Some problems of Ščepin on openly generated Boolean algebras

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June 15,1993
revised on August 31,1993

Abstract

In [21], E.V. Ščepin gave a list of open problems on openly generated and some other related spaces. In this note, we give solutions to the problems (*3*), (*7*), (*8*), (*9*) of the list. For problems (*3*), (*9*), we give negative solutions in ZFC. We show that the assertion of problem (*7*) is independent from ZFC (under existence of some large cardinal). For problem (*8*) we give a negative answer under $V = L$. It is still open if also for this problem an independence result can be obtained similary to the problem (*7*). In the last section we mention some results connected to this question.

This note is a revised version of author's paper with the same title appeared in:
the proceedings of the 10th Easter Conference on Model Theory, Humboldt Universität Berlin, 1993.

1 The Problems

In [21] E.V. Ščepin listed some open problems concerning openly generated spaces and some other related topological spaces. In the present note we shall give solutions to four of them. Some other problems in the list were also recently solved by L.B. Shapiro (still unpublished). For further informations the reader may consult [13].

The following are the problems we are going to deal with. The problems were originally presented in terms of topology. The reformulation of the problems in terms of Boolean algebras was done by L. Heindorf [13]. The numbering of the problems is the same as in [21]. The definition of the notions used here can be found in the next section.

(*3*) *Let B be a Boolean algebra such that the family of relatively complete subalgebras of B is absorbing. Does it follow that B is openly generated? — No. See Corollary 3.3.*

(*7*) *Let B be a Boolean algebra such that $B \models C(\aleph_2)$ holds. Does it follow that B is openly generated? — The answer depends on axioms of set theory: No, under $V = L$. See Corollary 3.4. Yes, under* MA^+(*σ-closed*). *See Theorem 4.3. (See also Theorem 4.1.)*

(*8*) *Let B be a Boolean algebra such that $B \models C(\aleph_3)$ holds. Does it follow that B is projective? — No, under $V = L$. See Corollary 3.4. (See also Theorem 5.3 and Theorem 5.4.)*

(*9*) *Let B be a Boolean algebra such that $B \oplus B$ has the Bockstein separation property. Does it follow that B is openly generated? — No. See Corollary 3.5.*

2 Preliminaries

Let A be a subalgebra of a Boolean algebra B (Notation: $A \leq B$). A is a *relatively complete subalgebra of* B (Notation: $A \leq_{\mathrm{rc}} B$) if, for every $b \in B$, the set $\{\, a \in A : a \leq b \,\}$ has the greatest element c. This c is called *the projection of b into A* and is denoted by $\mathrm{pr}^B_A(b)$. If $A \leq B$ but A is not relatively complete in B , this is denoted by $A \leq_{\neg\mathrm{rc}} B$.

For any set X, $[X]^{\aleph_0}$ denotes the set $\{\, Y \subseteq X \ :\ |\,Y\,| = \aleph_0 \,\}$. A subset $\mathcal{C}$ of $[X]^{\aleph_0}$ is said to be *closed unbounded* (club) if $\mathcal{C}$ is closed with respect to union of an increasing chain of countable length and cofinal in $[X]^{\aleph_0}$ (with respect to $\subseteq$). An $\mathcal{S} \subseteq [X]^{\aleph_0}$ is said to be *stationary* if $\mathcal{S} \cap \mathcal{C} \neq \emptyset$ holds for any club $\mathcal{C} \subseteq [X]^{\aleph_0}$. For basic facts about closed unbounded and stationary sets of $[X]^{\aleph_0}$ the reader may consult e.g. [3] or [15].

For a Boolean algebra B let $\mathrm{Sub}^{\aleph_0}_{\mathrm{rc}}(B) = \{\, A \in [B]^{\aleph_0} \ :\ A \leq_{\mathrm{rc}} B \,\}$. A Boolean algebra B is said to be *openly generated* if the set $\mathrm{Sub}^{\aleph_0}_{\mathrm{rc}}(B)$ includes a club subset of $[B]^{\aleph_0}$.

A Boolean algebra B is said to be *projective over a subalgebra A* of B (Notation: $A \leq_{\mathrm{proj}} B$) if $B \oplus \mathrm{Fr}\,\kappa \cong_A A \oplus \mathrm{Fr}\,\kappa$ holds for $\kappa = |\,B\,| + \aleph_0$. B is said to be *projective* if B is projective over 2, i.e. if $B \oplus \mathrm{Fr}\,\kappa$ is free for large enough κ. Note, that this definition differs from the original one (for the original definition and the equivalence of our definition to it see e.g. [17]). A Boolean algebra B is *countably generated* over a subalgebra A of B if there exists a countable set $X \subseteq B$ such that $B = A[X]$ holds.

Projective Boolean algebras have the following nice characterization:

Theorem 2.1 *For a Boolean algebra B and $A \leq B$ the following are equivalent:*

(a) *B is projective over A;*

(b) *(Haydon [12], Koppelberg [17]) There exists a continuously increasing sequence $(B_\alpha)_{\alpha<\rho}$ of subalgebras of B such that $B_0 = A$; $B_\alpha \leq_{\mathrm{rc}} B_{\alpha+1}$, $B_{\alpha+1}$ is countably generated over B_α for every $\alpha < \rho$ and $\bigcup_{\alpha<\rho} B_\alpha = B$;*

(c) *(Bandlow [1]) There exists a closed unbounded set $\mathcal{C} \subseteq [B]^{\aleph_0}$ such that $A \leq_{\mathrm{rc}} B$ holds for every $A \in \mathcal{C}$ and $\langle \bigcup \mathcal{C}' \rangle \leq_{\mathrm{rc}} B$ holds for every $\mathcal{C}' \subseteq \mathcal{C}$.* ∎

