

Strong downward Löwenheim-Skolem theorems for stationary logics, III — mixed support iteration

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Abstract

Continuing [Fuchino, Ottenbreit and Sakai[9, 10]] and [Fuchino and Ottenbreit[11]], we further study reflection principles in connection with the Löwenheim-Skolem Theorems of stationary logics. In this paper, we mainly analyze the situations in the models obtained by mixed support iteration of a supercompact length and then collapsing another supercompact cardinal to make it $(2^{\aleph_0})^+$. We show, among other things, that the reflection down to $< 2^{\aleph_0}$ of the non-metrizability of topological spaces with small character is independent from the reflection properties studied in [Fuchino, Ottenbreit and Sakai[9, 10]] and [Fuchino and Ottenbreit[11]].

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1 Introduction

III:intro

Reflection properties of the following type are considered in various mathematical contexts:

- (1.1) If a structure \mathfrak{A} in the class \mathcal{C} has the property \mathcal{P} , then there is a structure \mathfrak{B} in relation \mathcal{Q} to \mathfrak{A} such that \mathfrak{B} has the cardinality $< \kappa$ and \mathfrak{B} also has the property \mathcal{P} .

III:Ref-0

We shall call “ $< \kappa$ ” above the *reflection point* of the reflection property (1.1). If κ is a successor cardinal μ^+ , we shall also say that the reflection point of the reflection property is $\leq \mu$.

An instance of (1.1) is when \mathcal{C} = “first countable topological spaces”, \mathcal{P} = “non-metrizable”, \mathcal{Q} = “subspace” and $\kappa = \aleph_2$, that is, with the reflection point $\leq \aleph_1$. In this setting, the obtained reflection statement is:

- (1.2) For any first countable topological space X , if X is non-metrizable, then there is a subspace Y of X of cardinality $< \aleph_2$ such that Y is also non-metrizable.

III:Ref-1

The consistency of the statement above is still unknown. This persistently open problem about the consistency of the assertion (1.2) is called Hamburger’s Problem after Peter Hamburger who asked a related question (see [Hajnal-Juhász[12]]).

The naturalness of the question can be seen in the following known partial solutions: With “first countable” replaced by “compact”, the assertion (1.2) is a theorem in ZFC [Dow[6]]. With “first countable” replaced by “locally-compact”, the assertion (1.2) is independent from ZFC (for the consistency we need some

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This is an extended version of the paper with the same title. Some extra remarks and details omitted in the published version, as well as further corrections, may be found in this version. The additional stuff is typeset in dark electric blue like this paragraph. The most recent edition of this version is downloadable as:

<https://fuchino.ddo.jp/papers/SDLS-III-xx.pdf>

very large cardinal since \square_κ for some κ implies the negation of the statement, see [Fuchino-Juhász-Szentmiklóssy-Usuba[8]].

We shall call the following principle “Hamburger’s Hypothesis” (with the reflection point $< \kappa$):

HH($< \kappa$): For any topological space X with $\chi(x, X) < \kappa$ for all $x \in X$, if X is non-metrizable then there is a subspace Y of X of cardinality $< \kappa$ which is also non-metrizable.

Recall that the character $\chi(x, X)$ of a point x in a topological space X is the minimal possible cardinality of a neighborhood base of x in X . Without the condition on the character of points, we easily obtain a counter-example to the reflection of non-metrizability (see [Hajnal-Juhász[12]]).

Note that the original Hamburger’s Problem (1.2) is equivalent to **HH($< \aleph_2$)** ([Hajnal-Juhász[12]]). [If **HH($< \aleph_2$)** holds, then we clearly have (1.2).

Assume that (1.2) holds. To see that **HH($< \aleph_2$)** holds, suppose that X is a non-metrizable space with $\chi(x, X) < \aleph_2$ for all $x \in X$. If $\chi(x, X) = \aleph_1$ for some $x \in X$, then there is a subspace Y of X of cardinality \aleph_1 with $x \in Y$ and $\chi(x, Y) = \aleph_1$ (an elementary submodel argument proves this easily: Let θ be sufficiently large and let $M \prec \mathcal{H}(\theta)$ be such that $|M| = \aleph_1$, $\omega_1 \subseteq M$, and $\langle X, \tau \rangle \in M$. Then, $Y = X \cap M$ is as desired). By this x , Y is not metrizable. If $\chi(x, X) < \aleph_1$ for all $x \in X$, then, by the assumption, there is a non-metrizable subspace Y of X of cardinality $\leq \aleph_1$.]

HH($< \aleph_1$) does not hold: ω_1 in order topology as well as $X_{\mathcal{F}}$ in the proof of Theorem 2.1 for an unbounded $\mathcal{F} \subseteq {}^\omega\omega$ is a counterexample.

The following fact will be used in the proofs of Corollary 1.8 and Proposition 4.7:

Theorem 1.1 ([Dow, Tall and Weiss[7]]) *Suppose that X is a non-metrizable space, $\delta \in \text{Card}$ and $\mathbb{P} = \text{Fn}(\delta, 2)$, the poset with finite conditions adding δ many Cohen reals. Then we have*

$$(1.3) \quad \Vdash_{\mathbb{P}} \text{“}\check{X} \text{ is non-metrizable”}.$$

III:P-refl-a

□ III:refl-a-0

Topological space X is considered here as a pair $X = \langle X, \tau \rangle$ where τ is the open base of the topology. Note that the family \mathcal{O} of all open sets in the ground model need not to satisfy the axioms of open sets in a generic extension, while an open base remains to be an open base in the generic extension.

Let us call the posets of the form $\text{Fn}(\delta, 2)$ for some ordinal δ *generalized Cohen posets*.

For a class \mathcal{P} of posets, a cardinal κ is said to be *generically supercompact by* \mathcal{P} , if, for any $\lambda \geq \kappa$, there is a poset $\mathbb{P} \in \mathcal{P}$ such that, for a (\mathbb{V}, \mathbb{P}) -generic \mathbb{G} , there are classes $j, M \subseteq \mathbb{V}[\mathbb{G}]$ such that

(1.4) M is transitive in $\mathbf{V}[\mathbb{G}]$ and $j : V \xrightarrow{\simeq} M$,

III:ref-a-2

(1.5) $\text{crit}(j) = \kappa$,

III:ref-a-3

(1.6) $j(\kappa) > \lambda$, and

III:ref-a-4

(1.7) $j''\lambda \in M$.

III:ref-a-5

III:P-refl-a-0

Corollary 1.2 *If κ is generically supercompact by generalized Cohen posets, then $\text{HH}(< \kappa)$ holds.*

Proof. Suppose that X is a non-metrizable space with

(1.8) $\chi(x, X) < \kappa$ for all $x \in X$.

III:ref-a-1

Without loss of generality, $X = \langle \theta, \tau \rangle$ for some ordinal θ and an open base τ on θ . Let $\lambda \geq \theta$ be sufficiently large and let $\mathbb{P} = \text{Fn}(\mu, 2)$ for some cardinal μ such that, for a (\mathbf{V}, \mathbb{P}) -generic filter \mathbb{G} , there are classes $j, M \subseteq \mathbf{V}[\mathbb{G}]$ satisfying (1.4), (1.5), (1.6), and (1.7) for this λ .

Let $\tau'' = \{j(O) \cap j''\theta : O \in \tau\}$. Then we have $\langle j''\theta, \tau'' \rangle, \langle \theta, \tau \rangle \in M$, and $M \models \langle \theta, \tau \rangle \cong \langle j''\theta, \tau'' \rangle$ by (1.7) (see, e.g. Lemma 2.5 in [Fuchino, Ottenbreit and Sakai[10]]).

By Theorem 1.1, $\mathbf{V}[\mathbb{G}] \models \langle j''\theta, \tau'' \rangle$ is non-metrizable". By (1.8), $M \models \langle j''\theta, \tau'' \rangle$ is a sub-space of $\langle j(\theta), j(\tau) \rangle$ ".

Thus, $M \models$ "there is a non-metrizable subspace Y of $j(X)$ of cardinality $< j(\kappa)$ ". By elementarity, it follows that $\mathbf{V} \models$ "there is a non-metrizable subspace Y of X of cardinality $< \kappa$ ". □ (Corollary 1.2)

In a model obtained as the generic extension by $\text{Fn}(\kappa, 2)$ where κ is a supercompact cardinal, we have $2^{\aleph_0} = \kappa$ and κ is generically supercompact by generalized Cohen posets. Thus,

III:P-refl-a-1

Corollary 1.3 ([Dow, Tall and Weiss[7]]) *If $\text{ZFC} +$ "there is a supercompact cardinal" is consistent, then so is $\text{ZFC} + \text{HH}(< 2^{\aleph_0})$.* □

The Strong Downward Löwenheim-Skolem Theorem $\text{SDLS}^-(\mathcal{L}_{stat}^{\aleph_0}, < \kappa)$ for the stationary logic $\mathcal{L}_{stat}^{\aleph_0}$ down to $< \kappa$ is another natural reflection property. Here, the stationary logic $\mathcal{L}_{stat}^{\aleph_0}$ is a monadic second order logic whose second order variables run over countable subsets of the underlining set of the structure in question. The only second-order quantifier in the logic is 'stat' (as well as its dual 'aa' where the quantification "aa X " is introduced as the abbreviation of " $\neg \text{stat } X \neg$ "). The semantics of the logic is introduced by the following step in the recursion in addition to the usual recursive definition of the semantics for first order part of the

logic: for a structure $\mathfrak{A} = \langle A, \dots \rangle$ and $\mathcal{L}_{stat}^{\aleph_0}$ -formula $\varphi = \varphi(x_0, \dots, X_0, \dots, X)$ in the corresponding signature, where X_0, \dots, X are the second order variables in φ , as well as for $a_0, \dots \in A$ and $U_0, \dots \in [A]^{\aleph_0}$,

$$(1.9) \quad \begin{aligned} \mathfrak{A} \models_{stat} X \varphi(a_0, \dots, U_0, \dots, X) \\ \Leftrightarrow \{U \in [A]^{\aleph_0} : \mathfrak{A} \models \varphi(a_0, \dots, U_0, \dots, U)\} \text{ is stationary in } [A]^{\aleph_0}. \end{aligned}$$

III:ref-a-6

For a substructure $\mathfrak{B} = \langle B, \dots \rangle$ of \mathfrak{A} , the weak variant of elementary submodel relation $\prec_{\mathcal{L}_{stat}^{\aleph_0}}^-$ between \mathfrak{B} and \mathfrak{A} is defined by

$$(1.10) \quad \begin{aligned} \mathfrak{B} \prec_{\mathcal{L}_{stat}^{\aleph_0}}^- \mathfrak{A} \Leftrightarrow \\ \mathfrak{B} \models \varphi(b_0, \dots, b_{n-1}) \text{ holds if and only if } \mathfrak{A} \models \varphi(b_0, \dots, b_{n-1}) \text{ holds for all } \\ \mathcal{L}_{stat}^{\aleph_0}\text{-formulas } \varphi = \varphi(x_0, \dots, x_{n-1}) \text{ without free second-order variables, and} \\ \text{for all } b_0, \dots, b_{n-1} \in B. \end{aligned}$$

III:ref-a-7

The reflection principle $\text{SDLS}^-(\mathcal{L}_{stat}^{\aleph_0}, < \kappa)$ for a cardinal $\kappa \geq \aleph_2$ is defined by:

$\text{SDLS}^-(\mathcal{L}_{stat}^{\aleph_0}, < \kappa)$: For any structure \mathfrak{A} in a countable signature, there is a substructure \mathfrak{B} of \mathfrak{A} of cardinality $< \kappa$ such that $\mathfrak{B} \prec_{\mathcal{L}_{stat}^{\aleph_0}}^- \mathfrak{A}$.

In [Fuchino, Ottenbreit and Sakai[9]], we also considered the version of SDLS without ‘-’ by allowing second order free variables and second order parameters in the formulas φ in (1.10). However, it is proved there that the principle $\text{SDLS}(\mathcal{L}_{stat}^{\aleph_0}, < \kappa)$ obtained in this way for a regular κ is simply the conjunction of $\text{SDLS}^-(\mathcal{L}_{stat}^{\aleph_0}, < \kappa)$ and $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$.

In the standard model of PFA or under strongly Laver-generically supercompactness of a cardinal κ for proper posets (for definition of Laver-generic supercompactness, see p.16), we have the reflection principle $\text{SDLS}^-(\mathcal{L}_{stat}^{\aleph_0}, < \aleph_2)$. Actually, $\text{MA}^{+\omega_1}(\sigma\text{-closed})$ already implies this principle, and strongly Laver-generically supercompactness for properness of κ implies $\kappa = \aleph_2$ and $\text{PFA}^{+\omega_1}$.

If $\text{MA}^{+\omega_1}(\sigma\text{-closed})$ (or $\text{PFA}^{+\omega_1}$, or $\text{MM}^{+\omega_1}$, resp.) holds and \mathbb{P} is $< \aleph_2$ -directed closed, then we have $\Vdash_{\mathbb{P}} \text{“MA}^{+\omega_1}(\sigma\text{-closed)”}$ (or $\Vdash_{\mathbb{P}} \text{“PFA}^{+\omega_1}”$, or $\Vdash_{\mathbb{P}} \text{“MM}^{+\omega_1}”$ resp.) (Proposition 15 in [Fuchino and Ottenbreit[11]]).

Suppose that $\text{MA}^{+\omega_1}(\sigma\text{-closed})$ holds and $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. and there is a supercompact cardinal κ_1 . Let $\mathbb{P} = \text{Col}(2^{\aleph_0}, \kappa_1)$. In a generic extension by \mathbb{P} , we still have $\text{MA}^{+\omega_1}(\sigma\text{-closed})$ by the result mentioned above, and hence also $\text{SDLS}^-(\mathcal{L}_{stat}^{\aleph_0}, < \aleph_2)$. On the other hand, \mathbb{P} forces κ_1 to be $(2^{\aleph_0})^+$ and makes κ_1 generically supercompact by $< \aleph_2$ -closed posets (see, e.g. Lemma 4.10 in [Fuchino, Sakai and Ottenbreit[9]]). By Theorem 4.13 in [Fuchino, Sakai and Ottenbreit[9]], the assertion that $\kappa_1 = \kappa^+$ is generically supercompact by $< \kappa$ -closed posets is equivalent to the Game Reflection Principle $\text{GRP}^{< \kappa}(\leq \kappa)$ under $2^{< \kappa} = \kappa$. Thus, in this way, we obtain a model

of a very strong reflection property with the reflection point $< 2^{\aleph_0}$, together with an even stronger reflection property but with the reflection point $\leq 2^{\aleph_0}$.

$\text{SDLS}^-(\mathcal{L}_{stat}^{\aleph_0}, < 2^{\aleph_0})$ implies $2^{\aleph_0} = \aleph_2$ (Corollary 2.3 in in [Fuchino, Sakai and Ottenbreit[10]]). This means in particular that, if the continuum should be larger than \aleph_2 , this reflection statement is not available. In the model obtained by iterating ccc posets supercompact times with finite support along with a book-keeping provided by a Laver-function, the continuum is extremely large (e.g. in terms of existence of a saturated ideal) but the Strong Downward Löwenheim-Skolem Theorem $\text{SDLS}^{int}(\mathcal{L}_{stat}^{\aleph_0}, < 2^{\aleph_0})$ of the stationary logic $\mathcal{L}_{stat}^{\aleph_0}$ with internal interpretation (a weakening of $\text{SDLS}^-(\mathcal{L}_{stat}^{\aleph_0}, < 2^{\aleph_0})$) holds, as well as the Strong Downward Löwenheim-Skolem Theorem $\text{SDLS}_+^{int}(\mathcal{L}_{stat}^{\text{PKL}}, < 2^{\aleph_0})$ of internal interpretation of the PKL-logic with the reflection point $\leq 2^{\aleph_0}$ (Theorem 2.10 and Proposition 3.1 in [Fuchino, Sakai and Ottenbreit[10]] for $\text{SDLS}^-(\mathcal{L}_{stat}^{\aleph_0}, < 2^{\aleph_0})$; Proposition 4.1 and Theorem 4.5 in [Fuchino, Sakai and Ottenbreit[10]] for $\text{SDLS}_+^{int}(\mathcal{L}_{stat}^{\text{PKL}}, < 2^{\aleph_0})$) together with $\text{MA}^{+\mu}$ for all $\mu < 2^{\aleph_0}$. The significance of $\text{SDLS}_+^{int}(\mathcal{L}_{stat}^{\text{PKL}}, < 2^{\aleph_0})$ in this connection is that it implies that the continuum is very large (e.g. it implies that the continuum is at least weakly Mahlo).

For this model, there seems to be no way to force further to obtain a stronger reflection but with the reflection point $\leq 2^{\aleph_0}$ without destroying the reflection properties already existing in the model.

In the present paper, we show that the mixed support supercompact time iteration, roughly speaking, with Easton support mixed with the finite support, bookkept along with a Laver function together with a further collapse of the second supercompact cardinal creates a model in which “down to $< 2^{\aleph_0}$ ” type of reflection principles as mentioned above together with $\text{GRP}^{< 2^{\aleph_0}} (\leq 2^{\aleph_0})$ hold.

Modifying the finite support part of this iteration, we show the independence of $\text{HH}(< 2^{\aleph_0})$ from the other strong reflection properties.

For the definition of some of the set-theoretic principles and basic facts around them remained unexplained in the present paper, the reader should consult [Fuchino, Sakai and Ottenbreit[9, 10]]. These papers in extended version uploaded at the URLs given in the References may be also helpful since they contain some more details which were omitted in the submitted version of the papers.

In particular, we are going to drop the definition of $\text{SDLS}_+^{int}(\mathcal{L}_{stat}^{\text{PKL}}, < 2^{\aleph_0})$ and ask readers to consult [Fuchino, Sakai and Ottenbreit[10]] for details. However, we shall cite the following infinitary combinatorial characterization of this principle. This will be used in Proposition 4.7, (2) to show that this principle holds under certain instance of the two-dimensional Laver-generic large cardinal considered in

Section 4.

Extending the standard notation, for sets s and t , we denote with $\mathcal{P}_s(t)$ the set

$$(1.11) \quad [t]^{|s|} = \{a \in \mathcal{P}(t) : |a| < |s|\}.$$

III:refl-a-8

Lemma 1.4 (Proposition 4.1 in [Fuchino, Sakai and Ottenbreit[10]]) *For a regular cardinal $\kappa > \aleph_1$ $\text{SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\text{PKL}}, < \kappa)$ is equivalent to the assertion that $(*)_{< \kappa, \lambda}^{\text{int+PKL}}$ holds for all regular $\lambda \geq \kappa$ where*

$(*)_{< \kappa, \lambda}^{\text{int+PKL}}$: *For any countable expansion \mathfrak{A} of the structure $\langle \mathcal{H}(\lambda), \kappa, \in \rangle$ and any family $\langle S_a : a \in \mathcal{H}(\lambda) \rangle$ such that S_a is a stationary subset of $\mathcal{P}_\kappa(\mathcal{H}(\lambda))$ for all $a \in \mathcal{H}(\lambda)$, there are stationarily many $M \in \mathcal{P}_\kappa(\mathcal{H}(\lambda))$ such that $|\kappa \cap M|$ is regular, $\mathfrak{A} \upharpoonright M \prec \mathfrak{A}$ and $S_a \cap \mathcal{P}_{\kappa \cap M}(M) \cap M$ is stationary in $\mathcal{P}_{\kappa \cap M}(M)$ for all $a \in M$. \square*

We shall use freely the following “bullet notation” of names in forcing construction, introduced by Asaf Karagila ¹⁾.

If $t(x_0, \dots)$ is a term in some conservative expansion of the language and the axiom system of the set theory by definitions then for a poset \mathbb{P} and \mathbb{P} -names $\underline{a}_0, \dots, t(\underline{a}_0, \dots)^\bullet$ denotes the standard \mathbb{P} -name \underline{u} such that

$$(1.12) \quad \underline{u}[\mathbb{G}] = t^{V[\mathbb{G}]}(\underline{a}_0[\mathbb{G}], \dots) \text{ for any } (V, \mathbb{P})\text{-generic filter } \mathbb{G}$$

III:refl-a-9

(or, more syntactically, $\Vdash_{\mathbb{P}} \underline{u} \equiv t(\underline{a}_0, \dots)$).

For example, $\langle \underline{a}, \underline{b} \rangle^\bullet$ is denoted as $\text{op}(\underline{a}, \underline{b})$ in [Kunen[17]]. $t(\underline{a}_0, \dots)$ may have infinitely many parameters. For example, if $\underline{a}_\xi, \xi < \delta$ is a sequence of \mathbb{P} -names in the ground model, $\{\underline{a}_\xi : \xi < \delta\}^\bullet$ may be introduced as the \mathbb{P} -name $\{\langle \underline{a}_\xi, \mathbb{1}_{\mathbb{P}} \rangle : \xi < \delta\}$, while $\langle \underline{a}_\xi : \xi < \delta \rangle^\bullet$ may be introduced as the \mathbb{P} -name $\{\langle \langle \check{\xi}, \underline{a}_\xi \rangle^\bullet, \mathbb{1}_{\mathbb{P}} \rangle : \xi < \delta\}$. The choice of the exact definition of each bullet name is left to the reader. We only assume that the choice is done in a consistent way. If we want to emphasize that the bullet name $t(\underline{a}_0, \dots)^\bullet$ is a \mathbb{P} -name, we put the subscript \mathbb{P} and write $t(\underline{a}_0, \dots)_{\mathbb{P}}^\bullet$.

For a poset \mathbb{P} , \mathbb{P} -check names of a ground model set a is represented either simply by a or with a check as \check{a} . If it is necessary to make clear which poset is involved, we shall also write $(a)_{\mathbb{P}}^\check{\vee}$. This representation is used, in particular, if a ground model set is given by a term. Thus we write, e.g. $(\mathcal{P}(a))_{\mathbb{P}}^\check{\vee}$, $(a \cup b)_{\mathbb{P}}^\check{\vee}$, $(\{x \in a : \varphi(x, \dots)\})_{\mathbb{P}}^\check{\vee}$ etc.

A part of the results in the following, most of the materials in Section 5 in particular, have been presented in the PhD thesis [Ottenbreit Maschio Rodrigues[18]]

¹⁾The authors learned this extremely helpful notation in a tutorial lectures by Asaf Karagila in Kyoto at the RIMS Set Theory Workshop 2019.

of the second author, although some arguments and details are treated differently from those in the PhD thesis.

2 Reflection number of Hamburger's Hypothesis

III:hamburger

In the following, we shall examine the details of the example of the topological space given on p.158 in [van Douwen[5]].

A topological space $X = \langle X, \tau \rangle$ where τ is an open base of the topology is said to be a *Moore space*, if X is a regular Hausdorff space such that

(N2.1) there is a sequence $\mathcal{O}_n \subseteq \tau$ of open covers of X (i.e. $\bigcup \mathcal{O}_n = X$ for all $n \in \omega$) with the property that, for any closed $C \subseteq X$ and $x \in X \setminus C$, there is $n \in \omega$ such that all $O \in \mathcal{O}_n$ with $x \in O$ is disjoint with C .

III:vDw-0

The property (N2.1) is called the *developability* of X . If (N2.1) holds, we say that $\langle \mathcal{O}_n : n \in \omega \rangle$ is a *development* of X and X is *developable*.

The following is a warm-up exercise:

Lemma A 2.1 (1) *If X is a metrizable space, then X is a Moore space.*

(2) *If X is a Moore space then it is first countable.*

Proof. (1): Suppose that X is a metrizable space. Then X is Hausdorff and normal.

Let d be a metric on X which induces the topology of X . Then $\mathcal{O}_n = \{S_d(x, \frac{1}{n+1}) : x \in X\}$, for $n \in \omega$ form a development of X .

(2): Suppose that $\mathcal{O}_n, n \in \omega$ witness that X is a Moore space. Let $x \in X$. For each $n \in \omega$, let $O_{x,n}$ for $n \in \omega$ be such that $x \in O_{x,n}$ and $O_{x,n} \in \mathcal{O}_n$. By the property (N2.1) this sequence is well-defined and $\{O_{x,n} : n \in \omega\}$ is an open neighborhood basis for x .

□ (Lemma A 2.1)

A topological space $X = \langle X, \tau \rangle$ is *collectionwise Hausdorff* if, for any discrete closed set $D \subseteq X$, there is a family $\mathcal{U} = \{U_d : d \in D\}$ of pairwise disjoint open sets such that the mapping $D \ni d \mapsto U_d \in \mathcal{U}$ is 1-1 and $d \in U_d$ for all $d \in D$.

