

The first-order definability of generic large cardinals

Sakaé Fuchino (渕野 昌), and Hiroshi Sakai (酒井 拓史)

Abstract

We show that the notions of generic and Laver-generic supercompactness are first-order definable in the language of ZFC. This also holds for generic and Laver-generic (almost) hugeness as well as for generic versions of other large cardinals.

Contents

1. Introduction	1
2. V -normal ultrafilters	4
3. Sufficiently large initial segment of elementary embeddings	12
References	21

1 Introduction

defbility:intro

For a class of posets \mathcal{P} , a cardinal κ is said to be *generically supercompact by \mathcal{P}* if, for any regular $\lambda \geq \kappa$, there is a poset $\mathbb{P} \in \mathcal{P}$ such that, for a (V, \mathbb{P}) -generic \mathbb{G} ,

* Graduate School of System Informatics, Kobe University
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan
fuchinodiamond.kobe-u.ac.jp, hsakai@people.kobe-u.ac.jp

Date: June 27, 2021 *Last update:* April 5, 2022 (07:56 JST)

2020 Mathematical Subject Classification: 03E35, 03E50, 03E55, 03E65

Keywords: generic supercompactness, Laver generic supercompactness, saturated ideal, elementary embedding

The first author was partially supported by JSPS Kakenhi Grant No. 20K03717. The second author is supported by JSPS Kakenhi Grant No. 18K03397.

This is an extended version of the paper with the same title. Some extra remarks and details omitted in the final version for publication, as well as further corrections after the submission of the paper, may be found in this version. The additional stuff is typeset in the color “bulue(munsell)” like this paragraph. The most recent edition of this extended version is downloadable as:

<https://fuchino.ddo.jp/papers/definability-of-glc-x.pdf>

there are $M, j \subseteq V[G]$ such that

$$(1.1) \quad j : V \xrightarrow{\cong} M \subseteq V[G], \quad (1)$$

$$(1.2) \quad \text{crit}(j) = \kappa, j(\kappa) > \lambda,$$

$$(1.3) \quad j''\lambda \in M.$$

def:x-intro-a

def:x-intro-0

def:x-intro-1

We shall call the class mapping j as above a λ -generically supercompact embedding for κ (in $V[G]$).

It is easy to see that a generically supercompact cardinal κ for any class \mathcal{P} of posets is regular. Even so, a generically supercompact cardinal can be a successor cardinal: If we collapse all cardinals below a supercompact cardinal κ by $\text{Col}(\omega_1, \kappa)$,⁽²⁾ in the generic extension, $\kappa = \aleph_2$ and \aleph_2 is generically supercompact by σ -closed posets.

A generically supercompact cardinal can κ be also weakly inaccessible. Actually κ can be even really supercompact for any \mathcal{P} as far as this \mathcal{P} contains the trivial poset. However, a generically supercompact κ can also be weakly inaccessible (and much more) while it is not strongly inaccessible: If κ is supercompact and κ many Cohen reals are added, then κ is still a regular inaccessible cardinal (and actually much more) and it remain generically supercompact by c.c.c. posets, while it is the continuum in the generic extension.

Similarly to the genuine supercompactness, it is not immediately clear if the notion of generic supercompactness is definable in the language of ZFC. In most of the cases, this does not bother. This is because the generically supercompactness may be used in many applications merely as a schematic framework in which arguments in different settings are put together to obtain a better perspective.

However, the circumstances become different if we would like to think generic supercompactness as a set-theoretic axiom.

In [6], Bernhard König gave a characterization of the statement “ ω_2 is generically supercompact by σ -closed posets” in terms of the reflection of the non-existence of winning strategy of the second player in certain type of two player games. Since this reflection principle which König called “Strong Game Reflection Principle” is first-order definable, the statement mentioned above is also first-order formalizable.

In [1], König’s characterization is generalized to a characterization of the statement “ κ^+ is generically supercompact for $< \kappa$ -closed forcing” for arbitrary regular

⁽¹⁾ When we write $j : V \xrightarrow{\cong} M \subseteq V[G]$, we assume that M is a transitive class in (and thus an inner model of) $V[G]$.

⁽²⁾ We use here Kanamori’s notation in [5] of Lévy collapse.

uncountable κ . By the same argument as above, we conclude from this result that the statement is also first-order formalizable.

Based on the main idea in the proof of these results, we show in the following Section 2 that the generically supercompactness for any class \mathcal{P} of posets is first-order definable.

We say that a class \mathcal{P} of posets *iterable*, if \mathcal{P} is closed with respect to restriction (i.e., if $\mathbb{P} \in \mathcal{P}$ and $\mathbb{p} \in \mathbb{P}$, then $\mathbb{P} \upharpoonright \mathbb{p} \in \mathcal{P}$)⁽³⁾, and, for any $\mathbb{P} \in \mathcal{P}$ and \mathbb{P} -name $\tilde{\mathbb{Q}}$, we have

$$\text{if } \Vdash_{\mathbb{P}} \text{“}\tilde{\mathbb{Q}} \in \mathcal{P}\text{” then } \mathbb{P} * \tilde{\mathbb{Q}} \in \mathcal{P}.$$

For a cardinal κ and an iterable class \mathcal{P} of posets, we call κ a *Laver-generically supercompact for \mathcal{P}* (or *L-g supercompact*, for short) if, for any $\lambda \geq \kappa$ and any $\mathbb{P} \in \mathcal{P}$, there is a \mathbb{P} -name of a poset $\tilde{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}} \text{“}\tilde{\mathbb{Q}} \in \mathcal{P}\text{”}$ such that, for any $(\mathbb{V}, \mathbb{P} * \tilde{\mathbb{Q}})$ -generic filter \mathbb{H} , there are $M, j \subseteq \mathbb{V}[\mathbb{H}]$ such that

$$(1.4) \quad j : \mathbb{V} \xrightarrow{\sim} M,$$

$$(1.5) \quad \text{crit}(j) = \kappa, j(\kappa) > \lambda,$$

$$(1.6) \quad \mathbb{P}, \mathbb{H} \in M \text{ and}$$

$$(1.7) \quad j''\lambda \in M.$$

defbility:x-

gen-sc-13

defbility:x-

gen-sc-14

defbility:x-

gen-sc-15

defbility:x-

gen-sc-16

We shall call j as above a λ *L-g supercompact embedding (with the critical point κ , associated with \mathbb{H} over \mathbb{V})*.

For $\mathcal{P} =$ all the σ -closed posets, the supercompact κ in the ground model collapsed to be \aleph_2 by $\text{Col}(\omega_1, \kappa)$ is L-g supercompact for \mathcal{P} . For $\mathcal{P} =$ all the proper posets, the continuum in the standard model of PFA obtained by starting from a supercompact κ and by iterating with proper posets with countable support along with a Laver diamond is L-g supercompact for \mathcal{P} .

In these two models the L-g supercompact cardinal is \aleph_2 . This is not a coincidence: If all elements of \mathcal{P} preserves ω_1 and $\text{Col}(\omega_1, \{\omega_1\}) \in \mathcal{P}$ then κ being L-g supercompact for \mathcal{P} implies $\kappa = \aleph_2$ ([2]).

For $\mathcal{P} =$ all the ccc posets, a L-g supercompact cardinal for \mathbb{P} is obtained by starting from a supercompact κ and then iterating κ -times by ccc posets with finite support along with a Laver diamond.

The method in Section 2 cannot be applied (at least not in a straightforward way) to show the definability of Laver-generic large cardinals since apparently it cannot cover the condition (1.6).

⁽³⁾For the use of this condition, see the argument around (2.12)

In Section 3, we show that the existence of generic elementary embedding can be recovered from a large enough initial segment of a generic elementary embedding (Proposition 3.3). Using this, we can establish the definability of Laver-generic supercompactness for any iterable class of posets (Theorem 3.4).

The results discussed in this paper can be easily modified to adopt to other generic and Laver-generic large cardinals like those corresponding to super almost huge or super-huge cardinals.

In the following, we assume that our formal framework is that of ZFC and \mathcal{L}_ε denotes the language of set theory with the sole binary relation symbol ε . Nevertheless, when we consider generic elementary embeddings which may not be first-order definable, we go over to the second-order framework of the axiom system of von Neumann-Bernays-Gödel (NBGC) e.g. by adding an appropriate axiom Ψ claiming the existence of certain (class) names of elementary embeddings in a generic extension over each posets in a given class of posets.

We say that such system is first-order definable if we can find an axiom ψ in \mathcal{L}_ε such that the original second-order axiom $\text{NBGC} + \Psi$ is a conservative extension of the the axiom system $\text{ZFC} + \psi$.

In the framework of ZFC, when we are talking about a class \mathcal{P} of posets, we assume that we fix an \mathcal{L}_ε -formula $P(\cdot)$ which describes the elements of \mathcal{P} in such a way that $\mathcal{P} = \{\mathbb{P} : P(\mathbb{P})\}$. In this respect, when we said $\Vdash_{\mathbb{P}} \text{“}\tilde{Q} \varepsilon \mathcal{P}\text{”}$ in connection with iterability of \mathcal{P} above, we actually meant $\Vdash_{\mathbb{P}} \text{“}P(\tilde{Q})\text{”}$.