From (b) above we obtain the following characterization of openly generated Boolean algebras in terms of forcing:

Theorem 2.2 *A Boolean algebra B of cardinality $\leq \aleph_1$ is openly generated if and only if B is projective. More generally, a Boolean algebra B is openly generated if and only if $\Vdash_P$ "B is projective Boolean algebra" holds for some resp. any σ-closed p.o.-set P collapsing $\mid B \mid$ to $\leq \aleph_1$.*

Proof Let B be an openly generated Boolean algebra. Let $\mathcal{C} \subseteq \mathrm{Sub}^{\aleph_0}_{\mathrm{rc}}(B)$ be a club subset of $[B]^{\aleph_0}$. Let P be a σ-closed p.o.-set collapsing $\mid B \mid$ to $\leq \aleph_1$. Then by the σ-closedness of B we have

$$\Vdash_P \text{“}\mathcal{C} \text{ is closed in } [B]^{\aleph_0}\text{”}.$$

Hence we have $\Vdash_P$ "B is openly generated". Since $\Vdash_P$ "$\mid B \mid \leq \aleph_1$" it follows form Theorem 2.1 (b) that $\Vdash_P$ "B is projective".

For the converse recall that every σ-closed p.o.-set is proper (see e.g. Theorem 2.3 in [3]). Now, if B is not openly generated then the set $\mathcal{S} = [B]^{\aleph_0} \setminus \mathrm{Sub}^{\aleph_0}_{\mathrm{rc}}(B)$ is stationary in $[B]^{\aleph_0}$. For any σ-closed p.o.-set P collapsing $\mid B \mid$ to $\leq \aleph_1$ we have that $\Vdash_P$ "$\mathcal{S}$ is stationary" by properness of P. Hence it follows from Theorem 2.1 (c) that $\Vdash_P$ "B is not projective". ∎ (Proposition 2.2)

This characterization is quite useful to show that some of the properties of projective Boolean algebras of cardinality $\aleph_1$ still hold for openly generated Boolean algebras as we shall see this in Propositions 2.3 – 2.5.

Proposition 2.3 *Every openly generated Boolean algebra satisfies the* ccc.

Proof Let B be an openly generated Boolean algebra and P a σ-closed p.o.-set collapsing $|B|$ to $\aleph_1$. Then it holds that $\Vdash_P$ "B satisfies the ccc". Hence B satisfies the ccc. ▮ (Proposition 2.3)

Proposition 2.4 (Ščepin [21]) *Every relatively complete subalgebra of an openly generated Boolean algebra is openly generated. In particular every relatively complete subalgebra of a free Boolean algebra is openly generated.*

Proof Let $B \leq_{\rm rc} C$ for an openly generated Boolean algebra C. Without loss of generality $\kappa = |C| \geq \aleph_1$. Let P be a σ-closed p.o.-set which collapses κ to $\aleph_1$. By Theorem 2.2 have

$\Vdash_P$ "B is a relatively complete subalgebra of a projective Boolean algebra."

Since $\Vdash_P$ "$|B| \leq \aleph_1$", it follows from Proposition 2.12 in [17] that $\Vdash_P$ "B is projective". ▮ (Proposition 2.4)

A Boolean algebra B has the *Bockstein separation property* (*BSP*) if every regular ideal over B is countably generated (see e.g. [17]).

Proposition 2.5 *Every openly generated Boolean algebra has the BSP.*

Proof Let B be an openly generated Boolean algebra and I a regular ideal over B. Let P be as in Theorem 2.2. Then we have

$\Vdash_P$ "B is projective Boolean algebra and I is a regular ideal over B".

Since projective Boolean algebras have the BSP (see e.g. Theorem 1.12 in [17]) it follows that

$\Vdash_P$ "I is countably generated".

Since P is σ-closed it follows that I is really countably generated.▮ (Proposition 2.5)

A family $\mathcal{F}$ of subalgebras of B is said to be *absorbing* if for any subalgebra A of B there exists $C \in \mathcal{F}$ such that $A \leq C$ and $|A| = |C|$.

Proposition 2.6 *For any openly generated Boolean algebra B, the family of relatively complete subalgebras of B is absorbing.*

Proof Let $\mathcal{C} \subseteq \mathrm{Sub}^{\aleph_0}_{\mathrm{rc}}(B)$ be a club subset of $[B]^{\aleph_0}$. Then every $C \leq B$ such that $[C]^{\aleph_0} \cap \mathcal{C}$ is unbounded in $[C]^{\aleph_0}$ is relatively complete in B: if $C \leq_{\neg\mathrm{rc}} B$ there exists a $b \in B$ without projection into C. Then we can construct an increasing sequence $(C_\alpha)_{\alpha<\omega_1}$ of subalgebras of C such that $C_\alpha \in [C]^{\aleph_0} \cap \mathcal{C}$ and $(\mathrm{pr}^B_{C_\alpha}(b))_{\alpha<\omega_1}$ is strictly increasing. This is a contradiction since, by Proposition 2.3, B satisfies the ccc. ∎ (Proposition 2.6)

There are also some theorems on openly generated Boolean algebras which could be reduced to a theorem on projective Boolean algebras of cardinality $\aleph_1$ using Theorem 2.2 but whose actual proof must be done directly with some more complicated arguments. By Theorem 2.2 such theorems give also new results on projective Boolean algebras of cardinality $\aleph_1$. The following two theorems by Ščepin are examples of this kind:

Theorem 2.7 (Proposition 2.8 and Theorem 2.6 in [21]) *A subalgebra A of an openly generated Boolean algebra B is openly generated if and only if B satisfies the BSP. In particular a dyadic Boolean algebra B is openly generated if and only if B satisfies the BSP.* ∎

Theorem 2.8 (Theorem 2.3 in [21]) *If a Boolean algebra B can be represented as the union of continuous chain $(B_\alpha)_{\alpha<\delta}$ of openly generated subalgebras such that $B_\alpha \leq_{\mathrm{rc}} B$ for all $\alpha < \delta$, then B is also openly generated.* ∎

A sequence $(B_\alpha)_{\alpha<\kappa}$ of subalgebras of B is called a *filtration of* B if $\kappa = \mathrm{cof}|\ B\ |$; $(B_\alpha)_{\alpha<\kappa}$ is increasing and continuous; $\bigcup_{\alpha<\kappa} B_\alpha = B$ and $|\ B_\alpha\ | < |\ B\ |$ for all $\alpha < \kappa$. Note that for a Boolean algebra B of regular cardinality κ any two filtration coincide on a closed unbounded subset of the index set κ.

