLCA** for hyperhuge.*

(2) *Suppose that  $\mathcal{P}$  is a relevant and transfinately iterable class of posets (its definition may be more complex than  $\Sigma_2$ ). **ZFC** + super- $C^{(\infty)}$ - $\mathcal{P}$ -**LgLCA** for extendible is consistent. Then **ZFC** + super- $C^{(\infty)}$ - $\mathcal{P}$ -**LgLCA** for extendible does not imply  $\mathcal{P}$ -**LgLCA** for hyperhuge.*

**Proof.** (1): Suppose otherwise. Then, since a hyperhuge cardinal is ultrahuge, Theorem 7.3, (1) and Theorem 7.4 imply

$$(7.1) \quad \text{ZFC} + \mathcal{P}\text{-LgLCA for superhuge} \vdash \text{consis}(\text{ZFC} + “\exists \text{ a superhuge cardinal}”).$$

Since  $\mathcal{P}$  is transfinately iterable and  $\Sigma_2$ -definable, it follows (much like in the proof of Theorem 5.2, (1)) that

$$(7.2) \quad \text{ZFC} + \mathcal{P}\text{-LgLCA for superhuge} \vdash \text{consis}(\text{ZFC} + \mathcal{P}\text{-LgLCA for superhuge}).$$

By the Second Incompleteness Theorem, it follows that “ZFC +  $\mathcal{P}$ -LgLCA for superhuge” is inconsistent. This is a contradiction to our assumption.

(2): Similarly to (1) using Lemma 3.1, Proposition 3.2 as well as an analogue of Theorem 5.2, (2) in place of Theorem 7.4 and Theorem 5.2, (1).  $\square$  (Theorem 7.5)

**Lemma 7.6** *Suppose that  $(\mathcal{P}, \emptyset)_{\Pi_2}$ -RcA holds for a class of posets  $\mathcal{P}$  such that either  $\mathcal{P}$  collapses arbitrary large cardinal making it an ordinal of small cardinality, or  $\mathcal{P}$  adds more reals than any given cardinal. If there is at least a  $C^{(n)}$  and inaccessible cardinal. Then there are class many  $C^{(2)}$  and inaccessible cardinals.*

**Proof.** Assume towards a contradiction, that there are at least one but only set many  $C^{(2)}$  and inaccessible cardinals. By assumption on  $\mathcal{P}$ , there is a  $\mathbb{P} \in \mathcal{P}$  such that  $\Vdash_{\mathbb{P}}$  “there is no  $C^{(2)}$  and inaccessible cardinal”. Since the statement is  $\Pi_2$  without any parameter,  $(\mathcal{P}, \emptyset)_{\Pi_2}$ -RcA implies that there is a ground  $M$  without any inaccessible cardinal. But this is a contradiction since no inaccessible cardinal can resurrect by a set forcing.  $\square$  (Lemma 7.6)

**Proposition 7.7** *Suppose that  $\mathcal{P}$  is either the class of all  $\sigma$ -closed posets or the class of all ccc posets. Then  $\mathcal{P}$ -LgLCA for supercompact does not imply  $\mathcal{P}$ -LgLCA for extendible.*

**Proof.** Recall that the predicates “ $\underline{\kappa}$  is supercompact” and “ $\underline{\lambda}$  is  $C^{(2)}$ ” are  $\Pi_2$ .

