Two further aspects of Laver-generic Large Cardinal Axioms

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Abstract

Following an overview of the developments in the theory of set theoretic multiverse and geology in Section 1 and Section 2, we show in Section 3 that the Super-$C^{(\infty)}$-LgLCAA — a strong variant of Laver-generic Large Cardinal Axiom (LgLCA) for all posets — for hyperhuge implies that a certain family of generic extensions of an initial segment of bedrock makes up a model of multiverse axioms of Steel ([39],[40]) and that this model connects closely to the (true) multiverse over the (real) universe V (which strictly speaking, does not exist).

In Section 4, we introduce a new notion of generic Laver diamond and propose a strong variants of this principle that imply instances of LgLCA.

Finally in Section 5, we reexamine the idea of Laver Generic Maximums (LGMs) in connection with the results presented in Sections 3 and 4. We end the section by proving a theorem saying that a weak form of generic Laver diamond at the continuum (but strong enough to imply $\diamondsuit_{2^{\aleph_0}}$) follows from LgLCA for hyperhuge.

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1 Introduction

In modern set theory, models of set theory are often constructed starting from a chosen "universe" of set theory by taking its generic extension, or by taking an inner model of it, or else by some combination of these operations. There are also many other construction methods, including ultraproducts and class forcings, which can be also used in some combination with other methods. The collection of all models obtained by these constructions can be regarded as the (possibly ultimate) cosmos of mathematics, and is often referred to as the "*set-theoretic multiverse*". intro

Usually we take a countable transitive model M of set theory as the chosen initial universe M so that we can actually construct the M-generic set $\mathbb{G}$ for a poset $\mathbb{P} \in M$ (we shall also say $\mathbb{G}$ is an $(M, \mathbb{P})$-generic set in this context). Class forcing can also be treated in a similar way as the real extension of the initial universe.

However we are also often obliged to talk about generic extensions of the (real) universe V. This is merely a sort of modus operandi which actually makes no sense, since V, being the class of *all* sets, can not afford any sets outside it. We do know several ways to handle this apparent paradox (see e.g. Kunen [33], [34], the author's lecture note [6] contains a couple of further narrations slightly different from those by Kunen).

For the purpose of the following discussions, we will simply interpret assertions of the form "for a/ all $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\mathsf{V}[\mathbb{G}] \models \varphi$ holds" simply as rewording of the statements of the form "$\mathbb{p} \Vdash_{\mathbb{P}}$ "φ" for some $\mathbb{p} \in \mathbb{P}$" / "$\Vdash_{\mathbb{P}}$ "φ" ".

For a (set or class) model M of ZFC, $N \subseteq M$ is a *ground* of M if N is an inner model of ZFC in M (i.e. $N \models \mathsf{ZFC}$, N is transitive, and $\mathrm{On}^N = \mathrm{On}^M$) and there is a poset $\mathbb{P} \in N$ and $(N, \mathbb{P})$-generic $\mathbb{G} \in M$ such that $M = N[\mathbb{G}]$. In such situation we shall say that we can *return* to M from N by $\mathbb{P}$, and call $\mathbb{P}$ the *machine* for the return from N to M. Actually a uniform treatment of grounds of any model M of ZFC is known to be possible:

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All additional details not intended be included in the version for RIMS Kôkyûroku are either typeset in dark electric blue (the color of this paragraph) or put in a separate appendices.

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Theorem 1.1 (Woodin, Laver, independently, see e.g. Fuchs, Hamkins, and Reitz [22]) *All grounds N in a transitive (set or class) model M of* ZFC *is uniformly definable using a parameter from each N. More precisely, there is a Δ_2-formula $\Phi(\cdot,\cdot)$ such that for all ground N of M we have $N = \{x \in M : M \models \Phi(x, \mathcal{P}(\kappa)^N)\}$ where κ is the cardinality (in N) of a machine $\mathbb{P} \in N$ for the return.* ❑ p-intro-0

Fuchs, Hamkins, and Reitz [22] called the study of grounds of the universe V (possibly together with other (definable) inner models of V), the *set-theoretic geology*.

Note that, by Theorem 1.1, we can talk e.g. about a set-indexed family $\mathcal{F}$ of grounds in V. Note also that in contrast to the situation with generic extensions of V, the grounds W and the generic sets $\mathbb{G} \in \mathsf{V}$ as a machine to return to $V = \mathsf{W}[\mathbb{G}]$ in the set-theoretic geology are real objects in ZFC except that grounds are proper classes so that we actually have to talk about the formulas and parameters (as in Theorem 1.1) to define them instead of the grounds themselves.

The following theorem by Usuba tells us that the grounds of a given universe V have a strong directedness property in the downward direction.

Theorem 1.2 (Usuba, [43]) *For any set-indexed family $\mathcal{F}$ of grounds of V, there is a ground W of V such that W is a lower bound of all members of $\mathcal{F}$ (with respect to $\subseteq$).* ❑ p-intro-1

In a semantic narration, we call the universe in which we are "living" and from which we start the discussion of the set-theoretic multiverse, the *initial universe*. The initial universe can be the real universe V but it can also be a transitive (class or even, possibly set) model M of ZFC or a model of some large enough finite fragment of ZFC, that is chosen at the start of the argument.

The following is a direct consequence of Theorem 1.2:

Corollary 1.3 *If a model N is attained from an initial universe M by application of the operations of taking a generic extension and taking a ground, it can be represented as a generic extension of a ground of M.* ❑ p-intro-2

The "multiverse" obtained from the initial universe and taking closure with respect to the operations of taking a generic extension and taking a ground, is called *set-generic multiverse*.

If we take a countable transitive model M of ZFC as the initial universe, the natural interpretation of the set-generic multiverse over M should be the set

$$\mathcal{MV}_0^M := \{N : N \text{ is a generic extension of a ground of } M\}.$$

By Corollary 1.3, $\mathcal{MV}_0^M$ is closed under the operations of taking a generic extension, and taking a ground. Clearly, $\mathcal{MV}_0^M$ has cardinality $2^{\aleph_0}$. By Theorem 1.1, $\mathcal{MV}_0^M$ contains only countably many grounds of each $N \in \mathcal{MV}_0^M$.

$\mathcal{MV}_0^M$ could be seen as a miniature model of set-theoretic multiverse in which we could perform "Gedankenexperimenten" about the "real" multiverse. Unfortunately, there is one serious problem with $\mathcal{MV}_0^M$: $\mathcal{MV}_0^M$ does not have the amalgamation property.

Theorem 1.4 (Woodin, see e.g. Hamkins [25], or Fuchino [10], slide#8) *For a countable transitive model M of* ZFC *and $\mathbb{P} \in M$ with $M \models \mathbb{P} = \mathrm{Fn}(\omega, \omega)$, there are $(M, \mathbb{P})$-generic $\mathbb{G}_0$, $\mathbb{G}_1$ such that there is no common extension (nor common target of embedding over M) of $M[\mathbb{G}_0]$ and $M[\mathbb{G}_1]$ of the form $M[\mathbb{G}]$.* ❏ p-intro-3

John Steel [39], [40] introduced a natural model of multiverse that satisfies the strong amalgamation property.

Let M be an arbitrary countable transitive model of ZFC and let $\mathbb{G}$ be a $(M, \mathrm{Col}(\omega, {<}\,\mathrm{On}^M))$-generic filter (or $(M, \mathrm{Col}(\omega, \mathrm{On}^M))$-generic filter, in Kanamori's notation). Note that $\mathrm{Col}(\omega, {<}\,\mathrm{On}^M)$ is a class forcing in M and all ordinals in M are collapsed and become countable. In particular, $M[\mathbb{G}]$ is not a model of ZFC. Let

$$\mathcal{MV}_{\mathsf{ST}}^{M,\mathbb{G}} := \{N : N \text{ is a ground of } M[\mathbb{G} \restriction \alpha] \text{ for some } \alpha \in \mathrm{On}^M\}.$$

It is easy to see that $\mathcal{MV}_{\mathsf{ST}}^{M,\mathbb{G}}$ satisfies the strong amalgamation property (with respect to $\subseteq$). Steel considered the following theory of multiverse MV in the language $\mathcal{L}_{\mathsf{MV}} = \{\in, \mathcal{S}, \mathcal{W}\}$ where $\mathcal{S}$ and $\mathcal{W}$ are unary predicate symbols which probably stand for "sets" and "world".

Axioms of MV are:

(1.1) $\in$ satisfies the extendibility. x-intro-a

(1.2) $\forall x\, (\mathcal{S}(x) \lor \mathcal{W}(x))$, $\forall x\, (\mathcal{S}(x) \to \neg\mathcal{W}(x))$, x-intro-0
$\forall x\, (\mathcal{S}(x) \leftrightarrow \exists W\, (\mathcal{W}(W) \land x \in W))$.

(1.3) For every W with $\mathcal{W}(W)$, $\{x : \mathcal{S}(x), x \in W\}$ is a transitive proper subclass of $\mathcal{S}$ (for W as here we simply write "$W \in \mathcal{W}$" and identify W with $\{x : \mathcal{S}(x), x \in W\}$; also write "$a \in \mathcal{S}$" for $\mathcal{S}(a)$). x-intro-1

(1.4) (a) φ^W for all $W \in \mathcal{W}$, and for all (meta-mathematical quantification!) axiom φ of ZFC. x-intro-2

(b) For all $W \in \mathcal{W}$ and poset $\mathbb{P} \in W$, there is a $(W, \mathbb{P})$-generic $\mathbb{G} \in \mathcal{S}$ and $M[\mathbb{G}] \in \mathcal{W}$ for all such $\mathbb{G}$.

(c) For all $W \in \mathcal{W}$ if W' is a ground of W then $W' \in \mathcal{W}$ (this is formalizable by Theorem 1.1)

(d) $\mathcal{W}$ satisfies the strong amalgamation property, i.e. for all $M_0, N_1 \in \mathcal{W}$, there is $M_2 \in \mathcal{W}$ with $M_0, N_1 \subseteq M_2$.

Let

$$\widetilde{\mathcal{MV}}^{M,\mathbb{G}}_{\mathsf{ST}} := M[\mathbb{G}] \cup \mathcal{MV}^{M,\mathbb{G}}_{\mathsf{ST}}.\ \mathcal{S}^{\widetilde{\mathcal{MV}}^{M,\mathbb{G}}_{\mathsf{ST}}} := M[\mathbb{G}], \text{ and } \mathcal{W}^{\widetilde{\mathcal{MV}}^{M,\mathbb{G}}_{\mathsf{ST}}} := \mathcal{MV}^{M,\mathbb{G}}_{\mathsf{ST}}.$$

Then again by Corollary 1.3, we have

Lemma 1.5 (Steel [39], [40]) $\langle \widetilde{\mathcal{MV}}^{M,\mathbb{G}}_{\mathsf{ST}}, \in, \mathcal{S}^{\widetilde{\mathcal{MV}}^{M,\mathbb{G}}_{\mathsf{ST}}}, \mathcal{W}^{\widetilde{\mathcal{MV}}^{M,\mathbb{G}}_{\mathsf{ST}}} \rangle \models \mathsf{MV}$. ❑

Steel's multiverse $\widetilde{\mathcal{MV}}^{M,\mathbb{G}}_{\mathsf{ST}}$ depends on the initial countable universe M which might be totally disconnected from our real universe V. A better approximation of the real multiverse with the initial universe V may be attained by taking a large enough $n \in \mathbb{N}$ and letting M be the Mostowski collapse of some countable $N \prec_{\Sigma_n} \mathsf{V}$. So that M is a model of $\mathsf{ZFC} \cap \Sigma_n$ and M shares the same Σ_n-theory with V.

In Section 3, we work under ZFC + the Super-$C^{(\infty)}$-LgLCAA for the hyperhuge (see the next section for the definition) and show that, in this framework, we can introduce a new (set) model of multiverse which strongly reflects the situations of the multiverse generated from the initial universe V.

Note that although we said here "the *real* multiverse generated from the initial universe V", there can be no such thing as the theory of multiverse over the real universe V because of the illusory status of generic extensions of V we already mentioned above. For example it is not at all clear how the situation described in Theorem 1.1 should be interpreted/refuted in this "theory".

If $2^{\aleph_0}$ is a successor cardinal, MA implies $\diamondsuit_{2^{\aleph_0}}$. This follows from two classical results.

Theorem 1.6 (1) (Shelah, [38]) *If $\lambda = \chi^+ = 2^\chi$, $\diamondsuit_\lambda$ holds.* p-intro-4

(2) (Martin-Solovay, [35]) MA *implies that $2^\kappa = 2^{\aleph_0}$ for all $\aleph_0 \leq \kappa < 2^{\aleph_0}$,*

Since PFA implies MA and $2^{\aleph_0} = \aleph_2$, it implies $\diamondsuit_{2^{\aleph_0}}$. However, if the continuum is a limit cardinal, MA is not enough to get $\diamondsuit_{2^{\aleph_0}}$.