Corollary 2.9 *A Boolean algebra B is openly generated if and only if B has a filtration $(B_\alpha)_{\alpha<\delta}$ such that B_α is openly generated and $B_\alpha \leq_{\mathrm{rc}} B$ holds for all $\alpha < \delta$.*

Proof If there exists a filtration of B as above then B is openly generated by Theorem 2.8. Conversely if B is openly generated then the proof of Proposition 2.6 shows that B has a filtration $\mathcal{F}$ consisting of relatively complete subalgebras. By Proposition 2.4 the subalgebras in $\mathcal{F}$ are openly generated. ∎ (Corollary 2.9)

The problem (*7*) asks if $C(\aleph_2)$, a weakening of the property in the Corollary 2.9, already characterize the openly generated Boolean algebras. Here we say that Boolean algebra B satisfies the condition $C(\kappa)$ when there exist a partial ordering $I = (I, \leq)$ and an indexed system $(B_i)_{i\in I}$ of subalgebras of B such that

B_i is projective for every $i \in I$;

$B_i \leq B_j$ for every $i, j \in I$ such that $i \leq j$;

Every (upward) directed set $X \subseteq I$ of cardinality $< \kappa$ has its supremum $i_X \in I$ and

$B_{i_X} = \bigcup_{i\in X} B_i$ and $B = \bigcup_{i\in I} B_i$.

We shall call $(B_i)_{i\in I}$ as above a $C(\kappa)$-filtration of B. B satisfies the condition $C^*(\kappa)$ if there exists a $C(\kappa)$-filtration $(B_i)_{i\in I}$ of B such that $\mid B_i \mid < \kappa$ for all $i \in I$ holds. We call $(B_i)_{i\in I}$ then a $C^*(\kappa)$-filtration. By Theorem 2.2 it can be easily shown that B satisfies $C(\aleph_2)$ if and only if B satisfies $C^*(\aleph_2)$.

A Boolean algebra B is $L_{\infty\kappa}$-free if $B \equiv_{L_{\infty\kappa}} \mathrm{Fr}\,\kappa$ holds. By a theorem of D. Kueker as below, $L_{\infty\kappa}$-free Boolean algebras can be characterized purely algebraically. For Boolean algebras A, B such that $A \leq B$ we shall write $B|A$ if $B \cong_A A \oplus F$ holds for a free Boolean algebra F. Clearly we have $A \leq_{\mathrm{proj}} B$ if $B|A$ holds. An elegant proof of the following theorem can be found in [6]:

Theorem 2.10 (D. Kueker, [18]) *A Boolean algebra B is $L_{\infty\kappa}$-free if and only if there exists a family $\mathcal{F}$ of subalgebras of B of cardinality $< \kappa$ such that every $F \in \mathcal{F}$ is free; $\bigcup \mathcal{F} = B$; for every $\mathcal{F}' \subseteq \mathcal{F}$ with $\mid \mathcal{F}' \mid < \kappa$ there exists an $F_0 \in \mathcal{F}$ such that $F_0|F$ holds for every $F \in \mathcal{F}$.* ∎

3 A construction of Boolean algebras

In this section we introduce a construction of Boolean algebras which gives negative answers to all of four problems listed in Section 1 (For problems (*7*) and (*8*) we need additionally some consequences of $V = L$.). The construction, in particular for the case $\kappa = \aleph_1$, is due to S. Koppelberg. A stationary subset of a cardinal κ is called non-reflecting if $S \cap \alpha$ is not stationary for every $\alpha < \kappa$.

Theorem 3.1 *Let κ be regular and $\mathcal{S}$ be a non-reflecting stationary subset of κ such that $\mathcal{S} \subseteq \{\, \delta < \kappa \,:\, \mathrm{cof}(\delta) = \omega \,\}$. Then there exists a Boolean algebra A with a filtration $(A_\alpha)_{\alpha<\kappa}$ such that* i) *$A_\alpha \cong \mathrm{Fr}(\mid A_\alpha \mid)$;* ii) *$A_\alpha | A_\beta$ for every α, $\beta < \kappa$ with $\alpha \leq \beta$ and $\alpha \notin \mathcal{S}$;* iii) *$\{\, \alpha < \kappa \,:\, A_\alpha \leq_{\neg\mathrm{rc}} A \,\} = \mathcal{S}$.*

For the proof of Theorem 3.1 we need the following lemma:

Lemma 3.2 *Let $(B_\alpha)_{\alpha<\xi}$ be a continuously increasing sequence of Boolean algebras such that $B_\alpha <_{\rm proj} B_{\alpha+1}$ holds for all $\alpha < \xi$. Let $B = \bigcup_{\alpha<\xi} B_\alpha$ and let B' be such that $B \leq B'$ and $B' = B(x)$ for some $x \in B'$. If $B_\alpha \leq_{\rm rc} B'$ for every $\alpha < \xi$ holds then we have that $B_\alpha \leq_{\rm proj} B'$ for every $\alpha < \xi$.*

Proof By Theorem 2.1 (b) there exists a continuously increasing sequence $(C_\eta)_{\eta<\delta}$ of subalgebras of B such that $C_0 = B_0$; $C_\eta \leq_{\rm rc} C_{\eta+1}$ for every $\eta < \delta$; $C_{\eta+1}$ is countably generated over C_η for every $\eta < \delta$ and for every $\alpha < \kappa$ there exists an $\eta_\alpha < \delta$ such that $B_\alpha = C_{\eta_\alpha}$. We show: $B_{\eta_\alpha} <_{\rm proj} B(x)$.