2 V-normal ultrafilters

defbility:normal

In the context of generic supercompactness, the condition (1.3) implies a certain kind of closedness of M . This can be seen in the following Lemma:

Lemma A 2.1 (Lemma 2.5 in [2]) *Suppose that \mathbb{G} is a (\mathbb{V}, \mathbb{P}) -generic filter for a poset $\mathbb{P} \in \mathbb{V}$, and $j : \mathbb{V} \xrightarrow{\sim} M \subseteq \mathbb{V}[\mathbb{G}]$ is such that, for cardinals κ, λ in \mathbb{V} with $\kappa \leq \lambda$, $\text{crit}(j) = \kappa$ and $j''\lambda \in M$. Then, we have the following:*

def:P-gen-sc-

- (1) For any set $A \in \mathbb{V}$ with $\mathbb{V} \models |A| \leq \lambda$, we have $j''A \in M$.
- (2) $j \upharpoonright \lambda, j \upharpoonright \lambda^2 \in M$.
- (3) For any $A \in \mathbb{V}$ with $A \subseteq \lambda$ or $A \subseteq \lambda^2$ we have $A \in M$.
- (4) $(\lambda^+)^M \geq (\lambda^+)^{\mathbb{V}}$, Thus, if $(\lambda^+)^{\mathbb{V}} = (\lambda^+)^{\mathbb{V}[\mathbb{G}]}$, then $(\lambda^+)^M = (\lambda^+)^{\mathbb{V}}$.
- (5) $\mathcal{H}(\lambda^+)^{\mathbb{V}} \subseteq M$.
- (6) $j \upharpoonright A \in M$ for all $A \in \mathcal{H}(\lambda^+)^{\mathbb{V}}$. □

In the following, we use Kanamori's notation of collapsing posets (see §10 of [5]).

As it is already noticed in the introduction, it is consistent (modulo a supercompact cardinal) that a successor cardinal of a regular uncountable cardinal is generically supercompact.

Fact A 2.2 *Suppose that κ is a (really) supercompact cardinal, $\mu < \kappa$ a regular uncountable cardinal, and $\mathbb{P}_0 = \text{Col}(\mu, \kappa)$. Then, for a $(\mathbb{V}, \mathbb{P}_0)$ -generic \mathbb{G}_0 ,*

$$\mathbb{V}[\mathbb{G}_0] \models \text{“}\mu^+ \text{ is a generically supercompact cardinal by } < \mu\text{-closed posets”}.$$

Proof. Note that $\mathbb{V}[\mathbb{G}_0] \models \text{“}\mu^+ = \kappa\text{”}$.

For $\lambda \geq \kappa$, let $j : \mathbb{V} \xrightarrow{\sim} M$ be a λ -supercompact embedding for κ . Then we have

$$j(\mathbb{P}_0) \underset{\text{by elementarity}}{=} \underbrace{\text{Col}(j(\mu), j(\kappa))}_M \overset{\text{by closedness of } M}{=} \text{Col}(\mu, j(\kappa))^\mathbb{V}.$$

For a $(\mathbb{V}[\mathbb{G}_0], \text{Col}(\mu, j(\kappa) \setminus \kappa))$ -generic filter \mathbb{G} , the lifting

$$\tilde{j} : \mathbb{V}[\mathbb{G}_0] \xrightarrow{\sim} \underbrace{M[\mathbb{G}_0][\mathbb{G}]}_{\subseteq \mathbb{V}[\mathbb{G}_0][\mathbb{G}]}; \quad \underset{\sim}{a}^{\mathbb{G}_0} \mapsto j(\underset{\sim}{a})^{\mathbb{G}_0 * \mathbb{G}}$$

witnesses the generic λ -supercompactness of $\underbrace{\kappa}_{= (\mu^+)^{\mathbb{V}[\mathbb{G}_0]}}$ by μ -closed posets in $\mathbb{V}[\mathbb{G}_0]$. □ (Fact 2.0)

For a class \mathcal{P} of posets such that no $\mathbb{P} \in \mathcal{P}$ adds any new ω -sequence of ground model sets, the first-order definability of the generic supercompactness by \mathcal{P} can be seen in the following Proposition. The Proposition can be shown by a direct imitation of the proof of the characterization of supercompactness by Solovay and Reinhardt in terms of the existence of normal ultrafilters (see e.g. Theorem 22.7 in [5]).

Theorem 2.1 *Suppose that \mathcal{P} is a class of posets such that no $\mathbb{P} \in \mathcal{P}$ adds any new ω -sequence of ground model sets, and \mathcal{P} is closed with respect to restriction (i.e, if $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P} \in \mathbb{P}$, then $\mathbb{P} \upharpoonright \mathbb{P} \in \mathcal{P}$).*

An uncountable cardinal κ is generically supercompact by \mathcal{P} if and only if, for any $\lambda \geq \kappa$, there is a $\mathbb{P} \in \mathcal{P}$ such that

$$\Vdash_{\mathbb{P}} \text{“there is a } \mathbb{V}\text{-normal ultrafilter on } \mathcal{P}^\mathbb{V}(\mathcal{P}_\kappa(\lambda)^\mathbb{V})\text{”}.$$

Here, the notion of \mathbf{V} -normal ultrafilter is defined as follows: Suppose that we are living in a universe \mathbf{W} and \mathbf{V} is an inner model in \mathbf{W} . Let λ be an ordinal in \mathbf{V} , $\mathcal{I} \in \mathbf{V}$, $\mathcal{I} \subseteq \mathcal{P}^{\mathbf{V}}(\lambda)$ a σ -ideal with $\{\xi\} \in \mathcal{I}$ for all $\xi < \lambda$, and $\mathcal{B} \in \mathbf{V}$ the sub-Boolean algebra $\mathcal{B} = \mathcal{P}^{\mathbf{V}}(\mathcal{I})$ of $\mathcal{P}^{\mathbf{W}}(\mathcal{I})$.

In \mathbf{W} , $U \subseteq \mathcal{B}$ is a \mathbf{V} -normal ultrafilter if

(2.1) U is a ultrafilter on the Boolean algebra \mathcal{B} . I.e.,

- (i) $\emptyset \notin U$;
- (ii) $A \cap A' \in U$ for any $A, A' \in U$;
- (iii) if $A \in U$, $A \subseteq A' \in \mathcal{B}$, then $A' \in U$; and
- (iv) for any $A \in \mathcal{B}$, either $A \in U$ or $\mathcal{I} \setminus A \in U$;

(2.2) For any $x_0 \in \mathcal{I}$, we have $\{x \in \mathcal{I} : x_0 \subseteq x\} \in U$;

(2.3) For any $\langle A_\xi : \xi \in \lambda \rangle \in \mathbf{V}$, if $\{A_\xi : \xi < \lambda\} \subseteq U$, we have

$\Delta_{\xi \in \lambda} A_\xi \in U$. Here, $\Delta_{\xi \in \lambda} A_\xi$ is the diagonal intersection of A_ξ 's defined by

$$(2.4) \quad \Delta_{\xi \in \lambda} A_\xi := \{x \in \mathcal{I} : x \in A_\xi \text{ for all } \xi \in x\}.$$

Lemma 2.2 *Suppose that $U \subseteq \mathcal{B}$ is a \mathbf{V} -normal ultrafilter.*

(1) For $\delta < \lambda$ such that $\delta \in \mathcal{I}$, and $\langle A_\xi : \xi \in \delta \rangle \in \mathbf{V}$ with $A_\xi \in U$ for all $\xi \in \delta$, we have $\bigcap_{\xi \in \delta} A_\xi \in U$.

(2) (Pressing Down Lemma) For any $f \in \mathbf{V}$ with $f : \mathcal{I} \rightarrow \mathbf{V}$, if $\{x \in \mathcal{I} : f(x) \in x\} \in U$, then there is $\xi < \lambda$ such that $\{x \in \mathcal{I} : f(x) = \xi\} \in U$.

Proof. (1): Let $A_\xi := \mathcal{I}$ for all $\xi \in \lambda \setminus \delta$. Then

$$\underbrace{\underbrace{\Delta_{\xi \in \lambda} A_\xi}_{\in U \text{ by (2.3)}} \cap \underbrace{\{x \in \mathcal{I} : \delta \subseteq x\}}_{\in U \text{ by (2.2)}}}_{\in U \text{ by (2.1), (ii)}} \subseteq \bigcap_{\xi \in \delta} A_\xi.$$

Hence, $\bigcap_{\xi \in \delta} A_\xi \in U$ by (2.1), (iii).

(2): Suppose that f is a counter-example to the assertion. That is,

$$(2.5) \quad A := \{x \in \mathcal{I} : f(x) \in x\} \in U, \text{ but}$$

$$(2.6) \quad A_\xi := \{x \in \mathcal{I} : f(x) \neq \xi\} \in U \text{ for all } \xi \in \lambda.$$

Then $\Delta_{\xi < \lambda} A_\xi \cap A \in U$ by (2.3) and (2.1), (ii). By (2.1), (i), there is an element x^* of this set. $f(x^*) \in x^*$ by (2.5) but $f(x^*) \neq \xi$ for all $\xi \in x^*$ by (2.6) and the definition (2.4) of diagonal intersection. This is a contradiction. \square (Lemma 2.2)

Proof of Theorem 2.1: “ \Rightarrow ”: Let $\lambda \geq \kappa$ and let \mathbb{P} be a $< \mu$ -closed poset with (\mathbb{V}, \mathbb{P}) -generic \mathbb{G} and classes $j, M \subseteq \mathbb{V}[\mathbb{G}]$ such that $j : \mathbb{V} \xrightarrow{\simeq} M$ is a λ -generically supercompact embedding for κ . In particular, we have $j''\lambda \in M$. Note that

$$(2.7) \quad M \models j''\lambda \in \mathcal{P}_{j(\kappa)}(j(\lambda)) = j(\mathcal{P}_\kappa(\lambda)^\mathbb{V}).$$

defbilty:x-

In $\mathbb{V}[\mathbb{G}]$, let

gen-sc-5-2

$$(2.8) \quad U_j := \{A \in \mathbb{V} : A \subseteq \mathcal{P}_\kappa(\lambda)^\mathbb{V}, j''\lambda \in j(A)\}.$$

defbilty:x-

Claim 2.2.1 U_j is a \mathbb{V} -normal ultrafilter on $\mathcal{P}^\mathbb{V}(\mathcal{P}_\kappa(\lambda)^\mathbb{V})$.

gen-sc-5-2-a

Cl-gen-sc-0

$\vdash U_j \models (2.1)$, (i): $j(\emptyset) = \emptyset$ by elementarity (and transitivity of M). Thus $\emptyset \notin U_j$ by definition.