Start from a universe with a supercompact  $\kappa$  and at least two  $C^{(2)}$  and inaccessible cardinals above  $\kappa$ . Let  $\lambda_0 < \lambda_1$  be the first two  $C^{(2)}$  and inaccessible cardinals above  $\kappa$ . Then, in  $V_{\lambda_1}$ ,  $\kappa$  is supercompact and  $\lambda_0$  is the unique  $C^{(2)}$  and inaccessible cardinal. By Theorem 5.2 in [12], there is a poset  $\mathcal{P}$  of cardinality  $\kappa$  and  $(V_{\lambda_1}, \mathbb{P})$ -generic  $\mathbb{G}$  such that

$$\begin{aligned} V_{\lambda_1}[\mathbb{G}] \models & \text{“}\mathcal{P}\text{-LgLCA for super compact} \\ & + \lambda \text{ is the unique } C^{(n)} \text{ and inaccessible cardinal”}. \end{aligned}$$

By Lemma 7.6, it follows

$$V_{\lambda_1}[\mathbb{G}] \models \text{“}\mathcal{P}\text{-LgLCA for super compact} + \neg(\mathcal{P}, \emptyset)_{\Pi_2}\text{-RcA”}.$$

Thus, by Theorem 4.3, we have

$$V_{\lambda_1}[\mathbb{G}] \models \text{“}\mathcal{P}\text{-LgLCA for super compact} + \neg\mathcal{P}\text{-LgLCA for extendible”}.$$

$\square$  (Proposition 7.7)

## 8 Laver-generic Large Cardinal Axioms for all posets

In this section, we want to examine the principles which we will call the *Laver-generic Large Cardinal Axioms for all posets* and abbreviate it as **LgLCAAs**:

For a notion  $LC$  of large cardinal, **LgLCAA for  $LC$**  is the following assertion:

- (8.1) For any  $\lambda > 2^{\aleph_0}$  and for any poset  $\mathbb{P}$ , there is a  $\mathbb{P}$ -name  $\mathbb{Q}$  of a poset with the property that, for a  $(\mathbb{V}, \mathbb{P} * \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $j, M \subseteq \mathbb{V}[\mathbb{H}]$  such that (a)  $j : \mathbb{V} \xrightarrow{\sim}_{2^{\aleph_0}} M$ , (b)  $j(2^{\aleph_0}) > \lambda$ , (c)  $\mathbb{P}, \mathbb{P} * \mathbb{Q}, \mathbb{H} \in M$ , (d)  $|RO(\mathbb{P} * \mathbb{Q})| \leq j(2^{\aleph_0})$ , and (e)  $M$  satisfies the closure property corresponding to  $LC$ .

This version of Laver-genericity has been already discussed in [17] and [8]. Below, we examine it in connection with extendibility.

The following Lemma was given in [17] as a hidden consequence of its Theorem 3.3, (5) and Theorem 4.10. We give here an easy direct proof:

**Lemma 8.1** *LgLCAA for any notion of large cardinal implies CH.*

**Proof.** Suppose that **LgLCAA** and  $\neg\text{CH}$  hold. Let  $\mathbb{P} = \text{Col}(\omega, \{\omega_1\})$ . Then there are  $\mathbb{P}$ -name  $\mathbb{Q}$  of a poset such that for  $(\mathbb{V}, \mathbb{P} * \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $j, M \subseteq \mathbb{V}[\mathbb{H}]$  such that  $j : \mathbb{V} \xrightarrow{\sim}_{2^{\aleph_0}} M$  and  $\mathbb{H} \in M$ . Since  $\aleph_1 < 2^{\aleph_0}$ ,  $j$  sends  $\omega_1$  to itself. Since  $M \models \text{“}\omega_1^{\mathbb{V}} \text{ is countable”}$  it follows  $V \models \text{“}\omega_1^{\mathbb{V}} \text{ is countable”}$  by elementarity of  $j$ . This is a contradiction. □ (Lemma 8.1)

Note that by the lemma above, the **LgLCAA** is different from “the **LgLCA** for all posets” in the sense of previous sections in that “the **LgLCA**’ refers to  $\kappa_{\text{refl}}$  being the critical point of generic elementary embeddings while the present **LgLCAA** refers to  $2^{\aleph_0} = \aleph_1 < \kappa_{\text{refl}}$  being the critical point. Note also that by Theorem 4.6, “the **LgLCA** for all posets” in the sense of previous sections is inconsistent.