A (pairwise disjoint) family \mathcal{C} of closed subsets of a space X is said to be *discrete* if, for any $x \in X$, there is a neighborhood U of x such that U intersects with at most one element of \mathcal{C} .

$X = \langle X, \tau \rangle$ is *collectionwise normal* if, for any discrete family \mathcal{C} of closed sets, there is a family $\mathcal{U} = \{U_C : C \in \mathcal{C}\}$ of pairwise disjoint open sets such that $\mathcal{C} \ni C \mapsto U_C \in \mathcal{U}$ is 1-1 and $C \subseteq U_C$ for each $C \in \mathcal{C}$.

The following is immediate from the definitions above.

Lemma A 2.2 *For a Hausdorff space $X = \langle X, \tau \rangle$, if X is collectionwise normal then X is collectionwise Hausdorff.*

III:P-vDw-0

□

The following well-known facts are also used in the proof of Theorem 2.1 below.

III:P-vDw-1

Fact A 2.3 (1) *Any metrizable space X is collectionwise normal. In particular, by Lemma 2.2, any metrizable space X is collectionwise Hausdorff.*

(2) ([Bin[2]]) *A collectionwise normal Moore space is metrizable.* \square

As usual, \mathfrak{b} denotes the bounding number which is defined as the minimal possible cardinality of a subset of ${}^\omega\omega$ which is unbounded with respect to \leq^* (coordinate-wise comparison modulo finite).

III:P-vDw-2

Theorem 2.1 ([van Douwen[5]]) *There is a Moore space X of cardinality \mathfrak{b} such that X is not collectionwise Hausdorff (and hence non-metrizable by Fact A 2.3,(1)) but all subspaces of X of cardinality $< \mathfrak{b}$ are metrizable.*

Proof. Let $\mathcal{F} \subseteq {}^\omega\omega$ and let

$$(N2.2) \quad X_{\mathcal{F}} = \mathcal{F} \dot{\cup} \omega \dot{\cup} \mathcal{F} \times \omega \times \omega.$$

III:vDw-1

We define the topology on $X_{\mathcal{F}}$ by declaring that

$$(N2.3) \quad \text{elements of } \mathcal{F} \times \omega \times \omega \text{ are discrete;}$$

III:vDw-2

$$(N2.4) \quad \text{each } f \in \mathcal{F} \text{ has a neighborhood basis consisting of sets of the form } O_{f,s} = \{f\} \cup \{f\} \times f \setminus s \text{ where } s \text{ is a finite subset of } \omega \times \omega; \text{ and}$$

III:vDw-3

$$(N2.5) \quad \text{for } k \in \omega (\subseteq X_{\mathcal{F}}), U_{k,n} = \{k\} \cup \mathcal{F} \times \{k\} \times (\omega \setminus n) \text{ for } n \in \omega \text{ form a neighborhood basis of } k \in \omega \subseteq X_{\mathcal{F}}.$$

III:vDw-4

Cl-III:vDw-0

Claim 2.1.1 *$X_{\mathcal{F}}$ is a normal Hausdorff space.*

\vdash To show that $X_{\mathcal{F}}$ is normal, one of the cases to be checked is that any closed $F \subseteq X_{\mathcal{F}}$ and $\langle f, m, n \rangle \in \mathcal{F} \times \omega \times \omega \setminus F$ can be separated by open sets. $\{\langle f, m, n \rangle\}$ is the minimal open neighborhood of $\langle f, m, n \rangle$ by (N2.3).

For $g \in F \cap \mathcal{F}$, if $g \neq f$, then $O_{g,s}$ for any $s \in [\omega \times \omega]^{< \aleph_0}$ does not contain $\langle f, m, n \rangle$ and hence disjoint from $\{\langle f, m, n \rangle\}$. If $g = f$, then letting $s = \{\langle m, n \rangle\}$, $O_{g,s}$ does not contain $\langle f, m, n \rangle$ and hence disjoint from the open set $\{\langle f, m, n \rangle\}$.

For $k \in F \cap \omega$, $U_{k,n+1}$ is disjoint from $\{\langle f, m, n \rangle\}$.

For $\langle f', m', n' \rangle \in F \cap \mathcal{F} \times \omega \times \omega$, Since $\langle f', m', n' \rangle \neq \langle f, m, n \rangle$, the open neighborhood $\{\langle f', m', n' \rangle\}$ of $\langle f', m', n' \rangle$ is disjoint from $\{\langle f, m, n \rangle\}$.

Thus, we find an open superset of F disjoint from $\{\langle f, m, n \rangle\}$ by taking union of all the open sets as above.

The rest of the proof can be done similarly.

\dashv (Claim 2.1.1)

Cl-III:vDw-1

Claim 2.1.2 *$X_{\mathcal{F}}$ is developable.*

⊢ Let $\langle s_n : n \in \omega \rangle$ be an increasing sequence of finite subsets of $\omega \times \omega$ such that $\omega \times \omega = \bigcup_{n \in \omega} s_n$.

For each $n \in \omega$, let

$$(N2.6) \quad \mathcal{O}_n = \{ \{ \langle f, k, l \rangle \} : \langle f, k, l \rangle \in \mathcal{F} \times \omega \times \omega \} \\ \cup \{ O_{f, s_n} : f \in \mathcal{F} \} \cup \{ U_{k, n} : k \in \omega \}.$$

III:vDw-5

Then $\langle \mathcal{O}_n : n \in \omega \rangle$ is a development of X .

⊢ (Claim 2.1.2)

Cl-III:vDw-2

Claim 2.1.3 *If $\mathcal{F} \subseteq {}^\omega\omega$ is unbounded (with respect to \leq^*), then $X_{\mathcal{F}}$ is not collectionwise Hausdorff. In particular, \mathcal{F} is non-metrizable.*

⊢ $D = \mathcal{F} \cup \omega$ as a subset of $X_{\mathcal{F}}$ is discrete and closed. We show that this set is a counter-example to the collectionwise Hausdorffness. Suppose, toward a contradiction, that \mathcal{U} is a family of pairwise disjoint open sets in $X_{\mathcal{F}}$ which separates elements of D . Without loss of generality, we may assume that elements of \mathcal{U} are of the form either $O_{f, s}$ or $U_{k, n}$.

Let $f^* : \omega \rightarrow \omega$ be defined by

$$(N2.7) \quad f^*(k) = n \text{ if } U_{k, n} \in \mathcal{U}.$$

III:vDw-6

f^* is well-defined since \mathcal{U} is pairwise disjoint. Since \mathcal{F} is unbounded, there is $g^* \in \mathcal{F}$ such that $g^* \not\leq^* f^*$. Thus $g^*(k) > f^*(k)$ for infinitely many $k \in \omega$. Let $s \in [\omega \times \omega]^{< \aleph_0}$ be such that $g^* \in O_{g^*, s} \in \mathcal{U}$ and let $k \in \omega \setminus \{m \in \omega : \langle m, n \rangle \in s \text{ for some } n \in \omega\}$ be such that $g^*(k) > f^*(k)$. Then, since $U_{k, f^*(k)} \in \mathcal{U}$, we have $\langle g^*, k, g^*(k) \rangle \in O_{g^*, s} \cap U_{k, f^*(k)} \neq \emptyset$. This is a contradiction to the pairwise disjointness of \mathcal{U} .

Thus $X_{\mathcal{F}}$ is not collectionwise Hausdorff. $X_{\mathcal{F}}$ is non-metrizable by Fact 2.3, (1).

⊢ (Claim 2.1.3)

Cl-III:vDw-3

Claim 2.1.4 *If $\mathcal{F} \subseteq {}^\omega\omega$ is bounded, then $X_{\mathcal{F}}$ is collectionwise normal, and hence $X_{\mathcal{F}}$ is metrizable by Fact 2.3, (2).*

⊢ Suppose that \mathcal{C} is a discrete family of closed sets in $X_{\mathcal{F}}$. Let $g^* \in {}^\omega\omega$ be such that $f <^* g^*$ for all $f \in \mathcal{F}$. For each $f \in \mathcal{F}$, let $s_f \in [\omega]^{< \aleph_0}$ be such that $f \upharpoonright \omega \setminus s_f < g^* \upharpoonright \omega \setminus s_f$ (point-wise).

Since \mathcal{C} is discrete, for each $x \in X$ there is a neighborhood V_x of x such that V_x intersects at most one element of \mathcal{C} .

For $C \in \mathcal{C}$, let

$$(N2.8) \quad O_C = (C \cap \mathcal{F} \times \omega \times \omega) \\ \cup \bigcup \{ O_{f, f \upharpoonright s_f} \cap V_f : f \in C \cap \mathcal{F} \} \\ \cup \bigcup \{ U_{k, g^*(k)} \cap V_k : k \in C \cap \omega \}.$$

III:vDw-7

Then $\mathcal{U} = \{O_C : C \in \mathcal{C}\}$ separates elements of \mathcal{C} .

This shows that $X_{\mathcal{F}}$ is collectionwise normal. By Fact 2.3, (2) and since $X_{\mathcal{F}}$ is a Moore space by Claim 2.1.1 and Claim 2.1.2, it follows that $X_{\mathcal{F}}$ is metrizable. \dashv (Claim 2.1.4)

Now, suppose that $\mathcal{F} \subseteq {}^\omega\omega$ is unbounded with $|\mathcal{F}| = \mathfrak{b}$.

By Claim 2.1.3, $X_{\mathcal{F}}$ is non-metrizable for any $\mathcal{F}_0 \subseteq \mathcal{F}$ of cardinality $< \mathfrak{b}$, the subspace $X_{\mathcal{F}_0}$ of $X_{\mathcal{F}}$ is metrizable by Claim 2.1.4. Since subspaces of $X_{\mathcal{F}}$ of the form $X_{\mathcal{F}_0}$ for $\mathcal{F}_0 \in [\mathcal{F}]^{< \mathfrak{b}}$ are cofinal in $[\mathcal{F}]^{< \mathfrak{b}}$ and since any subspace of a metrizable space is metrizable, it follows that all subspaces of $X_{\mathcal{F}}$ of cardinality $< \mathfrak{b}$ are metrizable. \square (Theorem 2.1)

III:P-vDw-3

Corollary 2.2 *There is a non-metrizable Moore space $X = \langle X, \tau \rangle$ such that $\Vdash_{\mathbb{P}}$ “ \check{X} is metrizable” for a σ -centered poset \mathbb{P} .*

Proof. Let $X = X_{\mathcal{F}}$ for an unbounded family $\subseteq {}^\omega\omega$. Let \mathbb{P} be the Hechler forcing then $\Vdash_{\mathbb{P}}$ “ $\check{\mathcal{F}}$ is bounded”. Thus, by Claim 2.1.4, $\Vdash_{\mathbb{P}}$ “ $X_{\check{\mathcal{F}}}$ is metrizable”. By the absoluteness of the definition of $X_{\mathcal{F}}$, we have $\Vdash_{\mathbb{P}}$ “ $\check{X} = X_{\check{\mathcal{F}}}$ ”. \square (Corollary 2.2)

The reflection number $\mathfrak{Rfl}_{\text{HP}}$ of Hamburger’s Hypothesis is defined by:

$$(2.1) \quad \mathfrak{Rfl}_{\text{HP}} = \begin{cases} \text{the minimal cardinal } \kappa \text{ such that,} \\ \text{for any first countable non-metrizable} \\ \text{topological space } X, \text{ there is a non-} \\ \text{metrizable subspace } Y \text{ of } X \text{ of} \\ \text{cardinality } < \kappa; & \text{if such } \kappa \text{ exists,} \\ \infty; & \text{otherwise.} \end{cases} \quad \text{III:vDw-8}$$

Lemma 2.3 (1) $\mathfrak{b} < \mathfrak{Rfl}_{\text{HP}} \leq \infty$. III:P-vDw-4

(2) $\mathfrak{Rfl}_{\text{HP}} = \infty$ is consistent.

(3) ([Bagaria and Magidor[1]]) $\mathfrak{Rfl}_{\text{HP}} \leq$ the least ω_1 -strongly compact cardinal (if it exists).

Proof. (1): By Theorem 2.1.

For $\aleph_1 < \mathfrak{Rfl}_{\text{HP}}$, we have more direct examples: ω_1 with the order topology or E_{ω}^{κ} for any cardinal of uncountable cofinality (also with the order topology) are among the examples showing the inequality $\aleph_1 < \mathfrak{Rfl}_{\text{HP}}$.

(2): This holds if \square_{κ} holds for cofinally many κ (in *Card*) — actually $\text{ADS}^{-}(\kappa)$ for class many regular uncountable κ is enough (see Proposition 6.3 in [Fuchino, Juhász et al.[8]]).

(3): Suppose that (X, \mathcal{O}) is a first countable topological space such that all subspaces $Y \in [X]^{< \kappa}$ are metrizable. For each $x \in X$, let $\{O_{x,n} : n \in \omega\}$ be an open neighborhood base of x .

Let T be the $\mathcal{L}_{\omega_1, \omega}$ theory in the language with the binary relation symbols $O_n(x, y)$ for all $n \in \omega$ coding “ $y \in O_{x, n}$ ” and the binary symbols $d_q(x, y)$ for all $q \in \mathbb{Q}_{\geq 0}$ which should code “ $d(x, y) \leq q$ ”:

$$\begin{aligned}
(\aleph_{2.9}) \quad T = & \{O_n(c_a, c_b) : a, b \in X, b \in O_{a, n}\} \\
& \cup \{\neg O_n(c_a, c_b) : a, b \in X, b \notin O_{a, n}\} \\
& \cup \{\forall x \forall y (d_q(x, y) \rightarrow d_q(y, x)) : q \in \mathbb{Q}_{\geq 0}\} \\
& \cup \{\forall x \forall y (d_q(x, y) \rightarrow d_{q'}(x, y)) : q, q' \in \mathbb{Q}_{\geq 0}, q \leq q'\} \\
& \cup \{\forall x \forall y (d_0(x, y) \rightarrow x \equiv y)\} \\
& \cup \{\forall x \forall y \forall z (d_q(x, y) \wedge d_{q'}(y, z) \rightarrow d_{q+q'}(x, z)) : q, q' \in \mathbb{Q}_{\geq 0}\} \\
& \cup \{\forall x \bigwedge_{q \in \mathbb{Q}_{> 0}} \forall y (d_q(x, y) \rightarrow O_n(x, y)) : n \in \omega\} \\
& \cup \{\forall x \bigwedge_{q \in \mathbb{Q}_{> 0}} \bigwedge_{n \in \omega} \forall y (O_n(x, y) \rightarrow d_q(x, y))\}
\end{aligned}$$

Clearly all $T' \in [T]^{< \kappa}$ are satisfiable.

Since κ is ω_1 -strongly compact, it follows that T is also satisfiable. Let M be a model of T . Then $d : X^2 \rightarrow \mathbb{R}$ defined by

$$(\aleph_{2.10}) \quad d(a, b) = \inf\{q \in \mathbb{Q} : M \models d_q(c_a, c_b)\} \text{ for } a, b \in X$$

is a metric on X generating the topology of (X, \mathcal{O}) . □ (Lemma 2.3)

3 Preservation and non-preservation of stationarity of subsets of $\mathcal{P}_\kappa(\lambda)$

In the following, we show that the closedness of posets cannot be used to establish reflection principles concerning the stationarity of subsets of $\mathcal{P}_\kappa(\lambda)$ for $\kappa > \aleph_1$ in the generic extensions. At least, not in a straight-forward generalization of the usage of σ -closed posets in a forcing argument to obtain reflection properties on stationarity of subsets of $\mathcal{P}_{\aleph_1}(\lambda)$ in the generic extensions.

Actually, the examples of preservation and non-preservation of stationarity of subsets of $\mathcal{P}_\kappa(\lambda)$ in this section explain, why we need a mixed support iteration plus one further step with chain condition in connection with the following Lemma 3.1 to establish (some of the) results in Section 6 but not in a much simpler way.

It is well-known that ccc posets and σ -closed posets are proper. This means that such posets preserve stationarity of subsets of $\mathcal{P}_{\aleph_1}(\lambda)$ for any uncountable λ . For posets with κ -cc for regular cardinal $\kappa > \aleph_1$ we still have a corresponding lemma:

Lemma 3.1 *Suppose that κ is a regular uncountable cardinal and $\lambda \geq \kappa$. If $S \subseteq \mathcal{P}_\kappa(\lambda)$ is stationary and \mathbb{P} is a κ -cc poset, then we have $\Vdash_{\mathbb{P}}$ “ \check{S} is stationary”.*

Proof. Suppose that \mathcal{C} is a \mathbb{P} -name with $\Vdash_{\mathbb{P}} \text{“}\mathcal{C} \text{ is a club in } \mathcal{P}_{\kappa}(\lambda)\text{”}$.

In V , let $\mathcal{C} = \{C \in \mathcal{P}_{\kappa}(\lambda) : \Vdash_{\mathbb{P}} \check{C} \in \mathcal{C}\}$. Then \mathcal{C} is club by the κ -cc of \mathbb{P} . Hence $S \cap \mathcal{C} \neq \emptyset$. Since $\Vdash_{\mathbb{P}} \check{C} \subseteq \mathcal{C}$, it follows that $\Vdash_{\mathbb{P}} \check{S} \cap \mathcal{C} \neq \emptyset$. □ (Lemma 3.1)

In contrast, κ -closed poset can destroy stationarity of ground model stationary set $\subseteq \mathcal{P}_{\kappa}(\lambda)$ if $\kappa > \aleph_1$. This makes consistency proofs of stationary reflection of stationary subsets of $\mathcal{P}_{\kappa}(\lambda)$ for $\kappa \geq \aleph_2$ more involved. In the following, we shall examine situations where the stationarity of some subset of $\mathcal{P}_{\kappa}(\lambda)$ for $\kappa \geq \aleph_2$ is not preserved by a standard κ -closed poset.

For cardinal κ and a regular cardinal $\nu < \kappa$ we denote

$$(3.1) \quad E_{\nu}^{\kappa} = \{\alpha \in \kappa : cf(\alpha) = \nu\}.$$

III:ref-1-0

The following Lemma is used for our first example of non-preservation of stationarity in Proposition 3.3.

Lemma 3.2 *Suppose that κ is a regular cardinal with $\kappa \geq \aleph_2$ and $X \supseteq \kappa^+$. Then, for any distinct regular $\nu, \mu < \kappa$,*

III:L-preserv-

0

$$(3.2) \quad S = \{x \in \mathcal{P}_{\kappa}(X) : \kappa \cap x \in E_{\nu}^{\kappa}, cf(\sup(\kappa^+ \cap x)) = \mu\}$$

III:preserv-0

is stationary in $\mathcal{P}_{\kappa}(X)$.

Proof. Suppose that $C \subseteq \mathcal{P}_{\kappa}(X)$ is a club. Let $f : [X]^{<\aleph_0} \rightarrow X$ be such that $C\ell^*(f) = \{x \in \mathcal{P}_{\kappa}(X) : x \cap \kappa \in \kappa, x \text{ is closed with respect to } f\} \subseteq C$.

Let $X_0 \subseteq X$ be such that X_0 is closed with respect to f and $\kappa^+ \cap X_0 \in E_{\mu}^{\kappa^+}$. Let $\delta = \kappa^+ \cap X_0$.

Let $\langle x_{\xi} : \xi < \nu \rangle$ be a continuously increasing sequence in $\mathcal{P}_{\kappa}(X_0)$ such that

$$(3.3) \quad x_{\xi} \text{ is closed with respect to } f \text{ for all } \xi < \nu;$$

III:preserv-1

$$(3.4) \quad \sup(\kappa^+ \cap x_0) = \delta; \text{ and}$$

III:preserv-2

$$(3.5) \quad \sup(\kappa \cap x_{\xi}) + 1 \subseteq x_{\xi+1} \text{ for all } \xi < \nu.$$

III:preserv-3

Note that this construction is possible since κ is regular and $\nu, \mu < \kappa$.

Let $x = \bigcup_{\xi < \nu} x_{\xi}$. Then

$$(3.6) \quad x \text{ is closed with respect to } f; \quad (\text{by (3.3)})$$

III:preserv-4

$$(3.7) \quad \sup(\kappa^+ \cap x) = \delta; \quad (\text{by (3.4)})$$

III:preserv-5

$$(3.8) \quad \kappa \cap x \in \kappa \text{ and } cf(\kappa \cap x) = \nu. \quad (\text{by (3.5)})$$

III:preserv-6

$x \in S$ by (3.7) and (3.8). $x \in C\ell^*(f)$ by (3.6) and (3.8). Thus we have $\emptyset \neq S \cap C\ell^*(f) \subseteq S \cap C$. □ (Lemma 3.2)

For a regular cardinal, $\text{Add}(\kappa)$ denotes the set ${}^{\kappa}>2$ with the reverse inclusion. We denote with $\text{Col}(\kappa, \kappa^+)$ the set ${}^{\kappa}>\kappa^+$ with the reverse inclusion. $\text{Add}(\kappa)$ and $\text{Col}(\kappa, \kappa^+)$ are forcing equivalent to $\text{Fn}(\kappa, 2, \kappa)$ and $\text{Fn}(\kappa, \kappa^+, \kappa)$ in Kunen's notation in [Kunen[17]], respectively. The posets isomorphic to latter two posets are also denoted as $\text{Col}(\kappa, \{\kappa\})$ and $\text{Col}(\kappa, \{\kappa^+\})$ respectively, in the notation of [Kanamori[14]]. Both of the posets are κ -closed. $\text{Add}(\kappa)$ adds a new subset of κ while κ^+ is preserved if $\kappa^{<\kappa} = \kappa$. $\text{Col}(\kappa, \kappa^+)$ collapses κ^+ and makes it of cardinality and cofinality κ .

Proposition 3.3 *Suppose that κ is a regular cardinal $\geq \aleph_2$ and $X \supseteq \kappa^+$. Then, there is a stationary $S \subseteq \mathcal{P}_\kappa(X)$, such that $\Vdash_{\text{Col}(\kappa, \kappa^+)} \check{S}$ is not stationary in $\mathcal{P}_\kappa(X)$.*

Proof. In \mathbb{V} , let

$$(3.9) \quad S = \{x \in \mathcal{P}_\kappa(X) : x \cap \kappa \text{ and } \sup(x \cap \kappa^+) \text{ are limit ordinals, and } cf(x \cap \kappa) \neq cf(\sup(x \cap \kappa^+))\}.$$

S is a stationary subset of $\mathcal{P}_\kappa(X)$ by Lemma 3.2. We show that $\text{Col}(\kappa, \kappa^+)$ forces that S is not stationary.

Suppose that \mathbb{G} is a $(\mathbb{V}, \text{Col}(\kappa, \kappa^+))$ -generic filter. Note that, by $<\kappa$ -closedness, $\text{Col}(\kappa, \kappa^+)$ does not add any new sets of size $<\kappa$. Thus $\mathcal{P}_\kappa(X)^\mathbb{V} = \mathcal{P}_\kappa(X)^{\mathbb{V}[\mathbb{G}]}$, all cofinalities $<\kappa$ are preserved in the generic extension $\mathbb{V}[\mathbb{G}]$, and $cf(\mu) = \kappa$ in $\mathbb{V}[\mathbb{G}]$ for $\mu = (\kappa^+)^\mathbb{V}$.

In $\mathbb{V}[\mathbb{G}]$, let $\langle \gamma_\alpha : \alpha < \kappa \rangle$ be a continuously increasing sequence of ordinals cofinal in the ordinal μ . Let

$$(3.10) \quad C = \{x \in \mathcal{P}_\kappa(X) : x \cap \kappa \text{ and } \sup(x \cap \mu) \text{ are limit ordinals, and } \sup(x \cap \mu) = \gamma_{x \cap \kappa}\}.$$

Then C is a club in $\mathcal{P}_\kappa(X)$ and $C \cap S = \emptyset$.