(ii): Suppose $A, A' \in U_j$. By definition this means that $j''\lambda \in j(A)$ and $j''\lambda \in j(A')$. It follows that $j''\lambda \in j(A) \cap j(A') = j(\underbrace{A \cap A'}_{\text{by elementarity}})$. This shows that

by elementarity

$A \cap A' \in U_j$.

(iii): Suppose that $A \in U_j$ and $A' \in \mathbb{V}$ is such that $A \subseteq A' \subseteq \mathcal{P}_\kappa(\lambda)^\mathbb{V}$. Then by elementarity we have $M \models j(A) \subseteq j(A')$. Hence $j''\lambda \in j(A) \subseteq j(A')$, and $A' \in U_j$.

(iv): If $A \in \mathcal{P}^\mathbb{V}(\mathcal{P}_\kappa(\lambda)^\mathbb{V}) \setminus U_j$, then by (2.7), $j''\lambda \in j(\mathcal{P}_\kappa(\lambda)^\mathbb{V}) \setminus j(A) = j(\mathcal{P}_\kappa(\lambda)^\mathbb{V} \setminus A)$. Thus $\mathcal{P}_\kappa(\lambda)^\mathbb{V} \setminus A \in U_j$.

$U_j \models (2.2)$: Suppose $x_0 \in \mathcal{P}_\kappa(\lambda)^\mathbb{V}$ and let $A := \{x \in \mathcal{P}_\kappa(\lambda)^\mathbb{V} : x_0 \subseteq x\}$. Clearly $A \in \mathcal{P}^\mathbb{V}(\mathcal{P}_\kappa(\lambda)^\mathbb{V})$. By elementarity, and noting that $j(x_0) = j''x_0$ since $|x_0| < \kappa$, we have

$$M \models j(A) = \{x \in \mathcal{P}_{j(\kappa)}(j(\lambda)) : \underbrace{j(x_0)}_{= j''x_0} \subseteq x\}.$$

Thus $M \models j''\lambda \in j(A)$. Hence $A \in U_j$.

$U_j \models (2.3)$: Suppose that $\vec{A} := \langle A_\xi : \xi \in \lambda \rangle \in \mathbb{V}$ is such that $A_\xi \in U_j$, i.e.

$$(2.9) \quad j''\lambda \in j(A_\xi)$$

defbilty:x-

for all $\xi < \lambda$.

gen-sc-5-2-0

By elementarity, we have

$$(2.10) \quad j(\Delta_{\xi \in \lambda} A_\xi) = \{x \in \mathcal{P}_{j(\kappa)}(j(\lambda))^M : \forall \eta \in x (x \in j(\vec{A}(\eta)))\}$$

defbilty:x-

gen-sc-5-3

For $\eta \in j''\lambda$, there is $\eta_0 \in \lambda$ such that $\eta = j(\eta_0)$. Thus

$$(2.11) \quad j(\vec{A})(\eta) = j(\vec{A})(j(\eta_0)) \stackrel{\text{by elementarity}}{=} \underbrace{j(\vec{A}(\eta_0))}_{= j(A_{\eta_0})} \stackrel{(2.9)}{\ni} j''\lambda.$$

defbilty:x-

gen-sc-5-4

By (2.10) and (2.11), it follows that $j''\lambda \in j(\Delta_{\xi \in \lambda} A_\xi)$, and thus $\Delta_{\xi \in \lambda} A_\xi \in U_j$.

⊢ (Claim 2.2.1)

It follows that there is $\mathbb{p} \in \mathbb{G}$ such that

(2.12) $\mathbb{p} \Vdash_{\mathbb{P}}$ “there is a \mathbf{V} -normal ultrafilter on $\mathcal{P}^{\mathbf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathbf{V}})$ ”.

def:x-gen-sc-0

Since $\mathbb{P} \upharpoonright \mathbb{p} \in \mathcal{P}$ by the assumption on \mathcal{P} , we obtain the desired situation for λ by replacing \mathbb{P} with $\mathbb{P} \upharpoonright \mathbb{p}$.

“ \Leftarrow ”: Let $\lambda \geq \kappa$ and let \mathbb{P} be a $< \mu$ -closed poset with a (\mathbf{V}, \mathbb{P}) -generic \mathbb{G} and \mathbf{V} -normal ultrafilter $U \in \mathbf{V}[\mathbb{G}]$ on $\mathcal{P}^{\mathbf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathbf{V}})$.

Let

(2.13) $\mathcal{W} := \{f \in \mathbf{V} : f : \mathcal{P}_\kappa(\lambda)^{\mathbf{V}} \rightarrow \mathbf{V}\}$

(2.14) For $f, g \in \mathcal{W}$, $f \sim_U g \iff \{x \in \mathcal{P}_\kappa(\lambda)^{\mathbf{V}} : f(x) = g(x)\} \in U$;

defibility:x-

$f \in_U g \iff \{x \in \mathcal{P}_\kappa(\lambda)^{\mathbf{V}} : f(x) \in g(x)\} \in U$.

gen-sc-5-5

\sim_U is a congruence relation to \in_U . Thus may consider \in_U as a binary relation on \mathcal{W}/\sim_U and simply write

(2.15) $f/\sim_U \in_U g/\sim_U \iff f \in_U g$.⁽⁴⁾

Let $i_U : \mathbf{V} \rightarrow \mathcal{W}/\sim_U$ be defined by

(2.16) $i_U(a) := \text{const}_a/\sim_U$

defibility:x-

for $a \in \mathbf{V}$ where const_a denote the function on $\mathcal{P}_\kappa(\lambda)^{\mathbf{V}}$ whose value is constantly a . Łoś's Theorem holds:

gen-sc-6

Cl-gen-sc-0-0

Claim 2.2.2 For any formula $\varphi = \varphi(x_0, \dots, x_{n-1})$ in \mathcal{L}_ε (the language of ZF), and $f_0, \dots, f_{n-1} \in \mathcal{W}$, we have $\langle \mathcal{W}/\sim_U, \in_U \rangle \models \varphi(f_0/\sim_U, \dots, f_{n-1}/\sim_U)$, if and only if $\{x \in \mathcal{P}_\kappa(\lambda)^{\mathbf{V}} : \mathbf{V} \models \varphi(f_0(x), \dots, f_{n-1}(x))\} \in U$.

⊢ By induction on φ .

⊢ (Claim 2.2.2)

By Claim 2.2.2, the class mapping i_U above is an elementary embedding of \mathbf{V} into $\langle \mathcal{W}/\sim_U, \in_U \rangle$.

Cl-gen-sc-0-1

Claim 2.2.3 \in_U is (i) an extensional, (ii) well-founded and (iii) set-like relation on \mathcal{W}/\sim_U .

⁽⁴⁾ Here we apply the common trick to handle the equivalence classes by defining

$$f/\sim_U := \{g \in \mathcal{W} : g \sim_U f \text{ and } g \text{ is of minimal } \in\text{-rank} \\ \text{among elements of } \mathcal{W} \text{ with this property}\}$$

to make each equivalence class f/\sim_U a set.

⊢ (i): The extensionality of \in_U follows from the elementarity of i_U .

(ii): Assume, toward a contradiction, that there is a sequence $\langle f_n : n \in \omega \rangle$ in \mathcal{W} such that $f_{n+1} \in_U f_n$ for all $n \in \omega$. By the definition of \in_U , this means that $A_n := \{x \in \mathcal{P}_\kappa(\lambda)^\vee : f_{n+1}(x) \in f_n(x)\} \in U$ for all $n \in \omega$. Since \mathbb{P} does not add any new ω -sequence, $\langle f_n : n \in \omega \rangle \in \mathbb{V}$. Thus, we also have $\langle A_n : n \in \omega \rangle \in \mathbb{V}$. By Lemma 2.2, (1), it follows that $\bigcap_{n \in \omega} A_n \in U$. For an element x of this intersection, we have

$$f_0(x) \ni f_1(x) \ni f_2(x) \ni f_3(x) \ni \dots$$

by definition of A_n 's. This is a contradiction.

(iii): Let $f \in \mathcal{W}$ be arbitrary, and let $S := \bigcup_{x \in \mathcal{P}_\kappa(\lambda)^\vee} f(x)$. Then, by Łos's Theorem, we have

$$\{g/\sim_U : g/\sim_U \in_U f/\sim_U\} \subseteq \{g/\sim_U : g : \mathcal{P}_\kappa(\lambda)^\vee \rightarrow S\}$$

The right side of the inclusion is clearly a set.

⊢ (Claim 2.2.3)

Let $\mu_U : \langle \mathcal{W}/\sim_U, \in_U \rangle \rightarrow \langle M, \in \rangle$ be the Mostowski-collapse, and let $[\cdot]_U : \mathcal{W} \rightarrow M$; $f \mapsto [f]_U := \mu_U(f/\sim_U)$.

