Generic Absoluteness Revisited

Sakaé Fuchino[*,‡] Takehiko Gappo[†] Francesco Parente[*,§]
渕野 昌 合浦 岳彦 パレンテ・フランチェスコ

Abstract

The present paper is concerned with the relation between recurrence axioms and Laver-generic large cardinal axioms in light of principles of generic absoluteness and the Ground Axiom.

M. Viale proved that Martin's Maximum^{++} together with the assumption that there are class many Woodin cardinals implies $\mathcal{H}(\aleph_2)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}(\aleph_2)^{\mathsf{V}[\mathbb{G}]}$ for a generic $\mathbb{G}$ on any stationary preserving $\mathbb{P}$ which also preserves Bounded Martin's Maximum. We show that a similar but more general conclusion follows from each of $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_2}$-$\mathsf{RcA}^+$ (which is a fragment of a reformulation of the Maximality Principle for $\mathcal{P}$ and $\mathcal{H}(\kappa)$), and the existence of the tightly $\mathcal{P}$-Laver-generically huge cardinal.[⋆]

While under "$\mathcal{P}$ = all stationary preserving posets", our results are not very much more than Viale's Theorem, for other classes of posets, "$\mathcal{P}$ = all proper posets" or "$\mathcal{P}$ = all ccc posets", for example, our theorems are not at all covered by his theorem.

The assumptions (and hence also the conclusion) of Viale's Theorem are compatible with the Ground Axiom. In contrast, we show that the assumptions of our theorems (for most of the common settings of $\mathcal{P}$ and with a modification of the large cardinal property involved) imply the negation of the Ground Axiom. This fact is used to show that fragments of Recurrence Axiom $(\mathcal{P}, \mathcal{H}(\kappa))_\Gamma$-$\mathsf{RcA}^+$ can be different from the corresponding fragments of Maximality Principle $\mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa))_\Gamma$ for $\Gamma = \Pi_2$.

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[*] Graduate School of System Informatics, Kobe University
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan
[‡] `fuchino@diamond.kobe-u.ac.jp` [§] `francesco.parente@people.kobe-u.ac.jp`

[†] Technische Universität Wien Institut für Diskrete Mathematik und Geometrie
Wiedner Hauptstraße 8-10/104 1040 Wien, Austria
`takehiko.gappo@tuwien.ac.at`

[⋆] We use here the definite article since it is known that a tightly $\mathcal{P}$-Laver-generic large cardinal, Contents

1 Introduction and preliminaries

In the following, we tried very hard to make the present paper as self-contained as possible. For notions and notation which remain unexplained, the reader may refer to [28], [29], or [30]. Set-theoretic forcing is treated here just as in [30] with the exception that $\mathbb{P}$-names for a poset $\mathbb{P}$ are reresented with an under-tilde, e.g. as $\underset{\sim}{\mathbb{Q}}$ or $\underset{\sim}{S}$. We adopt the (fake but consistently interpretable) narration that generic filters "exist" though otherwise we remain in the ZFC narrative so that all classes mentioned here are (meta-mathematically) definable classes. theintro

The main theorem of Viale [39] states:

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if it exists, is the unique cardinal $\kappa_{\mathfrak{refl}}$ $(= \max(\{\aleph_2, 2^{\aleph_0}\}))$ for (almost?) all reasonable non-trivial instances of $\mathcal{P}$ and notions of large cardinal, see [15].

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All additional details not contained in the submitted version of the paper are either typeset in dark electric blue (the color in which this paragraph is typeset) or put in a separate appendices.

The most up-to-date pdf-file of this extended version is downloadable as:

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Theorem 1.1 (M. Viale, Theorem 1.4 in [39]) *Assume that* MM^{++} *holds, and there are class many Woodin cardinals. Then, for any stationary preserving poset* $\mathbb{P}$ *with* $\Vdash_{\mathbb{P}}$ "BMM", *we have* p-theintro-a

$$\mathcal{H}(\aleph_2)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}(\aleph_2)^{\mathsf{V}[\mathbb{G}]} \quad \textit{for } (\mathsf{V},\mathbb{P})\textit{-generic } \mathbb{G}. \qquad \square$$

Here MM^{++} is the following strengthening of the Martin's Axiom (MM):

(MM^{++}): For any stationary preserving $\mathbb{P}$, any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \aleph_2$, and any family $\mathcal{S}$ of $\mathbb{P}$-names of stationary subsets of ω_1 with $|\mathcal{S}| < \aleph_2$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ on $\mathbb{P}$ such that $\underset{\sim}{S}[\mathbb{G}]$ is a stationary subset of ω_1 for all $\underset{\sim}{S} \in \mathcal{S}$.