□ (Proposition 3.3)

In Proposition 3.3, the crucial fact which made the set S non-stationary in the generic extension was that the cardinal κ^+ is collapsed to be an ordinal of cofinality κ . However, stationarity of $\mathcal{P}_\kappa(\lambda)$ can be also destroyed by a $<\kappa$ -closed forcing without collapsing cardinals:

Proposition 3.4 *Suppose that κ is a supercompact and $|X| \geq 2^\kappa$. Then there is a stationary $S \subseteq \mathcal{P}_\kappa(X)$ such that $\Vdash_{\text{Add}(\kappa)} \check{S}$ is not a stationary subset of $\mathcal{P}_\kappa(X)$.*

Note that $|\text{Add}(\kappa)| = \kappa$ since κ is inaccessible and hence $\text{Add}(\kappa)$ is κ^+ -cc. Thus $\text{Add}(\kappa)$ here preserves cardinals and cofinality.

Proof. Let $\lambda = |X|$. Without loss of generality, we may assume that $X = \lambda$. In \mathbb{V} , let $\vec{B} = \langle B_\alpha : \alpha < \lambda \rangle$ be an enumeration of $\mathcal{P}(\kappa)$ and let

$$(3.11) \quad S = \{x \in \mathcal{P}_\kappa(\lambda) : \begin{array}{l} \text{(a)} \quad \kappa \cap x \in \kappa, \text{ and} \\ \text{(b)} \quad \{B_\alpha \cap (x \cap \kappa) : \alpha \in x\} = \mathcal{P}(\kappa \cap x)\}. \end{array}\} \quad \text{III:preserv-7}$$

Claim 3.4.1 $S \in \mathcal{U}$ for any normal ultrafilter \mathcal{U} over $\mathcal{P}_\kappa(\lambda)$. III:Cl-preserv-0

⊢ Suppose that \mathcal{U} is a normal ultrafilter over $\mathcal{P}_\kappa(\lambda)$. It is enough to show that $j_{\mathcal{U}}''\lambda \in j_{\mathcal{U}}(S)$ where $j_{\mathcal{U}} : \mathbb{V} \xrightarrow{\cong} M$ is the elementary embedding induced by \mathcal{U} .

We have

$$(3.12) \quad \begin{aligned} (j_{\mathcal{U}}''\lambda) \cap j_{\mathcal{U}}(\kappa) &= \{j_{\mathcal{U}}(\alpha) : \alpha < \lambda, j_{\mathcal{U}}(\alpha) < j_{\mathcal{U}}(\kappa)\} \\ &= \{j_{\mathcal{U}}(\alpha) : \alpha < \kappa\} = \{\alpha : \alpha \in \kappa\} = \kappa \in j_{\mathcal{U}}(\kappa). \end{aligned} \quad \begin{array}{l} \text{III:preserv-7-} \\ 0 \end{array}$$

For $\beta \in j_{\mathcal{U}}''\lambda$ with $\beta = j_{\mathcal{U}}(\alpha)$ for $\alpha \in \lambda$, $j_{\mathcal{U}}(\vec{B})(\beta) \cap (j_{\mathcal{U}}''\lambda \cap j_{\mathcal{U}}(\kappa)) = j_{\mathcal{U}}(B_\alpha) \cap \kappa = B_\alpha$.

Thus we have

$$(3.13) \quad \begin{aligned} \{j_{\mathcal{U}}(\vec{B})(\beta) \cap (j_{\mathcal{U}}''\lambda \cap j_{\mathcal{U}}(\kappa)) : \beta \in j_{\mathcal{U}}''\lambda\} &= \{B_\alpha : \alpha < \lambda\} \\ &= \mathcal{P}(\kappa) = \mathcal{P}(j_{\mathcal{U}}''\lambda \cap j_{\mathcal{U}}(\kappa)). \end{aligned} \quad \begin{array}{l} \text{III:preserv-7-} \\ 1 \end{array}$$

By elementarity, (3.12) and (3.13) imply $j_{\mathcal{U}}''\lambda \in j(S)$. ⊢ (Claim 3.4.1)

Since any normal filter over $\mathcal{P}_\kappa(\lambda)$ contains all club sets and hence it consists of stationary sets, and since there are normal ultrafilters over $\mathcal{P}_\kappa(\lambda)$ because κ is supercompact, we conclude that S is a stationary subset of $\mathcal{P}_\kappa(\lambda)$.

Thus the next Claim shows that S is as desired:

Claim 3.4.2 $\Vdash_{\text{Add}(\kappa)} \text{“} S \text{ is not stationary in } \mathcal{P}_\kappa(\lambda)\text{”}$.

⊢ Suppose that \mathbb{G} is a $(\mathbb{V}, \text{Add}(\kappa))$ -generic filter. In $\mathbb{V}[\mathbb{G}]$, we have $\bigcup \mathbb{G} : \kappa \rightarrow 2$. Let $A = (\bigcup \mathbb{G})^{-1} \{1\}$. By genericity, A is a new subset of κ . Let $F : \lambda \rightarrow \kappa$ be defined by $F(\alpha) = \min(B_\alpha \triangle A)$ for $\alpha < \lambda$. F is well-defined since $B_\alpha \neq A$ for all $\alpha < \lambda$. We show that $S \cap C_F = \emptyset$ where C_F is the club set defined by

$$(3.14) \quad C_F = \{x \in \mathcal{P}_\kappa(\lambda) : x \text{ is closed with respect to } F\}. \quad \text{III:preserv-8}$$

Suppose that $x \in S$. By (b) in (3.11) and since $A \cap x \in \mathcal{P}(x \cap \kappa)^\mathbb{V}$, there is an $\alpha^* \in x$ such that $B_{\alpha^*} \cap x = A \cap x$. But this implies that $F(\alpha^*) = \min(B_{\alpha^*} \triangle A) \notin x$. Thus, x is not closed with respect to F and $x \notin C_F$. ⊢ (Claim 3.4.2)

□ (Proposition 3.4)

The non-preservation of stationarity of subsets of $\mathcal{P}_\kappa(\lambda)$ along the line of the results above is further studied in [Sakai[19]].

4 Two dimensional Laver-generic large cardinals

III:two

For properties \mathfrak{P} and \mathfrak{Q} of posets, a cardinal κ is *Laver-generically supercompact* for $(\mathfrak{P}, \mathfrak{Q})$ if, for any poset \mathbb{P} with $\mathbb{P} \models \mathfrak{P}$, (\mathbb{V}, \mathbb{P}) -generic \mathbb{G} , and a cardinal λ , there are a \mathbb{P} -name \mathbb{Q} of a poset with $\Vdash_{\mathbb{P}} \mathbb{Q} \models \mathfrak{Q}$, and a $(\mathbb{V}, \mathbb{P} * \mathbb{Q})$ -generic \mathbb{H} with $i''\mathbb{G} \subseteq \mathbb{H}$ where $i : \mathbb{P} \rightarrow \mathbb{P} * \mathbb{Q}$ is the canonical complete embedding, such that there are $j, M \subseteq \mathbb{V}[\mathbb{H}]$ with

$$(4.1) \quad M \text{ is a transitive class in } \mathbb{V}[\mathbb{H}];$$

III:gen-large-

$$(4.2) \quad j : \mathbb{V} \xrightarrow{\simeq} M;$$

1-0

$$(4.3) \quad \text{crit}(j) = \kappa \text{ and } j(\kappa) > \lambda;$$

III:gen-large-

$$(4.4) \quad \mathbb{P}, \mathbb{H} \in M, \text{ and}$$

2

$$(4.5) \quad j''\lambda \in M.$$

III:gen-large-

3

III:gen-large-

κ is *strongly Laver-generically supercompact* for $(\mathfrak{P}, \mathfrak{Q})$ if M in the definition of the Laver-generic supercompactness for $(\mathfrak{P}, \mathfrak{Q})$ additionally satisfies

4

III:gen-large-

6

$$(4.6) \quad ([M]^{\aleph_0})^{\mathbb{V}[\mathbb{H}]} \subseteq M.$$

III:gen-large-

6-0

If $\{\mathbb{P} : \mathbb{P} \models \mathfrak{P}\}$ contains only trivial posets, then the (strongly) Laver-generic supercompactness for $(\mathfrak{P}, \mathfrak{Q})$ coincides with the (strongly) generic supercompactness by posets satisfying \mathfrak{Q} . If \mathfrak{P} and \mathfrak{Q} are equivalent, and \mathfrak{P} is iterable, that is, if for every $\mathbb{P} \models \mathfrak{P}$ and \mathbb{P} -name \mathbb{Q} with $\Vdash_{\mathbb{P}} \mathbb{Q} \models \mathfrak{P}$, we have $\mathbb{P} * \mathbb{Q} \models \mathfrak{P}$, then the (strongly) Laver-generic supercompactness for $(\mathfrak{P}, \mathfrak{P})$ is closely related to the (strongly) Laver generic supercompactness for \mathbb{P} in the sense of [10] but may not be exactly the same notion.

In the following, both of the properties \mathfrak{P} and \mathfrak{Q} considered in connection with the Laver-generic supercompactness imply the properness of the poset. In such a case, the model of the Laver-generic supercompactness constructed by forcing starting from a supercompact cardinal usually satisfies this strong version of Laver-generic supercompactness as well. This is because of the following well-known fact:

Lemma 4.1 *Suppose that $M \subseteq \mathbb{V}$ is an inner model with*

III:P-gen-

large-a-0

$$(4.7) \quad [M]^{\aleph_0} \subseteq M.$$

III:gen-large-

If $\mathbb{P} \in M$ is proper and \mathbb{G} is a (\mathbb{V}, \mathbb{P}) -generic filter, then we have

6-1

$$(4.8) \quad ([M[\mathbb{G}]]^{\aleph_0})^{\mathbb{V}[\mathbb{G}]} \subseteq M[\mathbb{G}].$$

III:gen-large-

Proof. Let $a \in ([M[\mathbb{G}]]^{\aleph_0})^{\mathbb{V}[\mathbb{G}]}$ and \mathbb{a} be a \mathbb{P} -name of a . In \mathbb{V} , let θ be a sufficiently large regular cardinal, and let $N \prec \mathcal{H}(\theta)$ and $\mathbb{p} \in \mathbb{G}$ be such that

6-2

- (4.9) $|N| = \aleph_0$; III:gen-large-
- (4.10) $\underset{\sim}{a}, \mathbb{P} \in N$; 6-3
- (4.11) for any maximal antichain $A \subseteq \mathbb{P}$ with $A \in N$, $A \cap N$ is predense below \mathbb{P} . III:gen-large-
6-4
6-5

Let $f \in N$ be a \mathbb{P} -name such that $\Vdash_{\mathbb{P}} "f : \omega \rightarrow \underset{\sim}{a} \text{ is a surjection}"$. For each $n \in \omega$, let $A_n \in M$ be a maximal antichain such that, for each $\mathfrak{r} \in A_n$, there is a \mathbb{P} -name $\underset{\sim}{b}_{n,\mathfrak{r}} \in M$ such that $\mathfrak{r} \Vdash_{\mathbb{P}} "f(n) \equiv \underset{\sim}{b}_{n,\mathfrak{r}}"$. Note that we have $\underset{\sim}{b}_{n,\mathfrak{r}} \in M \cap N$ if $\mathfrak{r} \in N$, since $\underset{\sim}{b}_{n,\mathfrak{r}}$ is uniquely determined for each $\mathfrak{r} \in A_n$.

Let $\underset{\sim}{a}^* = \{ \langle \underset{\sim}{b}_{n,\mathfrak{r}}, \mathfrak{r} \rangle : n \in \omega, \mathfrak{r} \in A_n \cap N \}$. Then $\underset{\sim}{a}^* \in M$ by (4.7) and $\underset{\sim}{a}^*[\mathbb{G}] = a[\mathbb{G}]$. Thus, $a = a[\mathbb{G}] \in M[\mathbb{G}]$. □ (Lemma 4.1)

The condition (4.6) can be even replaced with

$$(4.6') \quad ([M]^{j(\kappa)})^{\mathbb{V}[\mathbb{H}]} \subseteq M, \quad \text{III:gen-large-}$$

if we consider Laver-generic superhugeness instead of Laver-generic supercompactness. 6-0

This can be seen by means of the following:

Lemma 4.2 *Suppose that M is an inner model of \mathbb{V} with* III:P-gen-

$$(4.12) \quad V \models "[M]^\mu \subseteq M" \quad \text{large-a-1}$$

for a regular μ . If $\mathbb{P} \in M$ is μ^+ -cc, then, for any (\mathbb{V}, \mathbb{P}) -generic \mathbb{G} , we have III:gen-large-
6-6

$$(4.13) \quad ([M[\mathbb{G}]]^\mu)^{\mathbb{V}[\mathbb{G}]} \subseteq M[\mathbb{G}]. \quad \text{III:gen-large-}$$

Proof. Note that $\mathbb{P} \subseteq M$ since M is transitive. Suppose $g \in (\mu M[\mathbb{G}])^{\mathbb{V}[\mathbb{G}]}$. We show that $g \in M[\mathbb{G}]$. Let $\underset{\sim}{g}$ be a \mathbb{P} -name of g . For each $\xi < \mu$, there is a maximal pairwise incompatible $A_\xi \subseteq \mathbb{P}$ such that, for each $\mathfrak{p} \in A_\xi$, there is a \mathbb{P} -name $\underset{\sim}{a}_{\xi,\mathfrak{p}} \in M$ such that $\mathfrak{p} \Vdash_{\mathbb{P}} "g(\xi) \equiv \underset{\sim}{a}_{\xi,\mathfrak{p}}"$. By the μ^+ -cc of \mathbb{P} , we have $|A_\xi| \leq \mu$ and hence $A_\xi \in M$ by (4.12). 6-7

Let

$$(4.14) \quad \underset{\sim}{a}_\xi = \{ \langle \underset{\sim}{b}, \mathfrak{q} \rangle : \mathfrak{q} \leq_{\mathbb{P}} \mathfrak{p} \text{ for some } \mathfrak{p} \in A_\xi, \underset{\sim}{b} \text{ is a canonical } \mathbb{P}\text{-name with } \mathfrak{q} \Vdash_{\mathbb{P}} "b \varepsilon \underset{\sim}{a}_{\xi,\mathfrak{p}}" \}. \quad \text{III:gen-large-}$$

$\underset{\sim}{a}_\xi \in M$ since it is definable from $\leq \mu$ many parameters from M and by (4.12). It is also clear by the definition above that $\Vdash_{\mathbb{P}} "g(\xi) \equiv \underset{\sim}{a}_\xi"$. Let 6-8

$$(4.15) \quad g^* = \{ \langle \langle \check{\xi}, a_\xi \rangle_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}} \rangle : \xi < \mu \}.$$

III:gen-large-

Then $g^* \in M$ by (4.12) and $M[\mathbb{G}] \ni g^*[\mathbb{G}] = \vec{g}[\mathbb{G}] = g$.

□ (Lemma 4.2)

6-9

Lemma 4.2 is used with a generic superhuge cardinal to produce the strong generic superhugeness. This can be seen as follows:

Suppose that κ is a superhuge cardinal and $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be an iteration with

$$(4.16) \quad |\mathbb{P}_\alpha| < \kappa \text{ for all } \alpha < \kappa.$$

III:gen-large-

Let \mathbb{G}_κ be a $(\mathbb{V}, \mathbb{P}_\kappa)$ -generic filter.

6-10

For a given cardinal λ , let $j : \mathbb{V} \xrightarrow{\cong} M$ be an elementary embedding into an inner model M of \mathbb{V} such that $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $[M]^{j(\kappa)} \subseteq M$. Let $\mathbb{P}^* = j(\mathbb{P}_\kappa)$. By elementarity \mathbb{P}^* is the $j(\kappa)$ th iterand of the iteration $j(\vec{\mathbb{P}})$ in M . There is the canonical complete embedding $i : \mathbb{P}_\kappa \xrightarrow{\cong} \mathbb{P}^*$. Let \mathbb{G}^* be $(\mathbb{V}, \mathbb{P}^*)$ -generic filter with

$$(4.17) \quad i''\mathbb{G}_\kappa \subseteq \mathbb{G}^*.$$

III:gen-large-

By (4.16) and elementarity, we have $M \models “|\mathbb{P}^*| \leq j(\kappa)”$. Hence $|\mathbb{P}^*| \leq j(\kappa)$ and thus \mathbb{P}^* has $j(\kappa)^+$ -cc. By (4.17), j can be lifted to

6-11

$$(4.18) \quad \tilde{j} : V[\mathbb{G}_\kappa] \xrightarrow{\cong} M[\mathbb{G}^*]; \check{a}[\mathbb{G}_\kappa] \mapsto j(\check{a})[\mathbb{G}^*]$$

III:gen-large-

for \mathbb{P}_κ -names \check{a} . Now by Lemma 4.2, we have $([M[\mathbb{G}^*]]^{j(\kappa)})^{\mathbb{V}[\mathbb{G}^*]} \subseteq M[\mathbb{G}^*]$.

6-12

Based on the observations above, we define κ to be *strongly Laver-generically superhuge for the pair of properties* $(\mathfrak{P}, \mathfrak{Q})$ if, for any poset \mathbb{P} with $\mathbb{P} \models \mathfrak{P}$ and (\mathbb{V}, \mathbb{P}) -generic \mathbb{G} , there are a \mathbb{P} -name \mathbb{Q} of a poset with $\Vdash_{\mathbb{P}} “\mathbb{Q} \models \mathfrak{Q}”$ and a $(\mathbb{V}, \mathbb{P} * \mathbb{Q})$ -generic \mathbb{H} with $i''\mathbb{G} \subseteq \mathbb{H}$ where $i : \mathbb{P} \rightarrow \mathbb{P} * \mathbb{Q}$ is the canonical complete embedding, such that there are $j, M \subseteq \mathbb{V}[\mathbb{H}]$ with (4.1) \sim (4.4) and (4.6').

If $\{ \mathbb{P} : \mathbb{P} \models \mathfrak{P} \}$ contains only trivial posets, we shall say “*strongly superhuge for* \mathfrak{Q} ” instead of “strongly Laver-generically superhuge for $(\mathfrak{P}, \mathfrak{Q})$ ”.

The following is trivial.

III:P-gen-

Lemma 4.3 *Suppose that $\mathbb{P} \models \mathfrak{P}_0$ implies $\mathbb{P} \models \mathfrak{P}_1$ and $\mathbb{P} \models \mathfrak{Q}_1$ implies $\mathbb{P} \models \mathfrak{Q}_0$ for all posets \mathbb{P} (i.e. these implications are theorems in ZFC). If κ is (strongly) Laver-generically supercompact/superhuge for $(\mathfrak{P}_1, \mathfrak{Q}_1)$, then κ is (strongly) Laver-generically supercompact/huge for $(\mathfrak{P}_0, \mathfrak{Q}_0)$. □*

large-0

We call a property \mathfrak{P} of posets *iterable* if we can prove in ZFC that

$$(4.19) \quad \mathbb{P} * \mathbb{Q} \models \mathfrak{P} \text{ for any poset } \mathbb{P} \text{ with } \mathbb{P} \models \mathfrak{P} \text{ and } \mathbb{P}\text{-name } \mathbb{Q} \text{ of a poset with } \Vdash_{\mathbb{P}} “\mathbb{Q} \models \mathfrak{P}”.$$

Proposition 4.4 *Suppose that \mathfrak{P} is the property “forcing equivalent to a poset of the form $\text{Col}(\kappa, \mu)$ for some μ ”²⁾, \mathfrak{Q} is iterable and we can prove (in ZFC) that*

III:P-gen-
large-1

$$(4.20) \quad \forall P (P \text{ is a } \sigma\text{-directed closed poset} \rightarrow P \models \mathfrak{Q}).$$

III:gen-large-
6-13

If κ is strongly Laver-generically supercompact for $(\mathfrak{P}, \mathfrak{Q})$, then, for any $\mu \geq \kappa$, and for $\mathbb{P} = \text{Col}(\kappa, \mu)$, we have

$$(4.21) \quad \Vdash_{\mathbb{P}} \text{“}\kappa \text{ is generically supercompact by posets satisfying } \mathfrak{Q}\text{”}.$$

III:gen-large-

Proof. The proof is a typical application of the master condition argument.

7

Let $\mathbb{P} = \text{Col}(\kappa, \mu)$ for some $\mu \geq \kappa$. Note that $\mathbb{P} \models \mathfrak{P}$. Let \mathbb{G} be an arbitrary (\mathbb{V}, \mathbb{P}) -generic filter. We have to show that

$$(N4.1) \quad \mathbb{V}[\mathbb{G}] \models \text{“}\kappa \text{ is generic supercompact by posets with } \mathfrak{Q}\text{”}.$$

III:gen-large-

Let $\theta \geq \kappa$ be arbitrary and let $\lambda = \max\{\theta, \mu^{<\kappa}\}$. Let \mathbb{Q} be a \mathbb{P} -name with $\Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \text{ is a poset with } \mathfrak{Q}\text{”}$ such that there is a $(\mathbb{V}, \mathbb{P} * \mathbb{Q})$ -generic filter \mathbb{H} such that $i''\mathbb{G} \subseteq \mathbb{H}$ for the canonical complete embedding $i : \mathbb{P} \rightarrow \mathbb{P} * \mathbb{Q}$, with $j, M \subseteq \mathbb{V}[\mathbb{H}]$ such that (4.1) \sim (4.5) and (4.6) hold.

7-a

By (4.4), we have $\mathbb{G} \in M$. Let $\mathbb{R} = j(\mathbb{P})$. By elementarity,

$$(4.22) \quad M \models \text{“}\mathbb{R} \text{ is } < j(\kappa)\text{-directed closed”}.$$

III:gen-large-

By (4.22) and (4.6), $\mathbb{V}[\mathbb{H}] \models \text{“}\mathbb{R} \text{ is } \sigma\text{-directed closed”}$. Thus, $\mathbb{V}[\mathbb{H}] \models \text{“}\mathbb{R} \models \mathfrak{Q}\text{”}$ by (4.20).

7-0

By (4.3), $M \models |j''\mathbb{G}| < j(\kappa)$. Hence, by (4.22), there is (a master condition) $\mathfrak{r} \in \mathbb{R}$ such that $M \models \mathfrak{r} \leq_{\mathbb{R}} j''\mathbb{G}$.

Let \mathbb{K} be a $(\mathbb{V}[\mathbb{H}], \mathbb{R})$ -generic filter with $\mathfrak{r} \in \mathbb{K}$. Then

$$(4.23) \quad \tilde{j} : \mathbb{V}[\mathbb{G}] \xrightarrow{\sim} M[\mathbb{K}]; a^{\mathbb{G}} \mapsto j(a)^{\mathbb{K}}$$

III:gen-large-

is well defined and $j \subseteq \tilde{j}$. In particular we have $\kappa = \text{crit}(\tilde{j})$, $\tilde{j} > \lambda$ and $\tilde{j}''\lambda \in M[\mathbb{K}]$. we have $\mathbb{V}[\mathbb{H}] \models \text{“}\mathbb{R} \text{ is } \sigma\text{-directed closed”}$ by (4.22), (4.6), and since $M[\mathbb{K}] \models \text{“}\mathbb{R} \text{ is } \sigma\text{-directed closed”}$ by elementarity. Thus, in $\mathbb{V}[\mathbb{G}]$, letting $\mathbb{Q} = \mathbb{Q}[\mathbb{G}]$, with the \mathbb{Q} -name \mathbb{R} corresponding to \mathbb{R} such that $\Vdash_{\mathbb{Q}} \text{“}\mathbb{R} \text{ is } < \theta\text{-directed closed”}$, $\mathbb{Q} * \mathbb{R}$ satisfies \mathfrak{Q} and it induces a generic elementary embedding for generic λ -supercompactness. Since θ was arbitrary, it follows that (N4.1) holds. □ (Proposition 4.4)

8

III:P-gen-

large-2

²⁾In this section, we are back to Kanamori’s notation of collapsing posets. $\text{Col}(\kappa, \lambda)$ for an inaccessible λ is thus the poset collapsing all cardinals strictly between κ and λ by conditions of size $< \kappa$.

Corollary 4.5 *Suppose that κ is strongly Laver-generically supercompact for $(\mathfrak{P}, \mathfrak{Q})$ where \mathfrak{P} is the property “forcing equivalent to a poset of the form $\text{Col}(\kappa, \mu)$ for some μ ” and \mathfrak{Q} is “proper”. Suppose further that $\kappa_1 > \kappa$ is a supercompact and let $\mathbb{P} = \text{Col}(\kappa, \kappa_1)$. Then*

- (a) $\Vdash_{\mathbb{P}}$ “ κ is generically supercompact by proper posets”;
- (b) $\Vdash_{\mathbb{P}}$ “ κ^+ is generically supercompact by $< \kappa$ -closed posets”;
- (c) $\Vdash_{\mathbb{P}}$ “ $\text{SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\aleph_0} < \kappa)$ ”; and
- (d) $\Vdash_{\mathbb{P}}$ “ $\text{GRP}^{< \kappa}(\leq \kappa)$ ”.