Łos's Theorem (Claim 2.2.2) translates to the following:

Claim 2.2.4 *For any formula $\varphi = \varphi(x_0, \dots, x_{n-1})$ in \mathcal{L}_ε (the language of ZF), and $f_0, \dots, f_{n-1} \in \mathcal{W}$, we have $M \models \varphi([f_0]_U, \dots, [f_{n-1}]_U)$, if and only if $\{x \in \mathcal{P}_\kappa(\lambda)^\vee : \mathbb{V} \models \varphi(f_0(x), \dots, f_{n-1}(x))\} \in U$.*

Cl-gen-sc-1

Let

$$j_U : \mathbb{V} \xrightarrow{\cong} M; a \mapsto [a]_U := \mu_U(i_U(a)) = [const_a]_U.$$

We show that $j_U : \mathbb{V} \xrightarrow{\cong} M$ is a λ -generically supercompact embedding for κ .

Cl-gen-sc-2

Claim 2.2.5 (1) $j_U(\xi) = \xi$ for all $\xi \in \kappa$.

(2) $j_U''\lambda \in M$.

(3) $j_U(\kappa) > \lambda$.

⊢ (1): Note that $j_U(\xi) = \mu_U(i_U(\xi)) = [const_\xi]_U$. Thus, for $\xi < \kappa$ and $f \in \mathcal{W}$,

$$\begin{aligned} [f]_U \in j_U(\xi) &\Leftrightarrow [f]_U \in [const_\xi]_U \\ &\Leftrightarrow \underbrace{\{x \in \mathcal{P}_\kappa(\lambda)^\vee : f(x) \in \xi\}}_{\text{Claim 2.2.4}} \in U \\ &\Leftrightarrow \underbrace{\{x \in \mathcal{P}_\kappa(\lambda)^\vee : f(x) = \eta^*\}}_{\text{by Lemma 2.2, (2) and (2.2)}} \in U \text{ for some } \eta^* \in \xi \\ &\Leftrightarrow \underbrace{\{x \in \mathcal{P}_\kappa(\lambda)^\vee : f(x) = \eta^*\}}_{= const_{\eta^*}(x)} \in U \end{aligned}$$

$$\underbrace{\Leftrightarrow [f]_U = j_U(\eta^*) \text{ for some } \eta^* \in \xi.}_{\text{Claim 2.2.4}}$$

Thus, by induction on $\xi < \kappa$, we obtain $j_U(\xi) = \xi$ for all $\xi < \kappa$.

(2): We show that $[id_{\mathcal{P}_\kappa(\lambda)^\vee}]_U = j_U''\lambda$.

For an arbitrary $f \in \mathcal{W}$

$$\begin{aligned} [f]_U \in [id_{\mathcal{P}_\kappa(\lambda)^\vee}]_U &\Leftrightarrow \{x \in \mathcal{P}_\kappa(\lambda)^\vee : f(x) \in \underbrace{x}_{= id_{\mathcal{P}_\kappa(\lambda)^\vee}(x)}\} \in U \\ &\stackrel{\text{by Claim 2.2.4}}{\Leftrightarrow} \{x \in \mathcal{P}_\kappa(\lambda)^\vee : f(x) = \underbrace{\xi^*}_{= const_{\xi^*}(x)}\} \in U \text{ for some } \xi^* < \lambda \\ &\stackrel{\text{by Lemma 2.2, (2)}}{\Leftrightarrow} [f]_U = j_U(\xi^*) \text{ for some } \xi^* < \lambda. \\ &\stackrel{\text{by Claim 2.2.4}}{\Leftrightarrow} [f]_U = j_U(\xi^*) \text{ for some } \xi^* < \lambda. \end{aligned}$$

(3): We have

$$M \models \text{“ } otp([id_{\mathcal{P}_\kappa(\lambda)^\vee}]_U) < j(\kappa) \text{”}$$

by Łoś's Theorem (Claim 2.2.4) since $\{z \in \mathcal{P}_\kappa(\lambda)^\vee : otp(x) < \underbrace{\kappa}_{= const_\kappa(x)}\} = \mathcal{P}_\kappa(\lambda)^\vee \in U$.

On the other hand:

$$M \models \text{“ } otp([id_{\mathcal{P}_\kappa(\lambda)^\vee}]_U) = \underbrace{\lambda}_{\text{by (2)}} \text{”} \quad \dashv \quad \text{(Claim 2.2.5)}$$

□ (Theorem 2.1)

Note that the proof of Claim 2.2.3 relies on the condition on \mathcal{P} that no $\mathbb{P} \in \mathcal{P}$ adds any new ω -sequence ground model sets. Note also that the argument using the fact that the well-foundedness of a relation is Δ_1 is irrelevant here since the relation \in_U is not in the ground model.

Thus, the proof of Theorem 2.1 cannot simply be applied to the generic supercompactness by a class of posets \mathcal{P} whose elements might add new ω -sequences of ground model sets.

By Theorem 2.1 we obtain another characterization of generic supercompactness by a \mathcal{P} as in Theorem 2.1:

Corollary 2.3 *Suppose that \mathcal{P} is a class of posets such that no $\mathbb{P} \in \mathcal{P}$ adds any new ω -sequence of ground model sets, and \mathcal{P} is closed with respect to restriction. Then, the following are equivalent:*

- (a) κ is generically supercompact by \mathcal{P} .
- (b) For any $\lambda \geq \kappa$, there is a $\mathbb{P} \in \mathcal{P}$ such that

def:P-gen-sc-

2-0

$\Vdash_{\mathbb{P}}$ “there is a \mathbf{V} -normal ultrafilter on $\mathcal{P}^{\mathbf{V}}(\mathcal{P}_{\kappa}(\lambda)^{\mathbf{V}})$ ”.

(c) For any $\lambda \geq \kappa$, there is a $\mathbb{P} \in \mathcal{P}$ such that for any (\mathbf{V}, \mathbb{P}) -generic \mathbb{G} , there are classes $j, M \subseteq \mathbf{V}[\mathbb{G}]$ such that $j : \mathbf{V} \xrightarrow{\cong} M \subseteq \mathbf{V}[\mathbb{G}]$; $\text{crit}(j) = \kappa$; $j(\kappa) > \lambda$ and $j''\lambda \in M$. \square

For a class \mathcal{P} of posets which may contain posets adding a new ω sequence of ground model sets, we have to modify the argument above to obtain the following theorem which also implies the definability of generic supercompactness by \mathcal{P} .

We shall call a \mathbf{V} -normal ultrafilter U on $\mathcal{P}^{\mathbf{V}}(\mathcal{P}_{\kappa}(\lambda)^{\mathbf{V}})$ *steep* if \in_U defined as in (2.14) is well-founded.

Theorem 2.4 *Suppose that \mathcal{P} is a class of posets such that \mathcal{P} is closed with respect to restriction. Then, the following are equivalent:*

def:P-gen-sc-

3

- (a) κ is generically supercompact by \mathcal{P} .
- (b) For any regular $\lambda \geq \kappa$, there is a $\mathbb{P} \in \mathcal{P}$ such that

$\Vdash_{\mathbb{P}}$ “there is a steep \mathbf{V} -normal ultrafilter on $\mathcal{P}^{\mathbf{V}}(\mathcal{P}_{\kappa}(\lambda)^{\mathbf{V}})$ ”.

(c) For any $\lambda \geq \kappa$, there is a $\mathbb{P} \in \mathcal{P}$ such that for any (\mathbf{V}, \mathbb{P}) -generic \mathbb{G} , there are classes $j, M \subseteq \mathbf{V}[\mathbb{G}]$ such that $j : \mathbf{V} \xrightarrow{\cong} M \subseteq \mathbf{V}[\mathbb{G}]$, $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and $j''\lambda \in M$.

Proof of Theorem 2.4: A slight modification the proof of Theorem 2.1 will do: it is enough to show that, for U_j in the proof of “ \Rightarrow ” of Theorem 2.1, the relation \in_{U_j} defined in (2.14) is well-founded. This follows from the next Claim:

Cl-gen-sc-3

Claim 2.4.1 *In $\mathbf{V}[\mathbb{G}]$, the class mapping*

$$(2.17) \quad \iota : \mathcal{W}/\sim_{U_j} \rightarrow \mathbf{V}[\mathbb{G}]; \quad f/\sim_{U_j} \mapsto j(f(j''\lambda))$$

is well-defined, and it is an embedding of $\langle \mathcal{W}/\sim_{U_j}, \in_{U_j} \rangle$ into $\langle \mathbf{V}[\mathbb{G}], \in \rangle$.

\vdash For $f, g \in \mathcal{W}$, we have

$$\begin{aligned} f/\sim_{U_j} \sim_{U_j} g/\sim_{U_j} &\Leftrightarrow \underbrace{\{x \in \mathcal{P}_{\kappa}(\lambda)^{\mathbf{V}} : f(x) = g(x)\}}_{\text{by the definition (2.14) of } \sim_{U_j}} \in U_j \Leftrightarrow \underbrace{\{x \in \mathcal{P}_{\kappa}(\lambda)^{\mathbf{V}} : f(x) = g(x)\}}_{\text{by the definition (2.8) of } U_j} \\ &\Leftrightarrow j(\{x \in \mathcal{P}_{\kappa}(\lambda)^{\mathbf{V}} : f(x) = g(x)\}) \ni j''\lambda \Leftrightarrow \underbrace{j(f)(j''\lambda)}_{= \iota(f/\sim_{U_j})} = \underbrace{j(g)(j''\lambda)}_{= \iota(g/\sim_{U_j})}. \end{aligned}$$

This shows the well-definedness and the injectivity of ι .