Lemma 1.7 MA + "$2^{\aleph_0}$ *is a limit cardinal*" + $\neg\diamondsuit_{2^{\aleph_0}}$ *is consistent.* p-intro-4-0

A sketch of Proof. It is known that under the consistency of some large cardinal, $\neg\diamondsuit_\kappa$ for an inaccessible cardinal κ is consistent. Actually it is also known that $\neg\diamondsuit_\kappa$ for a Mahlo κ is consistent (H. Woodin).

Starting from a model with such κ, we force $\mathsf{MA} + \kappa = 2^{\aleph_0}$ by the standard ccc finite support iteration. Then $\neg\diamondsuit_\kappa$ is preserved. ❑ (Lemma 1.7)

In Section 4, we introduce the generic Laver diamond and show that its stronger versions imply both $\diamondsuit_{2^{\aleph_0}}$ and LgLCAs.

In Section 5, we discuss the impact of the results we presented in this note to the mathematical philosophical standpoint which is named "Laver Generic Maximum" (LGM) in Fuchino, and Usuba [20].

2 Maximality Principle, Bedrock Axiom, and Laver-generic Large Cardinal Axiom

At the beginning of this millennium, Joel Hamkins and his co-authors, among other authors, introduced several families of set-theoretic axioms and principles which claim in each of their own way that our universe V is rich and saturated among the universes in the (set-)generic multiverse (see e.g. [2], [21], [22], [24], [26], [27], [28], [36], [37], [23]). The Maximality Principle is one of such families of axioms. MP-BA-LgLCA

In the following, we give a definition of the Maximality Principle which was originally one of the characterizations of the principle given in Barton, Caicedo, Fuchs, Hamkins, Reitz, and Schindler [2], and also independently by Fuchino, and Usuba [20].

We often assume that a class $\mathcal{P}$ of posets is (normal and) iterable in the following sense:

We call a class $\mathcal{P}$ of posets *normal* if it satisfies (2.1) : $\{\mathbb{1}\} \in \mathcal{P}$, and x-intro-r-0-0
(2.2) : $\mathcal{P}$ is closed with respect to forcing equivalence (i.e. if $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P} \sim \mathbb{P}'$ then $\mathbb{P}' \in \mathcal{P}$). x-intro-r-1

A class $\mathcal{P}$ of posets is said to be *iterable* if it is normal, (2.3) : closed with respect to restriction (i.e. if $\mathbb{P} \in \mathcal{P}$ then $\mathbb{P} \restriction \mathbb{p} \in \mathcal{P}$ for any $\mathbb{p} \in \mathbb{P}$), and (2.4) : for any $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$, $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" implies $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathcal{P}$. x-intro-r-2 x-intro-r-3

For an iterable class $\mathcal{P}$ of posets, an inner model W of V is said to be a *$\mathcal{P}$-ground* if W is a ground in V and its machine to return to V is in $\mathcal{P}^{\mathsf{W}}$.

For an iterable class $\mathcal{P}$ of posets and a set S the *Maximality Principle* for $\mathcal{P}$ and S (notation: $\mathsf{MP}(\mathcal{P}, S)$) is the statement:

(2.5) For any formula $\varphi = \varphi(\overline{x})$, $\mathbb{P} \in \mathcal{P}$ and $\overline{a} \in S$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$" then there is a $\mathcal{P}$-ground W of V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$. x-intro-3

In the next section, $\mathsf{MP}(\text{all posets}, \mathcal{H}(2^{\aleph_0}))$ is going to be one of the principles used in the construction of a new model of the multiverse theory MV of Steel.

Lemma 2.1 (Fuchino, and Usuba [20], Theorem 3.3, (5), (5')) p-intro-5

(1) $\mathsf{MP}(\textit{all posets}, \mathcal{H}(2^{\aleph_0}))$ *implies* CH.

(2) $\mathsf{MP}(\textit{all posets}, \mathcal{H}(2^{\aleph_0}))$ *is a maximum in the sense that* $\mathsf{MP}(\textit{all posets}, \mathcal{H}(2^{\aleph_0})^+)$ *is inconsistent.*

Proof. (1): Assume that $\mathsf{MP}(\text{all posets}, \mathcal{H}(2^{\aleph_0}))$ holds. If $\aleph_1 < 2^{\aleph_0}$, then $\aleph_1 \in \mathcal{H}(2^{\aleph_0})$. Let $\mathbb{P}$ be a poset which collapses ${\aleph_1}^{\mathsf{V}}$. That is $\Vdash_{\mathbb{P}}$ "${\aleph_1}^{\mathsf{V}}$ is countable". By $\mathsf{MP}(\text{all posets}, \mathcal{H}(2^{\aleph_0}))$, there is a ground W of V such that $\mathsf{W} \models$ "${\aleph_1}^{\mathsf{V}}$ is countable". This is a contradiction.

(2): Similarly to (1). Assume toward a contradiction that $\mathsf{MP}(\text{all posets}, \mathcal{H}((2^{\aleph_0})^+))$ holds. Note that $\aleph_1 \in \mathcal{H}((2^{\aleph_0})^+)$. Let $\mathbb{P}$ be a poset which collapses ${\aleph_1}^{\mathsf{V}}$. That is $\Vdash_{\mathbb{P}}$ "${\aleph_1}^{\mathsf{V}}$ is countable". By $\mathsf{MP}(\text{all posets}, \mathcal{H}((2^{\aleph_0})^+))$, there is a ground W of V such that $\mathsf{W} \models$ "${\aleph_1}^{\mathsf{V}}$ is countable". This is a contradiction.

❑ (Lemma 2.1)

Another ingredient of the construction of the model of multiverse in the next section is the Bedrock Axiom.

The *bedrock* is the minimal ground of V (if it exists). By Usuba's Theorem 1.2, if the bedrock exists then it exists uniquely and it is the mantle, i.e. the intersection of all grounds. The mantle always exists, and it also follows from Usuba's Theorem 1.2 that the mantle is a model of ZFC (see Fuchs, Hamkins, and Reitz [22], Theorem 22. (3)).

For a cardinal κ in V, the *$<\kappa$-ground* W of V is a ground of V with a machine $\mathbb{P}$ for return to V such that $\mathsf{W} \models |\mathbb{P}| < \kappa$ ($\Leftrightarrow \mathsf{V} \models |\mathbb{P}| < \kappa$). A *$<\kappa$-mantle* of V is the intersection of all $<\kappa$-grounds of V.

Bedrock Axiom (BA) claims that the bedrock exists.

Lemma 2.2 *If* BA *holds then there are only set many grounds of* V. p-intro-6

Proof. Let $\overline{\mathsf{W}}$ be the bedrock. Then there is a poset $\mathbb{P} \in \overline{\mathsf{W}}$ and $(\overline{\mathsf{W}}, \mathbb{P})$-generic $\mathbb{G} \in \mathsf{V}$ such that $\mathsf{V} = \mathsf{W}[\mathbb{G}]$. Any ground W of V is an intermediate model of $\overline{\mathsf{W}}$ and V in terms of forcing extension (Grigorieff's theorem, see e.g. Friedman, Fuchino, and Sakai[5]). Such W is a $\mathbb{B}$-generic extension of $\overline{\mathsf{W}}$ for some complete subalgebra $\mathbb{B}$ of $\mathsf{RO}(\mathbb{P})$ in $\overline{\mathsf{W}}$ (see Jech [29], Lemma 15.43). This implies that there are only set many grounds. ❑ (Lemma 2.2)

While the Maximality Principle suggests that we have a rich "underworld" of grounds of our universe V (see the definition above), the Bedrock Axiom restricts the family of grounds of V to be set-indexed (Lemma 2.2). Thus it may initially appear that these two axioms are possibly incompatible to each other. However, the Super-$C^{(\infty)}$-LgLCAA for hyperhuge, which will be discussed subsequently, implies both axioms. In [36], Minden discusses the coexistence of Resurrection Axiom and Maximality Principle. This coexistence is also realized under the Super-$C^{(\infty)}$-LgLCAA for hyperhuge (see Fuchino [7], Section 6 and Section 7, see also [9]).

In the following sections, we are going to deal with variations of notions of generic large cardinals corresponding to several different kinds of (genuine) large cardinals. To be specific, we consider the variation of generic large cardinals in connection with supercompactness. The closure property of the target model M of the generic elementary embedding in the following definitions corresponding to the supercompactness is "$j''\lambda \in M$". We obtain the notions of generic large cardinals for other large cardinal properties by changing this condition with one of the another conditions as listed in the **table 1.** below.

The compound symbol "$j : N \overset{\preceq}{\to}_\kappa M$" means that "$j$ is an elementary embedding of transitive model N to the transitive model M of (sufficiently large fragment of) ZFC and κ is the critical point of j." Here "model" can mean both set or class model.

The notion of (tightly) $\mathcal{P}$-Laver generic LC for a class $\mathcal{P}$ of posets and the notion LC of large cardinal we are going to discuss below, were first introduced in Fuchino, Ottenbreit Maschio Rodrigues, and Sakai [16]. The initial motivation of the introduction of these generic LCs and axioms claiming the existence of these generic LCs was to formulate strong reflection statements (in terms of generic elementary embedding) which should give a uniform picture of the landscape with (mathematical) reflection statements (in the sense of footnote 1) , on p.25). This background can be traced in Fuchino, Ottenbreit Maschio Rodrigues, and Sakai [15], [16], [17], and Fuchino, and Ottenbreit Maschio Rodrigues [18].

The following definitions around (tight) Laver-genericity are slight modifications of the original definitions given in [16]. The definitions in the present paper were gradually crystallized in the papers, Fuchino, and Usuba [20], Fuchino, Gappo, and Parente [12], and finally in [9]. Since our present definitions are stronger than the original ones while the standard models of LgLCs in the original definitions are still models of LgLCs in the present definitions, all the results in the papers mentioned above are valid with the present definition (in a few cases with minimal modifications).

Let $\mathcal{P}$ be an iterable class of posets. A cardinal κ is said to be *tightly $\mathcal{P}$-generic supercompact* (tightly $\mathcal{P}$-gen. supercompact, for short) if, for any $\lambda > \kappa$, there is

$\mathbb{Q} \in \mathcal{P}$ such that, for $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$, $\lambda < j(\kappa)$, $|\,\mathsf{RO}(\mathbb{Q})^{\mathsf{V}}\,| \leq j(\kappa)$, $\mathbb{H} \in M$, and $j''\lambda \in M$.

The conditions "$|\,\mathsf{RO}(\mathbb{Q})^{\mathsf{V}}\,| \leq j(\kappa)$" and "$\mathbb{H} \in M$" are what is called "tightness" of the generic elementary embedding j here. Note that, in principle, $\mathbb{Q}$ can be reconstructed from $\mathbb{H}$ (e.g. this is the case if $\mathbb{Q}$ is Boolean i.e. positive elements of a Boolean algebra) so we also assume $\mathbb{Q} \in M$ in this context.

As is already said "$j''\lambda \in M$" is the condition corresponding to the supercompactness. Since a (genuine) supercompact cardinal is just tightly $\{\{\mathbb{1}\}\}$-generic supercompact (see Kanamori [31], Proposition 22.4, (b)), and since $\{\mathbb{1}\} \in \mathcal{P}$ for all iterable class $\mathcal{P}$ of posets, any (genuine) supercompact cardinal is a tightly $\mathcal{P}$-gen. supercompact for any iterable $\mathcal{P}$.

A cardinal κ is *tightly $\mathcal{P}$-Laver-generic supercompact* if, for any $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" such that, for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$, $\lambda < j(\kappa)$, $|\,\mathsf{RO}(\mathbb{Q})^{\mathsf{V}}\,| \leq j(\kappa)$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, and $j''\lambda \in M$.

Let us say the class $\mathcal{P}$ is *stationary preserving* if all $\mathbb{P} \in \mathcal{P}$ provably preserves the stationarity of a ground model stationary set $\subseteq \omega_1$. Of course, all stationary preserving $\mathcal{P}$ preserves ω_1 as well.

It is known that, if iterable $\mathcal{P}$ is stationary preserving and provably contains posets collapsing $\aleph_2$ and/or posets adding a new real, then κ being tightly $\mathcal{P}$-Laver-generic supercompact implies that $\kappa = \kappa_{\mathfrak{refl}}$ where $\kappa_{\mathfrak{refl}} := \max\{\aleph_2, 2^{\aleph_0}\}$, and $2^{\aleph_0} \leq \aleph_2$ or $2^{\aleph_0}$ is weakly inaccessible (Fuchino, Ottenbreit Maschio Rodrigues, and Sakai [16], Section 5).