For a notion  $LC$  of large cardinal, we define *super- $C^{(\infty)}$ -LgLCAA for  $LC$*  to be the following assertion:

- (8.2) For any  $n \in \mathbb{N}$ ,  $\lambda > 2^{\aleph_0}$  with  $\lambda \in C^{(\infty)}$ , and for any poset  $\mathbb{P}$ , there is a  $\mathbb{P}$ -name  $\mathbb{Q}$  of a poset with the property that, for a  $(\mathbb{V}, \mathbb{P} * \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $j, M \subseteq \mathbb{V}[\mathbb{H}]$  such that (a)  $j : \mathbb{V} \xrightarrow{\sim}_{2^{\aleph_0}} M$ , (b)  $j(2^{\aleph_0}) > \lambda$ , (c)  $\mathbb{P}, \mathbb{P} * \mathbb{Q}, \mathbb{H} \in M$ , (d)  $|RO(\mathbb{P} * \mathbb{Q})| \leq j(2^{\aleph_0})$ , (e)  $M$  satisfies the closure property corresponding to  $LC$ , and  $j(\lambda) \in C^{(n)}$ .

The following theorem can be proved similarly to Theorem 5.2 by constructing  $\mathbb{P}_\kappa$  as the direct limit of a FS iteration of length  $\kappa$ .

**Theorem 8.2** (1) *If  $\kappa$  is extendible, there is a poset  $\mathbb{P}_\kappa$  such that  $\Vdash_{\mathbb{P}_\kappa} \text{“}\kappa = 2^{\aleph_0}\text{”}$  and **LgLCAA** for extendible holds”.*

(2) *Suppose  $V_\mu \models \text{“}\kappa \text{ is strongly super-}C^{(\infty)} \text{ extendible”}$  for an inaccessible cardinal  $\mu$  and  $\kappa < \mu$ . Then there is  $\mathbb{P}_\kappa \in V_\mu$  such that  $V_\mu \models \Vdash_{\mathbb{P}_\kappa} \text{“}\kappa = 2^{\aleph_0}\text{”}$  and **super-}C^{(\infty)}\text{-LgLCAA}** for extendible holds”.*  $\square$

The **LgLCAA** for extendible and the **super-}C^{(\infty)}\text{-LgLCAA}** for extendible satisfy the following theorems corresponding to Theorems 4.2, 4.3, 4.7, and 4.8. The proof of the following theorems can be obtained simply by replacing “ $\kappa := \kappa_{\text{refl}}$ ” with “ $\kappa := 2^{\aleph_0}$ ”.

**Theorem 8.3** (A variation of Theorem 4.2) *Assume that the **LgLCAA** for extendible holds. Then for the class  $\mathcal{P}$  of all posets,  $\mathbb{R}A_{\aleph_1}^{\mathcal{P}}$  holds.*  $\square$

**Theorem 8.4** (A variation of Theorem 4.3) *Assume the **LgLCAA** for extendible. Then for the class  $\mathcal{P}$  of all posets,  $(\mathcal{P}, \mathcal{H}(\aleph_1))_\Gamma\text{-RcA}^+$  holds where  $\Gamma$  is the set of all formulas which are conjunctions of a  $\Sigma_2$ -formula and a  $\Pi_2$ -formula.*  $\square$

**Theorem 8.5** (A variation of Theorem 4.7) *Suppose that the **super-}C^{(\infty)}\text{-LgLCAA}** for extendible holds. Then for the class  $\mathcal{P}$  of all posets,  $\text{MP}(\mathcal{P}, \mathcal{H}(\aleph_1))$  holds.*  $\square$

**Theorem 8.6** (A variation of Theorem 4.8) *Assume that **LgLCAA** for extendible holds. Then, for any poset  $\mathbb{P}$ ,  $\mathcal{H}(\aleph_1)^\vee \prec_{\Sigma_2} \mathcal{H}(\aleph_1)^{\vee[G]}$  holds.*  $\square$

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