Similarly we can show

$$f/\sim_{U_j} \in U_j \quad g/\sim_{U_j} \in U_j \quad \Leftrightarrow \quad \underbrace{j(f)(j''\lambda)}_{= \iota(f/\sim_{U_j})} \in \underbrace{j(g)(''\lambda)}_{= \iota(g/\sim_{U_j})}. \quad \dashv \text{ (Claim 2.4.1)}$$

□ (Theorem 2.4)

3 Sufficiently large initial segment of elementary embeddings

In this section, we prove a characterization of Laver-generic supercompactness from which the first-order definability of this notion follows.

def:bility:initial

Lemma 3.1 *Suppose that \mathbb{P} is a poset (in \mathbb{V}), and \mathbb{G} a (\mathbb{V}, \mathbb{P}) -generic set. Suppose that $j, M \subseteq \mathbb{V}[\mathbb{G}]$ are such that $j : \mathbb{V} \xrightarrow{\cong} M \subseteq \mathbb{V}[\mathbb{G}]$.*

def:P-is-0-0

Then, for a cardinal θ (in \mathbb{V}), we have:

$$\mathbb{V}[\mathbb{G}] \models \text{“} j \upharpoonright \mathcal{H}(\theta)^{\mathbb{V}} : \mathcal{H}(\theta)^{\mathbb{V}} \xrightarrow{\cong} \mathcal{H}(j(\theta))^M \text{”}.$$

Proof. For any \mathcal{L}_ε -formula $\varphi = \varphi(x_0, \dots, x_{k-1})$ and $u_0, \dots, u_{k-1} \in \mathcal{H}(\theta)^{\mathbb{V}}$, we have

$$\begin{aligned} \mathcal{H}(\theta)^{\mathbb{V}} \models \varphi(u_0, \dots, u_{k-1}) &\Leftrightarrow \mathbb{V} \models \text{“} \mathcal{H}(\theta)^{\mathbb{V}} \models \varphi(u_0, \dots, u_{k-1}) \text{”} \\ &\quad \underbrace{\hspace{10em}}_{\mathcal{H}(\theta)^{\mathbb{V}} \text{ is transitive (in } \mathbb{V}\text{)}} \\ &\Leftrightarrow M \models \text{“} \mathcal{H}(j(\theta))^M \models \varphi(j(u_0), \dots, j(u_{k-1})) \text{”} \\ &\quad \underbrace{\hspace{10em}}_{\text{by elementarity of } j} \\ &\Leftrightarrow \mathbb{V}[\mathbb{G}] \models \text{“} \mathcal{H}(j(\theta))^M \models \varphi(j(u_0), \dots, j(u_{k-1})) \text{”}. \end{aligned} \quad \square \text{ (Lemma 3.1)}$$

Note that, in the Lemma above, $\mathcal{H}(j(\theta))^M$ is transitive in $\mathbb{V}[\mathbb{G}]$, since M is transitive in $\mathbb{V}[\mathbb{G}]$.

def:P-is-0-1

Lemma 3.2 *Suppose that \mathbb{P} is a poset (in \mathbb{V}), and \mathbb{G} a (\mathbb{V}, \mathbb{P}) -generic set. Suppose further that θ is a *regular* cardinal in \mathbb{V} and $j_0, N \in \mathbb{V}[\mathbb{G}]$ be such that N is transitive and $j_0 : \mathcal{H}(\theta)^{\mathbb{V}} \xrightarrow{\cong} N$.*

Let $N_0 = \bigcup j_0''\mathcal{H}(\theta)^{\mathbb{V}}$. Then, we have:

- (1) N_0 is transitive.
- (2) (i) $N_0 \prec N$, (ii) $j_0''\mathcal{H}(\theta) \subseteq N_0$, and (iii) $j_0 : \mathcal{H}(\theta)^{\mathbb{V}} \xrightarrow{\cong} N_0$.
- (3) For any $b \in N_0$, there is $a \in \mathcal{H}(\theta)^{\mathbb{V}}$ such that $b \in j_0(a)$.
- (4) If $\theta_0 < \theta$ is such that $\mathcal{H}(\theta_0)^{\mathbb{V}} \in \mathcal{H}(\theta)^{\mathbb{V}}$, then $\mathcal{H}(j_0(\theta_0))^N \subseteq N_0$.

Proof. Note that we have $\mathcal{H}(\theta)^\vee \models \text{ZFC}^-$.

(1): Suppose that $b \in N_0$ and $c \in b$. We have to show that $c \in N_0$.

Let $a \in \mathcal{H}(\theta)^\vee$ be such that $b \in j_0(a)$. Let $a^* = \text{trcl}(a)$. Then $a^* \in \mathcal{H}(\theta)^\vee$. Since $\mathcal{H}(\theta)^\vee \models a^*$ is transitive and $a \subseteq a^*$, we have

$$N \models j_0(a^*) \text{ is transitive and } j_0(a) \subseteq j_0(a^*)$$

by elementarity. Since N is transitive, $j_0(a^*)$ is really transitive. Since $c \in b \in j_0(a^*)$, it follows that $c \in j_0(a^*) \subseteq \bigcup j_0''\mathcal{H}(\theta)^\vee = N_0$.

(2), (i): We check that N_0 satisfies Vaught's criterion.

Suppose that $b_1, \dots, b_n \in N_0$ and $\varphi(x_0, \dots, x_n)$ is an \mathcal{L}_ε -formula such that

$$(3.1) \quad N \models \exists x \varphi(x, b_1, \dots, b_n).$$

def:x-is-2-a-0

We have to show that there is $b \in N_0$ such that $N \models \varphi(b, b_1, \dots, b_n)$.

Let $a_i \in \mathcal{H}(\theta)^\vee$ for $i \in (n+1) \setminus 1$ be such that $b_i \in j_0(a_i)$ for all $i \in (n+1) \setminus 1$. Then we have

$$(3.2) \quad \mathcal{H}(\theta)^\vee \models \exists x \forall y_1 \in a_1 \cdots \forall y_n \in a_n \left(\begin{aligned} &\exists y \varphi(y, y_1, \dots, y_n) \\ &\rightarrow \exists y \in x \varphi(y, y_1, \dots, y_n) \end{aligned} \right).$$

def:x-is-2-a

Let $a \in \mathcal{H}(\theta)^\vee$ be a witness of (3.2). That is,

$$\mathcal{H}(\theta)^\vee \models \forall y_1 \in a_1 \cdots \forall y_n \in a_n \left(\begin{aligned} &\exists y \varphi(y, y_1, \dots, y_n) \\ &\rightarrow \exists y \in a \varphi(y, y_1, \dots, y_n) \end{aligned} \right).$$

By elementarity, it follows that

$$(3.3) \quad N \models \forall y_1 \in j_0(a_1) \cdots \forall y_n \in j_0(a_n) \left(\begin{aligned} &\exists y \varphi(y, y_1, \dots, y_n) \\ &\rightarrow \exists y \in j_0(a) \varphi(y, y_1, \dots, y_n) \end{aligned} \right).$$

def:x-is-2-a-1

By (3.3) and (3.1), there is $b \in j_0(a) \subseteq \bigcup j_0''\mathcal{H}(\theta)^\vee = N_0$ such that

$$N \models \varphi(b, b_1, \dots, b_n).$$

(2), (ii): Suppose that $a \in \mathcal{H}(\theta)^\vee$. Then $\{a\} \in \mathcal{H}(\theta)^\vee$ and $j_0(a) \in \{j_0(a)\} = j_0(\{a\}) \subseteq \bigcup j_0''\mathcal{H}(\theta)^\vee = N_0$.

(2), (iii): This follows from (2), (i), (ii).

(3): This is clear by definition of N_0 .

(4): Suppose that $\theta_0 < \theta$ is such that $\mathcal{H}(\theta_0)^\vee \in \mathcal{H}(\theta)^\vee$. Let $a = \mathcal{H}(\theta_0)^\vee$. By elementarity, $N \models j_0(a)$ is $\mathcal{H}(j(\theta_0))$. Thus $j_0(a) = \mathcal{H}(j(\theta_0))^N$ and $j_0(a) \in N_0$ by (2), (ii). By (1), it follows that $\mathcal{H}(j(\theta_0))^N \subseteq N_0$. \square (Lemma 3.2)

Proposition 3.3 *Suppose that \mathbb{P} is a poset (in \mathbf{V}) and \mathbb{G} a (\mathbf{V}, \mathbb{P}) -generic filter. Suppose further that θ is a regular cardinal and $j_0 : \mathcal{H}(\theta)^\mathbf{V} \xrightarrow{\cong} N$ for a transitive set $N \in \mathbf{V}[\mathbb{G}]$ such that,*

$$(3.4) \quad \mathbb{P} \in \mathcal{H}(\theta)^\mathbf{V}; \text{ and,}$$

def:x-is-3

$$(3.5) \quad \text{for any } b \in N, \text{ there is } a \in \mathcal{H}(\theta)^\mathbf{V} \text{ such that } b \in j_0(a).$$

def:x-is-3-0

Then there are $j, M \subseteq \mathbf{V}[\mathbb{G}]$ such that

$$(3.6) \quad j : \mathbf{V} \xrightarrow{\cong} M \subseteq \mathbf{V}[\mathbb{G}],$$

def:x-is-4

$$(3.7) \quad N \subseteq M \text{ and } j \upharpoonright \mathcal{H}(\theta)^\mathbf{V} = j_0.$$

def:x-is-5

Proof. We mainly work in $\mathbf{V}[\mathbb{G}]$. Let

$$(3.8) \quad \mathcal{F} := \{f \in \mathbf{V} : f : \text{dom}(f) \rightarrow \mathbf{V}, \text{dom}(f) \in \mathcal{H}(\theta)^\mathbf{V}\}, \text{ and}$$

def:x-is-6

$$(3.9) \quad \Pi := \{\langle f, a \rangle : f \in \mathcal{F}, a \in j_0(\text{dom}(f))\}.$$

def:x-is-7

For $\langle f, a \rangle, \langle g, b \rangle \in \Pi$, let

$$(3.10) \quad \langle f, a \rangle \sim \langle g, b \rangle \iff \langle a, b \rangle \in j_0(S_{f(x)=g(y)}),$$

def:x-is-8

where $S_{f(x)=g(y)} := \{\langle x, y \rangle : x \in \text{dom}(f), y \in \text{dom}(g), f(x) = g(y)\}$; and

$$(3.11) \quad \langle f, a \rangle E \langle g, b \rangle \iff \langle a, b \rangle \in j_0(S_{f(x) \varepsilon g(y)}),$$

def:x-is-9

where $S_{f(x) \varepsilon g(y)} := \{\langle x, y \rangle : x \in \text{dom}(f), y \in \text{dom}(g), f(x) \in g(y)\}$.

def:Cl-0

Claim 3.3.1 (1) \sim is an equivalence relation on Π .