Against this back-ground, we say for an iterable stationary preserving $\mathcal{P}$, that the *$\mathcal{P}$-Laver-generic Large Cardinal Axiom for supercompactness* ($\mathcal{P}$-LgLCA *for supercompact*, for short) holds if $\kappa_{\mathfrak{refl}}$ is tightly $\mathcal{P}$-Laver-generic supercompact cardinal.

The $\mathcal{P}$-LgLCA for supercompact is a very strong axiom. For example, this axiom implies strong variants of double-plus version of forcing axiom for $\mathcal{P}$ for families of dense sets of size $< \kappa_{\mathfrak{refl}}$. So in particular if $\mathcal{P}$ is the class of all proper posets, the class of all semi-proper posets, or the class of all c.c.c. posets, then $\mathcal{P}$-LgLCA implies PFA^{++}, MM^{++}, and MA^{++} respectively (see Fuchino, Ottenbreit Maschio Rodrigues, and Sakai [16], Section 5, a slightly improved version in the case of very large continuum scenario is to be found in Fuchino [9], Theorem 6.4). On the other, hand it can be also proved that none of these forcing axioms implies LgLCA for supercompact (see [11]).

If $\mathcal{P}$ is the class of all posets, the existence of a tightly $\mathcal{P}$-Laver-generic supercompact cardinal shows a pattern different from that of stationary preserving $\mathcal{P}$: if κ is tightly $\mathcal{P}$-Laver-generic supercompact cardinal for the class $\mathcal{P}$ of all posets, then this implies that κ is (not $\kappa_{\mathfrak{refl}}$ but) $\aleph_1$ and CH holds (see Fuchino [9], Section 8)

We shall call the axiom claiming that there is a tightly $\mathcal{P}$-Laver-generic supercompact cardinal for the class $\mathcal{P}$ of all posets, the Laver-generic Large Cardinal Axiom for all posets and for supercompactness (LgLCAA for supercompact, for short). As mentioned above, LgLCAA for supercompact implies CH and $\aleph_1 = 2^{\aleph_0}$ is the critical point of the generic elementary embeddings it refers.

The the notions of generic large cardinals and Laver-generic Large Cardinal Axioms, we introduced so far, can be also formulated for other notions of large cardinals. This can be done by replacing the closedness condition "$j''\lambda \in M$" on the target model by other conditions. This is summarized in the following table:

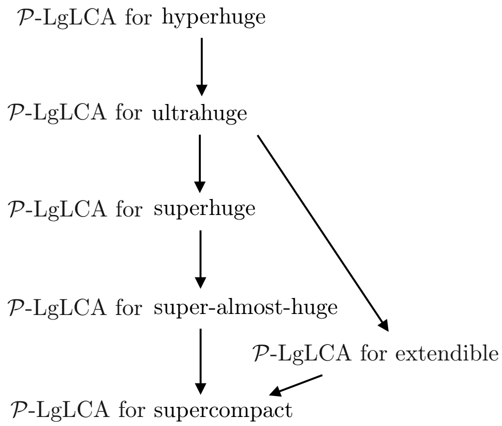
The $\mathcal{P}$-LgLCA/LgLCAA for	The condition "$j''\lambda \in M$" in the definition of "the $\mathcal{P}$-LgLCA for supercompact" is replaced with:
hyperhuge	$j''j(\lambda) \in M$
ultrahuge	$j''j(\kappa) \in M$ and ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$
superhuge	$j''j(\kappa) \in M$
super-almost-huge	$j''\mu \in M$ for all $\mu < j(\kappa)$
extendible	$V_{j(\lambda)}^{\mathsf{V}[\mathbb{H}]} \in M$ for all $\mu < j(\kappa)$

table 1.

By Kanamori [31] Proposition 22.4, (b), the closure properties listed in the **table 1.** above characterize the corresponding (genuine) large cardinal properties as far as the elementary embeddings in connection with the large cardinal properties can be introduced by ultrafilters.

It is not immediately clear that the notion of Laver generic large cardinal is formalizable in a single formula in the language of set theory. This can be done e.g. by using an abstract version of extender (or names of such extenders in generic extensions), see Fuchino, and Sakai [19].

The following implications are easy to prove by the definitions:



B ⟵ A in the diagram denotes "A implies B".

figure 1.

In [7], it is proved that any notions of generic large cardinals we mentioned above do not imply Maximality Principles for the corresponding class of posets. The super-$C^{(\infty)}$-Laver-generic large cardinal was invented in search of generic large cardinal which may imply Maximality Principle. I shall introduce this notion of generic large cardinal in connection with the notion of extendible cardinal since this is the notion of large cardinal considered in connection with super-$C^{(\infty)}$-Laver-genericity in the next section.

For $n \in \mathbb{N}$, a cardinal κ is *tightly super-$C^{(n)}$-$\mathcal{P}$-Laver-generic extendible* if, for any $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$, there are $\lambda' \in C^n$ with $\lambda' > \lambda$, and a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" such that, for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$, $\lambda < j(\kappa)$, $|\,\mathsf{RO}(\mathbb{Q})^{\mathsf{V}}\,| \leq j(\kappa)$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, $j(\lambda) \in (C^{(n)})^{\mathsf{V}[\mathbb{H}]}$, and ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$.

Similarly to the previous definitions, we define: the *Super-$C^{(\infty)}$-$\mathcal{P}$-Laver-generic Large Cardinal Axiom for extendibility* (the *Super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for extendible*, for short) for a stationary preserving iterative $\mathcal{P}$ holds, if $\kappa_{\mathfrak{refl}}$ is tightly super-$C^{(n)}$-$\mathcal{P}$-Laver-generic extendible for all $n \in \mathbb{N}$.

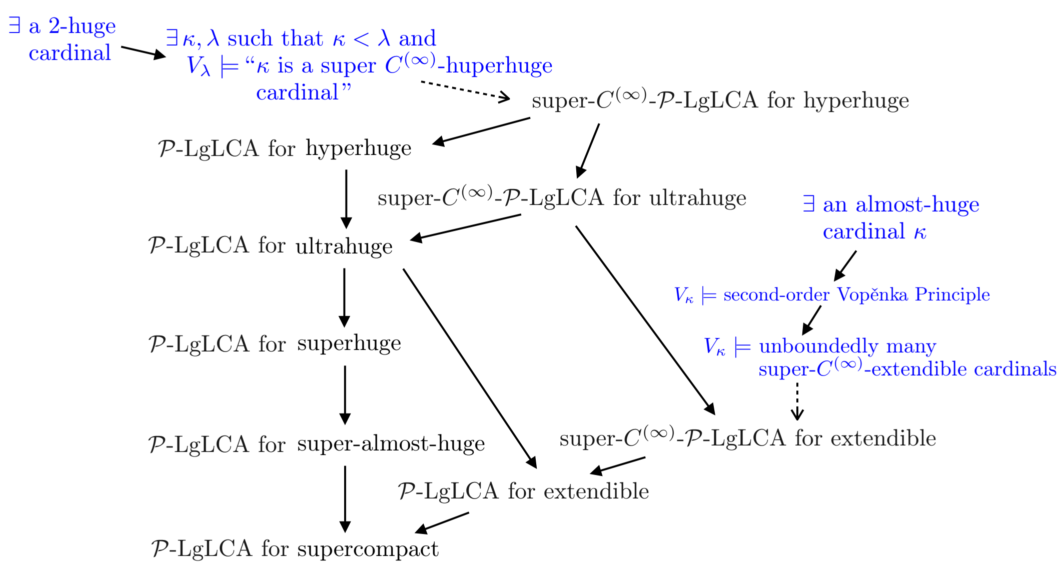
The *Super-$C^{(\infty)}$-Laver-generic Large Cardinal Axiom for all posets and for extendibility* (the *Super-$C^{(\infty)}$-LgLCAA for extendible*, for short) holds, if $2^{\aleph_0}$ is tightly super-$C^{(n)}$-$\mathcal{P}$-Laver-generic extendible for all $n \in \mathbb{N}$ where $\mathcal{P}$ is the class of all posets.

Note that a super-$C^{(0)}$-$\mathcal{P}$-Laver gen. LC is just $\mathcal{P}$-Laver gen. LC.

Super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for hyperhuge and *Super-$C^{(\infty)}$-Laver-generic Large Cardinal Axiom for all posets and for hyperhuge* are another set of axioms we consider in the following sections. The definition of these axioms are also created by applying the change in the previous **table 1.** to the corresponding "extendible" versions.

Note that these axioms are formulated as an axiom schema with infinitely many formulas. This is possible since the Laver-generic large cardinal that these axioms refer, is definable either as $\kappa_{\mathfrak{refl}}$ or as $\aleph_1$. In contrast, the genuine large cardinal version of "super-$C^{(\infty)}$" large cardinals are not formalizable in ZFC in general. There are, however, several circumstances where we can talk about super-$C^{(\infty)}$ large cardinals. One of them is when we are talking about a cardinal in a set model e.g. V_κ for some large cardinal κ. In this case, we can transfer the meta-mathematics into the logic inside V_κ and talk about "for all $n \in \omega$" instead of "for all $n \in \mathbb{N}$". Another situation is when we are talking about κ in an inner model and κ is definable without parameter in a definable outer model (e.g. as $2^{\aleph_0}$ in the outer model). In this case we can handle the formulation of "super-$C^{(\infty)}$" by providing infinitely many formulas in meta-mathematics each of which refers to the definable cardinal in the outer model and altogether describes the super-$C^{(\infty)}$ large cardinal situation of κ in the inner model.

For genuine large cardinals the corresponding notion of super-$C^{(n)}$ large cardinals is closely connected to that of $C^{(n)}$ large cardinals of Joan Bagaria [1]. E.g. in case of extendibility, a cardinal κ is super-$C^{(n)}$-extendible if and only if it is $C^{(n)}$-extendible (Lietz see e.g. [9]). However, the notion of $C^{(n)}$ large cardinals is irrelevant in the context of Laver-generic large cardinals since V_κ for cardinals κ such as $\aleph_2$ or $2^{\aleph_0}$ is never Σ_1-elementary submodel of V.



B ⟵ A in the diagram denotes: "A implies B"

B ⇠ A denotes: "A model of B can be created starting from A".

See also the remark below.

figure 2.

In the figure 2. above, the arrows with broken lines are only available for *transfinitely iterable* $\mathcal{P}$. That is, for such an iterable class $\mathcal{P}$ that is closed with respect to transfinite iteration with an appropriate kind of support such that $\mathcal{P}$ satisfies suitable iteration and factor lemmas for the iteration with the support.

The following theorem was first proved under the Super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for hyperhuge and the Super-$C^{(\infty)}$-LgLCAA for hyperhuge in Fuchino and Usuba [20] when Laver-generic version of extendibility was not yet in the scope.

Theorem 2.3 (Fuchino [9], Theorem 4.7) (1) *Suppose that $\mathbb{P}$ is an iterable class of posets, and the Super-$C^{(\infty)}$-LgLCA for extendible holds. Then $\mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))$ holds.* p-intro-7

(2) *Suppose that the Super-$C^{(\infty)}$-LgLCAA for extendible holds. Then $\mathsf{MP}(all, \mathcal{H}(\aleph_1))$ holds where "all" denotes the class of all posets.* ❑

The strongest instances of LgLCA also imply the BA. For genuine large large cardinals (this is not a typo), this phenomena has been observed by Usuba:

Theorem 2.4 (Usuba [44], see also Fuchino [11] Theorem 20.11) *If there is an extendible cardinal κ then BA holds and the bedrock is a $\leq\kappa$-ground of V.* ❑ usuba-ba-thm

Similar situation holds under a tightly $\mathcal{P}$-Laver-gen. hyperhuge cardinal:

Theorem 2.5 (Fuchino, and Usuba [20], Theorem 5.2) *If there is a tightly $\mathcal{P}$-gen. hyperhuge cardinal κ, then BA holds and the bedrock is a $\leq\kappa$-ground of V. More over, the tightly $\mathcal{P}$-Laver-gen. hyperhuge cardinal κ in V is hyperhuge in the bedrock.* p-intro-8

❑

Corollary 2.6 (1) *The $\mathcal{P}$-LgLCA for hyperhuge for a stationary preserving iterable $\mathcal{P}$ implies BA and the bedrock is a $\leq\kappa_{\mathfrak{refl}}$-ground of V.* p-intro-9

(2) *The LgLCAA for hyperhuge implies BA and the bedrock is a $\leq\aleph_1$-ground of V.*

(3) *The LgLCAA for hyperhuge and the $\mathcal{P}$-LgLCA for hyperhuge for all transfinitely iterable stationary preserving $\mathcal{P}$ are equiconsistent to the existence of a hyperhuge cardinal.* ❑

3 Multiverse in set-theoretic geology

We show that under the Bedrock Axiom and Maximality Principle, there is a natural (set) model of Steel's multiverse theory which connects more closely to the "real" multiverse over the universe V than it is the case with Steel's model we discussed in Section 1. The fundamental result for the construction is the following: multiverse

Proposition 3.1 *Suppose that* $\mathsf{MP}(\textit{all posets}, \mathcal{H}(2^{\aleph_0}))$ *and* BA *hold. Let* $\kappa = (2^{\aleph_0})^{\mathsf{V}}$*, and let* $\overline{\mathsf{W}}$ *be the bedrock.* p-multiv-0

(1) *For any* $\mathbb{P} \in \overline{\mathsf{W}}$ *with* $|\,\mathbb{P}\,| < \kappa$ *there is a* $(\overline{\mathsf{W}}, \mathbb{P})$*-generic* $\mathbb{G}$ *in* V.