Proof. (a): By Proposition 4.4.

(b): By Lemma 4.10 in [Fuchino, Sakai and Ottenbreit[9]].

(c): By (a) and, Theorem 2.10 and Propositions 3.1 in [Fuchino, Sakai and Ottenbreit[10]].

(d): By (b) above and Lemma 4.11 in [Fuchino, Sakai and Ottenbreit[9]].

□ (Corollary 4.5) !!!!

III:P-gen-
large-2-0

Proposition 4.6 *Suppose that \mathfrak{P} is the property “forcing equivalent to a poset of the form $\text{Col}(\kappa, \mu)$ for some μ ” and \mathfrak{Q} the property of posets such that*

$$(4.24) \quad \forall P \forall Q (\underset{\sim}{P} \models \mathfrak{P} \wedge \Vdash_P \text{“} \underset{\sim}{Q} \text{ is a } < \theta\text{-directed closed poset”} \\ \rightarrow P * \underset{\sim}{Q} \models \mathfrak{Q})$$

III:gen-large-
8-a

for a cardinal θ .

If κ is Laver-generically strongly superhuge for $(\mathfrak{P}, \mathfrak{Q})$, then, for any cardinal $\mu > \kappa$ and $\mathbb{P} = \text{Col}(\kappa, \mu)$, we have

$$(4.25) \quad \Vdash_{\mathbb{P}} \text{“} \kappa \text{ is strongly generically superhuge for } \mathfrak{Q}\text{”}.$$

III:gen-large-
8-a-a

Proof. Let \mathbb{G} be a (\mathbb{V}, \mathbb{P}) -generic filter. For cardinals λ , let $\lambda' = \max\{(\mu^{< \kappa})^+, \lambda, \theta\}$. In $\mathbb{V}[\mathbb{G}]$, let \mathbb{Q} be a poset with $\mathbb{Q} \models \mathfrak{Q}$ and \mathbb{H} a $(\mathbb{V}[\mathbb{G}], \mathbb{Q})$ -generic filter such that there are j , $M \subseteq \mathbb{V}[\mathbb{G}][\mathbb{H}]$ satisfying: M is a transitive class in $\mathbb{V}[\mathbb{G}][\mathbb{H}]$; $j : \mathbb{V} \xrightarrow{\cong} M$; $\text{crit}(j) = \kappa$; $j(\kappa) > \lambda'$; $\mathbb{G}, \mathbb{H} \in M$; and

$$(4.26) \quad ([M]^{j(\kappa)})^{\mathbb{V}[\mathbb{G}][\mathbb{H}]} \subseteq M.$$

III:P-gen-
large-2-1

Let $\mathbb{P}^* = j(\mathbb{P})$. By elementarity and (4.26), we have $\mathbb{P}^* = \text{Col}(j(\kappa), j(\mu))$ in $\mathbb{V}[\mathbb{G}][\mathbb{H}]$. Since \mathbb{P}^* is $j(\kappa)$ -directed closed (in M or in $\mathbb{V}[\mathbb{G}][\mathbb{H}]$), there is (a master condition) $\mathfrak{r} \in \mathbb{P}^*$ with $\mathfrak{r} \leq_{\mathbb{P}^*} j(\mathfrak{p})$ for all $\mathfrak{p} \in \mathbb{G}$. Let \mathbb{G}^* be $(\mathbb{V}[\mathbb{G}][\mathbb{H}], \mathbb{P}^*)$ -generic filter with $\mathfrak{r} \in \mathbb{G}^*$. Then j is lifted to

$$(4.27) \quad \tilde{j} : \mathbb{V}[\mathbb{G}] \xrightarrow{\cong} M[\mathbb{G}^*]; \underset{\sim}{a}[\mathbb{G}] \mapsto j(\underset{\sim}{a})[\mathbb{G}^*]$$

III:P-gen-
large-2-2

for \mathbb{P} -name a . $j \subseteq \tilde{j}$ and hence it is clear that \tilde{j} satisfies: $\text{crit}(\tilde{j}) = \kappa$; $\tilde{j}(\kappa) > \lambda$; $\mathbb{G}^* \in M[\mathbb{G}^*]$ since $\mathbb{G}^* \in M$ by (4.26) and $M \models \text{ZFC}$ where \mathbb{G}^* is the standard \mathbb{P}^* -name of \mathbb{G}^* . We also have

$$(4.28) \quad ([M[\mathbb{G}^*]]^{j(\kappa)})^{\mathbb{V}[\mathbb{G}][\mathbb{H}][\mathbb{G}^*]} \subseteq M[\mathbb{G}^*]$$

III:P-gen-

by Lemma 4.2. This shows that (4.25) holds. \square (Proposition 4.6)

large-2-3

III:P-gen-

Proposition 4.7 (1) *Let \mathfrak{P} be the property “forcing equivalent to a poset of the form $\text{Col}(\kappa, \mu)$ for some μ ” and \mathfrak{Q}_θ is the property “forcing equivalent to a regular sub-poset of the completion of a poset of the form ‘generalized Cohen poset \times $<$ θ -closed poset’”. If κ is strongly Laver-generically supercompact for $(\mathfrak{P}, \mathfrak{Q}_\theta)$ for all $\theta \in \text{Card}$, then, for any cardinal μ and $\mathbb{P} = \text{Col}(\kappa, \mu)$, we have*

large-3

$$(4.29) \quad \Vdash_{\mathbb{P}} \text{“HH}(< \kappa)\text{”}.$$

(2) *Let \mathfrak{P} be the property “forcing equivalent to a poset of the form $\text{Col}(\kappa, \mu)$ for some μ ” and \mathfrak{Q}_θ is the property “forcing equivalent to a regular sub-poset of the completion of the poset of the form ‘ccc poset \times $<$ θ -closed poset’”. If κ is strongly Laver-generically supercompact for $(\mathfrak{P}, \mathfrak{Q}_\theta)$ for all $\theta \in \text{Card}$, then, for any cardinal μ and $\mathbb{P} = \text{Col}(\kappa, \mu)$, we have*

$$(4.30) \quad \Vdash_{\mathbb{P}} \text{“SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\text{PKL}}, < \kappa)\text{”}.$$

Proof. (1): Suppose that $\mathbb{P} = \text{Col}(\kappa, \mu)$ and \mathbb{G} is a (\mathbb{V}, \mathbb{P}) -generic filter. In $\mathbb{V}[\mathbb{G}]$, let $X = \langle X, \tau \rangle$ be a non-metrizable topological space such that

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p.19,20

$$(4.31) \quad \chi(a, X) < \kappa \text{ for all } a \in X.$$

III:gen-large-

Let $\lambda_0 = |X|$, $\theta = \lambda = \max\{(2^{\lambda_0})^+, \mu^{< \kappa}\}$ and let $j : V \xrightarrow{\cong} M \subseteq \mathbb{V}[\mathbb{G}][\mathbb{H}]$ be such that $\text{crit}(j) = \kappa$,

8-a-0

$$(4.32) \quad j(\kappa) > \lambda, j''\lambda \in M, \text{ and}$$

III:gen-large-

$$(4.33) \quad ([M]^{\aleph_0})^{\mathbb{V}[\mathbb{G}][\mathbb{H}]} \subseteq M,$$

8-a-1

III:gen-large-

where $\mathbb{H} = \tilde{\mathbb{H}} \cap \mathbb{Q}$ for a $(\mathbb{V}[\mathbb{G}], \tilde{\mathbb{Q}})$ -generic filter $\tilde{\mathbb{H}}$ for a poset $\tilde{\mathbb{Q}}$ in $\mathbb{V}[\mathbb{G}]$ of the form

8-0

$$(4.34) \quad \tilde{\mathbb{Q}} \sim \text{generalized Cohen poset} \times < \theta\text{-closed poset}$$

III:gen-large-

and $\mathbb{Q} \leq \tilde{\mathbb{Q}}$.

9

Let $\mathbb{P}^* = j(\mathbb{P})$. By elementarity $M \models \text{“}\mathbb{P}^* \text{ is } j(\kappa)\text{-directed closed”}$. Since $|\mathbb{P}| \leq \lambda$, $j''\mathbb{G} \in M[\mathbb{G}]$ by (4.32) (see Lemma 2.5 in [Fuchino, Ottenbreit and Sakai[10]]). Since $\lambda < j(\kappa)$ there is $\mathfrak{r} \in \mathbb{P}^*$ such that $\mathfrak{r} \leq_{\mathbb{P}^*} j(\mathfrak{p})$ for all $\mathfrak{p} \in \mathbb{G}$. Let \mathbb{G}^* be a $(\mathbb{V}[\mathbb{G}][\tilde{\mathbb{H}}], \mathbb{P}^*)$ -generic filter with $\mathfrak{r} \in \mathbb{G}^*$. j is then lifted to

$$(4.35) \quad \tilde{j} : \mathbf{V}[\mathbb{G}] \xrightarrow{\cong} M[\mathbb{G}^*] \subseteq \mathbf{V}[\mathbb{G}][\mathbb{H}][\mathbb{G}^*] \subseteq \mathbf{V}[\mathbb{G}][\tilde{\mathbb{H}}][\mathbb{G}^*];$$

$$\tilde{a}[\mathbb{G}] \mapsto j(\tilde{a})[\mathbb{G}^*].$$

III:gen-large-
10

Let $\tau_0 = \{j(O) \cap j''X : O \in \tau\}$. By (4.31) and since κ is the critical point of j , we have

$$(4.36) \quad M \models \langle j''X, \tau_0 \rangle \text{ is a subspace of } \langle j(X), j(\tau) \rangle.$$

III:gen-large-
11

We also have

$$(4.37) \quad M[\mathbb{G}^*] \models \langle j''X, \tau_0 \rangle \text{ is homeomorphic to } \langle X, \tau \rangle.$$

III:gen-large-
12

Hence the same property holds in $\mathbf{V}[\mathbb{G}][\tilde{\mathbb{H}}][\mathbb{G}^*]$.

Now generalized Cohen poset part of $\tilde{\mathbb{H}}$ preserve the non-metrizability of $\langle X, \tau \rangle$ by Theorem 1.1. By the $< \theta$ -closed part of $\tilde{\mathbb{H}}$ no new metric on X is added. Hence $\mathbf{V}[\mathbb{G}][\tilde{\mathbb{H}}] \models \langle X, \tau \rangle \text{ is non-metrizable}$. It follows that $M \models \langle X, \tau \rangle \text{ is non-metrizable}$ and hence by $< \lambda$ -closedness of \mathbb{P}^* , it follows that $M[\mathbb{G}^*] \models \langle X, \tau \rangle \text{ is non-metrizable}$. Thus by (4.37), $M[\mathbb{G}^*] \models \langle \tilde{j}''X, \tau_0 \rangle \text{ is non-metrizable}$. Thus

$$(4.38) \quad M[\mathbb{G}^*] \models \tilde{j}(X) \text{ has a non-metrizable subspace } Y$$

$$\text{of cardinality } < \tilde{j}(\kappa).$$

By elementarity of \tilde{j} it follows that

$$(4.39) \quad \mathbf{V}[\mathbb{G}] \models X \text{ has a non-metrizable subspace } Y \text{ of cardinality } < \kappa.$$

(2): The proof is done similarly to (1), by using Lemma 3.1 in place of Theorem 1.1. □ (Proposition 4.7)

III:P-gen-
large-3-0

Lemma 4.8 *Let \mathfrak{Q}_θ for a cardinal θ be as in Proposition 4.7, (2) and assume that κ is strongly generically superhuge for \mathfrak{Q}_θ for all $\theta \in \text{Card}$. Then, for any $\lambda \geq \kappa$, $\mathcal{P}_\kappa(\lambda)$ carries a σ -saturated normal ideal.*

Proof. Let $\lambda \geq \kappa$ and let $\mathbb{Q} \leq \text{RO}(\mathbb{S} \times \mathbb{T})$ be such that \mathbb{S} is $< (2^{2^{\lambda < \kappa}})^+$ -closed poset, \mathbb{T} is ccc, and that there are a (\mathbf{V}, \mathbb{Q}) -generic filter \mathbb{H} and $j, M \subseteq \mathbf{V}[\mathbb{H}]$ such that M is an inner model in $\mathbf{V}[\mathbb{H}]$, $j : \mathbf{V} \xrightarrow{\cong} M$, $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, $(j''\lambda \in M)$ and $([M]^{j(\kappa)})^{\mathbf{V}[\mathbb{H}]} \subseteq M$. Let \mathbb{K} be a (\mathbf{V}, \mathbb{S}) -generic filter and \mathbb{L} a $(\mathbf{V}[\mathbb{K}], \mathbb{T})$ -generic filter such that $\mathbf{V}[\mathbb{H}] \subseteq \mathbf{V}[\mathbb{K}][\mathbb{L}]$. In $\mathbf{V}[\mathbb{K}][\mathbb{L}]$,

$$(4.40) \quad \mathcal{I} = \{X \in (\mathcal{P}(\mathcal{P}_\kappa(\lambda)^\mathbf{V}))^\mathbf{V} : j''\lambda \notin j(X)\}$$

is a V -normal ideal. Since $\mathbf{V}[\mathbb{K}] \models \mathbb{T} \text{ is ccc}$, it follows that, in $\mathbf{V}[\mathbb{K}]$, $\mathcal{I}' = \{X \in (\mathcal{P}(\mathcal{P}_\kappa(\lambda)^\mathbf{V}))^\mathbf{V} : \Vdash_{\mathbb{T}} \check{X} \in \tilde{\mathcal{I}}\}$ is a σ -saturated V -normal ideal for \mathbb{T} -name $\tilde{\mathcal{I}}$ of \mathcal{I} . Now, by the closedness of \mathbb{S} , $\mathcal{I}' \in \mathbf{V}$ and \mathcal{I}' is a σ -saturated normal ideal in \mathbf{V} .

□ (Lemma 4.8) !!!!

Proposition 4.9 *Let \mathfrak{P} and \mathfrak{Q}_θ for a cardinal θ be as in Proposition 4.7, (2). If κ is strongly Laver-generically superhuge for $(\mathfrak{P}, \mathfrak{Q}_\theta)$ for all cardinal θ , then, for any $\lambda \geq \kappa$, and $\mathbb{P} = \text{Col}(\kappa, \lambda)$,*

$$(4.41) \quad \Vdash_{\mathbb{P}} \text{“} \mathcal{P}_\kappa(\lambda) \text{ carries a } \sigma\text{-saturated normal ideal”}.$$

Proof. By Proposition 4.6 and Lemma 4.8.

□ (Corollary 4.9)

5 Mixed support iteration

The construction of the mixed support iteration we give here is similar to the one given in [Krueger[15, 16]]. Nevertheless, we will examine the details of our construction in the following, since there are a couple of points organized differently from [Krueger[15, 16]].

In this section, κ is always a fixed supercompact cardinal and $f : \kappa \rightarrow \mathbf{V}_\kappa$ is a Laver function, i.e. a function satisfying:

$$(5.1) \quad \text{for any set } a \text{ and any } \lambda \geq \kappa, \text{ there is } j : V \overset{\cong}{\rightarrow} M \text{ such that } \text{crit}(j) = \kappa, \text{ III:msi-a-0}$$

$$j(\kappa) > \lambda, [M]^\lambda \subseteq M \text{ and } j(f)(\kappa) = a$$

(see e.g. Theorem 20.21 in [Jech[13]]).

Let $\bar{f} : \kappa \rightarrow \kappa$ be defined by

$$(5.2) \quad \bar{f}(\alpha) = | \text{trcl}(f(\alpha)) | \text{ for } \alpha < \kappa. \text{ III:msi-a-1}$$

Let

$$(5.3) \quad S = \{ \alpha < \kappa : \alpha \text{ is a strongly Mahlo cardinal} \text{ III:msi-0}$$

$$\text{closed with respect to } \bar{f} \}, \text{ and let}$$

$$(5.4) \quad T = \kappa \setminus S. \text{ III:msi-1-a}$$

Let $\nu : \kappa \rightarrow \kappa$ be the mapping defined by

$$(5.5) \quad \nu(\alpha) = \min(S \setminus (\alpha + 1)) \text{ for } \alpha \in \kappa. \text{ III:msi-1-0}$$

We treat iterations here as in [Jech[13]] such that elements of α th step \mathbb{P}_α of an iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ are sequences of length α .

Let $\langle \mathbb{O}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be a finite support iteration of ccc posets which will be further specified later. This preparatory iteration should satisfy the following conditions:

$$(5.6) \quad \mathbb{R}_\alpha \in V_{\nu(\alpha)}, \text{ and} \quad \text{III:msi-2}$$

$$(5.7) \quad \Vdash_{\mathbb{O}_\alpha} \text{“} \mathbb{R}_\alpha \equiv \{\mathbb{1}_{\mathbb{R}_\alpha}\} \text{”}, \text{ for all } \alpha \in S. \quad \text{III:msi-3}$$

We denote the canonical embeddings of \mathbb{O}_α into \mathbb{O}_β for $\alpha \leq \beta \leq \kappa$ by $i_{\alpha,\beta}^*$. Thus, $i_{\alpha,\beta}^*$ is the mapping defined by $i_{\alpha,\beta}^*(\mathbb{P}) = \mathbb{P} \cup \vec{\mathbb{1}}_{\alpha,\beta}$, where $\vec{\mathbb{1}}_{\alpha,\beta}$ is the function g on $\beta \setminus \alpha$ with $g(\xi) = \mathbb{1}_{\mathbb{R}_\xi}$ for all $\xi \in \beta \setminus \alpha$.

$\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ is our final iteration which is specified once the preparatory iteration $\langle \mathbb{O}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ is fixed. The iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ is defined recursively in (A) and (B) below, together with the commutative systems of complete embeddings $\iota_\alpha : \mathbb{O}_\alpha \rightarrow \mathbb{P}_\alpha$, and $i_{\alpha,\beta} : \mathbb{P}_\alpha \rightarrow \mathbb{P}_\beta$ for $\alpha \leq \beta \leq \kappa$ which should satisfy

$$(5.8) \quad i_{\alpha,\alpha} = id_{\mathbb{P}_\alpha} \text{ for } \alpha \leq \kappa; \quad \text{III:msi-5}$$

$$(5.9) \quad \iota_\beta \circ i_{\alpha,\beta}^* = i_{\alpha,\beta} \circ \iota_\alpha, \text{ and} \quad \text{III:msi-5-0}$$

$$(5.10) \quad i_{\beta,\gamma} \circ i_{\alpha,\beta} = i_{\alpha,\gamma} \text{ for } \alpha \leq \beta \leq \gamma \leq \kappa; \quad \text{III:msi-5-1}$$

$$(5.11) \quad \text{supp}(\iota_\beta(\mathbb{O})) = \text{supp}(\mathbb{O}), \text{ where } \text{supp}(\cdot) \text{ is defined as in (5.17) below, and} \quad \text{III:msi-8-0}$$

$$(5.12) \quad \iota_\alpha(\mathbb{O} \upharpoonright \alpha) = \iota_\beta(\mathbb{O}) \upharpoonright \alpha \text{ for } \alpha < \beta \leq \kappa \text{ and } \mathbb{O} \in \mathbb{O}_\beta. \quad \text{III:msi-8-1}$$

We define now the *Easton-type mixed support* of the iteration as a sequence $\langle \mathcal{I}_\alpha : \alpha < \kappa \rangle$ of ideals where each \mathcal{I}_α for $\alpha \leq \kappa$ is an ideal over α .

$$(5.13) \quad \mathcal{I}_{\alpha+1} = \mathcal{I}_\alpha \cup \{s \cup \{\alpha\} : s \in \mathcal{I}_\alpha\} \text{ for all } \alpha < \kappa; \quad \text{III:msi-8-2}$$

$$(5.14) \quad \text{If } \gamma < \kappa \text{ is a limit ordinal but not a regular cardinal, then } \mathcal{I}_\gamma = \{s \subseteq \gamma : \text{III:msi-8-3} \\ s \cap \alpha \in \mathcal{I}_\alpha \text{ for all } \alpha < \gamma \text{ and } s \cap T \text{ is bounded in } \gamma\};$$

$$(5.15) \quad \text{If } \gamma \leq \kappa \text{ is a regular cardinal, then } \mathcal{I}_\gamma = \{s \subseteq \gamma : s \cap \alpha \in \mathcal{I}_\alpha \text{ for all} \quad \text{III:msi-8-4} \\ \alpha < \gamma \text{ and } s \text{ is bounded in } \gamma\}.$$

The following is easy to prove by induction on $\alpha \leq \kappa$:

Lemma 5.1 (1) \mathcal{I}_α is an ideal over α with $\{\{\beta\} : \beta < \alpha\} \subseteq \mathcal{I}_\alpha$ for all $\alpha \leq \kappa$.

(2) For all $\alpha \leq \kappa$, $s \in \mathcal{I}_\alpha \Leftrightarrow s \cap T$ is finite and $|s \cap \mu| < \mu$ for all regular infinite cardinal $\mu \leq \alpha$. \square

Now we are ready to define the iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ in the following (A) and (B):

(A) If $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha < \gamma, \beta < \gamma \rangle$ has been defined for a limit $\gamma \leq \kappa$, let

$$(5.16) \quad \mathbb{P}_\gamma = \{\mathbb{P} : \mathbb{P} \text{ is a sequence of length } \gamma, \quad \text{III:msi-8-5} \\ \mathbb{P} \upharpoonright \alpha \in \mathbb{P}_\alpha \text{ for all } \alpha < \gamma, \text{ and } \text{supp}(\mathbb{P}) \in \mathcal{I}_\gamma \}$$

where

$$(5.17) \quad \text{supp}(\mathbb{p}) = \{\alpha < \gamma : \mathbb{p}(\alpha) \neq \mathbb{1}_{\mathbb{Q}_\alpha}\}. \quad \text{III:msi-6}$$

For $\mathbb{p}, \mathbb{q} \in \mathbb{P}_\gamma$,

$$(5.18) \quad \mathbb{p} \leq_{\mathbb{P}_\gamma} \mathbb{q} \Leftrightarrow \mathbb{p} \upharpoonright \delta \leq_{\mathbb{P}_\delta} \mathbb{q} \upharpoonright \delta \text{ for all } \delta < \gamma. \quad \text{III:msi-8}$$

We assume that the complete embeddings $\iota_\alpha : \mathbb{O}_\alpha \rightarrow \mathbb{P}_\alpha$, $\alpha < \gamma$ have been defined such that (5.11) and (5.12) hold for all $\alpha \leq \beta < \gamma$. The mapping ι_γ is then defined by:

$$(5.19) \quad \iota_\gamma : \mathbb{O}_\gamma \rightarrow \mathbb{P}_\gamma; \quad \mathbb{p} \mapsto \bigcup_{\delta < \gamma} \iota_\delta(\mathbb{p} \upharpoonright \delta). \quad \text{III:msi-9}$$

For $\delta \leq \gamma$, let $\vec{\mathbb{1}}_{\delta, \gamma} = \{\langle \alpha, \mathbb{1}_{\mathbb{Q}_\alpha} \rangle : \delta \leq \alpha < \gamma\}$ as before and let $i_{\delta, \gamma}$ be defined by

$$(5.20) \quad i_{\delta, \gamma} : \mathbb{P}_\delta \rightarrow \mathbb{P}_\gamma; \quad \mathbb{p} \mapsto \mathbb{p} \cup \vec{\mathbb{1}}_{\delta, \gamma}. \quad \text{III:msi-9-0}$$

It is easy to check that ι_γ and $i_{\delta, \gamma}$, $\gamma \leq \delta$ are complete embeddings and (5.8) \sim (5.11) hold for all indices $\leq \gamma$.

(B) Now suppose that $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \gamma, \beta < \gamma \rangle$, $\langle \iota_\beta : \beta < \gamma \rangle$ and $\langle i_{\alpha, \beta} : \alpha \leq \beta \leq \gamma \rangle$ have been defined for some $\gamma < \kappa$.