For simplicity let us assume that $\mathrm{cof}(\eta_\alpha) < \mathrm{cof}(\delta)$ — in the other case the proof can be done by a little more complicated indexing as below.

Let δ' be such that $1+\eta_\alpha+\delta' = \delta$ holds. Let D_η, $\eta < \delta'$ be defined by: $D_0 = B_{\eta_\alpha}$ and $D_\eta = C_{1+\eta_\alpha+\eta}(x)$ for $0 < \eta < \delta'$. Then $(D_\eta)_{\eta<\delta'}$ is continuously increasing, $\bigcap_{\eta<\delta'} D_\eta = B(x)$ and $D_{\eta+1}$ is countably generated over D_η for every $\eta < \delta'$. So by Theorem 2.1 (b) we are done by showing $D_\eta \leq_{\rm rc} D_{\eta+1}$ for all $\eta < \delta'$.

For $\eta = 0$, this is clear since $D_0 = B_{\eta_\alpha} \leq_{\rm rc} B(x)$. For $\eta > 0$, let $\eta' > 1 + \eta_\alpha + \eta$ be such that $C_{\eta'} = B_{\alpha'}$ for some $\alpha' < \xi$. Then we have that

$$C_{1+\eta_\alpha+\eta} \leq_{\rm rc} C_{\eta'} = B_{\alpha'} \leq_{\rm rc} B(x).$$

Hence $C_{1+\eta_\alpha+\eta} \leq_{\rm rc} B(x)$. Since $C_{1+\eta_\alpha+\eta}(x) \leq_{\rm rc} B(x)$, it follows that $C_{1+\eta_\alpha+\eta} \leq_{\rm rc} C_{1+\eta_\alpha+1}(x) = D_{\eta+1}$. By Lemma 2.3 in [17] it follows that $D_\eta = C_{1+\eta_\alpha}(x) \leq_{\rm rc} D_{\eta+1}$.
∎ (Lemma 3.2)

Proof of Theorem 3.1 For $\alpha < \kappa$ we show that A_α can be defined inductively so that $A_\alpha \cong \mathrm{Fr}(\mid \alpha + \omega \mid)$ and

$$(*)_\alpha \qquad \forall\beta < \alpha\, [(\beta \notin \mathcal{S} \;\rightarrow\; A_\beta | A_\alpha) \wedge (\beta \in \mathcal{S} \;\rightarrow\; A_\beta \leq_{\neg\rm rc} A_\alpha)]$$

hold.

Let $A_0 = \mathrm{Fr}\,\omega$. Suppose that A_β, $\beta < \alpha$ for $\beta < \kappa$ have been constructed.

Case I. α is a limit: Let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Since $\alpha \cap \mathcal{S}$ is not stationary, there exists a closed unbounded $X \subseteq \alpha$ such that $X\mathcal{S} = \emptyset$. For every β, $\beta' \in X$ with $\beta < \beta'$ we have $A_{\beta'}|A_\beta$ by $(*)_\beta$ for all $\beta < \alpha$. As $A_\alpha = \bigcup_{\beta\in X} A_\beta$ it follows that $A_\alpha \cong \mathrm{Fr}\,\alpha$.

Case II. $\alpha = \gamma + 1$ for a $\gamma \notin \mathcal{S}$: Let $A_\alpha = A_\gamma \oplus \mathrm{Fr}\,\gamma$.

In both cases I and II it is easy to see that $(*)_\alpha$ holds.

Case III. $\alpha = \gamma+1$ for a $\gamma \in \mathcal{S}$: In this case we have that $\mathrm{cof}(\gamma) = \omega$. Let $(\beta_n)_{n\in\omega}$ be an increasing sequence of ordinals such that $\gamma = \bigcup_{n\in\omega} \beta_n$ and $\beta_n \notin \mathcal{S}$ for all $n \in \omega$. Then $A_\gamma = \bigcup_{n\omega} A_{\beta_n}$ By $(*)_\beta$ for all $\beta \leq \gamma$ we have that $A_{\beta_{n+1}}|A_{\beta_n}$ and $A_{\beta_n} \cong \mathrm{Fr}\,\beta$.

Let $(a_n)_{n<\omega}$ be a strictly increasing sequence of elements of B_γ such that $a_n \in A_{\beta_n}$ and $\mathrm{pr}^{A_{\beta_{n+1}}}_{A_{\beta_n}}(a_{n+1}) = a_n$ holds for all $n < \omega$. Let $I = \{ b \in A_\gamma : b \leq a_n$ for some $n < \omega \}$. Let x be (an element of a large enough Boolean algebra containing A_γ) such that $\{ b \in A_\gamma : b \leq x \} = I$ and $\{ b \in A_\gamma : b \leq -x \} = \{ 0 \}$. Then $A_{\beta_n} \leq_{\mathrm{rc}} A_\gamma(x)$ for all $n < \omega$ and $A_\gamma \leq_{\neg\mathrm{rc}} A_\gamma(x)$ hold. By Lemma 3.2 it follows that $A_{\beta_n} \leq_{\mathrm{proj}} A_\gamma(x)$ holds for every $n < \omega$.