(2) *For any* $\mathbb{P}_i \in \overline{\mathsf{W}}$ *with* $|\,\mathbb{P}_i\,| < \kappa$ *and* $(\overline{\mathsf{W}}, \mathbb{P}_i)$*-generic* $\mathbb{G}_i \in \mathsf{V}$ *for* $i \in 2$*, there is* $\mathbb{P} \in \overline{\mathsf{W}}$ *with* $|\,\mathbb{P}\,| < \kappa$ *and* $(\overline{\mathsf{W}}, \mathbb{P})$*-generic* $\mathbb{G} \in \mathsf{V}$ *such that* $\mathbb{G}_i \in \overline{\mathsf{W}}[\mathbb{G}]$ *for* $i \in 2$.

Proof. (1): Let $\mathbb{P} \in \overline{\mathsf{W}}$ be such that $|\,\mathbb{P}\,| < \kappa$. Without loss of generality, we may assume that the underlying set of $\mathbb{P}$ is an element of $\lambda \subseteq \mathcal{H}(\kappa)^{\mathsf{V}}$.

Since $\Vdash_{\mathbb{P}}$ "there is a $(\overline{\mathsf{W}}, \mathbb{P})$-generic filter", the Maximality Principle implies that there is some ground W of V such that $\mathbb{P} \in \mathsf{W}$ and there is a $(\overline{\mathsf{W}}, \mathbb{P})$-generic filter $\mathbb{G}$ in $\mathsf{W} \subseteq \mathsf{V}$.

(2): Suppose that $\mathbb{P}_i$ and $\mathbb{G}_i$, $i \in 2$ are as above. Let $\mathbb{P}^* \in \overline{\mathsf{W}}$ and $\mathbb{G}^*$ be such that $\mathsf{V} = \overline{\mathsf{W}}[\mathbb{G}^*]$. Let $\mathbb{Q}$ be a poset in V such that $\Vdash_{\mathbb{Q}}$ "$|\,\mathbb{P}^*\,| < 2^{\aleph_0}$". Then we have

$$\overline{\mathsf{V}} \models\ \Vdash_{\mathbb{Q}} \text{“}\exists\, \underline{\mathbb{P}} \in \overline{\mathsf{W}},\ |\,\underline{\mathbb{P}}\,| < 2^{\aleph_0}, \exists\, (\overline{\mathsf{W}}, \underline{\mathbb{P}})\text{-generic } \underline{\mathbb{G}} \text{ such that } \mathbb{G}_i \in \overline{\mathsf{W}}[\underline{\mathbb{G}}] \text{ for } i \in 2\text{”}.$$

By the Maximality Principle, the same statement holds in a ground of V, and hence also in V. ❑ (Proposition 3.1)

Lemma 3.2 (A) *Assume that the Super-*$C^{(\infty)}$*-*LgLCAA *for extendible holds and there is an extendible cardinal. Let* $\overline{\mathsf{W}}$ *be mantle of* V *and* $\kappa := (2^{\aleph_0})^{\mathsf{V}}$*. Then:* p-multiv-0-0

(0) CH *holds.*

(1) BA *holds, hence* $\overline{\mathsf{W}}$ *is the bedrock of* V*, and it is* $\leq \kappa$*-ground of* V.

(2) $\mathsf{MP}(\text{all posets}, \mathcal{H}(2^{\aleph_0}))$ *holds.*

(3) κ *is an inaccessible cardinal in* $\overline{\mathsf{W}}$.

(4) ${V_\kappa}^{\overline{\mathsf{W}}} \prec \overline{\mathsf{W}}$.

(B) *Assume that the Super-*$C^{(\infty)}$*-*LgLCAA *for hyperhuge holds. Then the assumptions of* (A) *hold. Thus all the conclusions of* (A) *hold.*

Sketch of the Proof. (A): (0): By Lemma 2.1, (1) and (2) of the present Lemma.

(1): By Theorem 2.4. Note that the assumption of the existence of an extendible cardinal is needed here.

(2): By Theorem 2.3, (2).

(3): Actually, we can prove a much stronger statement (see e.g. [11]). Under the Super-$C^{(\infty)}$-LgLCAA for extendible, if j is a generic elementary embedding witnessing the statement of the axiom then $j \restriction {V_\lambda}^{\overline{\mathsf{W}}} : {V_\lambda}^{\overline{\mathsf{W}}} \xrightarrow{\prec}_\kappa {V_{j(\lambda)}}^{\overline{\mathsf{W}}}$ witnesses the virtual extendibility of κ in $\overline{\mathsf{W}}$. This implies that κ is inaccessible in $\overline{\mathsf{W}}$.

(4): Similarly to (3) we can prove that κ is virtually super-$C^{(\infty)}$-extendible in $\overline{\mathsf{W}}$. ${V_\kappa}^{\overline{\mathsf{W}}} \prec \overline{\mathsf{W}}$ follows from this. (see [11])

(B): This follows from Corollary 4.2, Theorem 5.2, and Theorem 5.3 in Fuchino, and Usuba [20]. ❑ (Lemma 3.2)

Let us assume that the assumptions of Lemma 3.2, (A) hold. I.e., we assume that the Super-$C^{(\infty)}$-LgLCAA for extendible holds, $\kappa = (2^{\aleph_0})^{\mathsf{V}}$, and there is an extendible cardinal.

$$\mathcal{MV} := \{{V_\kappa}^{\overline{\mathsf{W}}}[\mathbb{G}] : \mathbb{G} \text{ is a } (\overline{\mathsf{W}}, \mathbb{P})\text{-generic filter } \in \mathsf{V} \text{ for some } \mathbb{P} \in {V_\kappa}^{\overline{\mathsf{W}}}\},$$

and let $\mathcal{S}^{\mathcal{MV}} := \bigcup \mathcal{MV}$.

Theorem 3.3 *Assume that the assumptions of Lemma 3.2, (A) hold. Then, by Lemma 3.2, (A), we have* $\mathsf{MP}($*all posets*$, \mathcal{H}(2^{\aleph_0}))$ *and* BA. *For* $\kappa = (2^{\aleph_0})^{\mathsf{V}}$, *and the bedrock* $\overline{\mathsf{W}}$, *let* $\mathcal{MV}$, *and* $\mathcal{S}^{\mathcal{MV}}$ *be as above. Then we have:* p-multiv-1

$$\widetilde{\mathcal{MV}} := \langle \mathcal{S}^{\mathcal{MV}} \cup \mathcal{MV}, \ \in, \ \mathcal{S}^{\mathcal{MV}}, \ \mathcal{MV}\rangle \models \mathsf{MV}.$$

Proof. Note that by Lemma 3.2, (3), all elements of $\mathcal{MV}$ are models of ZFC. Thus $\widetilde{\mathcal{MV}} \models$(1.4), (a). Proposition 3.1, (1) implies $\widetilde{\mathcal{MV}} \models$(1.4), (b), and Proposition 3.1, (2) implies $\widetilde{\mathcal{MV}} \models$(1.4), (d).

It is clear that $\widetilde{\mathcal{MV}}$ satisfies the rest of the axioms. ❑ (Theorem 3.3)

The multiverse $\widetilde{\mathcal{MV}}$ is closely connected with the "real multiverse" over V.

First, note that ${V_\kappa}^{\overline{\mathsf{W}}}$ is the bedrock (of any element of $\mathcal{MV}$). By Lemma 3.2, (4), we have ${V_\kappa}^{\overline{\mathsf{W}}} \prec \overline{\mathsf{W}}$. In particular, the bedrock of $\widetilde{\mathcal{MV}}$ satisfies the same theory as that of the bedrock $\overline{\mathsf{W}}$ of V.

Now, suppose that

$$\mathsf{V} \models \ \Vdash_{\mathbb{Q}} \text{“}\varphi\text{” for an } \mathcal{L}_\in\text{-sentence } \varphi \text{ and a poset } \mathbb{Q} \in \mathsf{V}.$$

Let $\mathsf{V} = \overline{\mathsf{W}}[\mathbb{G}]$ for a $(\overline{\mathsf{W}}, \mathbb{P})$-generic $\mathbb{G} \in \mathsf{V}$ for some $\mathbb{P} \in \overline{\mathsf{W}}$, and let $\underset{\sim}{\mathbb{Q}} \in \overline{\mathsf{W}}$ be a $\mathbb{P}$-name of $\mathbb{Q}$ such that $\overline{\mathsf{W}} \models \ \Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}} \text{“}\varphi\text{”}$.

In V, let $\mathbb{R}$ be a poset that forces the continuum to be $> |\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,|$. Then we have

$$\mathsf{V} \models \ \Vdash_{\mathbb{R}} \text{“}\exists \underline{P} \in \overline{\mathsf{W}}\,(\,|\,\underline{P}\,| < 2^{\aleph_0} \ \wedge \ \overline{\mathsf{W}} \models \ \Vdash_{\underline{P}} \text{“}\varphi\text{”})\text{”}.$$

By $\mathsf{MP}(\text{all-posets}, \mathcal{H}(2^{\aleph_0}))$, there is a ground W of V such that

$$\mathsf{W} \models \exists \underline{P} \in \overline{\mathsf{W}}\,(\,|\,\underline{P}\,| < 2^{\aleph_0} \ \wedge \ \overline{\mathsf{W}} \models \ \Vdash_{\underline{P}} \text{“}\varphi\text{”}).$$

Since $(2^{\aleph_0})^{\mathsf{W}} \leq (2^{\aleph_0})^{\mathsf{V}}$, and $\overline{\mathsf{W}}$ is absolute between W and V, it follows that

$$\mathsf{V} \models \exists \underline{P} \in \overline{\mathsf{W}}\,(\,|\,\underline{P}\,| < 2^{\aleph_0} \ \wedge\ \overline{\mathsf{W}} \models \ \Vdash_{\underline{P}} \text{“}\varphi\text{”}).$$

Let $\mathbb{P}^* \in \overline{\mathsf{W}}$ be such that $|\,\mathbb{P}^*\,| < \kappa = (2^{\aleph_0})^{\mathsf{V}}$, and $\overline{\mathsf{W}} \models\ \Vdash_{\mathbb{P}^*} \text{“}\varphi\text{”}$. Without loss of generality, we may assume that $\mathbb{P}^* \in {V_\kappa}^{\overline{\mathsf{W}}}$.

By Proposition 3.1, (1), there is a $(\overline{\mathsf{W}}, \mathbb{P}^*)$-generic $\mathbb{G}^* \in \mathsf{V}$. We have $\mathsf{V}[\mathbb{G}^*] \models \varphi$. Thus

$$(3.1) \qquad {V_\kappa}^{\overline{\mathsf{W}}[\mathbb{G}^*]} \in \mathcal{MV} \quad \text{and} \quad {V_\kappa}^{\overline{\mathsf{W}}[\mathbb{G}^*]} \models \varphi$$ x-multiv-0

by Lemma 3.2, (4).

In the same situation as above, if we have $\mathbb{Q} \in \mathcal{P}^*$ for a class $\mathcal{P}^*$ of posets and $\mathsf{V} \models \Phi$ for an $\mathcal{L}_\in$-sentence Φ in addition to the conditions we assumed above, then a slight modification of the argument above shows that we can find $W^* \in \mathcal{MV}$ with $W^* \models \Phi$, $\mathbb{Q}^* \in W^*$ with $W^* \models \text{“}\mathbb{Q}^* \in \mathcal{P}^*$ and $|\,\mathbb{Q}^*\,| < \kappa\text{”}$, and $(W^*, \mathbb{Q}^*)$-generic $\mathbb{H}^* \in \mathcal{S}$ such that $W^*[\mathbb{H}^*] \models \varphi$. Note that $W^*[\mathbb{H}^*] \in \mathcal{MV}$.