BMM is the Bounded Martin's Maximum, a weakening of MM which is an instance of Bounded Forcing Axioms: for a class $\mathcal{P}$ of posets closed under forcing equivalence and a cardinal κ, the *Bounded Forcing Axiom* for $\mathcal{P}$ and $<\kappa$ is the axiom stating:

($\mathsf{BFA}_{<\kappa}(\mathcal{P})$): For any complete Boolean[1)] $\mathbb{P} \in \mathcal{P}$, and a family $\mathcal{D}$ of maximal antichains in $\mathbb{P}$ such that $|\mathcal{D}| < \kappa$ and $|I| < \kappa$ for all $I \in \mathcal{D}$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ on $\mathbb{P}$.

The *Bounded Martin's Maximum* (BMM) is $\mathsf{BFA}_{<\aleph_2}$(stationary pres. posets). Bounded Forcing Axioms were introduced by Goldstern and Shelah [22] answering a problem asked by the first author of the present paper in [11].

In Theorem 1.1, the condition "$\Vdash_{\mathbb{P}}$ "BMM"" cannot be simply dropped. For example, the formula saying that there is a set which is the power set of ω is Σ_2. Since $\neg\mathsf{CH}$ holds in V under MM, if $\mathbb{P}$ forces CH then $\mathcal{H}(\aleph_2)^{\mathsf{V}} \not\prec_{\Sigma_2} \mathcal{H}(\aleph_2)^{\mathsf{V}[\mathbb{G}]}$ for $(\mathsf{V},\mathbb{P})$-generic $\mathbb{G}$.

We show that a conclusion similar to and more general than that of Theorem 1.1 follows from each of $(\mathcal{P},\mathcal{H}(\kappa))_{\Sigma_2}$-$\mathsf{RcA}^+$ which is a fragment of Recurrence Axiom (a reformulation of Maximality Principle introduced in [20]) for $\mathcal{P}$ and $\mathcal{H}(\kappa)$, see Section 2 below, and the existence of the tightly $\mathcal{P}$-Laver-gen. large cardinal (Theorem 4.1, Theorem 5.7).

The notion of Laver-generic large cardinal is introduced in Fuchino, Ottenbreit Maschio Rodrigues, and Sakai [15]. The definition we give here is the slightly modified version in later papers such as in Fuchino [13]:

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1) We say that a poset $\mathbb{P}$ is *complete Boolean* if $\mathbb{P} = \mathbb{B}\setminus\{\mathbb{0}_{\mathbb{B}}\}$ for a complete Boolean algebra. Note that the definition of $\mathsf{BFA}_{<\kappa}(\mathcal{P})$ makes sense only when $\mathbb{P}$ is complete Boolean (since otherwise it can be the case that $\mathbb{P}$ does not have any maximal antichains of size $<\kappa$).

For an iterable class $\mathcal{P}$ of posets (i.e. class $\mathcal{P}$ of posets satisfying (1.2) and (1.3) below) a cardinal κ is said to be (*tightly, resp.*) *$\mathcal{P}$-Laver-gen. supercompact* if, for any $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", such that for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are $j, M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$,[2)] $j(\kappa) > \lambda$, $\mathbb{P}, \mathbb{H}, j''\lambda \in M$ (and $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is of size $\leq j(\kappa)$, resp.).

This definition can be adopted to many other large cardinal notions other than supercompactness. The reader may refer to [13] for definitions of other variants of Laver-generic large cardinal. Defined as above, it is not obvious at first glance that the Laver-genericity is formalizable in the language $\mathcal{L}_{\in}$ of ZFC. That it is actually the case, is shown in Fuchino and Sakai [18].