Let $A_\alpha = A_\gamma(x) \oplus \mathrm{Fr}(|\,\gamma\,|)$. By definition of a projective extension it follows that $A_\alpha | A_{\beta_n}$ for every $n < \omega$. In particular A_α is free. For $\beta < \alpha$ such that $\beta \notin \mathcal{S}$ let $n < \omega$ be so that $\beta \leq \beta_n$ holds. By $(*)_{\beta_n}$ we have $A_{\beta_n} | A_\beta$. Hence $A_\alpha | A_\beta$ holds. If $\beta < \alpha$ and $\beta \in \mathcal{S}$ then we have either $\beta < \gamma$ or $\beta = \gamma$. In the first case $A_\beta \leq_{\neg\mathrm{rc}} A_\alpha$ holds by $(*)_\gamma$. In the second case the same follows from $A_\gamma \leq_{\neg\mathrm{rc}} A_\gamma(x)$.

▮ (Theorem 3.1)

Remarks. By i), the Boolean algebra A in Theorem 3.1 satisfies $C^*(\kappa)$. By i), ii) and Theorem 2.10, A is $L_{\infty\kappa}$-free. By a theorem in [23], A is productively ccc if $\mathcal{S}$ is costationary in κ. By iii) and Corollary 2.9, A in Theorem 3.1 is not openly generated.

Now we can give the solutions of the problems in section 1.

Corollary 3.3 *There exists a ccc non openly generated A such that the family of relatively complete subalgebras of A is absorbing.*

Proof Let $\mathcal{S} \subseteq \{ \delta < \omega_1 : \mathrm{cof}(\delta) = \omega \}$ be stationary and costationary. $\mathcal{S}$ is then non-reflecting stationary subset of ω_1. Let A be as in Theorem 3.1 for this $\mathcal{S}$. Then A is absorbing. ▮ (Corollary 3.3)

In [11] it is shown that for any κ there exists an $L_{\infty\aleph_1}$-free Boolean algebra not satisfying the κ-cc. Such Boolean algebra is absorbing but not openly generated.

Corollary 3.4 ($V = L$) *For every κ there exists a non openly generated A satisfying $C^*(\kappa)$.*

Proof Under $V = L$ there exists a non-reflecting stationary $\mathcal{S} \subseteq \{ \delta < \lambda : \mathrm{cof}(\delta) = \omega \}$ for every non-weakly compact regular λ, (see e.g. [5]). For any κ let $\lambda \geq \kappa$ be regular and non-weakly compact. Then A as in Theorem 3.1 for a non-reflecting stationary $\mathcal{S} \subseteq \{ \delta < \lambda : \mathrm{cof}(\delta) = \omega \}$ is non openly generated but satisfies $C^*(\kappa)$. ▮ (Corollary 3.4)

A.V. Ivanov [14] obtained the negative answer to the problem (*9*) under $V = L$. The following Corollary shows that the negative answer is already obtained in ZFC alone.

Corollary 3.5 *There exists a non-openly generated Boolean algebra A such that $A \oplus A$ satisfies the BSP.*

Proof Let $\mathcal{S} \subseteq \{\, \delta < \omega_1 \;:\; \mathrm{cof}(\delta) = \omega \,\}$ be stationary and costationary. Then, as remarked above, A is not openly generated and $A \oplus A$ satisfies the ccc. Since $A \oplus A \equiv_{L_{\infty\aleph_1}} \mathrm{Fr}\,\omega_1 \oplus \mathrm{Fr}\,\omega_1 \cong \mathrm{Fr}\,\omega_1$, $A \oplus A$ is $L_{\infty\aleph_1}$-free. Since $Fr\omega_1$ satisfies the BSP and there exists an $L_{\infty\aleph_1}$-sentence φ such that $B \models$ BSP if and only if $B \models \varphi$ holds for every ccc Boolean algebra B, it follows that $A \oplus A$ also satisfies the BSP.

▮ (Corollary 3.5)

Proposition 3.6 *For κ and $\mathcal{S}$ as in Theorem 3.1 if $\mathcal{S}$ is costationary then the Boolean algebra A constructed in Theorem 3.1 is not dyadic.*

Proof As we showed in the proof of Corollary 3.5 we have that $A \models$ BSP. Hence by Theorem 2.7 A would be openly generated when A were dyadic. ▮ (Proposition 3.6)

4 Openly generated Boolean algebras under the axiom MA^+(*σ-closed*)

MA^+(*σ-closed*) is the following axiom:

> *For any σ-closed p.o.-set P, family $\mathcal{D}$ of dense subsets of P such that $|\,\mathcal{D}\,| \leq \aleph_1$ and P-name $\dot{S}$ of stationary subset of ω_1, there exists a $\mathcal{D}$-generic filter G such that*
>
> $$\dot{S}^G = \{\, \alpha < \omega_1 \;:\; p \Vdash_P \text{“}\alpha \in \dot{S}\,\text{” for some } p \in G \,\}$$
>
> *is stationary in ω_1.*

Shelah [22] proved that MA^+(*σ-closed*) follows from MM (Martin's Maximum). MA^+(*σ-closed*) itself is already a rather strong axiom: e.g. MA^+(*σ-closed*) implies $\neg\Box_\kappa$ for every $\kappa > \omega$. In this section we shall prove that under MA^+(*σ-closed*) the problem (*7*) obtains the positive answer (Theorem 4.3). Some other results on openly generated Boolean algebras under MA^+(*σ-closed*) are to be found in [10].