4 Generic Laver diamonds

Suppose that κ is an uncountable regular cardinal. Recall that the *Diamond Principle for κ* (notation: $\diamondsuit_\kappa$) is the assertion saying that there is a sequence (called $\diamondsuit_\kappa$-sequence) $\langle a_\alpha : \alpha < \kappa \rangle$ such that $a_\alpha \subseteq \alpha$ for all $\alpha < \kappa$, and for any $X \subseteq \kappa$, the set $\{\alpha \in \kappa : X \cap \alpha = a_\alpha\}$ is stationary in κ. Laver-d

Laver diamond (also called Laver-function) *at κ for a notion LC of (large) large cardinal* is a mapping $f : \kappa \to V_\kappa$ such that for any set a, and $\lambda > \kappa$, there is an elementary embedding $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$ for some inner model $M \subseteq \mathsf{V}$ with $j(\kappa) > \lambda$ such that M satisfies the closure property corresponding to LC, and $j(f)(a) = a$ holds. Thus, if there is a Laver diamond at κ for LC then κ is a LC.

It is known that Laver diamond exists for most of large large cardinals (characterized in terms of elementary embedding) e.g. it is the case for supercompact, extendible, hyperhuge, etc.

Laver diamond at κ is a strengthening of $\diamondsuit_\kappa$-sequence: If $f : \kappa \to V_\kappa$ is a Laver diamond at κ for any notion LC of large cardinal, let $\langle a_\alpha : \alpha < \kappa \rangle$ be defined by

$$(4.1) \qquad a_\alpha := \begin{cases} f(\alpha); & \text{if } f(\alpha) \subseteq \alpha; \\ \emptyset; & \text{otherwise.} \end{cases}$$ x-Laver-d-0

Lemma 4.1 *The sequence $\langle a_\alpha : \alpha < \kappa \rangle$ defined in (4.1) is a $\diamondsuit_\kappa$ sequence. Thus the existence of a Laver diamond for κ implies $\diamondsuit_\kappa$.* p-Laver-d-0

Proof. Suppose $X \subseteq \kappa$. Then there is $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$ such that $j(f)(\kappa) = X$. Since $X = j(X) \cap \kappa$, it follows that $\kappa \in j(\{\alpha < \kappa : f(\alpha) = X \cap \alpha\}) = j(\{\alpha <$

$\kappa : a_\alpha = X \cap \alpha\}$). Thus $\{\alpha < \kappa : a_\alpha = X \cap \alpha\}$ is in the normal ultrafilter $\{U : U \subseteq \kappa, \kappa \in j(U)\}$. It follows that the set $\{\alpha < \kappa : a_\alpha = X \cap \alpha\}$ is stationary in κ as desired. □ (Lemma 4.1)

Actually the second half of Lemma 4.1 is superfluous since a result by Kunen says that κ has $\diamondsuit_\kappa$-sequence if κ is subtle (see Kanamori [30]).

We will examine several versions of generic Laver diamond that can live with a small cardinal, such as $2^{\aleph_0}$.

The most simple one is the following: Suppose that $\mathcal{P}$ is a class of iterable posets, and LC a notion of (large) large cardinal.

For an uncountable regular cardinal κ, *$\diamondsuit^{\mathcal{P},LC}_{Laver,\kappa}$-sequence* is a mapping $f : \kappa \to V_\kappa$ which satisfies:

(4.2) For any set a, and $\lambda > \kappa$, there is $\mathbb{Q} \in \mathcal{P}$ with $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$ such that there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, $j(\kappa) > \lambda$, j satisfying the closure property in **table 1.** corresponding to the LC, and $j(f)(\kappa) = a$. x-Laver-d-1

The *Generic Laver Diamond Principle* $\diamondsuit^{\mathcal{P},LC}_{Laver,\kappa}$ is the assertion of the existence of a $\diamondsuit^{\mathcal{P},LC}_{Laver,\kappa}$-sequence.

Laver diamond at κ is simply a $\diamondsuit^{\{\{\mathbb{1}\}\},LC}_{Laver,\kappa}$-sequence (for any notion LC of large cardinal).

A similar generic version of Laver diamond has been studied by Matteo Viale [46] and Sean Cox [4].

Lemma 4.2 *$\diamondsuit^{\mathcal{P},LC}_{Laver,\kappa}$ for any class $\mathcal{P}$ of posets implies $\diamondsuit_\kappa$.* *p-Laver-d-1*

Proof. Similarly to Lemma 4.1. Suppose that $f : \kappa \to V_\kappa$ is a $\diamondsuit^{\mathcal{P},LC}_{Laver,\kappa}$-sequence.

For $X \subseteq \kappa$ ($X \in \mathsf{V}$), there are $\mathbb{P} \in \mathcal{P}$, $\mathbb{G}$, j, and M as in the definition of $\diamondsuit^{\mathcal{P},LC}_{Laver,\kappa}$-sequence, such that $M \models j(f)(\kappa) = X$.

Then $j(\{\alpha < \kappa : f(\alpha) = X \cap \alpha\}) \ni \kappa$ since $j(X) \cap \kappa = X$.

It follows that $\mathsf{V} \models \{\alpha < \kappa : f(\alpha) = X \cap \alpha\}$ is stationary in κ.

Thus, by letting

$$a_\alpha := \begin{cases} f(\alpha), & \text{if } f(\alpha) \subseteq \alpha; \\ \emptyset, & \text{otherwise,} \end{cases}$$

$\langle a_\alpha : \alpha < \kappa \rangle$ is a $\diamondsuit_\kappa$-sequence. □ (Lemma 4.2)

Note that the proofs of Lemma 4.1 and Lemma 4.2 actually show that $\diamondsuit_\kappa(S)$ holds for $S := \{\alpha < \kappa : \mu \leq cf(\alpha)\}$ for all $\omega \leq \mu < \kappa$.

The proof of Proposition 4.3 below with slight modifications also proves the following Proposition 4.4 and Proposition 4.6.

Proposition 4.3 *Suppose that $\mathcal{P}$ is a Σ_2 transfinitely iterable class of posets containing a poset which provably adds a new real. If κ is an extendible cardinal, then there is a $\mathbb{P}_\kappa \in \mathcal{P}$ such that* p-Laver-d-2

$$\Vdash_{\mathbb{P}_\kappa} \text{“}\, 2^{\aleph_0} = \kappa,\ \kappa \textit{ is tightly } \mathcal{P}\textit{-Laver-gen. extendible, and } \diamondsuit^{\mathcal{P},\textit{extendible}}_{\textit{Laver},2^{\aleph_0}} \textit{ holds}\text{”}.$$

Proof. Assume that κ is an extendible cardinal, and $f : \kappa \to V_\kappa$ a Laver diamond for extendible at κ (see Corazza [3]).

Let $\vec{P} := \langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be an iteration in $\mathcal{P} \cap V_\kappa$ with the support suitable for $\mathcal{P}$ such that for $\beta < \kappa$:

$$\text{(4.3)} \qquad \underset{\sim}{\mathbb{Q}}_\beta := \begin{cases} \underset{\sim}{\mathbb{R}}_\beta, & \text{if } f(\beta) = \langle \underset{\sim}{\mathbb{R}}_\beta, \underset{\sim}{a}_\beta \rangle \text{ where } \underset{\sim}{\mathbb{R}}_\beta, \underset{\sim}{a}_\beta \text{ are } \mathbb{P}_\beta\text{-names and} \\ & \Vdash_{\mathbb{P}_\beta} \text{“}\underset{\sim}{\mathbb{R}}_\beta \in \mathcal{P}\text{”}; \\ \mathbb{P}_\beta\text{-name of the trivial forcing}, & \text{otherwise.} \end{cases}$$ x-Laver-d-2

Claim 4.3.1 $\mathbb{P}_\kappa$ *is as desired.* cl-Laver-d-0

$\vdash$ (a): $\Vdash_{\mathbb{P}_\kappa}$ “$\mathcal{P}$-LgLCA for extendible”: See e.g. the proof of Theorem 5.2 in Fuchino [9].

(b): $\Vdash_{\mathbb{P}_\kappa}$ “$\diamondsuit^{\mathcal{P},\textit{extendible}}_{\textit{Laver},\kappa}$”: Let $\underset{\sim}{g}$ be a $\mathbb{P}_\kappa$-name such that $\Vdash_{\mathbb{P}_\kappa}$ “$\underset{\sim}{g} : \kappa \to V_\kappa$” and

(4.4) If $f(\alpha) = \langle \underset{\sim}{\mathbb{R}}_\alpha, \underset{\sim}{a}_\alpha \rangle$ where $\underset{\sim}{\mathbb{R}}_\alpha$ and $\underset{\sim}{a}_\alpha$ are $\mathbb{P}_\alpha$-names, then $\Vdash_{\mathbb{P}_\kappa}$ “$\underset{\sim}{g}(\alpha) = \underset{\sim}{a}^*_\alpha$” where $\underset{\sim}{a}^*_\alpha$ is a $\mathbb{P}_\kappa$-name corresponding to the $\mathbb{P}_\alpha$-name $\underset{\sim}{a}_\alpha$. x-Laver-d-2-0

Then we have

Subclaim 4.3.1.1 $\Vdash_{\mathbb{P}_\kappa}$ “$\underset{\sim}{g}$ *is a* $\diamondsuit^{\mathcal{P},\textit{extendible}}_{\textit{Laver},\kappa}$*-sequence*”.

$\vdash$ Suppose that $\mathbb{G}_\kappa$ is a $(\mathsf{V}, \mathbb{P}_\kappa)$-generic filter. Let $X \in \mathsf{V}[\mathbb{G}_\kappa]$ and $\underset{\sim}{X}$ be a $\mathbb{P}_\kappa$-name of X.

Let $g = \underset{\sim}{g}[\mathbb{G}_\kappa]$. Then $g : \kappa \to {V_\kappa}^{\mathsf{V}[\mathbb{G}_\kappa]}$ and

$$\text{(4.5)} \qquad g(\alpha) = \begin{cases} \underset{\sim}{a}[\mathbb{G}_\alpha], & \text{if } f(\alpha) = \langle \underset{\sim}{\mathbb{Q}}_\alpha, \underset{\sim}{a}_\alpha \rangle \text{ for some } \underset{\sim}{\mathbb{Q}}_\alpha; \\ \emptyset, & \text{otherwise.} \end{cases}$$ x-Laver-d-2-0-0

Since f is a Laver diamond for extendible, there are j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, (4.6) : $V_{j(\lambda)} \in M$ and $j(f)(\kappa) = \langle \underset{\sim}{\mathbb{Q}}, \underset{\sim}{X} \rangle$ for some $\mathbb{P}_\kappa$-name $\underset{\sim}{\mathbb{Q}}$ such that $\Vdash_{\mathbb{P}_\kappa}$ “$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$”. x-Laver-d-2-1

Let $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \underset{\sim}{\mathbb{R}}$, and let $\mathbb{H}$ be $(\mathsf{V}[\mathbb{G}_\kappa], \underset{\sim}{\mathbb{R}}[\mathbb{G}_\kappa])$-generic. Let

$$\text{(4.7)} \qquad \tilde{j} : \mathsf{V}[\mathbb{G}_\kappa] \to \mathsf{V}[\mathbb{G}_\kappa * \mathbb{H}]; \;\; \underset{\sim}{a}[\mathbb{G}_\kappa] \;\mapsto\; j(\underset{\sim}{a})[\mathbb{G}_\kappa * \mathbb{H}].$$ x-Laver-d-2-2

Then, we have $\tilde{j} \supseteq j$, $\tilde{j} : \mathsf{V}[\mathbb{G}_\kappa] \xrightarrow{\preceq}_\kappa M[\mathbb{G}_\kappa * \mathbb{H}]$. Since $\mathbb{P}_\kappa$ has the κ-cc, $j(\mathbb{P}_\kappa)$ has the $j(\kappa)$-cc. Thus (4.6) implies that $V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{G}_\kappa * \mathbb{H}]} \in M[\mathbb{G}_\kappa][\mathbb{H}]$.

Also, we have $\tilde{j}(g)(\kappa) = j(\underset{\sim}{g})[\mathbb{G}_\kappa][\mathbb{H}](\kappa) = \underset{\sim}{X}[\mathbb{G}_\kappa] = X$ by (4.5) and (4.7).

Thus $V[\mathbb{G}_\kappa] \models$ "g is a $\diamondsuit^{\mathcal{P},extendible}_{Laver,\kappa}$-sequence". ⊣ (Subclaim 4.3.1.1)

⊣ (Claim 4.3.1)

❑ (Proposition 4.3)

The super-$C^{(n)}$-version of the Generic Laver Diamond Principle can be also introduced.