(2) \sim is a congruence relation to E .

\vdash (1): Clearly \sim is reflective and symmetric. We show that \sim is transitive. Suppose that $\langle f, a \rangle, \langle g, b \rangle, \langle h, c \rangle \in \Pi$, $\langle f, a \rangle \sim \langle g, b \rangle$ and $\langle g, b \rangle \sim \langle h, c \rangle$. By the definition (3.10), we have $\langle a, b \rangle \in j_0(S_{f(x)=g(y)})$ and $\langle b, c \rangle \in j_0(S_{g(y)=h(z)})$. Thus

$$\langle a, c \rangle \in j_0(S_{f(x)=g(y)}) \circ j_0(S_{g(y)=h(z)}) = j_0(S_{f(x)=g(y)} \circ S_{g(y)=h(z)})$$

by elementarity of j_0

$$\subseteq j_0(S_{f(y)=h(z)}).$$

by $S_{f(x)=g(y)} \circ S_{g(y)=h(z)} \subseteq S_{f(x)=h(z)}$ and elementarity

This shows that $\langle f, a \rangle \sim \langle h, c \rangle$.

(2): Suppose $\langle f_0, a_0 \rangle, \langle f_1, a_1 \rangle, \langle g, b \rangle \in \Pi$, $\langle f_0, a_0 \rangle \sim \langle f_1, a_1 \rangle$, and $\langle f_0, a_0 \rangle E \langle g, b \rangle$. Then

$$\begin{aligned} \langle a_1, b \rangle &\in j_0(S_{f_1(x_1)=f_0(x_0)}) \circ j_0(S_{f_0(x_0) \varepsilon g(y)}) = j_0(S_{f_1(x_1)=f_0(x_0)} \circ S_{f_0(x_0) \varepsilon g(y)}) \\ &\subseteq j_0(S_{f_1(x_1) \varepsilon g(y)}). \end{aligned}$$

Thus $\langle f_1, a_1 \rangle E \langle g, b \rangle$.

Similarly, we can show that, for $\langle f, a \rangle, \langle g_0, b_0 \rangle, \langle g_1, b_1 \rangle \in \Pi$, $\langle g_0, b_0 \rangle \sim \langle g_1, b_1 \rangle$ and $\langle f, a \rangle E \langle g_0, b_0 \rangle$ implies $\langle f, a \rangle E \langle g_1, b_1 \rangle$. Since \sim is an equivalence relation by (1), it follows that \sim is a congruence relation to E . \dashv (Claim 3.3.1)

Let Π/\sim be the class of the equivalence classes (in the sense of footnote (4)) of \sim . We denote the equivalence class of $\langle f, a \rangle \in \Pi$ modulo \sim by $\langle f, a \rangle/\sim$. For simplicity, we denote the binary relation on Π/\sim corresponding to E also by E . Thus, $\langle f, a \rangle/\sim E \langle g, b \rangle/\sim :\Leftrightarrow \langle f, a \rangle E \langle g, b \rangle$.

Generalizing the notation we already used in (3.10) and (3.11), we let

$$\begin{aligned} S_{\varphi(f_0(x_0), \dots, f_{n-1}(x_{n-1}))} \\ := \{ \langle u_0, \dots, u_{n-1} \rangle \in \mathbf{V} : u_0 \in \text{dom}(f_0), \dots, u_{n-1} \in \text{dom}(f_{n-1}), \\ \mathbf{V} \models \varphi(f_0(u_0), \dots, f_{n-1}(u_{n-1})) \} \end{aligned}$$

for each \mathcal{L}_ε -formula $\varphi = \varphi(x_0, \dots, x_{n-1})$.

We have the following ‘‘Łoś’s Theorem’’ for $\langle \Pi/\sim, E \rangle$.

def:Cl-is-a

Claim 3.3.2 *For any \mathcal{L}_ε -formula $\varphi = \varphi(x_0, \dots, x_{n-1})$ and $\langle f_0, a_0 \rangle, \dots, \langle f_{n-1}, a_{n-1} \rangle \in \Pi$, we have*

$$\begin{aligned} \langle \Pi/\sim, E \rangle \models \varphi(\langle f_0, a_0 \rangle/\sim, \dots, \langle f_{n-1}, a_{n-1} \rangle/\sim) \\ \Leftrightarrow \langle a_0, \dots, a_{n-1} \rangle \in j_0(S_{\varphi(f_0(x_0), \dots, f_{n-1}(x_{n-1}))}). \end{aligned}$$

\vdash By induction on φ . If φ is atomic, the claim follows from the definitions (3.10) and (3.11) of \sim and E .

The induction step for ‘‘ $\varphi = \neg\varphi_0$ ’’ is trivial.

Suppose $\varphi = \varphi(x_0, \dots, x_{n-1})$, $\varphi = \varphi_0 \vee \varphi_1$, and $\langle f_0, a_0 \rangle, \dots, \langle f_{n-1}, a_{n-1} \rangle \in \Pi$. Note that

$$(3.12) \quad S_{\varphi(f_0(x_0), \dots, f_{n-1}(x_{n-1}))} = S_{\varphi_0(f_0(x_0), \dots, f_{n-1}(x_{n-1}))} \cup S_{\varphi_1(f_0(x_0), \dots, f_{n-1}(x_{n-1}))}. \quad \text{def:x-is-10}$$

We have

$$\begin{aligned} \langle \Pi/\sim, E \rangle \models \varphi(\langle f_0, a_0 \rangle/\sim, \dots, \langle f_{n-1}, a_{n-1} \rangle/\sim) \\ \Leftrightarrow \langle \Pi/\sim, E \rangle \models \varphi_0(\langle f_0, a_0 \rangle/\sim, \dots, \langle f_{n-1}, a_{n-1} \rangle/\sim) \\ \text{or } \langle \Pi/\sim, E \rangle \models \varphi_1(\langle f_0, a_0 \rangle/\sim, \dots, \langle f_{n-1}, a_{n-1} \rangle/\sim) \end{aligned}$$

by induction hypothesis

$$\begin{aligned} \Leftrightarrow \langle a_0, \dots, a_{n-1} \rangle \in j_0(S_{\varphi_0(f_0(x_0), \dots, f_{n-1}(x_{n-1}))}) \\ \text{or } \langle a_0, \dots, a_{n-1} \rangle \in j_0(S_{\varphi_1(f_0(x_0), \dots, f_{n-1}(x_{n-1}))}) \\ \Leftrightarrow \langle a_0, \dots, a_{n-1} \rangle \in j_0(S_{\varphi_0(f_0(x_0), \dots, f_{n-1}(x_{n-1}))}) \cup j_0(S_{\varphi_1(f_0(x_0), \dots, f_{n-1}(x_{n-1}))}) \end{aligned}$$

by elementarity of j_0 and (3.12)

$$\Leftrightarrow \langle a_0, \dots, a_{n-1} \rangle \in j_0(S_{\varphi(f_0(x_0), \dots, f_{n-1}(x_{n-1}))}).$$

Finally, suppose $\varphi = \exists x \varphi_0(x, x_1, \dots, x_{n-1})$ and $\langle f_1, a_1 \rangle, \dots, \langle f_{n-1}, a_{n-1} \rangle \in \Pi$.

If $\langle \Pi / \sim, E \rangle \models \varphi(\langle f_1, a_1 \rangle / \sim, \dots, \langle f_{n-1}, a_{n-1} \rangle / \sim)$, then there is $\langle f, a \rangle \in \Pi$ such that $\langle \Pi / \sim, E \rangle \models \varphi_0(\langle f, a \rangle / \sim, \langle f_1, a_1 \rangle / \sim, \dots, \langle f_{n-1}, a_{n-1} \rangle / \sim)$. By induction hypothesis, it follows that

$$(3.13) \quad \langle a, a_1, \dots, a_{n-1} \rangle \in j_0(S_{\varphi_0(f(x_0), f_1(x_1), \dots)}).$$

def:x-is-10-0

Since

$$\begin{aligned} \mathcal{H}(\theta)^\forall \models \forall y_0 \cdots \forall y_{n-1} \left(\langle y_0, y_1, \dots, y_{n-1} \rangle \in S_{\varphi_0(f(x_0), f_1(x_1), \dots)} \right. \\ \left. \rightarrow \langle y_1, \dots, y_{n-1} \rangle \in S_{\varphi(f_1(x_1), \dots)} \right), \end{aligned}$$

it follows, by elementarity, that

$$(3.14) \quad N \models \forall y_0 \cdots \forall y_{n-1} \left(\langle y_0, y_1, \dots, y_{n-1} \rangle \in j_0(S_{\varphi_0(f(x_0), f_1(x_1), \dots)} \right. \\ \left. \rightarrow \langle y_1, \dots, y_{n-1} \rangle \in j_0(S_{\varphi(f_1(x_1), \dots)} \right).$$

def:x-is-10-1

By (3.13) and (3.14), we obtain $\langle a_1, \dots, a_{n-1} \rangle \in j_0(S_{\varphi(f_1(x_1), \dots, f_{n-1}(x_{n-1}))})$.