Before beginning with the theorem let us first remark that, for dyadic Boolean algebras, the answer to the problem (*7*) is positive:

Theorem 4.1 *A subalgebra B of an openly generated Boolean algebra is openly generated if and only if $B \models C^*(\aleph_2)$ holds.*

Proof If B is openly generated we have $B \models C^*(\aleph_2)$ by Corollary 2.9. Now assume that B is contained in an openly generated Boolean algebra and $B \models C^*(\aleph_2)$. To prove that B is openly generated it is enough by Theorem 2.7 to show that B satisfies the BSP. This follows from the next Lemma. ▮ (Theorem 4.1)

Lemma 4.2 ([10]) *If $B \models C^*(\aleph_2)$ then B satisfies the BSP.*

Proof Let $(B_i)_{i\in I}$ be a $C^*(\aleph_2)$-filtration of B. Let

$$I^* = \{\, (i, A) \,:\, i \in I,\ A \leq_{\rm rc} B_i,\ |\, A \,| \leq \aleph_1 \,\}$$

be the p.o.-set with the partial ordering

$$(i, A) \leq (i', A') \quad \Leftrightarrow \quad i \leq i' \text{ and } A \leq A'.$$

For $i^* \in I^*$ with $i^* = (i, A)$, let $A_{i^*} = A$. We show that every increasing chain in I^* of length ω_1 has its supremum: Let $(i^*_\alpha)_{\alpha<\omega_1}$ be an increasing sequence in I^*, say $i^*_\alpha = (i_\alpha, A_\alpha)$ for $\alpha < \omega_1$. Let i be the supremum of $(i_\alpha)_{\alpha<\omega_1}$ and $A = \bigcup_{\alpha<\omega_1} A_\alpha$. Then $A \leq B_i$. We show that $A \leq_{\rm rc} B_i$ holds. For contradiction let us assume that there is some $b \in B_i$ without its projection into A. Without loss of generality we may assume that $b \in B_{i_0}$. Then $(\mathrm{pr}^{B_{i\alpha}}_{A_\alpha}(b))_{\alpha<\omega_1}$ is non-eventually constant increasing sequence of elements of B_i. This is a contradiction since B_i is projective and hence satisfies the ccc.

Now suppose that there would be a generated regular ideal J over B which is not countably generated. Let $(i^*_\alpha)_{\alpha<\omega_1}$ and $(C_\alpha)_{\alpha<\omega_1}$ be such that

$(i^*_\alpha)_{\alpha<\omega_1}$ is an increasing chain in I^* and $A_{i^*_\alpha}$ is countable for every $\alpha < \omega_1$;

$(C_\alpha)_{\alpha<\omega_1}$ is a continuously increasing sequence of countable subalgebras of B such that $(C_\alpha, J \cap C_\alpha) \prec (B, J)$ for every $\alpha < \omega_1$;

$C_\alpha \leq A_{i^*_\alpha} \leq C_{\alpha+1}$ for every $\alpha < \omega_1$;

$J \cap C_\alpha$ does not generate $J \cap C_{\alpha+1}$ for every $\alpha < \omega_1$.

Let $i^* = (i, A)$ be the supremum of $(i^*_\alpha)_{\alpha<\omega_1}$. Since we have $(A, J \cap A) \prec (B, J)$, $J \cap A$ is a regular ideal over A. Since $A \leq_{\rm rc} B_i$ and B_i is projective, A is openly generated by Corollary 2.4. Hence A has the BSP by Corollary 2.5. But this is a contradiction since $J \cap A$ is not countably generated by the last condition of the construction. ▮ (Lemma 4.2)

Now let us turn to the theorem we mentioned at the beginning of the section:

Theorem 4.3 (MA$^+$(σ-*closed*)) *A Boolean algebra B is openly generated if and only if B satisfies the condition $C^*(\aleph_2)$.*

This theorem was originally proved in [10] using some other characterizations of openly generated Boolean algebras available under MA$^+$(σ-*closed*). Here we shall give a direct proof of the theorem.

Proof If B is openly generated then $B \models C^*(\aleph_2)$ holds by Corollary 2.9.

Suppose that $B \models C^*(\aleph_2)$ holds and let $(B_i)_{i\in I}$ be a $C^*(\aleph_2)$-filtration of B. Toward a contradiction assume that B is not openly generated. Then there exists a stationary $\mathcal{S} \subseteq [B]^{\aleph_0}$ such that $A \leq_{\neg\mathrm{rc}} B$ for every $A \in \mathcal{S}$ holds. For each $i \in I$ let $\mathcal{C}_i$ be a club subset of $[B_i]^{\aleph_0}$ such that $2 \in \mathcal{C}_i$ and $A \leq_{\mathrm{rc}} B_i$ hold for every $A \in \mathcal{C}_i$. Let

$$
\begin{aligned}
P = \{\, (A,J,f) \,:\, \; & J \subseteq I,\ |\, J\,| \leq \aleph_0 \text{ and } J \text{ is well-ordered},\\
& A \leq B,\ |\, A\,| \leq \aleph_0,\ A \cap B_i \in \mathcal{C}_i \text{ for every } i \in J,\\
& f : \alpha \to A \text{ for some } \alpha < \omega_1 \text{ and } f \text{ is 1-1 onto} \quad \}.
\end{aligned}
$$

For (A_0, J_0, f_0), $(A_1, J_1, F_1) \in P$ let

$$
\begin{aligned}
&(A_0, J_0, f_0) \leq (A_1, J_1, F_1) \;\Leftrightarrow\\
&\qquad A_1 \leq A_0,\ J_0 \text{ is endextension of } J_1 \text{ and } f_1 \subseteq f_0.
\end{aligned}
$$

Claim 4.3.1 *P is σ-closed.*

$\vdash$ If (A_n, J_n, f_n), $n \in \omega$ is a decreasing chain in P then $(\bigcup_{n\in\omega} A_n, \bigcup_{n\in\omega} J_n, \bigcup_{n\in\omega} f_n)$ is the infimum of the chain. $\dashv$ (Claim 4.3.1)

For $\beta < \omega_1$ let

$$
\begin{aligned}
D_\beta = \{\, (A,J,f) \in P \,:\, \; & otp(J) \geq \beta,\ \beta \in \mathrm{dom}(f),\ f[\beta] \subseteq B_j \text{ for some } j \in J,\\
& \text{for every } \gamma < \beta \text{ if } f[\gamma] \in \mathcal{S} \text{ there exists}\\
& a \in A \text{ such that } a \text{ has no projection into } f[\gamma] \quad \}.
\end{aligned}
$$

Let $\mathcal{D} = \{\, D_\beta \,:\, \beta < \omega_1 \,\}$.