Let $n \in \mathbb{N}$, and let LC be a notion of large cardinal. For an uncountable regular cardinal κ, *$\diamondsuit^{+n,\mathcal{P},LC}_{Laver,\kappa}$-sequence* is a mapping $f : \kappa \to V_\kappa$ which satisfies:

(4.8) For any set a, and $\lambda_0 > \kappa$, there are $C^{(n)}$-cardinal $\lambda > \lambda_0$, $\mathbb{Q} \in \mathcal{P}$ with $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$ such that there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, $j(\kappa) > \lambda$, j satisfying the closure property in table 1 corresponding to the LC, $j(\lambda) \in (C^{(n)})^{\mathsf{V}[\mathbb{H}]}$, and $j(f)(\kappa) = a$. x-Laver-d-3

The *Super-$C^{(n)}$-Generic Laver Diamond Principle* $\diamondsuit^{+n,\mathcal{P},LC}_{Laver,\kappa}$ is the assertion of the existence of a $\diamondsuit^{+n,\mathcal{P},LC}_{Laver,\kappa}$-sequence.

The *Super-$C^{(\infty)}$-Generic Laver Diamond Principle* $\diamondsuit^{+\infty,\mathcal{P},LC}_{Laver}$ is then the assertion that $\diamondsuit^{+n,\mathcal{P},LC}_{Laver,2^{\aleph_0}}$ holds for all $n \in \mathbb{N}$.

Proposition 4.4 (1) *Suppose that $\mathcal{P}$ is a Σ_n transfinitely iterable class of posets containing a poset which provably adds a new real. If κ is a super-$C^{(n^*)}$-extendible cardinal for sufficiently large $n^* > n$, then there is a $\mathbb{P}_\kappa \in \mathcal{P}$ such that* p-Laver-d-3

$$\Vdash_{\mathbb{P}_\kappa} \text{“} 2^{\aleph_0} = \kappa,\ \kappa \textit{ is tightly super-}C^{(n)}\textit{-}\mathcal{P}\textit{-Laver-gen. extendible, and } \diamondsuit^{+n,\mathcal{P},extendible}_{Laver,2^{\aleph_0}} \textit{ holds”}.$$

(2) *Suppose that μ is an inaccessible cardinal and $\mathcal{P}$ is a transfinitely iterable class of posets containing a poset which provably adds a new real.*
If $V_\mu \models$ "κ is a super-$C^{(\infty)}$-extendible cardinal", then there is a $\mathbb{P}_\kappa \in \mathcal{P}^{V_\mu}$ such that

$$V_\mu \models\ \Vdash_{\mathbb{P}_\kappa} \text{“} 2^{\aleph_0} = \kappa,\ \kappa \textit{ is tightly super-}C^{(\infty)}\textit{-}\mathcal{P}\textit{-Laver-gen. extendible, and } \diamondsuit^{+\infty,\mathcal{P},extendible}_{Laver,2^{\aleph_0}} \textit{ holds”}.$$

Proof. Similarly to the proof of Proposition 4.3 using Lemma 5.1, (2) in Fuchino [9]. See the proof of Theorem 5.2, (2) in [9] for more details. ❑ (Proposition 4.4)

It is possible to further infuse the generic Laver diamond with the resurrection features of Laver-generic large cardinals.

For an $n \in \mathbb{N}$, an iterable class $\mathcal{P}$ of posets, a notion LC of large cardinal, and for an uncountable regular cardinal κ, *$\diamondsuit^{++n,\mathcal{P},LC}_{Laver,\kappa}$-sequence* is a mapping $f : \kappa \to V_\kappa$ which satisfies:

(4.9) For any set a, any $\mathbb{P} \in \mathcal{P}$ and $\lambda_0 > \kappa$, there are a cardinal $\lambda > \lambda_0$ with $\lambda \in C^{(n)}$, a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", and $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$ such that there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$, $j(\kappa) > \lambda$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, j satisfying the closure property in table 1 corresponding to the LC, $|\,\mathsf{RO}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| \leq j(\kappa)$, $j(\lambda) \in (C^{(n)})^{\mathsf{V}[\mathbb{H}]}$, and $j(f)(\kappa) = a$. x-Laver-d-4

The *Super-$C^{(n)}$-Generic Laver Diamond Principle^{++} $\diamondsuit^{++n,\mathcal{P},LC}_{Laver,\kappa}$* is the assertion of the existence of a $\diamondsuit^{++n,\mathcal{P},LC}_{Laver,\kappa}$-sequence.

The *Super-$C^{(\infty)}$-Generic Laver Diamond Principle^{++} $\diamondsuit^{++\infty,\mathcal{P},LC}_{Laver}$* is then the assertion that $\diamondsuit^{++n,\mathcal{P},LC}_{Laver,2^{\aleph_0}}$ holds for all $n \in \mathbb{N}$.

As it is clear from the definition, super-$C^{(\infty)}$-Generic Laver Diamond Principles with $^{++}$ is are amalgamations of strong diamond principles and Super-$C^{(\infty)}$-LgLCAs.

For a class $\mathcal{P}$ of posets, let

($\mathsf{MA}^{**}(\mathcal{P})$): For any $\mathbb{P} \in \mathcal{P}$ and $\langle D_\alpha : \alpha < \mu \rangle$, $\langle \underset{\sim}{S}_\beta : \beta < \nu \rangle$ where μ, $\nu < \kappa_{\mathfrak{refl}}$, $D_\alpha \subseteq \mathbb{P}$ is dense subset of $\mathbb{P}$ for all $\alpha < \mu$ and $\underset{\sim}{S}_\beta$ is a $\mathbb{P}$-name of a stationary subset of $[\lambda_\beta]^{\aleph_0}$ for some uncountable $\lambda_\beta < \kappa_{\mathfrak{refl}}$, for all $\beta < \nu$, there is a filter $\mathbb{G} \subseteq \mathbb{P}$ such that $\mathbb{G} \cap D_\alpha \neq \emptyset$ for all $\alpha < \mu$ and $\underset{\sim}{S}_\beta[\mathbb{G}]$ is a stationary subset of $[\lambda_\beta]^{\aleph_0}$ for all $\beta < \nu$ (Fuchino [9]).

Note that, if $\kappa_{\mathfrak{refl}} = \aleph_2$, $\mathsf{MA}^{**}(\mathcal{P})$ is simply equivalent to the usual $\mathsf{MA}^{++}(\mathcal{P})$.

Proposition 4.5 *Assume that LC is a notion of large cardinal which implies extendibility.* p-Laver-d-4 *(1) If $\mathcal{P}$ is ω_1-preserving, transfinitely iterable and it provably adds a real, then $\diamondsuit^{++0,\mathcal{P},LC}_{Laver}$ implies ① $\neg\mathsf{CH}$, ② $\diamondsuit_{2^{\aleph_0}}(S)$ for $S = \{\alpha : \mu \leq cf(\alpha)\}$ for all $\mu < 2^{\aleph_0}$, ③ the $\mathcal{P}$-LgLCA for LC, and ④ $\mathcal{P}$ is stationary preserving. In particular ⑤ $\mathsf{MA}^{++}(\mathcal{P}, {<}\, 2^{\aleph_0})$ holds, and ⑥ if $\mathcal{P}$ consists of proper posets then $\mathsf{MA}^{**}(\mathcal{P})$ holds.*

(2) If $\mathcal{P}$ is ω_1-preserving and iterable, and provably adds a real, then $\diamondsuit^{++\infty,\mathcal{P},LC}_{Laver}$ implies the Super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for LC holds in addition to ① $\sim$ ⑥ in (1).

(3) If $\mathcal{P}$ provably contains a poset which collapses $\aleph_1$, then $\diamondsuit^{++0,\mathcal{P},LC}_{Laver}$ implies $\diamondsuit_{\omega_1}$.

(4) If $\diamondsuit^{++\infty,\mathcal{P},all\ posets}_{Laver}$ implies, $\diamondsuit_{\omega_1}$, and the Super-$C^{(\infty)}$-LgLCAA.

Proof. (1), ① : $\diamondsuit^{++0,\mathcal{P},LC}_{Laver}$ implies that the continuum is tightly $\mathcal{P}$-Laver-generic extendible. Clearly this is not compatible with CH since, if CH would hold, ω_1^{V}

should be mapped to ${\omega_1}^M$ where we have ${\omega_1}^{\mathsf{V}} \leq {\omega_1}^M \leq {\omega_1}^{\mathsf{V}[\mathbb{H}]} = {\omega_1}^{\mathsf{V}}$. This is in contradiction to the requirement that ω_1 should be the critical point of the generic elementary embedding.

② : $\diamondsuit^{++0,\mathcal{P},LC}_{Laver}$ implies $\diamondsuit_{2^{\aleph_0}}(S)$ for $S = \{\alpha : \mu \leq cf(\alpha)\}$ for all $\mu < 2^{\aleph_0}$ by Lemma 4.2 and the remark after its proof.

③ : is clear from ① and the definitions involved.

④ : By Theorem 4.3 in Fuchino [9], and Lemma 5.1, (3).

⑤ : By ④ and Theorem 5.7 in Fuchino, Ottenbreit Maschio Rodrigues, and Sakai [16].

⑥ : By Theorem 6.4 in Fuchino [9].

(2): By (1), ①, we have $2^{\aleph_0} = \kappa_{\mathfrak{refl}}$. Thus by definition, $\diamondsuit^{++\infty,\mathcal{P},LC}_{Laver}$ implies the Super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for LC.

(3): $\diamondsuit^{++0,\mathcal{P},LC}_{Laver}$ implies that $2^{\aleph_0}$ is tightly $\mathcal{P}$-Laver gen. LC. Thus by Lemma 2.1, (1) , we have CH. By Lemma 4.2, it follows that $\diamondsuit_{\omega_1}$ holds.

(4): is a special case of (3) (and its proof). ❑ (Proposition 4.5)

Note that (1) and (2) in Proposition 4.4 for $\mathcal{P}$ = all c.c.c. posets give an answer to the question about $\diamondsuit_{2^{\aleph_0}}$ under Martin's axiom with large continuum mentioned in Section 1. Note also that (3) and (4) provide a $\mathsf{V} \neq \mathsf{L}$ scenario of a strengthening of $\diamondsuit_{\aleph_1}$.

The proof of Proposition 4.4 above actually proves the following proposition.

Proposition 4.6 (1) *Suppose that $\mathcal{P}$ is a Σ_n transfinitely iterable class of posets containing a poset which provably adds a new real. If κ is a super-$C^{(n^*)}$-extendible cardinal for sufficiently large $n^* > n$, then there is a $\mathbb{P}_\kappa \in \mathcal{P}$ such that* p-Laver-d-5

$$\Vdash_{\mathbb{P}_\kappa} \text{“}\, 2^{\aleph_0} = \kappa,\ \diamondsuit^{++n,\mathcal{P},extendible}_{Laver,2^{\aleph_0}} \text{ holds”}.$$

(2) *Suppose that μ is an inaccessible cardinal and $\mathcal{P}$ is a transfinitely iterable class of posets containing a poset which provably adds a new real.*
If $V_\mu \models$ “κ is a super-$C^{(\infty)}$-extendible cardinal”, then there is a $\mathbb{P}_\kappa \in \mathcal{P}^{V_\mu}$ such that

$$V_\mu \models \Vdash_{\mathbb{P}_\kappa} \text{“}\, 2^{\aleph_0} = \kappa,\ \kappa \text{ is tightly super-}C^{(\infty)}\text{-}\mathcal{P}\text{-Laver-gen. extendible, and } \diamondsuit^{++\infty,\mathcal{P},extendible}_{Laver,2^{\aleph_0}} \text{ holds”}.$$

❑

5 Laver Generic Maximums revisited

LGM

In Fuchino, and Usuba [43], the following weakening of Maximality Principle is named Recurrence Axiom. For an iterable class $\mathcal{P}$ of posets and a set S the *Recurrence Axiom for $\mathcal{P}$ and S* (notation: $(\mathcal{P}, S)$-RcA) is the statement:

(5.1) For any formula $\varphi = \varphi(\overline{x})$, $\mathbb{P} \in \mathcal{P}$ and $\overline{a} \in S$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$" then there is a ground W of V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$. x-max-0

The difference between Maximality Principle (defined by (2.5)) and Recurrence Axiom is merely that while Maximality Principle claims the existence of a $\mathcal{P}$-ground, Recurrence Axiom only claims the existence of a ground (which might not be a $\mathcal{P}$-ground).

For a set Γ of formulas, the Recurrence Axiom restricted to Γ (notation: $(\mathcal{P}, S)_\Gamma$-RcA) is defined as expected:

(5.2) For any formula $\varphi = \varphi(\overline{x}) \in \Gamma$, $\mathbb{P} \in \mathcal{P}$ and $\overline{a} \in S$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$" then there is a ground W of V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$. x-max-1

The same type of restriction of (2.5) to Γ is called $(\mathcal{P}, S)_\Gamma$-RcA^+.