(a) If $\gamma \in S$ and

$$(5.21) \quad f(\gamma) = \langle \mu, \theta, R \rangle \text{ for some cardinals } \mu, \theta > \gamma \text{ and a set } R, \quad \text{III:msi-9-1}$$

then let

$$(5.22) \quad \mathbb{Q}_\gamma = (\text{Col}(\gamma, \mu))_{\mathbb{P}_\gamma}^\bullet, \text{ and} \quad \text{III:msi-10}$$

$$(5.23) \quad \mathbb{P}_{\gamma+1} = \{\mathbb{p} \cup \{\langle \gamma, \mathbb{q} \rangle\} : \mathbb{p} \in \mathbb{P}_\gamma, \mathbb{q} \text{ is a canonical } \mathbb{P}_\gamma\text{-name} \text{ }^4\} \\ \text{such that } \Vdash_{\mathbb{P}_\gamma} \text{“} \mathbb{q} \in \mathbb{Q}_\gamma \text{”}. \quad \text{III:msi-11}$$

For $\mathbb{p}_0 \cup \{\langle \gamma, \mathbb{q}_0 \rangle\}, \mathbb{p}_1 \cup \{\langle \gamma, \mathbb{q}_1 \rangle\} \in \mathbb{P}_{\gamma+1}$,

$$(5.24) \quad \mathbb{p}_0 \cup \{\langle \gamma, \mathbb{q}_0 \rangle\} \leq_{\mathbb{P}_{\gamma+1}} \mathbb{p}_1 \cup \{\langle \gamma, \mathbb{q}_1 \rangle\} \Leftrightarrow \mathbb{p}_0 \leq_{\mathbb{P}_\gamma} \mathbb{p}_1 \text{ and} \\ \mathbb{p}_0 \Vdash_{\mathbb{P}_\gamma} \text{“} \mathbb{q}_0 \leq_{\mathbb{Q}_\gamma} \mathbb{q}_1 \text{”}. \quad \text{III:msi-12}$$

For $\mathbb{p} \in \mathbb{O}_{\gamma+1}$, let

$$(5.25) \quad \iota_{\gamma+1}(\mathbb{p}) = \iota_\gamma(\mathbb{p} \upharpoonright \gamma) \cup \{\langle \gamma, \mathbb{1}_{\mathbb{Q}_\gamma} \rangle\}, \quad \text{III:msi-12-0}$$

and, for $\alpha \leq \gamma$ and $\mathbb{p} \in \mathbb{P}_\alpha$, let

$$(5.26) \quad i_{\alpha, \gamma+1}(\mathbb{P}) = i_{\alpha, \gamma}(\mathbb{P}) \cup \{\langle \gamma, \mathbb{1}_{\mathbb{Q}_\gamma} \rangle\}.$$

III:msi-12-1

(b) If $\gamma \in S$ but (a) does not hold, then let \mathbb{Q}_γ be a \mathbb{P}_γ -name of trivial forcing and the rest is treated just as in the case (a).

In both of the cases (a) and (b), it is clear that the defined mappings are complete embeddings and satisfy (5.8) \sim (5.11).

(c) If $\gamma \notin S$, then let \mathbb{Q}_γ be the \mathbb{P}_γ -name $\iota_\gamma(\mathbb{R}_\gamma)$ and ⁵⁾

$$(5.27) \quad \mathbb{P}_{\gamma+1} = \{\mathbb{P} \cup \{\langle \gamma, \iota_\gamma(\mathbb{r}) \rangle\} : \mathbb{P} \in \mathbb{P}_\gamma, \mathbb{r} \text{ is a canonical } \mathbb{O}_\gamma\text{-name} \\ \text{such that } \Vdash_{\mathbb{O}_\gamma} \text{“} \mathbb{r} \in \mathbb{R}_\gamma \text{”}\}.$$

III:msi-13

For $\mathbb{P}_0 \cup \{\langle \gamma, \iota_\gamma(\mathbb{r}_0) \rangle\}$, $\mathbb{P}_1 \cup \{\langle \gamma, \iota_\gamma(\mathbb{r}_1) \rangle\} \in \mathbb{P}_{\gamma+1}$,

$$(5.28) \quad \mathbb{P}_0 \cup \{\langle \gamma, \iota_\gamma(\mathbb{r}_0) \rangle\} \leq_{\mathbb{P}_{\gamma+1}} \mathbb{P}_1 \cup \{\langle \gamma, \iota_\gamma(\mathbb{r}_1) \rangle\} \\ \Leftrightarrow \mathbb{P}_0 \leq_{\mathbb{P}_\gamma} \mathbb{P}_1 \text{ and there is } \mathbb{O} \in \mathbb{O}_\gamma \text{ such that } \mathbb{P}_0 \leq_{\mathbb{P}_\gamma} \iota_\gamma(\mathbb{O}) \\ \text{and } \mathbb{O} \Vdash_{\mathbb{O}_\gamma} \text{“} \mathbb{r}_0 \leq_{\mathbb{R}_\gamma} \mathbb{r}_1 \text{”}.$$

III:msi-14

For $\mathbb{O} \in \mathbb{O}_{\gamma+1}$, let

$$(5.29) \quad \iota_{\gamma+1}(\mathbb{O}) = \iota_\gamma(\mathbb{O} \upharpoonright \gamma) \cup \{\langle \gamma, \iota_\gamma(\mathbb{O}(\gamma)) \rangle\},$$

III:msi-15

and, for $\alpha \leq \gamma$ and $\mathbb{P} \in \mathbb{P}_\alpha$, let

$$(5.30) \quad i_{\alpha, \gamma+1}(\mathbb{P}) = i_{\alpha, \gamma}(\mathbb{P}) \cup \{\langle \gamma, \mathbb{1}_{\mathbb{Q}_\gamma} \rangle\}.$$

III:msi-16

Also in this case, the mapping introduced are complete embeddings and (5.8) \sim (5.11) are satisfied.

This finishes the construction of our Easton-type mixed support iteration.

The following three Lemmas can be proved easily with the standard argument in the order as we present them here.

III:P-msi-0-a-

Lemma 5.2 *For an ordinal $\gamma \leq \kappa$ and a γ -sequence \mathbb{P} ,*

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$$\begin{aligned} \mathbb{P} \in \mathbb{P}_\gamma &\Leftrightarrow \Vdash_{\mathbb{P}_\xi} \text{“} \mathbb{P}(\xi) \in \mathbb{Q}_\xi \text{” for all } \xi \in S \cap \gamma, \\ &\mathbb{P}(\xi) = \iota_\xi(\mathbb{r}) \text{ for a canonical } \mathbb{O}_\gamma\text{-name } \mathbb{r} \text{ with} \\ &\Vdash_{\mathbb{O}_\xi} \text{“} \mathbb{r} \in \mathbb{R}_\xi \text{” for all } \xi \in T \cap \gamma, \text{ and} \\ &\text{supp}(\mathbb{P}) = \{\xi < \gamma : \mathbb{P}(\xi) \neq \mathbb{1}_{\mathbb{Q}_\xi}\} \in \mathcal{I}_\gamma. \end{aligned}$$

□

Lemma 5.3 For $\delta \leq \gamma \leq \kappa$, $\mathbb{P}_0, \mathbb{P}_1 \in \mathbb{P}_\gamma$ and $s \subseteq \gamma$, if $\text{supp}(\mathbb{P}_0) \subseteq \delta \cup s$, $\text{supp}(\mathbb{P}_1) \subseteq \delta \cup (\gamma \setminus s)$ and $\mathbb{P}_0 \restriction \delta \leq_{\mathbb{P}_\delta} \mathbb{P}_1 \restriction \delta$, then

$$(5.31) \quad \mathbb{P}_2 = (\mathbb{P}_0 \restriction (\delta \cup s)) \cup (\mathbb{P}_1 \restriction (\gamma \setminus (\delta \cup s))) \\ (= (\mathbb{P}_0 \restriction \text{supp}(\mathbb{P}_0)) \cup (\mathbb{P}_1 \restriction (\gamma \setminus \text{supp}(\mathbb{P}_0)))) \in \mathbb{P}_\gamma$$

and \mathbb{p}_2 is a maximal element of \mathbb{P}_γ below \mathbb{p}_0 and \mathbb{p}_1 with respect to $\leq_{\mathbb{P}_\gamma}$.

Proof. We prove the assertion of the Lemma by induction on γ with $\delta \leq \gamma \leq \kappa$.

The rest will be written later. (Scan_2020-02-03... p.18) □ (Lemma 5.3)

Note that, in the Lemma above, we are talking about “a” maximal element since $\leq_{\mathbb{P}_\gamma}$ is merely a pre-ordering in general.

The following can be proved applying the Pressing-down Lemma and Lemma 5.3 above. Note that, for $\alpha \in S \cup \{\kappa\}$, $R = \{\beta < \alpha : \mathbb{P}_\beta \text{ is a direct limit of } \langle \mathbb{P}_\xi : \xi < \alpha \rangle\}$ is a stationary subset of α by (5.3) and (5.15).

Lemma 5.4 For $\nu \in S \cup \{\kappa\}$, we have $|\mathbb{P}_\mu| < \nu$ for all $\mu < \nu$, $\mathbb{P}_\nu \subseteq V_\nu$ and \mathbb{P}_ν has the ν -cc. □

For posets \mathbb{P}, \mathbb{Q} , a mapping $p : \mathbb{Q} \rightarrow \mathbb{P}$ is said to be a *projection* if

$$(5.32) \quad p(\mathbb{1}_\mathbb{Q}) = \mathbb{1}_\mathbb{P};$$

$$(5.33) \quad p \text{ is order-preserving; and}$$

$$(5.34) \quad \text{for any } \mathbb{p} \in \mathbb{P} \text{ and } \mathbb{q} \in \mathbb{Q}, \text{ if } \mathbb{p} \leq_{\mathbb{P}} p(\mathbb{q}), \text{ then there is } \mathbb{q}' \in \mathbb{Q} \text{ such that } \mathbb{q}' \leq_{\mathbb{Q}} \mathbb{q} \text{ and } p(\mathbb{q}') \leq_{\mathbb{P}} \mathbb{p}.$$

Note that we do not assume that a projection is a surjection. However:

Lemma A 5.1 If $p : \mathbb{Q} \rightarrow \mathbb{P}$ is a projection then $p''\mathbb{Q}$ is a dense subset of \mathbb{P} .

Proof. For $\mathbb{p} \in \mathbb{P}$, we have $\mathbb{p} \leq_{\mathbb{P}} \mathbb{1}_\mathbb{P} = p(\mathbb{1}_\mathbb{Q})$. Thus by (5.34), there is $\mathbb{q}' \in \mathbb{Q}$ such that $p(\mathbb{q}') \leq_{\mathbb{P}} \mathbb{p}$. □ (Lemma A 5.1)

The following is standard and also easy to check:

Lemma 5.5 Suppose that \mathbb{P}, \mathbb{Q} are posets and $p : \mathbb{Q} \rightarrow \mathbb{P}$ is a projection.

³⁾ At the moment, R does not play any role. This component is added here so that we can later modify the construction.

⁴⁾ Adopting the terminology of [Cummings[4]], we call a \mathbb{P} -name \tilde{a} is a *canonical* \mathbb{P} -name if, for any \mathbb{P} -name \tilde{b} with $\Vdash_{\mathbb{P}} \tilde{a} \equiv \tilde{b}$, we have $|\text{trcl}(\tilde{a})| \leq |\text{trcl}(\tilde{b})|$.

⁵⁾ With ι_γ , we also denote the embedding of $V^{\mathbb{Q}_\gamma}$ into $V^{\mathbb{P}_\gamma}$ canonically induced by ι_γ .

(1) If \mathbb{H} is a (\mathbf{V}, \mathbb{Q}) -generic filter, then $p''\mathbb{H}$ generates a (\mathbf{V}, \mathbb{P}) -generic filter.

(2) If \mathbb{G} is a (\mathbf{V}, \mathbb{P}) -generic filter, then letting

$$(5.35) \quad \mathbb{Q}/\mathbb{G} = \{q \in \mathbb{Q} : p(q) \in \mathbb{G}\}$$

III:proj-3

be poset with the pre-ordering $\leq_{\mathbb{Q}}$ restricted to it, any $(\mathbf{V}[\mathbb{G}], \mathbb{Q}/\mathbb{G})$ -generic filter \mathbb{H} is a (\mathbf{V}, \mathbb{Q}) -generic filter with $p''\mathbb{H} \subseteq \mathbb{G}$. \square

Suppose that $\langle \mathbb{P}_{\alpha}, \mathbb{Q}_{\beta} : \alpha \leq \kappa, \beta < \kappa \rangle$ is an Easton-type mixed support iteration with the Laver-function $f : \kappa \rightarrow V_{\kappa}$ and S as above over a finite support iteration $\langle \mathbb{O}_{\alpha}, \mathbb{R}_{\beta} : \alpha \leq \kappa, \beta < \kappa \rangle$.

Note that, for $\alpha \leq \beta \leq \kappa$,

$$(5.36) \quad p_{\beta, \alpha} : \mathbb{P}_{\beta} \rightarrow \mathbb{P}_{\alpha}; q \mapsto q \upharpoonright \alpha \text{ is a projection, and}$$

III:proj-4

$$(5.37) \quad p_{\beta, \alpha} \circ i_{\alpha, \beta} = id_{\mathbb{P}_{\alpha}}.$$

III:proj-5

For $\delta_0 < \kappa$, let \mathbb{G}_{δ_0} be a $(\mathbf{V}, \mathbb{P}_{\delta_0})$ -generic filter. Working in $\mathbf{V}[\mathbb{G}_{\delta_0}]$, let $\delta_0 \leq \gamma \leq \kappa$, and let

$$(5.38) \quad \mathbb{P}_{\gamma}/\mathbb{G}_{\delta_0} = \{p \in \mathbb{P}_{\gamma} : p \upharpoonright \delta_0 \in \mathbb{G}_{\delta_0}\}$$

III:msi-16-0

be the poset with the pre-ordering $\leq_{\mathbb{P}_{\gamma}}$ restricted to $\mathbb{P}_{\gamma}/\mathbb{G}_{\delta_0}$ and with the designated maximal element $\mathbb{1}_{\mathbb{P}_{\gamma}/\mathbb{G}_{\delta_0}} = \mathbb{1}_{\mathbb{P}_{\gamma}}$.

Lemma 5.6 (1) A $(\mathbf{V}[\mathbb{G}_{\delta_0}], \mathbb{P}_{\gamma}/\mathbb{G}_{\delta_0})$ -generic filter \mathbb{H} is also a $(\mathbf{V}, \mathbb{P}_{\gamma})$ -generic filter with $i_{\delta_0, \gamma}''\mathbb{G}_{\delta_0} \subseteq \mathbb{H}$.

III:P-msi-1

(2) If \mathbb{H} is a $(\mathbf{V}, \mathbb{P}_{\gamma})$ -generic filter with $i_{\delta_0, \gamma}''\mathbb{G} \subseteq \mathbb{H}$, then \mathbb{H} is a $(\mathbf{V}[\mathbb{G}_{\delta_0}], \mathbb{P}_{\gamma}/\mathbb{G}_{\delta_0})$ -generic filter.

Proof. By (5.36), (5.37) and Lemma 5.5. \square (Lemma 5.6)

It is well-known that projections and complete embeddings are two interchangeable notions for cBa ⁶⁾:

III:P-msi-1-a

Lemma 5.7 For cBa posets \mathbb{P} and \mathbb{Q} , there is a complete embedding $i : \mathbb{P} \rightarrow \mathbb{Q}$ if and only if there is a projection $p : \mathbb{Q} \rightarrow \mathbb{P}$.

For cBa posets complete embeddings are injections and projections are surjections.

⁶⁾ We call a poset $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ a cBa poset if (the underlying set) \mathbb{P} of the poset coincides with the positive elements of a complete Boolean algebra and $\leq_{\mathbb{P}}$ coincides with the ordering of the complete Boolean algebra.

Proof. Suppose that $\mathbb{P} = \mathbb{A}^+$ and $\mathbb{Q} = \mathbb{B}^+$ for complete Boolean algebras \mathbb{A} and \mathbb{B} .

If $i : \mathbb{P} \rightarrow \mathbb{Q}$ is a complete embedding, then $p : \mathbb{Q} \rightarrow \mathbb{P}$ defined by $p(\mathfrak{b}) = \prod^{\mathbb{A}} \{\mathfrak{a} \in \mathbb{P} : i(\mathfrak{a}) \geq_{\mathbb{Q}} \mathfrak{b}\}$ for $\mathfrak{b} \in \mathbb{Q}$ is a projection.

If $p : \mathbb{Q} \rightarrow \mathbb{P}$ is a projection, then the mapping $i : \mathbb{P} \rightarrow \mathbb{Q}$ defined by $i(\mathfrak{a}) = \sum^{\mathbb{B}} \{\mathfrak{b} \in \mathbb{Q} : p(\mathfrak{b}) \leq_{\mathbb{P}} \mathfrak{a}\}$ for $\mathfrak{a} \in \mathbb{P}$ is a complete embedding. Note that $i(\mathfrak{a}) \in \mathbb{Q}$ by Lemma A 5.1.

□ (Lemma 5.7)

For the analysis of the structure of the iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$, the following alternative treatment of the quotient $\mathbb{P}_\gamma / \mathbb{G}_{\delta_0}$ proves often to be more appropriate.

For $\mathfrak{p}_0, \mathfrak{p}_1 \in \mathbb{P}_\gamma$, with $\text{supp}(\mathfrak{p}_0) \cap \text{supp}(\mathfrak{p}_1) = \emptyset$, we denote with $\mathfrak{p}_0 \wedge_{\mathbb{P}_\gamma} \mathfrak{p}_1$, the element \mathbb{P}_γ defined by:

$$(5.39) \quad \mathfrak{p}_0 \wedge_{\mathbb{P}_\gamma} \mathfrak{p}_1 = \mathfrak{p}_0 \upharpoonright \text{supp}(\mathfrak{p}_0) \cup \mathfrak{p}_1 \upharpoonright (\gamma \setminus \text{supp}(\mathfrak{p}_0)).$$

III:msi-+18-a

If it causes no confusion, we drop the subscript \mathbb{P}_γ in this notation and simply write $\mathfrak{p}_0 \wedge \mathfrak{p}_1$ in place of $\mathfrak{p}_0 \wedge_{\mathbb{P}_\gamma} \mathfrak{p}_1$.

Suppose $\delta_0 < \gamma \leq \kappa$ and \mathbb{G}_{δ_0} is a $(\mathbb{V}, \mathbb{P}_{\delta_0})$ -generic filter. In $\mathbb{V}[\mathbb{G}_{\delta_0}]$, let $\mathbb{P}_\gamma | \mathbb{G}_{\delta_0} = \{\mathfrak{p} \in \mathbb{P}_\gamma : \text{supp}(\mathfrak{p}) \subseteq \gamma \setminus \delta_0\}$ be the poset with the pre-ordering $\leq_{\mathbb{P}_\gamma | \mathbb{G}_{\delta_0}}$ defined by

$$(5.40) \quad \mathfrak{q}_0 \leq_{\mathbb{P}_\gamma | \mathbb{G}_{\delta_0}} \mathfrak{q}_1 \Leftrightarrow i_{\delta_0, \gamma}(\mathfrak{p}) \wedge_{\mathbb{P}_\gamma} \mathfrak{q}_0 \leq_{\mathbb{P}_\gamma} i_{\delta_0, \gamma}(\mathfrak{p}) \wedge_{\mathbb{P}_\gamma} \mathfrak{q}_1 \text{ for some } \mathfrak{p} \in \mathbb{G}_{\delta_0}$$

III:msi-28

for $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathbb{P}_\gamma | \mathbb{G}_{\delta_0}$, and with the designated maximal element $\mathbb{1}_{\mathbb{P}_\gamma | \mathbb{G}_{\delta_0}} = \mathbb{1}_{\mathbb{P}_\gamma}$.

Note that, for $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathbb{P}_\gamma | \mathbb{G}_{\delta_0}$,

$$(5.41) \quad \mathfrak{q}_0 \leq_{\mathbb{P}_\gamma} \mathfrak{q}_1 \text{ implies } \mathfrak{q}_0 \leq_{\mathbb{P}_\gamma | \mathbb{G}_{\delta_0}} \mathfrak{q}_1,$$

III:msi-28-0

since $\mathbb{1}_{\mathbb{P}_{\delta_0}} \in \mathbb{G}_{\delta_0}$.

In the following, just for convenience, we shall often misuse the notation and write instead of $i_{\delta_0, \gamma}(\mathfrak{p}) \wedge_{\mathbb{P}_\gamma} \mathfrak{q}_0$ etc. simply $\mathfrak{p} \wedge \mathfrak{q}_0$ etc. The following Lemma is also formulated in this sloppy handling of the notation.

III:P-msi-1-0

Lemma 5.8 (1) For $\mathfrak{p} \in \mathbb{P}_{\delta_0}$ and $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathbb{P}_\gamma | \mathbb{G}_{\delta_0}$ If $\mathfrak{p} \wedge \mathfrak{q}_0 \leq_{\mathbb{P}_\gamma} \mathfrak{p} \wedge \mathfrak{q}_1$, then for $\mathfrak{p}' \leq_{\mathbb{P}_{\delta_0}} \mathfrak{p}$, we have $\mathfrak{p}' \wedge \mathfrak{q}_0 \leq_{\mathbb{P}_\gamma} \mathfrak{p}' \wedge \mathfrak{q}_1 \leq_{\mathbb{P}_\gamma} \mathfrak{p} \wedge \mathfrak{q}_1$.

(2) For $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathbb{P}_\gamma | \mathbb{G}_{\delta_0}$ with $\text{supp}(\mathfrak{q}_0) \cap \text{supp}(\mathfrak{q}_1) = \emptyset$, $\mathfrak{q}_0 \wedge \mathfrak{q}_1$ is a join of \mathfrak{q}_0 and \mathfrak{q}_1 both with respect to $\leq_{\mathbb{P}_\gamma}$ and with respect to $\leq_{\mathbb{P} | \mathbb{G}_{\delta_0}}$.

Proof. (1): By Lemma 5.3.

(2): $\mathfrak{q}_0 \wedge \mathfrak{q}_1$ is a join of \mathfrak{q}_0 and \mathfrak{q}_1 with respect to $\leq_{\mathbb{P}_\gamma}$ by Lemma 5.3. By (5.41), it follows that $\mathfrak{q}_0 \wedge \mathfrak{q}_1 \leq_{\mathbb{P}_\gamma | \mathbb{G}_{\delta_0}} \mathfrak{q}_0, \mathfrak{q}_1$. Suppose now that $\mathfrak{r} \leq_{\mathbb{P}_\gamma | \mathbb{G}_{\delta_0}} \mathfrak{q}_0, \mathfrak{q}_1$. Then

there are $\mathfrak{s}_0, \mathfrak{s}_1 \in \mathbb{G}_{\delta_0}$ such that $\mathfrak{s}_0 \wedge \mathfrak{r} \leq_{\mathbb{P}_\gamma} \mathfrak{s}_0 \wedge \mathfrak{q}_0$ and $\mathfrak{s}_1 \wedge \mathfrak{r} \leq_{\mathbb{P}_\gamma} \mathfrak{s}_1 \wedge \mathfrak{q}_1$. Let $\mathfrak{s}_2 \in \mathbb{G}_{\delta_0}$ be such that $\mathfrak{s}_2 \leq_{\mathbb{P}_{\delta_0}} \mathfrak{s}_0, \mathfrak{s}_1$. Then, by (1), we have $\mathfrak{s}_2 \wedge \mathfrak{r} \leq_{\mathbb{P}_\gamma} \mathfrak{s}_2 \wedge \mathfrak{q}_0, \mathfrak{s}_2 \wedge \mathfrak{q}_1$. By Lemma 5.3, it follows that $\mathfrak{s}_2 \wedge \mathfrak{r} \leq_{\mathbb{P}_\gamma} \mathfrak{s}_2 \wedge (\mathfrak{q}_0 \wedge \mathfrak{q}_1)$. Thus $\mathfrak{r} \leq_{\mathbb{P}_\gamma | \mathbb{G}_{\delta_0}} \mathfrak{q}_0 \wedge \mathfrak{q}_1$.

□ (Lemma 5.8)

For $\gamma \leq \kappa$, $\mathbb{p} \in \mathbb{P}_\gamma$ and $X \subseteq \kappa$, let $\mathbb{p} \upharpoonright X$ be the condition $\mathfrak{r} \in \mathbb{P}_\gamma$ defined by

$$(5.42) \quad \mathfrak{r}(\alpha) = \begin{cases} \mathbb{p}(\alpha), & \text{if } \alpha \in X; \\ \mathbb{1}_{\mathbb{Q}_\alpha}, & \text{otherwise} \end{cases}$$

III:msi-18-0

for all $\alpha < \gamma$.