Conversely, assume that $\langle a_1, \dots, a_{n-1} \rangle \in j_0(S_{\varphi(f_1(x_1), \dots, f_{n-1}(x_{n-1}))})$. Let $d = \text{dom}(f_1) \times \cdots \times \text{dom}(f_{n-1})$. Note that $d \in \mathcal{H}(\theta)^\forall$.

Let $f \in \mathcal{V}$ with $f : d \rightarrow \mathcal{V}$ be defined by

$$f(\langle u_0, \dots, u_{n-1} \rangle) = \begin{cases} \text{some } a \in \mathcal{V} \text{ such that } \mathcal{V} \models \varphi_0(a, f_1(u_1), \dots, f_{n-1}(u_{n-1})), \\ \text{if there is such } a \in \mathcal{V}; \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $\langle u_0, \dots, u_{n-1} \rangle \in d$. Then we have

$$\begin{aligned} \mathcal{H}(\theta)^\forall \models \forall x_1 \cdots \forall x_{n-1} \left(\langle x_1, \dots, x_{n-1} \rangle \in S_{\varphi(f_1(x_1), \dots)} \right. \\ \left. \rightarrow (\langle \langle x_1, \dots, x_{n-1} \rangle, x_1, \dots, x_{n-1} \rangle \in S_{\varphi_0(f(x), f_1(x_1), \dots)} \right). \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{H}(\theta)^\forall \models \forall x_1 \cdots \forall x_{n-1} \left(\langle x_1, \dots, x_{n-1} \rangle \in S_{\varphi(f_1(x_1), \dots)} \right. \\ \left. \rightarrow \exists x (\langle x, x_1, \dots, x_{n-1} \rangle \in S_{\varphi_0(f(x), f_1(x_1), \dots)} \right). \end{aligned}$$

By elementarity, it follows that

$$\begin{aligned} N \models \forall x_1 \cdots \forall x_{n-1} \left(\langle x_1, \dots, x_{n-1} \rangle \in j_0(S_{\varphi(f_1(x_1), \dots)} \right. \\ \left. \rightarrow \exists x (\langle x, x_1, \dots, x_{n-1} \rangle \in j_0(S_{\varphi_0(f(x), f_1(x_1), \dots)} \right). \end{aligned}$$

Hence, there is $a \in N$ such that $\langle a, a_1, \dots, a_{n-1} \rangle \in j_0(S_{\varphi_0(f(x), f_1(x_1), \dots)})$. By induction hypothesis, it follows that

$$\langle \Pi/\sim, E \rangle \models \varphi_0(\langle f, a \rangle/\sim, \langle f_1, a_1 \rangle/\sim, \dots, \langle f_{n-1}, a_{n-1} \rangle/\sim).$$

Thus $\langle \Pi/\sim, E \rangle \models \varphi(\langle f_1, a_1 \rangle/\sim, \dots, \langle f_{n-1}, a_{n-1} \rangle/\sim)$. ⊣ (Claim 3.3.2)

For $u \in \mathbf{V}$, let $\check{f}_u : 1 \rightarrow \mathbf{V}$ be defined by $\check{f}_u(\emptyset) = u$. Let $i : \mathbf{V} \rightarrow \Pi/\sim$ be defined by $i(u) := \langle \check{f}_u, \emptyset \rangle/\sim$ for $u \in \mathbf{V}$.

def:Cl-is-0

Claim 3.3.3 i is an elementary embedding of $\langle \mathbf{V}, \in \rangle$ into $\langle \Pi/\sim, E \rangle$.

⊢ Suppose that $\varphi = \varphi(x_0, \dots, x_{n-1})$ is an \mathcal{L}_ε -formula and $u_0, \dots, u_{n-1} \in \mathbf{V}$. Then we have

$$\begin{aligned} \langle \Pi/\sim, E \rangle \models \varphi(i(u_0), \dots, i(u_{n-1})) \\ \Leftrightarrow \underbrace{\langle \emptyset, \emptyset, \dots, \emptyset \rangle}_{\text{by Claim 3.3.2}} \in j_0(S_{\varphi(\check{f}_{u_0}(x_0), \dots, \check{f}_{u_{n-1}}(x_{n-1}))}) \\ \quad \text{by definition of } S_{\varphi(\dots)} \\ = j_0(\{\langle x_0, \dots, x_{n-1} \rangle : \mathbf{V} \models \varphi(\check{f}_{u_0}(x_0), \dots, \check{f}_{u_{n-1}}(x_{n-1}))\}) \\ = \begin{cases} \emptyset, & \text{if } \mathbf{V} \not\models \varphi(u_0, \dots, u_{n-1}); \\ \{\langle \emptyset, \dots, \emptyset \rangle\}, & \text{if } \mathbf{V} \models \varphi(u_0, \dots, u_{n-1}). \end{cases} \\ \Leftrightarrow \mathbf{V} \models \varphi(u_0, \dots, u_{n-1}). \end{aligned}$$

⊣ (Claim 3.3.3)

def:Cl-is-1

Claim 3.3.4 (1) E is well-founded.

(2) E is set like.

⊢ (1): Suppose not and let $\langle f_n, b_n \rangle \in \Pi$, $n \in \omega$ (in $V[\mathbb{G}]$) be such that

$$(3.15) \quad \langle f_0, b_0 \rangle \exists \langle f_1, b_1 \rangle \exists \langle f_2, b_2 \rangle \exists \dots \quad \text{def:x-is-11}$$

Let \check{f}_n , $n \in \omega$ be \mathbb{P} -names of f_n , $n \in \omega$ (note that we can choose \check{f}_n , $n \in \omega$ such that $\langle \check{f}_n : n \in \omega \rangle \in \mathbf{V}$ and $\Vdash_{\mathbb{P}} \text{“}\check{f}_n \in \mathbf{V}\text{”}$ for each $n \in \omega$), and let

$$(3.16) \quad \mathcal{Q} := \{ \langle \mathbb{P}, n, u \rangle : \mathbb{P} \in \mathbb{P}, n \in \omega, u \in \mathcal{H}(\theta)^\mathbf{V}, \\ \mathbb{P} \text{ decides } \check{f}_n, \text{ and } \mathbb{P} \Vdash_{\mathbb{P}} \text{“}u \in \text{dom}(\check{f}_n)\text{”} \}. \quad \text{def:x-is-12}$$

By (3.4) and since θ is regular, we have $\mathcal{Q} \in \mathcal{H}(\theta)^\mathbf{V}$.

For $\langle \mathbb{P}_0, n_0, u_0 \rangle, \langle \mathbb{P}_1, n_1, u_1 \rangle \in \mathcal{Q}$, let

$$\begin{aligned} \langle \mathbb{P}_0, n_0, u_0 \rangle \sqsubset \langle \mathbb{P}_1, n_1, u_1 \rangle &: \Leftrightarrow \mathbb{P}_0 \leq_{\mathbb{P}} \mathbb{P}_1, \quad n_0 = n_1 + 1, \\ &\text{and } \mathbb{P}_0 \Vdash_{\mathbb{P}} \text{“}\check{f}_{n_0}(u_0) \in \check{f}_{n_1}(u_1)\text{”}. \end{aligned}$$

In $V[\mathbb{G}]$, let $\langle \mathbb{P}_n : n \in \omega \rangle$ be a descending sequence in \mathbb{G} with respect to $\leq_{\mathbb{P}}$ such that each \mathbb{P}_n decides \check{f}_n to be f_n .

Subclaim 3.3.4.1 $\langle \langle j_0(\mathbb{P}_n), n, b_n \rangle : n \in \omega \rangle$ is a descending sequence in $j_0(\langle \mathcal{Q}, \sqsubset \rangle)$ with respect to $j_0(\sqsubset)$.

\vdash For $n \in \omega$, we have to show that

$$\langle j_0(\mathbb{P}_{n+1}), n+1, b_{n+1} \rangle j_0(\sqsubset) \langle j_0(\mathbb{P}_0), n, b_n \rangle$$

holds. By the choice of \mathbb{P}_n 's, we have $\mathbb{P}_{n+1} \leq_{\mathbb{P}} \mathbb{P}_n$, $\mathbb{P}_{n+1} \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{f}_{n+1} = f_{n+1} \text{”}$, and $\mathbb{P}_n \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{f}_n = f_n \text{”}$. Thus we have

$$(3.17) \quad \mathbb{P}_{n+1} \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{f}_{n+1} = f_{n+1} \wedge \underset{\sim}{f}_n = f_n \text{”}.$$

def:x-is-12-a

It follows that, for each $n \in \omega$,

$$\begin{aligned} \sqsubset &\supseteq \{ \langle \langle \mathbb{P}_{n+1}, n+1, u \rangle, \langle \mathbb{P}_n, n, v \rangle \rangle : \mathbb{P}_{n+1} \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{f}_{n+1}(u) \varepsilon \underset{\sim}{f}_n(v) \text{”} \} \\ &= \{ \langle \langle \mathbb{P}_{n+1}, n+1, u \rangle, \langle \mathbb{P}_n, n, v \rangle \rangle : f_{n+1}(u) \varepsilon f_n(v) \} \\ &\quad \underbrace{\hspace{10em}}_{\text{by (3.17)}} \\ &= \{ \langle \langle \mathbb{P}_{n+1}, n+1, u \rangle, \langle \mathbb{P}_n, n, v \rangle \rangle : \langle u, v \rangle \in S_{f_{n+1}(x_0) \varepsilon f_n(x_1)} \}. \end{aligned}$$

by the definition of $S_{\dots \varepsilon \dots}$ in (3.11)

Thus, for each $n \in \omega$,

$$\begin{aligned} N \models j_0(\sqsubset) &\supseteq \{ \langle \langle j_0(\mathbb{P}_{n+1}), n+1, u \rangle, \langle j_0(\mathbb{P}_n), n, v \rangle \rangle : \langle u, v \rangle \in j_0(S_{f_{n+1}(x_0) \varepsilon f_n(x_1)}) \} \\ &\supseteq \underbrace{\langle \langle j_0(\mathbb{P}_{n+1}), n+1, b_{n+1} \rangle, \langle j_0(\mathbb{P}_0), n, b_n \rangle \rangle}_{\text{by (3.15)}}. \end{aligned}$$

\dashv (Subclaim 3.3.4.1)

Since $\mathbb{V}[\mathbb{G}]$ can see the sequence $\langle \mathbb{P}_n : n \in \omega \rangle$, we have $\mathbb{V}[\mathbb{G}] \models \text{“} j_0(\langle \mathcal{Q}, \sqsubset \rangle)$ is not well-founded”.