$(\mathcal{P}, S)_{\Sigma_1}$-$\mathsf{RcA}$ or $(\mathcal{P}, S)_{\Sigma_1}$-$\mathsf{RcA}^+$ for $S = \mathcal{H}(\kappa_{\mathfrak{refl}})$ or $S = \mathcal{H}(2^{\aleph_0})$ decides the size of the continuum for may natural classes $\mathcal{P}$ of posets. This fact was already used in the proof of Proposition 4.5. (5) and (5') of the next lemma were already mentioned with proofs in Section 2 as Lemma 2.1, (1), (2).

Lemma 5.1 (Theorem 3.3 in Fuchino and Usuba [20], see also Lemma 20 in the extended version of [8]) *Assume that $\mathcal{P}$ is an iterable class of posets.* (1) *If $\mathcal{P}$ contains a poset which adds a real (over the universe), then $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-$\mathsf{RcA}$ implies $\neg\mathsf{CH}$.* p-Lg-RcA-0-1

(2) *Suppose that $\mathcal{P}$ contains a poset which forces ${\aleph_2}^{\mathsf{V}}$ to be equinumerous with ${\aleph_1}^{\mathsf{V}}$. Then $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-$\mathsf{RcA}$ implies $2^{\aleph_0} \leq \aleph_2$.*

(2′) *If $\mathcal{P}$ contains a posets which forces ${\aleph_2}^{\mathsf{V}}$ to be equinumerous with ${\aleph_1}^{\mathsf{V}}$, then $(\mathcal{P}, \mathcal{H}((\aleph_2)^+))_{\Sigma_1}$-$\mathsf{RcA}$ does not hold.*

(3) *If $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-$\mathsf{RcA}$ holds then all $\mathbb{P} \in \mathcal{P}$ preserve $\aleph_1$ and they are also stationary preserving.*

(4) *If $\mathcal{P}$ contains a poset which adds a real as well as a poset which collapses ${\aleph_2}^{\mathsf{V}}$, then $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-$\mathsf{RcA}$ implies $2^{\aleph_0} = \aleph_2$.*

(5) *If $\mathcal{P}$ contains a poset which collapses ${\aleph_1}^{\mathsf{V}}$, then $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-$\mathsf{RcA}$ implies CH.*

(5′) *If $\mathcal{P}$ contains a poset which collapses ${\aleph_1}^{\mathsf{V}}$ then $(\mathcal{P}, \mathcal{H}((2^{\aleph_0})^+))_{\Sigma_1}$-$\mathsf{RcA}$ does not hold.*

(6) *Suppose that all* $\mathbb{P} \in \mathcal{P}$ *preserve cardinals and* $\mathcal{P}$ *contains posets adding at least* κ *many reals for each* $\kappa \in \mathit{Card}$. *Then* $(\mathcal{P}, \emptyset)_{\Sigma_2}$-$\mathsf{RcA}^+$ *implies that* $2^{\aleph_0}$ *is very large.*

(6′) *Suppose that* $\mathcal{P}$ *is as in* (6). *Then* $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_2}$-$\mathsf{RcA}^+$ *implies that* $2^{\aleph_0}$ *is a limit cardinal. Thus if* $2^{\aleph_0}$ *is regular in addition, then* $2^{\aleph_0}$ *is weakly inaccessible.* ❑

The next lemma is an easy corollary of Lemma 5.1.

Let *all*, *proper*, *semi-proper*, and *c.c.c.* denote the class of all posets, the class of all proper posets, class of all semi-proper posets, and the class of all c.c.c. posets, respectively.

Lemma 5.2 (1) $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_2}$-$\mathsf{RcA}$ *for* $\mathcal{P}$ = *proper or* $\mathcal{P}$ = *semi-proper implies* $2^{\aleph_0} = \aleph_2$. *p-max-a*

(2) $(\mathit{c.c.c.}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_2}$-$\mathsf{RcA}^+$ *implies the continuum is weakly inaccessible.*

Proof. (1): By Lemma 5.1, (4).

(2): By Lemma 5.1, (6′). ❑ (Lemma 5.2)

An advantage of Recurrence Axiom over Maximality Principle is that it satisfies the following monotonicity lemma which is trivial by itself but makes comparisons between variations of the axiom much easier.

Lemma 5.3 (Monotonicity of Recurrence Axioms) *For classes of posets* $\mathcal{P}$, $\mathcal{P}'$ *and sets* A, A' *of parameters, if* $\mathcal{P} \subseteq \mathcal{P}'$ *and* $A \subseteq A'$, *then we have* *p-max-0*

$$(\mathcal{P}', A')\text{-}\mathsf{RcA} \;\Rightarrow\; (\mathcal{P}, A)\text{-}\mathsf{RcA}.$$ ❑

If we have decided that the Recurrence Axiom is a desirable or even indispensable essential feature of our set-theoretic universe, we would want to postulate a maximum amount of it. From this point of view, and guided by Lemma 5.1, three scenarios emerge:

(A) $(\mathit{all}, \mathcal{H}(2^{\aleph_0}))$-$\mathsf{RcA}$: *all* is apparently maximal among all transfinitely iterable families of posets. Note that CH follows from this Recurrence Axiom, see Lemma 2.1, (1).*

(B) $(\mathit{all}, \emptyset)$-$\mathsf{RcA}$ + $(\mathit{semi\text{-}proper}, \mathcal{H}(2^{\aleph_0}))$-$\mathsf{RcA}$: *semi-proper* is maximal among transfinitely iterable classes of posets which is stationary preserving (and makes $\mathsf{MA}(\mathcal{P}, {<}\aleph_2)$ consistent, cf. (B′) below).
Note that $(\mathit{semi\text{-}proper}, \mathcal{H}(\kappa_{\mathfrak{refl}}))$-$\mathsf{RcA}$ implies $2^{\aleph_0} = \aleph_2$, see Lemma 5.2, (1).*

(Γ) $(\mathit{all}, \emptyset)$-$\mathsf{RcA}$ + $(\mathit{c.c.c.}, \mathcal{H}(2^{\aleph_0}))$-$\mathsf{RcA}$ for $\mathcal{P}$ = all c.c.c. posets: *c.c.c.* is maximal(?) among transfinitely iterable classes of posets which are proper and preserving all cardinals. Note that $(\mathit{c.c.c.}, \mathcal{H}(2^{\aleph_0}))$-$\mathsf{RcA}^+$ implies $2^{\aleph_0}$ is weakly compact, see Lemma 5.2, (2).*

The statements marked by * suggest that natural strengthenings of the trichotomy (A), (B) and (Γ) should lead to a trichotomy solution of the continuum problem: the continuum is either $\aleph_1$, or $\aleph_2$, or else very large (at least weakly inaccessible).

The following Theorem together with Lemma 5.1 shows that this trichotomy of the continuum hypothesis is already established by a slight extension of (A), (B) and (Γ) .

Theorem 5.4 (Theorem 6.1 in Fuchino, Gappo and Parente [12] reformulated under LgLCAs for extendible, see Theorem 4.3 in Fuchino [9]) (1) *Suppose that $\mathcal{P}$ is an iterable class of posets, and assume that the $\mathcal{P}$-LgLCA for extendible holds. Then $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_\Gamma$-RcA$^+$ holds where Γ is the set of all formulas which are conjunctions of a Σ_2-formula and a Π_2-formula.* p-Lg-RcA-0

(2) *Assume that LgLCAA for extendible holds. Then we have $(\mathcal{P}, 2^{\aleph_0})_\Gamma$-RcA$^+$ for Γ as in* (1). ❑

Starting from (A), (B) and (Γ) , the following combinations of strongest forms of LgLCAs are identified which imply these alternatives:

(A$^+$) The Super-$C^{(\infty)}$-LgLCAA for hyperhuge.

(B$^+$) "For each $n \in \mathbb{N}$ there is a *semi-proper*-ground W of the universe V such that $(2^{\aleph_0})^{\mathsf{W}} = {\omega_1}^{\mathsf{W}}$ $(2^{\aleph_0})^{\mathsf{W}} = {\omega_1}^{\mathsf{W}} = {\omega_1}^{\mathsf{V}}$ is the tightly super-$C^{(n)}$-all-posets-Laver gen. hyperhuge cardinal in W" + the Super-$C^{(\infty)}$-*semi-proper*-LgLCA for hyperhuge.

(Γ$^+$) "For each $n \in \mathbb{N}$ there is a *c.c.c.*-ground W of the universe V such that $(2^{\aleph_0})^{\mathsf{W}} = {\omega_1}^{\mathsf{W}}$ $(2^{\aleph_0})^{\mathsf{W}} = {\omega_1}^{\mathsf{W}} = {\omega_1}^{\mathsf{V}}$ is the tightly super-$C^{(n)}$-all-posets-Laver gen. hyperhuge cardinal in W" + the Super-$C^{(\infty)}$-*c.c.c.*-LgLCA for hyperhuge.

All of these axiom schemas and all variations of them treated in this section have consistency strength (much) less than that of the existence of a 2-huge cardinal. By Theorem 2.5 and its variation (Theorem 5.8 in [20]), the exact consistency strengths of most of these axioms in terms of corresponding genuine large cardinals are known.

In Fuchino, and Usuba [20], these three axiom schemas are called *Laver Generic Maximums* (LGMs) because of the richness of the consequences of each of these axiom schemas. They imply strong forms of "desirable" axioms and principles such as Absoluteness Theorems, Maximality Principles, and Recurrence Axioms for the respective class of posets (see Fuchino [8], Fuchino, Usuba [20] and Fuchino [9]).

(B$^+$) and (Γ^+) also imply their versions of strong Forcing Axioms and Reflection Principles[1)] (see Fuchino, Ottenbreit Maschio Rodrigues, and Sakai [16].).

Even if some (consistent) mathematical statement does not follow from one of the LGMs or even incompatible with it, the maximal amount of Recurrence Axioms available under the LGM (see below) and Maximality Principles which extend them, imply in most of the cases that there are grounds of V (often even $\mathcal{P}$-grounds of V for respective $\mathcal{P}$) which satisfy the statement.

By Theorem 2.6, all of the LGMs imply BA. They also imply corresponding Maximality Principles in (A), (B) and (Γ) by Theorem 2.3. In particular, denoting the bedrock (under BA) by $\overline{\mathsf{W}}$, we have

Proposition 5.5 (1) *(A$^+$), (B$^+$) and (Γ^+) imply (A), (B) and (Γ) respectively.* p-max-1

(2) *Each of (B$^+$) and (Γ^+) implies $\mathsf{MP}(all, \mathcal{H}({\omega_1}^{\mathsf{V}})^{\overline{\mathsf{W}}})$.*

(3) *(A$^+$) admits the construction of the multiverse in grounds discussed in Section 3.*

(4) *(A$^+$), (B$^+$) and (Γ^+) imply CH, $2^{\aleph_0} = \aleph_2$, and $2^{\aleph_0}$ is weakly Mahlo, respectively.*

Proof. (1), (2): By Theorem 2.3. (3): see Section 3. (4): By Theorem 2.3 and Lemma 5.1. ❑ (Proposition 5.5)

The findings in Section 4 invite us to update the LGMs to the trichotomy of strong generic Laver diamonds:

(A^{++}) $\diamondsuit^{++\infty, all, hyperhuge}_{Laver, 2^{\aleph_0}}$.

(B^{++}) "For each $n \in \mathbb{N}$ there is a *semi-proper*-ground W of the universe V such that $\diamondsuit^{++n, all, hyperhuge}_{Laver, 2^{\aleph_0}}$ holds in W" + $\diamondsuit^{++\infty, semi\text{-}proper, hyperhuge}_{Laver, 2^{\aleph_0}}$.

(Γ^{++}) "For each $n \in \mathbb{N}$ there is a *semi-proper*-ground W of the universe V such that $\diamondsuit^{++n, all, hyperhuge}_{Laver, 2^{\aleph_0}}$ holds in W" + $\diamondsuit^{++\infty, c.c.c., hyperhuge}_{Laver, 2^{\aleph_0}}$.

There is another trichotomy in terms of very strong reflection principles (in the sense of footnote 1)) mentioned implicitly in Fuchino, Ottenbreit Maschio Rodrigues, and Sakai [16]. For this trichotomy, (A^{++}) should be replaced by

(A$'^{++}$) $\diamondsuit^{++\infty, \mathcal{P}, hyperhuge}_{Laver, \kappa_{\mathfrak{refl}}}$

for a transfinitely iterable $\mathcal{P}$ which consists of all σ-closes posets. The rational of this choice of $\mathcal{P}$ is the Game Reflection Principle of Bernhard Koenig in [32] (the

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1) With "Reflection Principles", we mean here such mathematical structural reflection principle down to $\kappa_{\mathfrak{refl}}$ as the statements discussed e.g. in [13] or [15]. refl-principles

principle called global Game Reflection Principle and denoted by GRP^+ in [32]) which is a very strong reflection principle down to $<\aleph_2$ characterized by $\aleph_2$ being a $\mathcal{P}$-generically supercompact cardinal (see also [15]).