Since $\text{supp}(\mathbb{p} \upharpoonright X) \subseteq \text{supp}(\mathbb{p})$, we have $\mathbb{p} \upharpoonright X \in \mathbb{P}_\gamma$ by Lemma 5.2. By definition, it is also clear that $\mathbb{p} \leq_{\mathbb{P}_\gamma} \mathbb{p} \upharpoonright X$.

For $X \subseteq \gamma$ and $P \subseteq \mathbb{P}_\gamma$, let us write

$$(5.43) \quad P \upharpoonright X = \{\mathbb{p} \upharpoonright X : \mathbb{p} \in P\}.$$

III:msi-28-1

Note that the underlying set of $\mathbb{P}_\gamma | \mathbb{G}_{\delta_0}$ could be also described as $\mathbb{P}_\gamma \upharpoonright (\gamma \setminus \delta_0)$ with this notation.

The poset $\mathbb{P}_\gamma | \mathbb{G}_{\delta_0}$ is forcing equivalent to $\mathbb{P}_\gamma / \mathbb{G}_{\delta_0}$.

III:P-msi-3

Lemma 5.9 *The mapping*

$$(5.44) \quad i_\upharpoonright : \mathbb{P}_\gamma / \mathbb{G}_{\delta_0} \rightarrow \mathbb{P}_\gamma | \mathbb{G}_{\delta_0} ; \quad \mathfrak{q} \mapsto \mathfrak{q} \upharpoonright (\gamma \setminus \delta_0)$$

III:msi-29

is a dense embedding.

Proof. i_\upharpoonright is surjective: If $\mathbb{p} \in \mathbb{P}_\gamma | \mathbb{G}_{\delta_0}$ then $\mathbb{p} \in \mathbb{P}_\gamma / \mathbb{G}_{\delta_0}$ and $i_\upharpoonright(\mathbb{p}) = \mathbb{p}$.

$$i_\upharpoonright(\mathbb{1}_{\mathbb{P}_\gamma / \mathbb{G}_{\delta_0}}) = i_\upharpoonright(\mathbb{1}_{\mathbb{P}_\gamma}) = \mathbb{1}_{\mathbb{P}_\gamma} \upharpoonright (\gamma \setminus \delta_0) = \mathbb{1}_{\mathbb{P}_\gamma} = \mathbb{1}_{\mathbb{P}_\gamma | \mathbb{G}_{\delta_0}}.$$

i_\upharpoonright is order preserving: Suppose that $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathbb{P}_\gamma / \mathbb{G}_{\delta_0}$ and $\mathfrak{q}_0 \leq_{\mathbb{P}_\gamma} \mathfrak{q}_1$. Then $\mathfrak{q}_0 \upharpoonright \delta_0, \mathfrak{q}_1 \upharpoonright \delta_0 \in \mathbb{G}_{\delta_0}$ and $\mathfrak{q}_0 \upharpoonright \delta_0 \leq_{\mathbb{P}_{\delta_0}} \mathfrak{q}_1 \upharpoonright \delta_0$. It follows that

$$(5.45) \quad \mathfrak{q}_0 \upharpoonright \delta_0 \wedge \underbrace{(\mathfrak{q}_0 \upharpoonright (\gamma \setminus \delta_0))}_{=i_\upharpoonright(\mathfrak{q}_0)} \leq_{\mathbb{P}_\gamma} \mathfrak{q}_0 \upharpoonright \delta_0 \wedge \underbrace{(\mathfrak{q}_1 \upharpoonright (\gamma \setminus \delta_0))}_{=i_\upharpoonright(\mathfrak{q}_1)}.$$

Thus, $i_\upharpoonright(\mathfrak{q}_0) \leq_{\mathbb{P}_\gamma | \mathbb{G}_{\delta_0}} i_\upharpoonright(\mathfrak{q}_1)$.

i_\upharpoonright is incompatibility preserving: Suppose that $\mathfrak{q}_0, \mathfrak{q}_1 \in \mathbb{P}_\gamma / \mathbb{G}_{\delta_0}$ and, $i_\upharpoonright(\mathfrak{q}_0)$ and $i_\upharpoonright(\mathfrak{q}_1)$ are compatible in $\mathbb{P}_\gamma | \mathbb{G}_{\delta_0}$. Then, there is $\mathfrak{r} \in \mathbb{P}_\gamma | \mathbb{G}_{\delta_0}$ such that $\mathfrak{r} \leq_{\mathbb{P}_\gamma | \mathbb{G}_{\delta_0}} i_\upharpoonright(\mathfrak{q}_0), i_\upharpoonright(\mathfrak{q}_1)$. By the definition of $\leq_{\mathbb{P}_\gamma | \mathbb{G}_{\delta_0}}$, this means that there are $\mathfrak{s}_0, \mathfrak{s}_1 \in \mathbb{G}_{\delta_0}$ such that $\mathfrak{s}_0 \wedge \mathfrak{r} \leq_{\mathbb{P}_\gamma} \mathfrak{s}_0 \wedge i_\upharpoonright(\mathfrak{q}_0)$ and $\mathfrak{s}_1 \wedge \mathfrak{r} \leq_{\mathbb{P}_\gamma} \mathfrak{s}_1 \wedge i_\upharpoonright(\mathfrak{q}_1)$.

Let $\mathfrak{s}_2 \in \mathbb{G}_{\delta_0}$ be such that $\mathfrak{s}_2 \leq_{\mathbb{P}_{\delta_0}} \mathfrak{s}_0, \mathfrak{s}_1, \mathfrak{q}_0 \upharpoonright \delta_0, \mathfrak{q}_1 \upharpoonright \delta_0$. Then $\mathfrak{s}_2 \wedge \mathfrak{r} \in \mathbb{P}_\gamma / \mathbb{G}_{\delta_0}$ and $\mathfrak{s}_2 \wedge \mathfrak{r} \leq_{\mathbb{P}_\gamma} \mathfrak{q}_0, \mathfrak{q}_1$ by Lemma 5.8, (1). □ (Lemma 5.9)

Working further in $\mathbb{V}[\mathbb{G}_{\delta_0}]$, let

$$(5.46) \quad \mathbb{S}_{\delta_0, \gamma} = (\mathbb{P}_\gamma \mid \mathbb{G}_{\delta_0}) \upharpoonright S = \{\mathbb{p} \in \mathbb{P}_\gamma : \text{supp}(\mathbb{p}) \subseteq S \setminus \delta_0\}$$

III:msi-17

be the poset with the pre-ordering $\leq_{\mathbb{P}_\gamma \mid \mathbb{G}_{\delta_0}}$ restricted to it and with the designated maximal element $\mathbb{1}_{\mathbb{S}_{\delta_0, \gamma}} = \mathbb{1}_{\mathbb{P}_\gamma}$. Let

$$(5.47) \quad \mathbb{T}_{\delta_0, \gamma} = (\mathbb{P}_\gamma \mid \mathbb{G}_{\delta_0}) \upharpoonright T = \{\mathbb{p} \in \mathbb{P}_\gamma : \text{supp}(\mathbb{p}) \subseteq T \setminus \delta_0\}$$

III:msi-18

be the poset with the pre-ordering $\leq_{\mathbb{P}_\gamma \mid \mathbb{G}_{\delta_0}}$ restricted to it and with the designated maximal element $\mathbb{1}_{\mathbb{T}_{\delta_0, \gamma}} = \mathbb{1}_{\mathbb{P}_\gamma}$.

3.14 in Ottenbreit

III:P-msi-2

Lemma 5.10 *In $V[\mathbb{G}_{\delta_0}]$, the mapping*

$$(5.48) \quad \pi_{\delta_0, \gamma} : \mathbb{S}_{\delta_0, \gamma} \times \mathbb{T}_{\delta_0, \gamma} \rightarrow \mathbb{P}_\gamma \mid \mathbb{G}_{\delta_0}; \quad \langle \mathfrak{s}, \mathfrak{t} \rangle \mapsto \mathfrak{s} \wedge \mathfrak{t}$$

III:msi-19

is a projection.

Proof. $\pi_{\delta_0, \gamma} \models (5.32)$ is clear by the definition of $\pi_{\delta_0, \gamma}$.

To show that $\pi_{\delta_0, \gamma}$ is order-preserving, suppose that $\mathfrak{s}' \leq_{\mathbb{P}_\gamma \mid \mathbb{G}_{\delta_0}} \mathfrak{s}$ and $\mathfrak{t}' \leq_{\mathbb{P}_\gamma \mid \mathbb{G}_{\delta_0}} \mathfrak{t}$. Then, there are $\mathfrak{u}_0, \mathfrak{u}_1 \in \mathbb{G}_{\delta_0}$ such that $\mathfrak{u}_0 \wedge \mathfrak{s}' \leq_{\mathbb{P}_\gamma} \mathfrak{u}_0 \wedge \mathfrak{s}$ and $\mathfrak{u}_1 \wedge \mathfrak{t}' \leq_{\mathbb{P}_\gamma} \mathfrak{u}_1 \wedge \mathfrak{t}$.

Let $\mathfrak{u}_2 \in \mathbb{G}_{\delta_0}$ be such that $\mathfrak{u}_2 \leq_{\mathbb{P}_{\delta_0}} \mathfrak{u}_0, \mathfrak{u}_1$. By Lemma 5.8, (1), we have $\mathfrak{u}_2 \wedge \mathfrak{s}' \leq_{\mathbb{P}_\gamma} \mathfrak{u}_2 \wedge \mathfrak{s}$ and $\mathfrak{u}_2 \wedge \mathfrak{t}' \leq_{\mathbb{P}_\gamma} \mathfrak{u}_2 \wedge \mathfrak{t}$.

By Lemma 5.3, it follows that $\mathfrak{u}_2 \wedge (\mathfrak{s}' \wedge \mathfrak{t}') \leq_{\mathbb{P}_\gamma} \mathfrak{u}_2 \wedge (\mathfrak{s} \wedge \mathfrak{t})$. Thus, $\pi_{\delta_0, \gamma}(\langle \mathfrak{s}', \mathfrak{t}' \rangle) = \mathfrak{s}' \wedge \mathfrak{t}' \leq_{\mathbb{P}_\gamma \mid \mathbb{G}_{\delta_0}} \mathfrak{s} \wedge \mathfrak{t} = \pi_{\delta_0, \gamma}(\langle \mathfrak{s}, \mathfrak{t} \rangle)$.

To show that $\pi_{\delta_0, \gamma}$ also satisfies (5.34), suppose that $\langle \mathfrak{s}, \mathfrak{t} \rangle \in \mathbb{S}_{\delta_0, \gamma} \times \mathbb{T}_{\delta_0, \gamma}$ and $\mathbb{p} \in \mathbb{P}_\gamma \mid \mathbb{G}_{\delta_0}$ are such that

$$(5.49) \quad \mathbb{p} \leq_{\mathbb{P}_\gamma \mid \mathbb{G}_{\delta_0}} \mathfrak{s} \wedge \mathfrak{t} = \tau_{\delta_0, \gamma}(\langle \mathfrak{s}, \mathfrak{t} \rangle).$$

III:msi-23-0

Let $\mathfrak{u} \in \mathbb{G}_{\delta_0}$ be such that $\mathfrak{u} \wedge \mathbb{p} \leq_{\mathbb{P}_\gamma} \mathfrak{u} \wedge (\mathfrak{s} \wedge \mathfrak{t})$.

Let \mathbb{p}_0 be a $\gamma \setminus \delta_0$ -sequence defined by

$$(5.50) \quad \mathbb{p}_0(\xi) = \begin{cases} \mathfrak{q}_\xi, & \text{if } \xi \in \text{supp}(\mathbb{p}); \\ \mathbb{1}_{\mathfrak{Q}_\xi}, & \text{otherwise} \end{cases}$$

III:msi-24

for $\xi \in \gamma \setminus \delta_0$, where \mathfrak{q}_ξ is a canonical \mathbb{P}_ξ -name of an element of \mathfrak{Q}_ξ such that

$$(5.51) \quad \mathfrak{u} \wedge \mathbb{p} \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} \text{“} \mathfrak{q}_\xi \equiv \mathbb{p}(\xi) \text{”}, \text{ and}$$

III:msi-25

$$(5.52) \quad \mathbb{p}' \Vdash_{\mathbb{P}_\xi} \text{“} \mathfrak{q}_\xi \equiv (\mathfrak{s} \wedge \mathfrak{t})(\xi) \text{”}, \quad \text{for all } \mathbb{p}' \in \mathbb{P}_\xi \\ \text{with } \mathbb{p}' \perp_{\mathbb{P}_\xi} \mathfrak{u} \wedge \mathbb{p} \upharpoonright \xi.$$

III:msi-25-0

Note that, by (5.51) and (5.52), we have

$$(5.53) \quad \Vdash_{\mathbb{P}_\xi} \text{“} \mathfrak{q}_\xi \leq_{\mathfrak{Q}_\xi} (\mathfrak{s} \wedge \mathfrak{t})(\xi) \text{” for all } \xi \in \gamma \setminus \delta_0.$$

III:msi-25-1

Let $\mathfrak{s}_0 = \mathbb{p}_0 \upharpoonright S$ and $\mathfrak{t}_0 = \mathbb{p}_0 \upharpoonright T$.

By the construction, it is clear that the following Claim holds, and this shows that $\langle \mathfrak{s}_0, \mathfrak{t}_0 \rangle$ is a witness for (5.34).

III:Cl-msi-0

Claim 5.10.1 (a) $\pi_{\delta_0, \gamma}(\langle \mathfrak{s}_0, \mathfrak{t}_0 \rangle) = \mathbb{p}_0 \leq_{\mathbb{P}_\gamma \mid \mathbb{G}_{\delta_0}} \mathbb{p}$,

(b) $\langle \mathfrak{s}_0, \mathfrak{t}_0 \rangle \leq_{\mathbb{S}_{\delta_0, \gamma} \times \mathbb{T}_{\delta_0, \gamma}} \langle \mathfrak{s}, \mathfrak{t} \rangle$.

\vdash (a): By (5.51). (b): By (5.53).

\dashv (Claim 5.10.1)

\square (Lemma 5.10)

Let \mathbb{G}_{δ_0} denote the standard \mathbb{P}_{δ_0} -name of a $(\mathbb{V}, \mathbb{P}_{\delta_0})$ -generic filter.

III:P-msi-3-0

Lemma 5.11 *Suppose that \mathfrak{q}_0 and \mathfrak{q}_1 are \mathbb{P}_{δ_0} -names of elements of $\mathbb{P}_\gamma \mid \mathbb{G}_{\delta_0}$. Then, we have*

$$(5.54) \quad \Vdash_{\mathbb{P}_{\delta_0}} \text{“} \mathfrak{q}_0 \leq_{\mathbb{P}_\gamma \mid \mathbb{G}_{\delta_0}} \mathfrak{q}_1 \text{”} \Leftrightarrow \Vdash_{\mathbb{P}_{\delta_0}} \text{“} \vec{\mathfrak{I}}_{\delta_0} \wedge \mathfrak{q}_0 \leq_{\mathbb{P}_\gamma} \vec{\mathfrak{I}}_{\delta_0} \wedge \mathfrak{q}_1 \text{”}.$$

III:msi-29-a-0

Proof. “ \Leftarrow ” is trivial since $\Vdash_{\mathbb{P}_\delta} \text{“} \vec{\mathfrak{I}}_{\delta_0} \in \mathbb{G}_{\delta_0} \text{”}$.

“ \Rightarrow ”: Suppose that the left side of (5.54) holds. This means that $\Vdash_{\mathbb{P}_{\delta_0}} \text{“} \exists p \in \mathbb{G}_{\delta_0} (p \wedge \mathfrak{q}_0 \leq_{\mathbb{P}_\gamma} p \wedge \mathfrak{q}_1) \text{”}$. By Lemma 5.8, it follows that

$$(5.55) \quad \Vdash_{\mathbb{P}_{\delta_0}} \text{“} \exists p \in \mathbb{G}_{\delta_0} \forall p' \leq_{\mathbb{P}_\gamma} p (p' \wedge \mathfrak{q}_0 \leq_{\mathbb{P}_\gamma} p' \wedge \mathfrak{q}_1) \text{”}.$$

III:msi-29-a-1

Suppose, toward a contradiction, that

$$(5.56) \quad \not\Vdash_{\mathbb{P}_{\delta_0}} \text{“} \vec{\mathfrak{I}}_{\delta_0} \wedge \mathfrak{q}_0 \leq_{\mathbb{P}_\gamma} \vec{\mathfrak{I}}_{\delta_0} \wedge \mathfrak{q}_1 \text{”}.$$

III:msi-29-a-2

Then, there are $\mathbb{p}_0 \in \mathbb{P}_{\delta_0}$ and $\delta_0 \leq \delta < \gamma$ such that, for any $\mathbb{p} \leq_{\mathbb{P}_{\delta_0}} \mathbb{p}_0$,

$$(5.57) \quad \mathbb{p} \Vdash_{\mathbb{P}_{\delta_0}} \text{“} \vec{\mathfrak{I}}_{\delta_0} \wedge \mathfrak{q}_0 \upharpoonright \delta \leq_{\mathbb{P}_\delta} \vec{\mathfrak{I}}_{\delta_0} \wedge \mathfrak{q}_1 \upharpoonright \delta \text{”}, \text{ but}$$

III:msi-29-a-3

$$(5.58) \quad \mathbb{p} \Vdash_{\mathbb{P}_{\delta_0}} \text{“} \vec{\mathfrak{I}}_{\delta_0} \wedge \mathfrak{q}_0 \upharpoonright \delta \Vdash_{\mathbb{P}_\delta} \text{“} \mathfrak{q}_0(\delta) \not\leq_{\mathbb{Q}_\delta} \mathfrak{q}_1(\delta) \text{”} \text{”}.$$

III:msi-29-a-4

By Lemma 5.8, (1), it follows that, for any $\mathbb{p} \leq_{\mathbb{P}_{\delta_0}} \mathbb{p}_0$, $\mathbb{p} \Vdash_{\mathbb{P}_{\delta_0}} \text{“} \check{\mathbb{p}} \in \mathbb{G}_{\delta_0} \text{ and } \check{\mathbb{p}} \wedge \mathfrak{q}_0 \not\leq_{\mathbb{P}_\gamma} \check{\mathbb{p}} \wedge \mathfrak{q}_1 \text{”}$. This is a contradiction to (5.55) by Lemma 5.8, (1). \square (Lemma 5.11)

III:P-msi-4

Lemma 5.12 *For $\delta_0 < \gamma \leq \kappa$ and $(\mathbb{V}, \mathbb{P}_{\delta_0})$ -generic filter \mathbb{G}_{δ_0} , we have $\mathbb{V}[\mathbb{G}_{\delta_0}] \models \text{“} \mathbb{S}_{\delta_0, \gamma} \text{ is } < \nu(\delta_0)\text{-closed”}$.*

Proof. In \mathbb{V} , let $\mathbb{S}_{\delta_0, \gamma}$ be a \mathbb{P}_{δ_0} -name of $\mathbb{S}_{\delta_0, \gamma}$ and \check{h} be a \mathbb{P}_{δ_0} -name of a descending δ -sequence in $\mathbb{S}_{\delta_0, \gamma}$ for some

$$(5.59) \quad \delta < \nu(\delta_0).$$

III:msi-29-0

By Lemma 5.11, we have $\Vdash_{\mathbb{P}_{\delta_0}} \vec{\mathbb{1}}_{\delta_0} \wedge \check{h}(\xi) \leq_{\mathbb{P}_\gamma} \vec{\mathbb{1}}_{\delta_0} \wedge \check{h}(\eta)$ for all $\xi < \eta < \delta$.

Let

$$(5.60) \quad D = \{\alpha < \gamma : \mathfrak{r} \Vdash_{\mathbb{P}_{\delta_0}} \text{“}\alpha \in \text{supp}(\check{h}(\xi)) \text{ for some } \xi < \delta\text{”}$$

III:msi-30

$$\text{for some } \mathfrak{r} \in \mathbb{P}_{\delta_0}\}.$$

Since $\Vdash_{\mathbb{P}_{\delta_0}} \text{“}(\forall \xi < \delta) \text{supp}(\xi) \subseteq S \setminus \nu(\delta_0)\text{”}$, we have

$$(5.61) \quad D \subseteq S \setminus \nu(\delta_0).$$

III:msi-31

III:Cl-msi-1

Claim 5.12.1 *For any regular $\delta_0 \leq \mu \leq \gamma$, we have $|D \cap \mu| < \mu$. Thus, $D \in \mathcal{I}_\gamma$.*

\vdash By (5.61), it is enough to show the inequality for all regular cardinal μ with $\nu(\delta_0) \leq \mu \leq \gamma$. For such μ , we have $\Vdash_{\mathbb{P}_{\delta_0}} \text{“}\text{supp}(\check{h}(\xi)) \cap \mu \text{ is a bounded subset of } \mu\text{”}$ for all $\xi < \delta$. Thus

$$(5.62) \quad D_{\mu, \xi, \mathfrak{r}} = \{\alpha < \mu : \mathfrak{r} \Vdash_{\mathbb{P}_{\delta_0}} \text{“}\alpha \in \text{supp}(\check{h}(\xi))\text{”}\}$$

III:msi-32

is a bounded subset of μ for each $\xi < \delta$ and $\mathfrak{r} \in \mathbb{P}_{\delta_0}$. By (5.59) and Lemma 5.4 for $\nu = \nu(\alpha)$, it follows that $D \cap \mu = \bigcup_{\xi < \delta, \mathfrak{r} \in \mathbb{P}_{\delta_0}} D_{\mu, \xi, \mathfrak{r}}$ is a bounded subset of μ .

\dashv (Claim 5.12.1)

Now we define, by induction on $\delta_0 \leq i \leq \gamma$, \mathbb{P}_{δ_0} -names \mathbb{P}_i , $i \in \gamma + 1 \setminus \delta_0$ such that

$$(5.63) \quad \Vdash_{\mathbb{P}_{\delta_0}} \text{“}\mathbb{P}_i \in \mathbb{P}_i \upharpoonright (S \setminus \delta_0)\text{”}$$

III:msi-33

$$(5.64) \quad \Vdash_{\mathbb{P}_{\delta_0}} \text{“}\text{supp}(\mathbb{P}_i) \subseteq \check{D}\text{”}$$

III:msi-36

$$(5.65) \quad \Vdash_{\mathbb{P}_{\delta_0}} \text{“}(\langle \mathbb{P}_i : i \in \gamma + 1 \setminus \delta_0 \rangle)^\bullet \text{ is an increasing sequence of sequences”};$$

III:msi-34

and

$$(5.66) \quad \Vdash_{\mathbb{P}_{\delta_0}} \text{“}\mathbb{P}_i \text{ is a lower bound of } \langle \check{h}(\xi) \upharpoonright i : \xi < \delta \rangle$$

III:msi-35

$$\text{with respect to } \leq_{\mathbb{P}_i \upharpoonright \mathcal{Q}_{\delta_0}}\text{”}$$

for all $i \in \gamma + 1 \setminus \delta_0$.

For $i = \delta_0$, $\mathbb{P}_i = \emptyset$ will do.

Suppose now that i is a limit ordinal and \mathbb{P}_j , $j < i$ has been defined such that

$$(5.63') \quad \Vdash_{\mathbb{P}_{\delta_0}} \text{“}\mathbb{P}_j \in \mathbb{P}_j \upharpoonright (S \setminus \delta_0)\text{”}$$

III:msi-33

$$(5.64') \quad \Vdash_{\mathbb{P}_{\delta_0}} \text{“}\text{supp}(\mathbb{P}_j) \subseteq \check{D}\text{”}$$

III:msi-36

(5.65') $\Vdash_{\mathbb{P}_{\delta_0}}$ “ $(\langle \mathbb{P}_j : j \in i \setminus \delta_0 \rangle)^\bullet$ is an increasing sequence of sequences” ; III:msi-34

and

(5.66') $\Vdash_{\mathbb{P}_{\delta_0}}$ “ \mathbb{P}_j is a lower bound of $\langle h(\xi) \upharpoonright j : \xi < \delta \rangle$ with respect to $\leq_{\mathbb{P}_j \upharpoonright \mathcal{Q}_{\delta_0}}$ ” III:msi-35
for all $j \in i \setminus \delta_0$.