Claim 4.3.2 *For every $\beta < \omega_1$, D_β is dense in P.*

This follows immediately from the following:

Claim 4.3.3 *For any $C \leq B$, $|\, C\,| \leq \aleph_0$ and any countable $J \subseteq I$ there exists $A \leq B$ such that $C \leq A$ and $A \cap B_i \in \mathcal{C}_i$ holds for all $i \in J$*

$\vdash$ Let $\{ i_n \}_{n\in\omega}$ be an enumeration of J such that each $i \in J$ appears infinitely often there. Let $(A_n)_{n\in\omega}$ be an increasing sequence of countable subalgebras of B such that $C \leq A_0$ and $A_n \cap G_{i_n}$ holds for every $n \in \omega$. Then $A = \bigcup_{n\in\omega} A_n$ is as required. $\dashv$ (Claim 4.3.3)

Let $\dot{G}$ be a P-name of the generic set over P and let $\dot{S}$ be a P-name such that

$$\Vdash_P \text{“}\, \dot{S} = \bigcup\{ f^{-1}[C] \,:\, C \in \mathcal{S},\ (A, J, f) \in \dot{G} \text{ for some } A \text{ and } J \,\}\text{”}.$$

Since $\Vdash_P$ “$\mathcal{S}$ is stationary” holds by Claim 4.3.1 it follows that $\Vdash_P$ “$\dot{S}$ is stationary subset of ω_1”. Now let G be a $\mathcal{D}$ generic filter over P such that

$$\dot{S}^G = \{ \alpha < \omega_1 \,:\, p \Vdash_P \text{“}\alpha \in \dot{S}\text{” for some } p \in G \}$$

is stationary. Let

$$\begin{aligned} A^* &= \bigcup\{ A \,:\, (A, J, f) \in G \text{ for some } J,\ f \},\\ J^* &= \bigcup\{ J \,:\, (A, J, f) \in G \text{ for some } A,\ f \},\\ f^* &= \bigcup\{ f \,:\, (A, J, f) \in G \text{ for some } A,\ J \}. \end{aligned}$$

By the $\mathcal{D}$-genericity, J^* has order type ω_1. Let j^* be the supremum of J^*. Again by $\mathcal{D}$-genericity we have $A^* \leq B_{j^*}$.

Claim 4.3.4 $A^* \leq_{\rm rc} B_{j^*}$

$\vdash$ For $j \in J^*$, we can represent $A^* \cap B_j$ as the union of a chain in $\mathcal{C}_j$. Hence we have $A^* \cap B_j \in \mathcal{C}_j$ for every $j \in J^*$. Suppose that there were a $b \in B_{j^*}$ without any projection into A^*. Then $({\rm pr}^{A^*}_{A^*\cap B_j}(b))_{j\in J}$ would be an uncountable non eventually constant sequence in B_{j^*}. But this is a contradiction as B_{j^*} satisfies the ccc. $\dashv$ (Claim 4.3.4)

It follows that A^* is openly generated. On the other hand $\{ f^*[\alpha] \,:\, \alpha \in \dot{S}^G \}$ is stationary in $[A^*]^{\aleph_0}$ and $f^*[\alpha] \leq_{\neg{\rm rc}} B$ holds for every $\alpha \in \dot{S}^G$. Hence $f^*[\alpha] \leq_{\neg{\rm rc}} A^*$ holds by the $\mathcal{D}$-genericity of G. This is a contradiction. ∎ (Theorem 4.3)

5 Projective Boolean algebras under the existence of a supercompact cardinal

Concerning problem (*8*) we still do not know if theorems analogous to Theorems 4.1, 4.3 hold for projective Boolean algebras. In particular the following problems are still open:

Problem 5.1 *For a dyadic Boolean algebra B does $B \models C^*(\aleph_3)$ implies that B is projective?*

Problem 5.2 *Is it consistent that, for every Boolean algebra B, $B \models C^*(\aleph_3)$ implies that B is projective?*

Note that $C^*(\aleph_2)$ in place of $C^*(\aleph_3)$ is not enough for Problem 5.1 since there exist (in ZFC) openly generated Boolean algebras which are not projective. By Ščepin [20] there exists a relatively complete subalgebra of $\mathrm{Fr}\,\omega_2$ which is not projective. Hence $C^*(\aleph_2)$ in place of $C^*(\aleph_3)$ is neither enough for Problem 5.2.

Up to now the best results known to the author in this connection are the following two theorems. Let us recall that a cardinal κ is supercompact if, for every cardinal λ, there exists an elementary embedding $j : V \rightarrow M$ for some inner model M with critical point κ such that $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$ hold.