A tightly $\mathcal{P}$-Laver-generic large cardinal, if it exists, is unique and decided to be $\kappa_{\mathfrak{refl}} := \max(\{\aleph_2, 2^{\aleph_0}\})$ for all known reasonable non-trivial instances of $\mathcal{P}$ with a strong enough large cardinal notion (see [15], or [13], [12]). This is the reason why we often simply talk about *the* tightly $\mathcal{P}$-Laver-generic large cardinal.

While under "$\mathcal{P}$ = all semi-proper posets", our results are not much more than slight variants of Viale's (but without relying on the stationary tower forcing technique), for other classes of posets, for example "$\mathcal{P}$ = all proper posets" or "$\mathcal{P}$ = all ccc posets", they are not at all covered by Viale's result in [39] nor by its proof.

In the following we shall always assume that the classes $\mathcal{P}$ of posets we consider are *normal*, that is,

(1.1) $\quad$ $\mathcal{P}$ is closed with respect to forcing equivalence, and $\{\mathbb{1}\} \in \mathcal{P}$. $\quad$ x-theintro-0-0

In particular, we assume that for any $\mathbb{P}_0 \in \mathcal{P}$ there is a complete Boolean[2)] $\mathbb{P} \in \mathcal{P}$ which is forcing equivalent to $\mathbb{P}_0$. In some cases like the case "$\mathbb{P}$ = all σ-closed posets" where the original class of posets is not normal we just replace $\mathcal{P}$ with its closure with respect to forcing equivalence without mention.

In some cases (like in the definition of Laver-genericity above) it is natural to consider (normal) classes of posets which are closed with respect to two-step iteration. A class $\mathcal{P}$ of posets is called *iterable* if

(1.2) $\quad$ $\mathcal{P}$ is closed with respect to restriction. That is, for $\mathbb{P} \in \mathcal{P}$ and $\mathbb{p} \in \mathbb{P}$, we always have $\mathbb{P} \restriction \mathbb{p} \in \mathcal{P}$, and $\quad$ x-theintro-0-0-a-0

(1.3) $\quad$ For any $\mathbb{P} \in \mathcal{P}$, and any $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of a poset with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", we have $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathcal{P}$. $\quad$ x-theintro-0-0-a-1

Viale's Absoluteness Theorem 1.1 is a result built upon the following Theorem 1.2. We shall use the following notation for the formulation of the Theorem:

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2) "$j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$" denotes the condition that j is an elementary embedding of V into a transitive M with the critical point κ.

For an ordinal α, let $\alpha^{(+)} := \sup(\{|\beta|^+ : \beta < \alpha\})$. Note that $\alpha^{(+)} = \alpha$ if α is a cardinal. Otherwise, we have $\alpha^{(+)} = |\alpha|^+$.

In Bagaria [4] the following theorem contains the extra assumption that, translated into the context of the following formulation, κ is a successor of a cardinal of uncountable cofinality. However we can eliminate this assumption by slightly modifying the proof in [4].

Theorem 1.2 (Bagaria's Absoluteness Theorem, Theorem 5 in [4]) *For an uncountable cardinal κ and a class $\mathcal{P}$ of posets closed under forcing equivalence, and restriction (in the sense of (1.2)) the following are equivalent:* (a) $\mathsf{BFA}_{<\kappa}(\mathcal{P})$. p-theintro-0

(b) *For any $\mathbb{P} \in \mathcal{P}$, Σ_1-formula φ in $\mathcal{L}_\in$ and $a \in \mathcal{H}(\kappa)$, $\Vdash_{\mathbb{P}}$ "$\varphi(a)$" $\Leftrightarrow \varphi(a)$.*

(c) *For any $\mathbb{P} \in \mathcal{P}$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, we have $\mathcal{H}(\kappa)^{\mathsf{V}} \prec_{\Sigma_1} \mathcal{H}((\kappa^{(+)})^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]}$.*

Proof. Note that (b) $\Leftrightarrow$ (c) is trivial since ZFC proves that

(1.4) $\quad \mathcal{H}(\mu) \prec_{\Sigma_1} \mathsf{V}$ for any uncountable cardinal μ x-theintro-0

(Lévy [32]). ⟦ For $a \in \mathcal{H}(\mu)$, let $\nu := |\,trcl^+(a)\,| < \mu$ (here, $trcl^+(a)$ denotes the variant of transitive closure which satisfies $a \in trcl^+(a)$). Then $\nu < \mu$. If $\mathcal{H}(\mu) \models \varphi(a)$ for a Σ_1-formula φ then it follows $\mathsf{V} \models \varphi(a)$.