By Lemma 5.11, (5.66') implies

(5.67) $\Vdash_{\mathbb{P}_{\delta_0}}$ “ $\vec{1}_{\delta_0} \wedge \mathbb{P}_j \leq_{\mathbb{P}_j} \vec{1}_{\delta_0} \wedge h(\xi) \upharpoonright j$ ” for all $\xi < \delta$ and $j < i$. III:msi-37

Let \mathbb{P}_i be the \mathbb{P}_{δ_0} -name such that

(5.68) $\Vdash_{\mathbb{P}_{\delta_0}}$ “ $\mathbb{P}_i \equiv \bigcup(\{\mathbb{P}_j : j < i\})^\bullet$ ”. III:msi-39

We show that \mathbb{P}_i together with $\mathbb{P}_j, j < i$ satisfies

(5.63'') $\Vdash_{\mathbb{P}_{\delta_0}}$ “ $\mathbb{P}_j \in \mathbb{P}_j \downarrow (S \setminus \delta_0)$ ” for all $j \in i + 1 \setminus \delta_0$; III:msi-33

(5.64'') $\Vdash_{\mathbb{P}_{\delta_0}}$ “ $\text{supp}(\mathbb{P}_j) \subseteq \check{D}$ ” for all $j \in i + 1 \setminus \delta_0$; III:msi-36

(5.65'') $\Vdash_{\mathbb{P}_{\delta_0}}$ “ $(\langle \mathbb{P}_j : j \in i + 1 \setminus \delta_0 \rangle)^\bullet$ is an increasing sequence of sequences” ; III:msi-34

and

(5.66'') $\Vdash_{\mathbb{P}_{\delta_0}}$ “ \mathbb{P}_j is a lower bound of $\langle h(\xi) \upharpoonright j : \xi < \delta \rangle$ with respect to $\leq_{\mathbb{P}_j \upharpoonright \mathcal{Q}_{\delta_0}}$ ” III:msi-35
for all $j \in i + 1 \setminus \delta_0$.

(5.64'') follows from (5.64') and (5.68). (5.63'') follows from this. (5.65'') is clear by (5.68) and (5.66'') follows from (5.67).

Finally, suppose that $\mathbb{P}_j, j \leq i$ has been defined for $\nu(\delta_0) \leq j < \gamma$ in accordance with (5.63) \sim (5.66). In particular, we have

(5.67') $\Vdash_{\mathbb{P}_{\delta_0}}$ “ $\vec{1}_{\delta_0} \wedge \mathbb{P}_i \leq_{\mathbb{P}_i} \vec{1}_{\delta_0} \wedge h(\xi) \upharpoonright i$ ” for all $\xi < \delta$. III:msi-37

If $i \notin S$, then let $\mathbb{P}_{i+1} = (\mathbb{P}_{i+1} \cup \{\langle i, \mathbb{1}_{\mathbb{Q}_i} \rangle\})_{\mathbb{P}_{\delta_0}}^\bullet$.

If $i \in S$, then we have

(5.69) $\Vdash_{\mathbb{P}_i}$ “ \mathbb{Q}_i is $< \nu(\delta_0)$ -closed” III:msi-40

by (B), (a) and (b) in the definition of our mixed support iteration. By (5.67') and by the choice of h , we have

$$(5.70) \quad \Vdash_{\mathbb{P}_{\delta_0}} \vec{\mathbb{1}}_{\delta_0} \wedge \mathbb{P}_i \Vdash_{\mathbb{P}_i} \langle \langle h(\xi)(i) : \xi < \delta \rangle \rangle_{\mathbb{P}_i}^\bullet$$

is a descending sequence in \mathbb{Q}_i ””.

III:msi-41

By (5.69), there is a \mathbb{P}_{δ_0} -name \mathbb{Q} of \mathbb{P}_{δ_0} -name such that

$$(5.71) \quad \Vdash_{\mathbb{P}_{\delta_0}} \vec{\mathbb{1}}_{\delta_0} \wedge \mathbb{P}_i \Vdash_{\mathbb{P}_i} \text{“}\mathbb{Q} \text{ is a lower bound of } \langle \langle h(\xi)(i) : \xi < \delta \rangle \rangle_{\mathbb{P}_i}^\bullet \text{””}.$$
III:msi-42

Let $\mathbb{P}_{i+1} = (\mathbb{P}_i \cup \{ \langle i, \mathbb{Q} \rangle \})_{\mathbb{P}_{\delta_0}}^\bullet$. Similarly to the previous case, we can show that \mathbb{P}_{i+1} together with \mathbb{P}_j , $j \leq i$ satisfies (5.63) \sim (5.66). □ (Lemma 5.12)

Lemma 5.13 *Suppose that \mathbb{P} is a poset, \mathbb{Q} a \mathbb{P} -name of a poset with*

III:P-msi-4-0

$$(5.72) \quad \Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \text{ is ccc”},$$
III:msi-42-a-a

and \mathbb{S} is a σ -closed poset. Then we have

$$(5.73) \quad \Vdash_{\mathbb{S}} \text{“}\Vdash_{\mathbb{P}} \check{\mathbb{Q}} \text{ is ccc””}.$$
III:msi-42-a

Proof. Suppose that \mathcal{G} is a \mathbb{S} -name of a \mathbb{P} -name such that

$$(5.74) \quad \Vdash_{\mathbb{S}} \text{“}\Vdash_{\mathbb{P}} \check{\mathcal{G}} \text{ is a subset of } \check{\mathbb{Q}} \text{ of cardinality } \aleph_1 \text{””}.$$
III:msi-42-a-0

We have to show

$$(5.75) \quad \Vdash_{\mathbb{S}} \text{“}\Vdash_{\mathbb{P}} \text{“there are compatible elements in } \mathcal{G} \text{””}.”$$
III:msi-42-a-

Let f be a \mathbb{S} -name of \mathbb{P} -name such that

$$(5.76) \quad \Vdash_{\mathbb{S}} \text{“}\Vdash_{\mathbb{P}} \text{“} f : \omega_1 \rightarrow \mathcal{G} \text{ and } f \text{ is an injective enumeration of } \mathcal{G} \text{””}.”$$
III:msi-42-a-1

Let $\mathfrak{s} \in \mathbb{S}$ and $\mathfrak{p} \in \mathbb{P}$ be arbitrary. By σ -closedness of \mathbb{S} , we can find a decreasing sequence $\langle \mathfrak{s}_\alpha : \alpha < \omega_1 \rangle$ of elements of \mathbb{S} and a sequence \mathbb{q}_α , $\alpha < \omega_1$ of \mathbb{P} -names such that

$$(5.77) \quad \mathfrak{s}_0 \leq_{\mathbb{S}} \mathfrak{s},$$
III:msi-42-a-2

$$(5.78) \quad \mathfrak{s}_\alpha \Vdash_{\mathbb{S}} \text{“}\Vdash_{\mathbb{P}} \check{\mathbb{q}}_\alpha \equiv f(\alpha) \text{””}.$$
III:msi-42-a-3

By (5.74), (5.76) and (5.78), we have

$$(5.79) \quad \mathfrak{s}_\alpha \Vdash_{\mathbb{S}} \text{“}\Vdash_{\mathbb{P}} \check{\mathbb{q}}_\alpha \varepsilon \check{\mathbb{Q}} \text{””}.$$
III:msi-42-a-

Since the relation $\cdot \Vdash \cdot \varepsilon \cdot$ is Δ_1 , it follows that

3-0

$$(5.80) \quad \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{\mathcal{Q}}_{\alpha} \in \underset{\sim}{\mathcal{Q}} \text{”}.$$

III:msi-42-a-4

By (5.72), there are $\mathbb{P}' \leq_{\mathbb{P}} \mathbb{P}$ and $\alpha_0 < \alpha_1 < \omega_1$ such that

$$(5.81) \quad \mathbb{P}' \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{\mathcal{Q}}_{\alpha_0} \top_{\underset{\sim}{\mathcal{Q}}} \underset{\sim}{\mathcal{Q}}_{\alpha_1} \text{”}.$$

III:msi-42-a-5

By (5.76) and (5.78), and since $\langle \mathfrak{s}_{\alpha} : \alpha < \omega_1 \rangle$ is decreasing,

$$(5.82) \quad \mathfrak{s}_{\alpha_1} \Vdash_{\mathfrak{S}} \text{“} \mathbb{P}' \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{f}(\alpha_0) \top_{\underset{\sim}{\mathcal{Q}}} \underset{\sim}{f}(\alpha_1) \text{”} \text{”}.$$

III:msi-42-a-6

Thus

$$(5.83) \quad \mathfrak{s}_{\alpha_1} \Vdash_{\mathfrak{S}} \text{“} \exists x \leq_{\check{\mathbb{P}}} \mathbb{P} x \Vdash_{\check{\mathbb{P}}} \text{“} \text{there are compatible elements in } \underset{\sim}{\mathcal{S}} \text{”} \text{”}.$$

III:msi-42-a-7

$\mathfrak{s}_{\alpha_1} \leq_{\mathfrak{S}} \mathfrak{s}$ by (5.77). Since \mathfrak{s} was arbitrary, it follows that

$$(5.84) \quad \Vdash_{\mathfrak{S}} \text{“} \exists x \leq_{\check{\mathbb{P}}} \check{\mathbb{P}} x \Vdash_{\check{\mathbb{P}}} \text{“} \text{there are compatible elements in } \underset{\sim}{\mathcal{S}} \text{”} \text{”}.$$

III:msi-42-a-8

Now, since \mathbb{P} was arbitrary, (5.75) follows. □ (Lemma 5.13)

III:P-msi-5

Lemma 5.14 (1) For $\delta \leq \kappa$, ι_{δ} is an isomorphism from \mathcal{O}_{δ} to $\mathbb{T}_{0,\delta}$. $\langle i_{\beta} : \beta \leq \delta \rangle$ forms a commutative system together with $\langle \mathcal{O}_{\beta}, i_{\beta,\gamma}^* : \beta \leq \gamma \leq \delta \rangle$ and $\langle \mathbb{T}_{0,\beta}, i_{\beta,\gamma} \upharpoonright \mathbb{T}_{\beta} : \beta \leq \gamma \leq \delta \rangle$. In particular, $\langle \mathbb{T}_{0,\beta} : \beta \leq \delta \rangle$ is homomorphic to the sequence of iterands of a finite support iteration of ccc posets.

(2) For $\delta_0 < \gamma \leq \kappa$ and $(\mathbb{V}, \mathbb{P}_{\delta_0})$ -generic filter \mathbb{G}_{δ_0} , we have

$$(5.85) \quad \mathbb{V}[\mathbb{G}_{\delta_0}] \models \text{“} \mathbb{T}_{\delta_0,\gamma} \text{ has the ccc”}.$$

III:msi-42-0

Proof. (1): By induction $\delta \leq \kappa$.

(2): Note first that, by Lemma 5.6, $\pi_{0,\delta_0} : \mathbb{S}_{0,\delta_0} \times \mathbb{T}_{0,\delta_0} \rightarrow \mathbb{P}_{\delta_0}$ is a projection. Let $\mathbb{G}_{\mathbb{S}} * \mathbb{G}_{\mathbb{T}}$ be a $(\mathbb{V}, \mathbb{S}_{0,\delta_0} \times \mathbb{T}_{0,\delta_0} / \mathbb{G}_{\delta_0})$ -generic filter in the sense of Lemma 5.5, (2) where we assume that $\mathbb{G}_{\mathbb{S}}$ and $\mathbb{G}_{\mathbb{T}}$ are the generic filters over \mathbb{S}_{0,δ_0} and \mathbb{T}_{0,δ_0} respectively. We have $\mathbb{G}_{\mathbb{T}} = \mathbb{G}_{\delta_0} \upharpoonright T$. Thus $\mathbb{T}_{\delta_0,\gamma} = \mathbb{T}_{0,\gamma} \upharpoonright \mathbb{G}_{\mathbb{T}}$. By the Factor Lemma for finite support iteration of ccc posets, we have $\mathbb{V}[\mathbb{G}_{\mathbb{T}}] \models \mathbb{T}_{\delta_0,\gamma}$ is ccc.

Since \mathbb{S}_{0,δ_0} is σ -closed by Lemma 5.12, $\mathbb{V}[\mathbb{G}_{\mathbb{S}}][\mathbb{G}_{\mathbb{T}}] \models \mathbb{T}_{\delta_0,\gamma}$ is ccc by Lemma 5.13. Since $\mathbb{V}[\mathbb{G}_{\delta_0}]$ is an inner model of $\mathbb{V}[\mathbb{G}_{\mathbb{S}}][\mathbb{G}_{\mathbb{T}}]$, it follows that $\mathbb{V}[\mathbb{G}_{\delta_0}] \models \mathbb{T}_{\delta_0,\gamma}$ is ccc.

□ (Lemma 5.14)

Summarizing what we have proved above, we obtain the following:

III:P-msi-6

Proposition 5.15 Suppose that κ is a supercompact cardinal, $f : \kappa \rightarrow \kappa$ a Laver function with $S, T \subseteq \kappa$ defined by (5.3), (5.4), and let $\nu : \kappa \rightarrow \kappa$ be defined by (5.5).

For the preparatory finite support ccc iteration $\langle \mathbb{O}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ satisfying (5.6), (5.7), let $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be the Easton-type mixed support iteration over $\langle \mathbb{O}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ as defined in (A) and (B) on pages 24 ~ 26 with the complete embeddings $i_{\delta,\gamma}$, ι_δ and projections $p_{\delta,\gamma}$ for $\delta \leq \gamma \leq \kappa$.

For any $(\mathbf{V}, \mathbb{P}_\kappa)$ -generic filter \mathbb{G}_κ , $\delta_0 < \kappa$ and $\mathbb{G}_{\delta_0} = p_{\kappa,\delta_0}'' \mathbb{G}_\kappa$, there are posets \mathbb{S} , \mathbb{T} , a regular subposet \mathbb{Q} of the completion of $\mathbb{S} \times \mathbb{T}$ in $\mathbf{V}[\mathbb{G}_{\delta_0}]$ such that

$$(5.86) \quad \mathbf{V}[\mathbb{G}_{\delta_0}] \models \text{“}\mathbb{S} \text{ is } \nu(\delta_0)\text{-closed and } \mathbb{T} \text{ is ccc”}, \text{ and}$$

III:msi-44

$$(5.87) \quad \mathbf{V}[\mathbb{G}_{\delta_0}] \models \text{“}\mathbb{Q} \sim \mathbb{P}_\kappa/\mathbb{G}_{\delta_0}\text{”}.$$

III:msi-45

In particular, there is a $(\mathbf{V}[\mathbb{G}_{\delta_0}], RO(\mathbb{S} \times \mathbb{T}))$ -generic filter $\tilde{\mathbb{H}}$ such that, letting $\mathbb{H} = \tilde{\mathbb{H}} \cap \mathbb{Q}$, we have $\mathbf{V}[\mathbb{G}_\kappa] = \mathbf{V}[\mathbb{G}_{\delta_0}][\mathbb{H}]$.

Proof. By Lemma 5.9, we have $\mathbf{V}[\mathbb{G}_{\delta_0}] \models \mathbb{P}_\kappa/\mathbb{G}_{\delta_0} \sim \mathbb{P}_\kappa|\mathbb{G}_{\delta_0}$.

In $\mathbf{V}[\mathbb{G}_{\delta_0}]$, $\mathbb{P}_\kappa|\mathbb{G}_{\delta_0}$ is forcing equivalent to a regular sub-poset of the completion of $\mathbb{S}_{\delta_0,\gamma} \times \mathbb{T}_{\delta_0,\gamma}$ by Lemma 5.10 (c.f. Lemma 5.7). By Lemma 5.12, $\mathbb{S}_{\delta_0,\gamma}$ is $\nu(\delta_0)$ -closed and by Lemma 5.14, $\mathbb{T}_{\delta_0,\gamma}$ is ccc. \square (Proposition 5.15)

III:P-msi-7

Theorem 5.16 Suppose that κ , f , S , T , ν , $\langle \mathbb{O}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$, $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$, $i_{\delta,\gamma}$, ι_δ , $p_{\delta,\gamma}$ for $\delta \leq \gamma \leq \kappa$, and \mathbb{G}_κ are as in Proposition 5.15.

(0) If $\langle \mathbb{O}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ adds κ many reals then $\mathbf{V}[\mathbb{G}_\kappa] \models \kappa = 2^{\aleph_0}$.

(1) In $\mathbf{V}[\mathbb{G}_\kappa]$, κ is strongly Laver-generically supercompact for $(\mathfrak{P}, \mathfrak{Q}_\theta)$ for all $\theta \in \text{Card}$ for the properties of posets \mathfrak{P} and \mathfrak{Q}_θ as in Proposition 4.7, (2).

(1') If κ is superhuge, then, in $\mathbf{V}[\mathbb{G}_\kappa]$, κ is strongly Laver-generically superhuge for $(\mathfrak{P}, \mathfrak{Q}_\theta)$ for all $\theta \in \text{Card}$ for the properties of posets \mathfrak{P} and \mathfrak{Q}_θ as in Proposition 4.7, (2).

(2) If the preparatory iteration $\langle \mathbb{O}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ is such that $\Vdash_{\mathbb{Q}_\alpha} \text{“}\mathbb{R}_\alpha \equiv \text{Fn}(\omega, 2)\text{”}$ for all $\alpha \in T$, then in $\mathbf{V}[\mathbb{G}_\kappa]$, κ is strongly Laver-generically supercompact for $(\mathfrak{P}, \mathfrak{Q}_\theta)$ for all $\theta \in \text{Card}$ for the properties of posets \mathfrak{P} and \mathfrak{Q}_θ as in Proposition 4.7, (1).

(2') If the preparatory iteration $\langle \mathbb{O}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ is such that $\Vdash_{\mathbb{Q}_\alpha} \text{“}\mathbb{R}_\alpha \equiv \text{Fn}(\omega, 2)\text{”}$ for all $\alpha \in T$ and κ is superhuge, then in $\mathbf{V}[\mathbb{G}_\kappa]$, κ is strongly Laver-generically superhuge for $(\mathfrak{P}, \mathfrak{Q}_\theta)$ for all $\theta \in \text{Card}$ for the properties of posets \mathfrak{P} and \mathfrak{Q}_θ as in Proposition 4.7, (1).

Proof. (0): By Lemma 5.4, κ is a regular cardinal in $\mathbf{V}[\mathbb{G}_\kappa]$ and $2^{\aleph_0} \leq \kappa$ in $\mathbf{V}[\mathbb{G}_\kappa]$. Since $\iota_\kappa : \mathbb{O}_\kappa \rightarrow \mathbb{P}_\kappa$ is a complete embedding, if \mathbb{O}_κ adds κ many reals then $\kappa \geq 2^{\aleph_0}$.

(1): In $V[\mathbb{G}_\kappa]$, let $\mathbb{P} = \text{Col}(\kappa, \mu)$ for some cardinal μ and, let λ and θ be two other cardinals. Without loss of generality, we may assume that $\mu^{<\kappa} \leq \lambda$. Since f is a Laver function there is an elementary embedding $j : V \xrightarrow{\cong} M$ such that

$$(5.88) \quad \text{crit}(j) = \kappa, \tag{III:msi-46}$$

$$(5.89) \quad j(\kappa) > \lambda, \tag{III:msi-47}$$

$$(5.90) \quad [M]^\lambda \subseteq M, \text{ and} \tag{III:msi-48}$$

$$(5.91) \quad j(f)(\kappa) = \langle \mu, \theta, \emptyset \rangle. \tag{III:msi-49}$$

Let $S^* = j(S)$, $\nu^* = j(\nu)$ and, let $\vec{\mathbb{P}}^* = j(\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle)$. Since $\vec{\mathbb{P}}^*$ is $j(\kappa)$ -(double) sequence by elementarity, we write

$$(5.92) \quad \vec{\mathbb{P}}^* = \langle \mathbb{P}_\alpha^*, \mathbb{Q}_\beta^* : \alpha \leq j(\kappa), \beta < j(\kappa) \rangle. \tag{III:msi-50}$$

By the elementarity of j , Lemma 5.4, (5.88) and (5.90), we have $\mathbb{P}_\alpha^* = \mathbb{P}_\alpha$ and $\mathbb{Q}_\beta^* = \mathbb{Q}_\beta$ for all $\alpha, \beta < \kappa$. $\mathbb{P}_\kappa^* = \mathbb{P}_\kappa$ by elementarity and (5.90). Note that $\kappa \in S^*$ by elementarity. So it follows that $\mathbb{Q}_\kappa^* = (\text{Col}(\kappa, \mu))_{\mathbb{P}_\kappa}^\bullet$ by (5.22) and (5.91). Thus $\mathbb{Q}_\kappa^*[\mathbb{G}_\kappa] = \mathbb{P}$.

Also by (5.91), we have $\nu^*(\kappa) \geq \theta$. Let \mathfrak{g} be a $(V[\mathbb{G}_\kappa], \mathbb{P})$ -generic filter. In $M[\mathbb{G}_\kappa][\mathfrak{g}]$, $\mathbb{P}_{j(\kappa)}^*/\mathbb{G}_\kappa * \mathfrak{g}$ is forcing equivalent to a regular sub-poset of the completion of a poset of the form “ccc poset \times θ -closed poset” by Proposition 5.15. Thus, $\mathbb{P}_{j(\kappa)}^*/\mathbb{G}_\kappa * \mathfrak{g} \models \mathfrak{Q}_\theta$.

Let \mathbb{H}^* be a $(V[\mathbb{G}_\kappa * \mathfrak{g}], \mathbb{P}_{j(\kappa)}^*/\mathbb{G}_\kappa * \mathfrak{g})$ -generic filter. Then we can find a $(V, \mathbb{P}_{j(\kappa)}^*)$ -generic filter \mathbb{H} such that $M[\mathbb{H}] = M[\mathbb{G}_\kappa * \mathfrak{g} * \mathbb{H}^*]$ and $i_{\kappa, j(\kappa)}^* \text{''} \mathbb{G}_\kappa \subseteq \mathbb{H}$ for the complete embedding $i_{\kappa, j(\kappa)}^* : \mathbb{P}_\kappa \xrightarrow{\cong} \mathbb{P}_{j(\kappa)}^*$ associated with $\vec{\mathbb{P}}^*$. j can be then lifted to

$$(5.93) \quad \tilde{j} : V[\mathbb{G}_\kappa] \xrightarrow{\cong} M[\mathbb{H}]; a[\mathbb{G}] \mapsto j(a)[\mathbb{H}]. \tag{III:msi-51}$$

It is easy to show that \tilde{j} with \mathbb{H} satisfies (4.1) \sim (4.5) and (4.6). The last condition holds by Lemma 4.1. This shows that κ is strongly Laver-generically supercompact for $(\mathfrak{P}, \mathfrak{Q}_\theta)$.

(2'): is proved similarly to (2) above. The condition (4.6') is shown using Lemma 4.2.

(3), (3'): can be proved similarly to (2) and (2').

□ (Theorem 5.16)

6 Models with strong reflection properties down to $< 2^{\aleph_0}$ and with even stronger reflection properties but down to $\leq 2^{\aleph_0}$

III:refl

As an application of the forcing constructions considered in the previous sections, we give two models of large continuum with strong reflection properties around the continuum. In one of the models, we have $\text{HH}(< 2^{\aleph_0})$ and, in the other, this reflection property is negated. Thus, we obtain the independence of $\text{HH}(< 2^{\aleph_0})$ from other strong reflection principles in the large continuum context.

In contrast, the most of the other reflection properties are situated in a tight web of implications which is (almost) upward directed (see e.g. the diagram in the last section of [Fuchino, Sakai and Ottenbreit[9]]). This suggests that the reflection of non-metrizability is a totally different kind of reflection statement from the other reflection principles.

With an arbitrary preparatory finite support ccc iteration $\langle \mathbb{O}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ we already have the following:

Theorem 6.1 *Let κ, κ_1 with $\kappa < \kappa_1$ be two supercompact cardinals and let $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be the Easton-type mixed support iteration over an arbitrary preparatory finite support ccc iterating $\langle \mathbb{O}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ which adds κ many reals. Let $\mathbb{P} = \mathbb{P}_\kappa * (\text{Col}(\kappa, \kappa_1))_{\mathbb{P}_\kappa}^\bullet$. Then, in the \mathbb{P} generic extension over \mathbb{V} , we have*

III:P-models-

$$(6.1) \quad 2^{\aleph_0} = \kappa;$$

III:models-0

$$(6.2) \quad \text{SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\aleph_0}, < 2^{\aleph_0}) \text{ and } \text{GRP}^{< 2^{\aleph_0}}(\leq 2^{\aleph_0}).$$

III:models-1

$$(6.3) \quad \text{SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\text{PKL}}, < 2^{\aleph_0}) \text{ and } \mathcal{P}_{2^{\aleph_0}}(\lambda) \text{ carries a } \sigma\text{-saturated normal ideal for all } \lambda \geq 2^{\aleph_0}.$$

III:models-2

Proof. (6.1): By Theorem 5.16, (0).