Theorem 5.3 *Let κ be supercompact. For every Boolean algebra B, if $B \models C^*(\kappa)$ then B is projective.*

Proof Suppose that there exists a non-projective Boolean algebra B such that $B \models C^*(\kappa)$ holds. Let $(B_i)_{i\in I}$ be a $C^*(\kappa)$-filtration of B and let $\lambda \geq |\, B \,|, |\, I \,|$. Let $j : V \rightarrow M$ be an elementary embedding with critical point κ such that $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$ hold. Let $j((B_i)_{i\in I}) = (B^*_{i^*})_{i^*\in I^*}$. By $|\, j[B] \,|, |\, j[I] \,| \leq \lambda$ we have that $j[B], j[I] \in M$. Since $j[B]$ ($\cong B$) is not projective, $M \models$"$j[B]$ is not projective". By $|\, B_i \,| < \kappa$ we have $B^*_{j(i)} = j[B_i]$ for all $i \in I$. Hence we have that $M \models$"$j[B] = \bigcup_{i^*\in j[I]} B^*_{i^*}$". Since $j(\kappa) > \lambda \leq |\, j[I] \,|^M$ and by the elementarity of j, it follows that $M \models$"$j[B] = B^*_{i^*}$ for some $i^* \in I^*$". Hence again by the elementarity of j, $M \models$"$j[B]$ is projective". This is a contradiction. ∎ (Theorem 5.3)

The idea of the proof of the next theorem is similar to Theorem 3.1 in [4].

Theorem 5.4 *If the existence of a supercompact cardinal is consistent with* ZFC *then the following assertion is also consistent with* ZFC*:*

For every Boolean algebra B, if $B \models C^*(2^{\aleph_0})$ then B is projective.

Proof Let κ be a supercompact cardinal. Let $P = \mathrm{Fn}(\kappa, 2)$ and let G be a V-generic filter over P where V is our ground model.

We shall show that $V[G]$ satisfies the assertion in the theorem: otherwise, in $V[G]$, there would be a non-projective B satisfying $C^*(\kappa)$ in $V[G]$. Let $(B_i)_{i\in I}$ be a $C^*(\kappa)$-filtration of B in $V[G]$. Without loss of generality we may assume that the underlying sets of B and I are some cardinals λ and μ such that $\lambda \geq \mu$. Further we may assume that 0 and 1 of the Boolean algebra B are the ordinals 0 and

1 respectively, and these facts for the corresponding P-names are already forced by 1_P. Let $\dot{a}$, $\dot{m}$, $\dot{c}$ be "nice" P-names (in the sense of [19], p.208) of addition, multiplication and complement operation in B respectively and $\dot{r}$ be a P-name of the partial ordering of I. By the ccc of P we have: $|\,\dot{a}\,| = |\,\dot{m}\,| = |\,\dot{c}\,| = \lambda$ and $|\,\dot{r}\,| \leq \lambda$. For $\alpha \in \mu$ let $\dot{B}_\alpha$ be a nice P-name of (the underlying set of) B_α. Again by the ccc of P, $\dot{B}_\alpha$ has cardinality $< \kappa$. Now let $j : V \to M$ be an elementary embedding with critical point κ such that $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$ hold. Let $\dot{a}^* = j(\dot{a})$, $\dot{m}^* = j(\dot{m})$, $\dot{c}^* = j(\dot{c})$, $\dot{r}^* = j(\dot{r})$, $\dot{S}^* = j((\dot{B}_\alpha)_{\alpha<\mu})$ and $P^* = j(P)$. Then we have that $M \models$"$P^* = \mathrm{Fn}(j(\kappa), 2)$" and $M \models \;\Vdash_P$ "$\dot{S}^*$ with the index set $\mu^* = (j(\mu), \dot{r}^*)$ is $C^*(j(\kappa))$-filtration of $(j(\lambda), \dot{a}^*, \dot{m}^*, \dot{c}^*, 0, 1)$".

Let $\dot{a}^\dagger = j[\dot{a}]$, $\dot{m}^\dagger = j[\dot{m}]$, $\dot{c}^\dagger = j[\dot{c}]$, $\dot{r}^\dagger = j[\dot{r}]$ and $P^\dagger = j[P]$. Since $|\,\dot{a}\,|, \ldots,$ $|\,P\,| \leq \lambda$ we have $\dot{a}^\dagger, \ldots,$ $P^\dagger \in M$. $P^\dagger = \mathrm{Fn}(j[\kappa], 2)$ and $\dot{a}^\dagger, \ldots,$ $\dot{r}^\dagger$ are $P^\dagger$-names. Since $\Vdash_{P^\dagger}$ "$(j[\lambda], \dot{a}^\dagger, \dot{m}^\dagger, \dot{c}^\dagger, 0, 1)$ is a non-projective Boolean algebra" we have that

$$M \models \;\Vdash_{P^\dagger} \text{"}(j[\lambda], \dot{a}^\dagger, \dot{m}^\dagger, \dot{c}^\dagger, 0, 1) \text{ is a non-projective Boolean algebra".}$$

Hence it follows by Corollary 2.3 in [9] that

$$M \models \;\Vdash_{P^*} \text{"}(j[\lambda], \dot{a}^\dagger, \dot{m}^\dagger, \dot{c}^\dagger, 0, 1) \text{ is a non-projective Boolean algebra".}$$

On the other hand since

$$M \models \;\Vdash_{P^*} \text{"}(j[\lambda], \dot{a}^\dagger, \dot{m}^\dagger, \dot{c}^\dagger, 0, 1) = \bigcup_{\alpha<\mu} j(\dot{B}_\alpha)\text{"}$$

and $M \models$ "$j[\mu]$ is directed and $|\,j[\mu]\,| < j(\kappa)$", it follows that

$$M \models \;\Vdash_{P^*} \text{"}(j[\lambda], \dot{a}^\dagger, \dot{m}^\dagger, \dot{c}^\dagger, 0, 1) \text{ is an element of } \dot{S}^*\text{".}$$

Hence

$$M \models \;\Vdash_{P^*} \text{"}(j[\lambda], \dot{a}^\dagger, \dot{m}^\dagger, \dot{c}^\dagger, 0, 1) \text{ is a projective Boolean algebra".}$$

This is a contradiction. ❚ (Theorem 5.4)

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