Since being well-founded is Δ_1 , it follows that $N \models \text{“} j_0(\langle \mathcal{Q}, \sqsubset \rangle)$ is not well-founded”.

By elementarity, it follows that $\mathcal{H}(\theta)^{\mathbb{V}} \models \text{“} \langle \mathcal{Q}, \sqsubset \rangle$ is not well-founded”. However, if $\langle \langle \mathfrak{q}_n, k_n, u_n \rangle : n \in \omega \rangle$ is a descending sequence in $\langle \mathcal{Q}, \sqsubset \rangle$, then we would have

$$g_{k_0}(u_0) \ni g_{k_1}(u_1) \ni g_{k_2}(u_2) \ni \dots$$

where g_{k_n} , for each $n \in \omega$, is the element of \mathcal{F} which is decided to be $\underset{\sim}{f}_{k_n}$ by \mathfrak{q}_n . This is a contradiction.

(2): Suppose that $\langle f, a \rangle, \langle g, b \rangle \in \Pi$ and

$$(3.18) \quad \langle f, a \rangle E \langle g, b \rangle.$$

Let $f_0 : \text{dom}(f) \rightarrow \bigcup g''\text{dom}(g) \cup \{\infty\}$, where ∞ is a set such that $\infty \notin g''\text{dom}(g)$, be defined by

$$f_0(u) = \begin{cases} f(u), & \text{if } f(u) \in \bigcup g''\text{dom}(g); \\ \infty, & \text{otherwise} \end{cases}$$

for all $u \in \text{dom}(f)$. By the definition of f_0 , we have $S_{f(x_0)\varepsilon g(x_1)} = S_{f_0(x_0)\varepsilon g(x_1)}$. Thus we have

$$(3.19) \quad \langle f, a \rangle \sim \langle f_0, a \rangle.$$

This implies that

$$\begin{aligned} & \{\pi \in \Pi/\sim : \pi E \langle g, b/\sim \rangle\} \\ & \subseteq \{\langle f, a \rangle/\sim : \text{dom}(f) \in \mathcal{H}(\theta)^\vee, \\ & \quad f : \text{dom}(f) \rightarrow \bigcup g''\text{dom}(g) \cup \{\infty\}, a \in j_0(\text{dom}(f))\}. \end{aligned}$$

The right side of the inclusion is clearly a set. \dashv (Claim 3.3.4)

$\langle \Pi/\sim, E \rangle$ is extensional by Claim 3.3.3. Hence, by Claim 3.3.4, there is the Mostowski collapse

$$m : \langle \Pi/\sim, E \rangle \rightarrow \langle \mathbf{V}[\mathbb{G}], \in \rangle.$$

Let $M := m''\Pi/\sim$ and $j := m \circ i$. By Claim 3.3.3, we have

$$j : \mathbf{V} \xrightarrow{\cong} M \subseteq \mathbf{V}[\mathbb{G}].$$

Note that, for $a \in \mathcal{H}(\theta)^\vee$,

$$(3.20) \quad j(a) = m \circ i(a) = m(\langle \check{f}_a, \emptyset \rangle/\sim).$$

def:x-is-12-0

For each $b \in N$, let $d_b \in \mathcal{H}(\theta)^\vee$ be such that $b \in j_0(d_b)$. We can always find such d_b by (3.5). Let

$$\iota : N \rightarrow \Pi/\sim; \quad b \mapsto \langle \text{id}_{d_b}, b \rangle/\sim.$$

def:Cl-is-2

Claim 3.3.5 ι is an embedding of $\langle N, \in \rangle$ into $\langle \Pi/\sim, E \rangle$, and $\iota''N$ is a full initial segment of Π/\sim with respect to E . In particular, for any $b \in N$, we have $m(\iota(b)) = m(\langle \text{id}_{d_b}, b \rangle/\sim) = b$.

\vdash Note that

$$(3.21) \quad j_0(\text{id}_{d_b}) = \text{id}_{j_0(d_b)}$$

def:x-is-13

by elementarity.

For $b, c \in N$

$$\iota(b) E \iota(c) \underset{\text{by definition of } \iota}{\Leftrightarrow} \langle id_{d_b}, b \rangle E \langle id_{d_c}, c \rangle \overset{\text{by the definition (3.11) of } E}{\Leftrightarrow} \underbrace{j_0(id_{d_b})(b)}_{= b, \text{ by (3.21)}} \in \underbrace{j_0(id_{d_c})(c)}_{= c, \text{ by (3.21)}}.$$

Suppose that $\langle f, a \rangle / \sim E \langle id_{d_b}, b \rangle = \iota(b)$ for $\langle f, a \rangle \in \Pi$. This means that

$$j_0(f)(a) \in \underbrace{j_0(id_{d_b})(b)}_{\text{by (3.21)}} = b.$$

Let $c := j_0(f)(a)$. Then we have $c \in b \in N$. Since N is transitive it follows that $c \in N$. By the definition (3.10) of \sim , we have

$$\iota(c) = \langle id_{d_c}, c \rangle / \sim = \langle f, a \rangle / \sim. \quad \dashv \text{ (Claim 3.3.5)}$$

Together with the previous Claim, the following Claim shows that our j and M are as desired:

def:Cl-is-3

Claim 3.3.6 $j \upharpoonright \mathcal{H}(\theta)^\vee = j_0$.

\vdash Suppose that $a \in \mathcal{H}(\theta)^\vee$. We show that $j(a) = j_0(a)$.

Note that $j(a) = m(\langle \check{f}_a, \emptyset \rangle / \sim)$. For $b := j_0(a)$, we have $\langle \check{f}_a, \emptyset \rangle \sim \langle id_{d_b}, b \rangle$ by (3.10). It follows that $j(a) = m(\langle id_{d_b}, b \rangle / \sim) \underset{\text{by Claim 3.3.5}}{=} b = j_0(a)$. \dashv (Claim 3.3.6)

\square (Proposition 3.3)

def:P-is-2

Theorem 3.4 *Suppose that \mathcal{P} is an iterable class of posets. Then the following are equivalent:*

- (a) κ is L - g supercompact for \mathcal{P} .
- (b) For any λ , and for any $\mathbb{P} \in \mathcal{P}$, there is a \mathbb{P} -name \mathbb{Q} with $\Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{P}\text{”}$ such that

$\Vdash_{\mathbb{P} * \mathbb{Q}} \text{“there are a regular cardinal } \theta > \kappa, \text{ a transitive set } N, \text{ and a mapping } j_0 \text{ such that}$

- (1) $j_0 : \mathcal{H}(\theta)^\vee \xrightarrow{\cong} N$,
- (2) $\text{crit}(j_0) = \kappa, \theta, j_0(\kappa) > \lambda$,
- (3) for any $b \in N$, there is $a \in \mathcal{H}(\theta)^\vee$ such that $b \in j_0(a)$
- (4) $\mathbb{P} * \mathbb{Q}, \mathbb{H} \in N$, and
- (5) $j_0 \upharpoonright \lambda \in N$.

Proof. “(a) \Rightarrow (b)”: By Lemma 3.1 and Lemma 3.2.

“(b) \Rightarrow (a)”: By Proposition 3.3.

\square (Theorem 3.4)

References

defbilty:ref

- [1] Sakaé Fuchino, André Ottenbreit Maschio Rodrigues, and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, I, in Archive for Mathematical Logic Vol.60, 1-2, (2021), 17–47.
Extended version of the paper: <https://fuchino.ddo.jp/papers/SDLS-x.pdf>
- [2] _____: Strong downward Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum, to appear in Archive for Mathematical Logic, Vol.60, 3-4, (2021), 495–523.
Extended version of the paper: <https://fuchino.ddo.jp/papers/SDLS-II-x.pdf>
- [3] Sakaé Fuchino and André Ottenbreit Maschio Rodrigues, Reflection principles, generic large cardinals, and the Continuum Problem, to appear in the Proceedings of the Symposium on Advances in Mathematical Logic 2018. Extended version of the paper:
https://fuchino.ddo.jp/papers/refl_principles_gen_large_cardinals_continuum_problem-x.pdf
- [4] Thomas Jech, *Theory*, The Third Millennium Edition, Springer (2001/2006).
- [5] Akihiro Kanamori, *The Higher Infinite*, Second Edition, Springer Monographs in Mathematics, Springer-Verlag, (2003/2009).
- [6] Bernhard König, Generic compactness reformulated, Archive of Mathematical Logic 43, (2004), 311 – 326.
- [7] K. Kunen, *Set Theory, An Introduction to Independence Proofs*, North-Holland (1980).