(6.2): By Theorem 5.16, (1), κ is strongly Laver-generically supercompact for $(\mathfrak{P}, \mathfrak{Q})$ for properties $\mathfrak{P}, \mathfrak{Q}$ as in Corollary 4.5. Thus, by Corollary 4.5, (2), we have $\text{SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\aleph_0}, < 2^{\aleph_0})$ and $\text{GRP}^{< 2^{\aleph_0}}(\leq 2^{\aleph_0})$ in the generic extension.

(6.3): By Theorem 5.16, (1), κ is strongly Laver-generically supercompact for $(\mathfrak{P}, \mathfrak{Q}_\theta)$ for any θ for \mathfrak{P} and \mathfrak{Q}_θ as in Proposition 4.7, (2). Thus (6.3) holds by Proposition 4.7, (2) and (3). □ (Theorem 6.1)

The following strengthening of the forcing axiom $\text{MA}(\mathcal{P})$ for a class \mathcal{P} of posets was studied in [Fuchino, Ottenbreit and Sakai[10]]:

For a poset \mathbb{P} , \mathbb{P} -name \mathcal{S} of a set of subsets of On and a filter \mathbb{G} on \mathbb{P} , let

$$(6.4) \quad \begin{aligned} \mathcal{S}(\mathbb{G}) = \{b : b = \{\alpha \in \text{On} : \mathbb{P} \Vdash_{\mathbb{P}} \check{\alpha} \in \check{s}\} \text{ for a } \mathbb{P} \in \mathbb{G}\} \\ \text{for a } \mathbb{P}\text{-name } \check{s} \text{ such that} \\ \Vdash_{\mathbb{P}} \check{s} \in \mathcal{S} \text{ and } \text{sup}(\check{s}) \equiv \text{sup}(b) \}. \end{aligned}$$

III:models-3

Note that if \mathbb{G} is a (\mathbb{V}, \mathbb{P}) -generic filter, then $\mathcal{S}(\mathbb{G}) = \mathcal{S}[\mathbb{G}]$. [...]

For uncountable cardinals μ and $\kappa > \aleph_1$, let $\text{MA}^{++\mu}(\mathcal{P}, < \kappa)$ be the strengthening of $\text{MA}^{+\mu}(\mathcal{P}, < \kappa)$ defined by:

$\text{MA}^{++\mu}(\mathcal{P}, < \kappa)$: For any $\mathbb{P} \in \mathcal{P}$, any family \mathcal{D} of dense subsets of \mathbb{P} with $|\mathcal{D}| < \kappa$ and any family \mathcal{S} of \mathbb{P} -names such that $|\mathcal{S}| \leq \mu$ and $\Vdash_{\mathbb{P}} \check{s}$ is a stationary subset of $\mathcal{P}_{\eta_{\check{s}}}(\theta_{\check{s}})$ for some $\omega < \eta_{\check{s}} \leq \theta_{\check{s}} < 2^{\aleph_0}$ with $\eta_{\check{s}}$ regular, for all $\check{S} \in \mathcal{S}$, there is a \mathcal{D} -generic filter \mathbb{G} over \mathbb{P} such that $\mathcal{S}(\mathbb{G})$ is stationary in $\mathcal{P}_{\eta_{\check{s}}}(\theta_{\check{s}})$ for all $\check{S} \in \mathcal{S}$.

In case of $\mu = \omega_1$, the principle $\text{MA}^{++\omega_1}(\mathcal{P}, < \kappa)$ is equivalent to the usual $\text{MA}^{+\omega_1}(\mathcal{P}, < \kappa)$.

Proposition 6.2 Suppose that κ is a supercompact cardinal and f a Laver function on κ . Let S and T be defined by (5.3) and (5.4).

III:P-models-
0-0

(1) Suppose that the preparatory finite support ccc iteration $\langle \mathbb{O}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ is defined by:

$$(6.5) \quad \mathbb{R}_\beta = \begin{cases} (P)_{\mathbb{O}_\beta}^\vee, & \text{if } \beta = \alpha + 1 \text{ for an } \alpha \in S \text{ and} \\ & f(\alpha) = \langle \mu, \theta, P \rangle \text{ for cardinals } \mu, \theta \text{ and a poset } P \\ & \text{such that } \Vdash_{\mathbb{O}_\beta} (P)_{\mathbb{O}_\beta}^\vee \text{ is a ccc poset}; \\ \{\mathbb{1}_{\mathbb{R}_\beta}\}, & \text{otherwise.} \end{cases}$$

III:models-4

Then, for the Easton-type mixed support iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ over $\langle \mathbb{O}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ and $\mathbb{P} = \mathbb{P}_\kappa * (\text{Col}(\kappa, \mu))_{\mathbb{P}_\kappa}^\bullet$ for a regular $\mu > \kappa$,

$$(6.6) \quad \begin{aligned} \Vdash_{\mathbb{P}} \text{“MA}^{++\eta}(\mathcal{P}, < 2^{\aleph_0}) \text{ for all cardinal } \eta < 2^{\aleph_0} \\ \text{holds for } \mathcal{P} = \{P : P \text{ is a ccc poset and } P \in \mathbb{V}\} \text{”}, \end{aligned}$$

III:models-5

where “ P is ccc poset” is meant “ccc poset in the \mathbb{P} -generic extension” while \mathbb{V} denotes here the ground model before extending generically by \mathbb{P} .

(2) Suppose that $\langle \mathbb{O}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ is a preparatory finite support iteration such that each \mathbb{Q}_α for $\alpha \in T$ is a \mathbb{O}_α -name of the Cohen poset $\text{Fn}(\omega_2)$. Then, for the Easton-type mixed support iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ over $\langle \mathbb{O}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ and $\mathbb{P} = \mathbb{P}_\kappa * (\text{Col}(\kappa, \mu))_{\mathbb{P}_\kappa}^\bullet$ for a regular $\mu > \kappa$,

$$(6.7) \quad \Vdash_{\mathbb{P}} \text{“MA}^{++\eta}(\mathcal{P}, < 2^{\aleph_0}) \text{ for all cardinal } \eta < 2^{\aleph_0} \text{ holds for} \quad \text{III:models-6}$$

$$\mathcal{P} = \{P : P \text{ is forcing equivalent to } \text{Fn}(\mu, 2) \text{ for some } \mu\},$$

Proof. (1): Suppose that \mathbb{G}_κ is a $(\mathbb{V}, \mathbb{P}_\kappa)$ -generic filter and \mathfrak{g} a $(\mathbb{V}[\mathbb{G}_\kappa], \text{Col}(\kappa, \mu)^{\mathbb{V}[\mathbb{G}_\kappa]})$ -generic filter. Let $\eta, \nu < \kappa$, $\mathbb{R} \in V$, and $\mathcal{D}, \mathcal{S} \in V[\mathbb{G}_\kappa * \mathfrak{g}]$ be such that

$$(6.8) \quad \mathbb{V}[\mathbb{G}_\kappa][\mathfrak{g}] \models \text{“}\mathbb{R} \text{ is a ccc poset,} \quad \text{III:models-7}$$

$$\mathcal{D} \text{ is a family of dense subsets of } \mathbb{R} \text{ with } |\mathcal{D}| = \nu,$$

$$\text{and } \mathcal{S} \text{ is a family of } \mathbb{R}\text{-names with } |\mathcal{S}| = \eta \text{ such}$$

$$\text{that each element } \check{s} \text{ of } \mathcal{S} \text{ is a } \mathbb{R}\text{-name of a station-}$$

$$\text{ary subset of } \mathcal{P}_{\eta_{\check{s}}}(\theta_{\check{s}}) \text{ for some } \omega < \eta_{\check{s}} \leq \theta_{\check{s}} < 2^{\aleph_0}$$

$$\text{with } \eta_{\check{s}} \text{ regular”}.$$

Let $|\mathbb{R}| = \lambda$. Without loss of generality, we may assume that the underlying set of \mathbb{R} is λ . Thus $\mathbb{R} = \langle \lambda \leq_{\mathbb{R}} \rangle$. Let θ be sufficiently large and let $j : V \xrightarrow{\cong} M$ be such that $\text{crit}(j) = \kappa$, $j(\kappa) > \theta$,

$$(6.9) \quad [M]^\theta \subseteq M, \text{ and} \quad \text{III:models-8}$$

$$(6.10) \quad j(f)(\kappa) = \langle \mu, \theta, \mathbb{R} \rangle. \quad \text{III:models-9}$$

Let $S^* = j(S)$, $\nu^* = j(\nu)$ and, let $\vec{\mathbb{P}}^* = j(\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle)$. As before, we write

$$(6.11) \quad \vec{\mathbb{P}}^* = \langle \mathbb{P}_\alpha^*, \mathbb{Q}_\beta^* : \alpha \leq j(\kappa), \beta < j(\kappa) \rangle. \quad \text{III:models-10}$$

We have $\mathbb{P}_\alpha = \mathbb{P}_\alpha^*$ for $\alpha \leq \kappa$.

Let \mathbb{G}_κ be a $(\mathbb{V}, \mathbb{P}_\kappa)$ -generic filter. Then $\mathbb{Q}_\kappa^*[\mathbb{G}_\kappa] = \text{Col}(\kappa, \mu)^{\mathbb{V}[\mathbb{G}_\kappa]}$ and $\nu^*(\kappa) \geq \theta$ by (6.10). Let \mathfrak{g} be a $(\mathbb{V}[\mathbb{G}_\kappa], \text{Col}(\kappa, \mu)^{\mathbb{V}[\mathbb{G}_\kappa]})$ -generic filter. By (6.10) and (6.8), $\mathbb{Q}_{\kappa+1}^*[\mathbb{G}_\kappa * \mathfrak{g}] \sim \mathbb{R}$.

Thus, in M , $\mathbb{P}_{j(\kappa)}^*$ is factored as

$$(6.12) \quad \mathbb{P}_{j(\kappa)}^* \sim \mathbb{P}_\kappa * \text{Col}(\kappa, \mu)_{\mathbb{P}_\kappa}^\bullet * \mathbb{R} * \mathbb{R}_1 \quad \text{III:models-11}$$

where \mathbb{R} corresponds to the ground model poset \mathbb{R} . We have

$$(6.13) \quad \Vdash_{\mathbb{P}_\kappa * \text{Col}(\kappa, \mu)_{\mathbb{P}_\kappa}^\bullet * \mathbb{R}} \text{“}\mathbb{R}_1 \text{ is a regular sub-poset of the completion of} \quad \text{III:models-12}$$

$$\text{a poset of the form ‘ccc poset } \times \nu^*(\kappa)\text{-closed}$$

$$\text{poset’ ”}$$

by (5.10), (5.12), and (5.14).

Now, let \mathbb{h} be $(M[\mathbb{G} * \mathfrak{g}], \mathbb{R})$ -generic filter and \mathbb{H} be $(M[\mathbb{G} * \mathfrak{g} * \mathbb{h}], \mathbb{R}_1[\mathbb{G} * \mathfrak{g} * \mathbb{h}])$ -generic filter. $j : V \xrightarrow{\cong} M$ is then lifted to

$$(6.14) \quad j^* : \mathbb{V}[\mathbb{G}_\kappa] \xrightarrow{\cong} M[\mathbb{G}_\kappa * \mathfrak{g} * \mathfrak{h} * \mathbb{H}]; \underset{\sim}{a}[\mathbb{G}_\kappa] \mapsto j^*(a)[\mathbb{G}_\kappa * \mathfrak{g} * \mathfrak{h} * \mathbb{H}]. \quad \text{III:models-13}$$

We have

$$(6.15) \quad M \models \text{“}\mathbb{C}^* = j^*(\text{Col}(\kappa, \mu)^{\mathbb{V}[\mathbb{G}_\kappa]}) \text{ is } < j(\kappa)\text{-directed closed”} \quad \text{III:models-13-0}$$

by elementarity and $(\mu^{<\kappa})^+ < \theta < j(\kappa)$. Thus, we can find an $(M[\mathbb{G}_\kappa * \mathfrak{g} * \mathfrak{h} * \mathbb{H}], \mathbb{C}^*)$ -generic filter \mathfrak{g}^* such that $j^{**}\mathfrak{g} \subseteq \mathfrak{g}^*$. j^* is then further lifted to

$$(6.16) \quad j^{**} : (\mathbb{V}[\mathbb{G}_\kappa])[\mathfrak{g}] \xrightarrow{\cong} (M[\mathbb{G}_\kappa * \mathfrak{g} * \mathfrak{h} * \mathbb{H}])[\mathfrak{g}^*]; \underset{\sim}{a}[\mathfrak{g}] \mapsto j^*(a)[\mathfrak{g}^*] \quad \text{III:models-14}$$

for $\text{Col}(\kappa, \mu)^{\mathbb{V}[\mathbb{G}_\kappa]}$ -names $\underset{\sim}{a}$ in $\mathbb{V}[\mathbb{G}_\kappa]$.

In $M[\mathbb{G}_\kappa * \mathfrak{g} * \mathfrak{h}]$, \mathfrak{h} is a filter on \mathbb{R} which intersects with each element of \mathcal{D} and each element $\underset{\sim}{S}$ of \mathcal{S} is interpreted as a stationary subset of $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$ by the genericity of \mathfrak{h} and since $\mathcal{D}, \mathcal{S} \in M$ by the closedness (6.9) of M . These interpretations of elements of \mathcal{S} remain stationary in $M[\mathbb{G}_\kappa * \mathfrak{g} * \mathfrak{h} * \mathbb{H} * \mathfrak{g}^*]$ by (6.13). and (6.15) (see Lemma 3.1).

Since \mathcal{D} and \mathcal{S} have cardinality $< \kappa$, we have

$$(6.17) \quad \begin{aligned} j^{**}(\mathcal{D}) &= \{j^{**}(D) : D \in \mathcal{D}\}, \text{ and} \\ j^{**}(\mathcal{S}) &= \{j^{**}(\underset{\sim}{S}) : \underset{\sim}{S} \in \mathcal{S}\}. \end{aligned} \quad \text{III:models-15}$$

It follows that \mathfrak{h} generates a filter on $j^{**}(\mathbb{R})$ which intersects each element of $j^{**}(\mathcal{D})$ and interprets each element $\underset{\sim}{S}$ of $j^{**}(\mathcal{S})$ as a superset of the corresponding interpretation of the element of \mathcal{S} by \mathfrak{h} is a stationary subset of $\mathcal{P}_{\eta_{\underset{\sim}{S}}}()$ in $M[\mathbb{G}_\kappa * \mathfrak{g} * \mathfrak{h} * \mathbb{H} * \mathfrak{g}^*]$.

Thus, we have

$$(6.18) \quad \begin{aligned} M[\mathbb{G}_\kappa * \mathfrak{g} * \mathfrak{h} * \mathbb{H} * \mathfrak{g}^*] \models \text{“there is a } j^{**}(\mathcal{D})\text{-generic filter on } j^{**}(\mathbb{R}) \\ \text{which interprets each element } \underset{\sim}{S} \text{ of } j^{**}(\mathcal{S}) \\ \text{as a stationary subset of } \mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})\text{”}. \end{aligned} \quad \text{III:models-16}$$

By the elementarity of j^{**} , it follows that

$$(6.19) \quad \begin{aligned} \mathbb{V}[\mathbb{G}_\kappa * \mathfrak{g}] \models \text{“there is a } \mathcal{D}\text{-generic filter on } \mathbb{R} \\ \text{which interprets each element } \underset{\sim}{S} \text{ of } \mathcal{S} \\ \text{as a stationary subset of } \mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})\text{”}. \end{aligned} \quad \text{III:models-17}$$

(2): can be proved similarly to (1). □ (Proposition 6.2)

Theorem 6.3 (1) *Suppose that the existence of two supercompact cardinals is consistent. Then the following combination of the principles is also consistent:*

$$(6.2) \quad \text{SDLS}_+^{int}(\mathcal{L}_{stat}^{\aleph_0}, < 2^{\aleph_0}), \text{ GRP}^{< 2^{\aleph_0}} (\leq 2^{\aleph_0}); \quad \text{III:models-1}$$

- (6.3) $\text{SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\text{PKL}}, < 2^{\aleph_0})$; III:models-2
- (6.20) $\text{MA}^{++\eta}(\mathcal{P}, < 2^{\aleph_0})$ for $\mathcal{P} = \{\mathbb{P} : \mathbb{P} \sim \text{Fn}(\lambda, 2) \text{ for some } \lambda\}$ III:models-21
for all $\eta < 2^{\aleph_0}$; and
- (6.21) $\text{HH}(< 2^{\aleph_0})$. III:models-22

(2) If there is a superhuge cardinal and a supercompact cardinal above it, then the combination of the principles (6.2) \sim (6.21) above together with

- (6.22) $\mathcal{P}_{2^{\aleph_0}}(\lambda)$ carries a σ -saturated normal ideal for all $\lambda \geq 2^{\aleph_0}$ III:models-22-0
- is consistent.

Proof. (1): For two supercompact cardinals $\kappa < \kappa_1$, let $\langle \mathbb{O}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ and $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be as in Proposition 6.2, (2).

Then, the generic extension of \mathbf{V} by $\mathbb{P} = \mathbb{P}_\kappa * (\text{Col}(\kappa, \kappa_1))_{\mathbb{P}_\kappa}^\bullet$ is as desired: $\Vdash_{\mathbb{P}}$ “(6.1), (6.2), (6.3)” by Theorem 6.1. $\Vdash_{\mathbb{P}}$ “(6.20)” by Proposition 6.2, (2) and $\Vdash_{\mathbb{P}}$ “(6.21)” by Theorem 5.16, (2) and Proposition 4.7.

(2): Let $\mathbb{P} = \mathbb{P}_\kappa * (\text{Col}(\kappa, \kappa_1))_{\mathbb{P}_\kappa}^\bullet$ be as in (1) but for a superhuge κ and a supercompact κ_1 above κ . Then $\Vdash_{\mathbb{P}}$ “(6.2) \sim (6.21)” as in (1) and $\Vdash_{\mathbb{P}}$ “(6.22)” by Theorem 5.16, (2') and Proposition 4.9. □ (Theorem 6.3)

Theorem 6.4 (1) Suppose that the existence of two supercompact cardinals is consistent. Then the following combination of principles is also consistent: III:P-models-2

- (6.2) $\text{SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\aleph_0}, < 2^{\aleph_0})$, $\text{GRP}^{< 2^{\aleph_0}}(\leq 2^{\aleph_0})$, III:models-1
- (6.3) $\text{SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\text{PKL}}, < 2^{\aleph_0})$; III:models-2
- (6.23) There is an inner model M of \mathbf{V} such that $(2^{\aleph_0})^{\mathbf{V}} = (2^{\aleph_0})^M$ and V is reached from M by the forcing with a regular sub-poset of the completion of the product of ccc and $< 2^{\aleph_0}$ -closed posets, and $\text{MA}^{++\eta}(\mathcal{P}, < 2^{\aleph_0})$ for $\mathcal{P} = \{\mathbb{P} : \mathbb{P} \text{ is a ccc poset } \mathbb{P} \in M\}$ III:models-23
for all $\eta < 2^{\aleph_0}$, and
- (6.24) $\neg \text{HH}(< 2^{\aleph_0})$. III:models-24

(2) If there is a superhuge cardinal and a supercompact cardinal above it, then the combination of the principles (6.2), (6.3), (6.23) and (6.24) above together with

- (6.22) $\mathcal{P}_{2^{\aleph_0}}(\lambda)$ carries a σ -saturated normal ideal for all $\lambda \geq 2^{\aleph_0}$ III:models-22-0
- is consistent.

Proof. (1): Let $\langle \mathbb{O}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be the following modification of the preparatory ccc finite support iteration (6.5) in Proposition 6.2, (1):

$$(6.25) \quad \mathbb{R}_\beta = \begin{cases} (P)_{\mathbb{O}_\beta}^\vee, & \text{if } \beta = \alpha + 1 \text{ for an } \alpha \in S \text{ and} \\ & f(\alpha) = \langle \mu, \theta, P \rangle \text{ for cardinals } \mu, \theta \text{ and a poset } P \\ & \text{such that } \Vdash_{\mathbb{O}_\beta} \text{“}(P)_{\mathbb{O}_\beta}^\vee \text{ is a ccc poset”}; \\ \text{Hechler real forcing over } \mathbb{O}_\beta, & \text{if } \beta \in T \text{ but} \\ & \beta \text{ is not a successor of an element of } S; \\ \{\mathbb{1}_{\mathbb{R}_\beta}\}, & \text{otherwise.} \end{cases}$$

Let $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be the Easton-type mixed support iteration over $\langle \mathbb{O}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ and $\mathbb{P} = \mathbb{P}_\kappa * (\text{Col}(\kappa, \kappa_1))_{\mathbb{P}_\kappa}^\bullet$.

Then, the generic extension of \mathbb{V} by $\mathbb{P} = \mathbb{P}_\kappa * (\text{Col}(\kappa, \kappa_1))_{\mathbb{P}_\kappa}^\bullet$ is as desired: $\Vdash_{\mathbb{P}}$ “(6.1), (6.2), (6.3)” by Theorem 6.1.

$\Vdash_{\mathbb{P}}$ “(6.23)” follows from (the proof of) Proposition 6.2, (1). Note that the proof of Proposition 6.2, (1) does not rely on the value of \mathbb{R}_β for $\beta \in T$ which is not a successor of the element of S .

Now the Hechler part of the preparatory iteration introduces an \leq^* -increasing sequence $\vec{h} = \langle f_\alpha : \alpha < 2^{\aleph_0} \rangle$ of functions of length 2^{\aleph_0} (in the generic extension by \mathbb{P}_κ) $\mathcal{F} = \{f_\alpha : \alpha < 2^{\aleph_0}\}$ is still unbounded in the generic extension by \mathbb{P}_κ by the genericity of f_α 's: f_α 's may no more Hechler reals above corresponding intermediate models in $\mathbb{V}^{\mathbb{P}_\kappa}$ but each of them adds a Cohen real as its coordinatewise summand (see [Truss[21]]). \mathcal{F} remains unbounded in \mathbb{P} -generic extension $\mathbb{V}[\mathbb{G}_\kappa * \mathbb{H}]$ since no new reals are added by $\text{Col}(\kappa, \kappa_1)$. Thus, in $\mathbb{V}[\mathbb{G}]$, the first countable topological space $X_{\mathcal{F}}$ constructed in Section 2 is non-metrizable but all subspaces of $X_{\mathcal{F}}$ of size $< 2^{\aleph_0}$ are metrizable. Thus $\Vdash_{\mathbb{P}}$ “(6.24)”.

(2): Let $\mathbb{P} = \mathbb{P}_\kappa * (\text{Col}(\kappa, \kappa_1))_{\mathbb{P}_\kappa}^\bullet$ be as in (1) with superhuge κ and supercompact κ_1 above κ . Then $\Vdash_{\mathbb{P}}$ “(6.2), (6.3), (6.23), (6.24)” as in (1) and $\Vdash_{\mathbb{P}}$ “(6.22)” by Theorem 5.16,(2') and Proposition 4.9. □ (Theorem 6.4)

We end up with mentioning some remaining open problems. As noted in Section 2, Hamburger's Problem i.e. the consistency of $\text{HH}(< \aleph_2)$ is still widely open. Galvin's Conjecture is also a persistingly open problem which can be discussed in our context (see e.g. [Todorcevic [20]]).

Both of the following two problems, which might be more at hand, are related to the last theorem in this section:

Problem 6.5 *Can we have the full $\text{MA}^{++\mu}(\text{ccc}, < 2^{\aleph_0})$ together with all other strong reflection properties in some modification of the model of Theorem 6.4?*

Problem 6.6 *What is the $\mathfrak{Refl}_{\text{HH}}$ in the model of Theorem 6.4? Can we make it $(2^{\aleph_0})^+$ or ∞ by some modification of the construction in the proof?*

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III:ref

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