Note that (A'^{++}) also implies CH (see Fuchino [8], Lemma 6).

It may be more natural to leave the two trichotomies, and to see instead, that the four sets of axioms (A^{++}), (A'^{++}), (B^{++}), (Γ^{++}) represent the tetrachotomy of significant universes over the bedrock.

In terms of the trichotomy of maximal reflection axioms, (Γ^{++}) does not possess the desired maximality since the reflection principle FRP (introduced in Fuchino, Juhász, Soukup Szentmiklóssy, and Usuba [13]) is still independent from it (see Proposition 6.7 in Fuchino [9]). FRP is known to be equivalent to many natural mathematical reflection statement about the structural reflection down to $<\aleph_2$ which follows from (A'^{+}) and (B^{+}). So we also want to retain this axiom under the extension of (Γ).

In Fuchino, and Usuba [20], the axiom suggested for the scenario corresponding to (Γ^{++}) which also imply FRP can be updated in our context with strong generic Laver diamonds as:

(Γ'$^{++}$) "For each $n \in \mathbb{N}$ there is a *semi-proper*-ground W of the universe V such that $\diamondsuit^{++n,all,hyperhuge}_{Laver,2^{\aleph_0}}$ holds in W, and a *c.c.c.*-ground W' of the universe V such that $\diamondsuit^{++n,semi\text{-}proper,hyperhuge}_{Laver,2^{\aleph_0}}$ holds in W'" + $\diamondsuit^{++\infty,c.c.c.,hyperhuge}_{Laver,2^{\aleph_0}}$.

Lemma 5.6 (Γ'$^{++}$) *implies* FRP. p-max-1-0

Proof. $\mathsf{W}' \models \mathsf{FRP}$ since $\mathsf{W}' \models \mathsf{MA}^+(\sigma\text{-closed})$ and FRP follows from $\mathsf{MA}^{++}(\sigma\text{-closed})$. In [13], it is shown that FRP is preserved under c.c.c.-forcing. Thus we have $\mathsf{V} \models \mathsf{FRP}$. ❑ (Lemma 5.6)

There are still many open problems in connection with the tetrachotomy (A^{++}), (A'^{++}), (B^{++}), (Γ'$^{++}$) . At the moment we even do not know if (X^{++}) already follows from (X^{+}) for $X = \mathrm{A}, \mathrm{A}', \mathrm{B}$, or Γ.

The following is a partial result to the question above.

Theorem 5.7 (1) *Suppose that $\mathcal{P}$ is stationary preserving iterable class of posets which provably contains a forcing adding a real. Then $\mathcal{P}$-LgLCA for hyperhuge implies $\diamondsuit^{all,hyperhuge}_{Laver,2^{\aleph_0}}$. In particular $\diamondsuit_{2^{\aleph_0}}(S)$ for $S = \{\alpha < 2^{\aleph_0} : cf(\alpha) \geq \mu\}$ for all $\omega \leq \mu < 2^{\aleph_0}$ hold under this axiom.* p-max-2

(2) LgLCAA *for hyperhuge implies $\diamondsuit^{all,hyperhuge}_{Laver,2^{\aleph_0}}$. In particular $\diamondsuit_{2^{\aleph_0}}$ holds under this axiom.*

For the proof of Theorem 5.7, we use the following:

Theorem 5.8 ([11], Lemma 20.12) *Suppose that κ is a tightly $\mathcal{P}$-generic extendible cardinal (for any family $\mathcal{P}$ of posets), and the bedrock $\overline{\mathsf{W}}$ exists. Then V is a generic extension of $\overline{\mathsf{W}}$ by a poset $\mathbb{P}^* \in \overline{\mathsf{W}}$ that is of cardinality $\leq \kappa$ and satisfies $<\kappa$-c.c.* p-ext-bedrock-3-a

Proof. Let $\mathbb{P} \in \overline{\mathsf{W}}$ be such that there is $(\overline{\mathsf{W}}, \mathbb{P})$-generic $\mathbb{G} \in \mathsf{V}$ with $\mathsf{V} = \overline{\mathsf{W}}[\mathbb{G}]$. Without loss of generality, we may assume that the underlining set of $\mathbb{P}$ is the cardinal $|\,\mathbb{P}\,|$. Let

(5.3) $\lambda > \kappa$, $|\,\mathbb{P}\,|$ be a limit of inaccessibles such that $\lambda \in C^{(n^*)}$ for a sufficiently large $n^* \in \mathbb{N}$. x-ext-bedrock-9-0-0

Note that there is such λ since κ is virtually extendible in $\overline{\mathsf{W}}$ hand hence there are class many inaccessible cardinals (in $\overline{\mathsf{W}}$ and hence also in V).

Since κ is tightly $\mathcal{P}$-generic extendible, there are $\mathbb{Q}$, $\mathbb{H}$, j, M such that $\mathbb{Q} \in \mathsf{V}$ is a poset, $\mathbb{H}$ is $(\mathsf{V}, \mathbb{Q})$-generic, j, $M \subseteq \mathsf{V}[\mathbb{H}]$. $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{Q}$, $\mathbb{H} \in M$,

(5.4) ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$, and x-ext-bedrock-9-1

(5.5) $|\,\mathsf{RO}(\mathbb{Q})\,|^{\mathsf{V}} \leq j(\kappa)$. x-ext-bedrock-10

Note that $j(\lambda)$ is a limit of inaccessible in M by the elementarity of j, and hence

(5.6) $j(\lambda)$ is a limit of inaccessibles in $\mathsf{V}[\mathbb{H}]$ x-ext-bedrock-11

by (5.4).

By (5.5), $|\,\mathbb{Q}\,| \leq j(\kappa)$ and $\mathbb{Q}$ has the $<j(\kappa)$-c.c. It follows that $\mathsf{V}[\mathbb{H}]$ is a generic extension of $\overline{\mathsf{W}}$ by a poset in ${V_{j(\lambda)}}^{\mathsf{V}}$ of size $\leq j(\kappa)$ and with $<j(\kappa)$-c.c. Thus, by (5.6), we have

$$
{V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \models \text{"the universe is a generic extension of the bedrock by a forcing of size } \leq j(\kappa) \text{ and with the } j(\kappa)\text{-c.c."}.
$$

We have ${V_{j(\lambda)}}^{M} = {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]}$ by (5.4). Thus,

$$
{V_{j(\lambda)}}^{M} \models \text{"the universe is a generic extension of the bedrock by a forcing of size } \leq j(\kappa) \text{ and with the } j(\kappa)\text{-c.c."}.
$$

Since ${V_{j(\lambda)}}^{M} \prec_{\Sigma_{n^*}} M$ by (5.3) and by the elementarity of j, it follows that

$$
M \models \text{"the universe is a generic extension of the bedrock by a forcing of size } \leq j(\kappa) \text{ and with the } j(\kappa)\text{-c.c."}.
$$

Again by the elementarity of j, we finally obtain

$$
\mathsf{V} \models \text{"the universe is a generic extension of the bedrock by a forcing of size } \leq \kappa \text{ and with the } \kappa\text{-c.c."}.
$$

□ (Theorem 5.8)

Lemma 5.9 *Let LC be one of the large cardinal properties in table 1. Suppose that κ is LC and $f:\kappa\to V_\kappa$ is a Laver diamond for LC. If $\mathbb{P}$ is a κ-c.c. poset of size κ, then we have* p-max-3

$$\Vdash_{\mathbb{P}} \text{“}\diamondsuit^{all,LC}_{Laver,\kappa}\text{”}.$$

Proof. Suppose that κ is LC and $f:\kappa\to {V_\kappa}^{\mathsf{V}}$ is a Laver diamond for LC κ. Suppose also that $\mathbb{G}$ is a $(\mathsf{V},\mathbb{P})$-generic filter. Without loss of generality, we may assume that (5.7): the underlying set of $\mathbb{P}$ is κ. x-max-2

Let $f_0:\kappa\to {V_\kappa}^{\mathsf{V}}$ be the mapping (in V) defined by

$$f_0(\alpha):=\begin{cases} f(\alpha), & \text{if } f(\alpha) \text{ is a } \mathbb{P}\text{-name;}\\ \emptyset, & \text{otherwise,}\end{cases}$$

and let $\tilde f:\kappa\to {V_\kappa}^{\mathsf{V}[\mathbb{G}]}$ be the mapping (in $\mathsf{V}[\mathbb{G}]$) defined by

$$\tilde f(\alpha):=f_0(\alpha)[\mathbb{G}].$$

We show that $\tilde f$ defined as above, is a $\diamondsuit^{all,LC}_{Laver,\kappa}$-sequence in $\mathsf{V}[\mathbb{G}]$.

Suppose $a\in\mathsf{V}[\mathbb{G}]$ and $a=\underset{\sim}{a}[\mathbb{G}]$ for a $\mathbb{P}$-name $\underset{\sim}{a}$, and $\lambda>\kappa$. Since f is a Laver diamond for LC κ, there are j, $M\subseteq\mathsf{V}$ such that $j:\mathsf{V}\overset{\preceq}{\rightarrow}_\kappa M$, $j(\kappa)>\lambda$, j and M satisfy the closure property for LC, and $j(f)(\kappa)=\underset{\sim}{a}$.

Claim 5.9.1 $\mathsf{V}[\mathbb{G}]\models$ “$\mathbb{P}\lessdot j(\mathbb{P})$”.

$\vdash$ Since “$\mathbb{P}\lessdot\mathbb{P}'$” is a first order property of $\langle\mathbb{P}',\mathbb{P}\rangle$, it is enough to show that $M\models$ “$\mathbb{P}\lessdot j(\mathbb{P})$”.

Suppose that $P\subseteq\mathbb{P}$ is a maximal pairwise incompatible family (in M and hence in V) then $|\,P\,|<\kappa$ by the $<\kappa$-.c.c. of $\mathbb{P}$. It follows that $j(P)=P$. Since we have $M\models$ “$j(P)\subseteq j(\mathbb{P})$ is a maximal pairwise incompatible family” by elementarity, it follows that P is maximal pairwise incompatible family $\subseteq j(\mathbb{P})$ (in M).

$\dashv$ (Claim 5.9.1)

Let $\mathbb{H}$ be a $(\mathsf{V},j(\mathbb{P}))$-generic filter such that $\mathbb{G}\subseteq\mathbb{H}$. Then

$$\tilde j:\mathsf{V}[\mathbb{G}]\to M[\mathbb{H}];\ \ \underset{\sim}{b}[\mathbb{G}]\ \mapsto\ j(\underset{\sim}{b})[\mathbb{H}]$$

is a well-defined elementary embedding with $j\subseteq\tilde j$ (and hence $crit(\tilde j)=crit(j)=\kappa$). Note that $\tilde j(\mathbb{G})=\mathbb{H}$, and $\underset{\sim}{a}$ is also a $j(\mathbb{P})$-name. Thus, we have $\tilde j(\tilde f)(\kappa)=j(f_0)(\kappa)[\mathbb{H}]=\underset{\sim}{a}[\mathbb{H}]=\underset{\sim}{a}[\mathbb{G}]=a$, $\tilde j$ and $M[\mathbb{H}]$ satisfy the closure property for the generic version of LC, and $\tilde j(\kappa)=j(\kappa)>\lambda$.

This shows that $\Vdash_{\mathbb{P}}$ “$\tilde f$ is a $\diamondsuit^{all,LC}_{Laver,\kappa}$”-sequence. ❑ (Lemma 5.9)

Proof of Theorem 5.7. Note that in both of (1) and (2), we have that the tightly $\mathcal{P}$-Laver-g. hyperhuge cardinal κ, is $2^{\aleph_0}$ ($\mathcal{P} = all$ in case (2)). For the case (1), this is because of Theorem 5.4, (1) and Lemma 5.1, (1). For the case (2) by definition.

By Theorem 2.5, $\kappa := (2^{\aleph_0})^{\mathsf{V}}$ is hyperhuge in the bedrock $\overline{\mathsf{W}}$. Thus there is a Laver diamond $f : \kappa \to V_\kappa{}^{\overline{\mathsf{W}}}$ for hyperhuge in $\overline{\mathsf{W}}$, and the machine to return to V from $\overline{\mathsf{W}}$ is of cardinality $\leq \kappa$ and satisfies $< \kappa$-c.c. by Theorem 5.8.

By Lemma 5.9, it follows that $\diamondsuit^{all,hyperhuge}_{Laver,2^{\aleph_0}}$ holds in V. ❑ (**Theorem 5.7**)

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