Suppose that $\varphi(x) = \exists y\, \psi(x, y)$ where $\psi(x, y)$ is a Σ_0-formula, and assume that $\mathsf{V} \models \varphi(a)$. Let b be such that $\mathsf{V} \models \psi(a, b)$. Let δ be large enough such that a, $b \in V_\delta$. Let $M \prec V_\delta$ be such that $trcl^+(a) \subseteq M$, $b \in M$, and $|\,M\,| = \nu$. Let $m : M \overset{\cong}{\to} M_0$ be the Mostowski-collapse. Note that $m \restriction trcl^+(a) = id_{\,trcl^+(a)}$. Thus $M_0 \models \psi(a, m(b))$, and hence $M_0 \models \phi(a)$. Since $M_0 \subseteq \mathcal{H}(\mu)$, it follows that $\mathcal{H}(\mu) \models \varphi(a)$. ⟧

Note also that if $\kappa = 2^{\aleph_0}$, we also have the equivalence of (a), (b), (c) with

(c') *For any $\mathbb{P} \in \mathcal{P}$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, we have $\mathcal{H}(2^{\aleph_0})^{\mathsf{V}} \prec_{\Sigma_1} \mathcal{H}((2^{\aleph_0})^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]}$.*

By the remark above, it is enough to prove (a) $\Leftrightarrow$ (b).

(a) $\Rightarrow$ (b): Let $\mathbb{P} \in \mathcal{P}$. Without loss of generality, $\mathbb{P}$ is completely Boolean with $\mathbb{P} = \mathbb{B} \setminus \{\mathbb{0}_{\mathbb{B}}\}$. Suppose $a \in \mathcal{H}(\kappa)$ and φ is a Σ_1-formula in $\mathcal{L}_\in$. If $\varphi(a)$ holds in V, then clearly we also have $\Vdash_{\mathbb{P}}$ "$\varphi(a)$".

Suppose now that $\varphi = \exists y\, \psi(x, y)$ for a bounded formula ψ in $\mathcal{L}_\in$, and $\Vdash_{\mathbb{P}}$ "$\varphi(a)$". Without loss of generality, we may assume that $a \subseteq \mu$ for some cardinal $\mu < \kappa$ (this is because a can be reconstructed from $trcl^+(a)$, and $trcl^+(a)$ can be coded by a subset a^* of $|\,trcl^+(a)\,|$. The formula $\varphi(a)$ can be replaced by the formula saying:

$$\exists x\,(\, x \text{ is the set "}a\text{" reconstructed from the transitive set coded by } a^* \text{ and } \varphi(x) \text{ holds}\,).$$

Note that this formula is Σ_1 with the parameter a^* if φ is Σ_1). We may also assume that a is not an ordinal (if necessary, we can replace a with a subset of μ with some redundant complexity to make $a \notin \mathrm{On}$).

Let $\underset{\sim}{b}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\psi(a, \underset{\sim}{b})$". Let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic filter and we work in $\mathsf{V}[\mathbb{G}]$. Letting $b = \underset{\sim}{b}[\mathbb{G}]$, we have $\psi(a, b)$.

Working further in $\mathsf{V}[\mathbb{G}]$, let λ be large enough such that V_λ satisfies large enough fragment of ZFC, a, $b \in V_\lambda$, and $V_\lambda \models \psi(a, b)$. Let $M \prec V_\lambda$ be such that $\mu \subseteq M$, a, $b \in M$, and $|\,M\,| = \mu$. Let $m : M \xrightarrow{\cong} M_0$ be the Mostowski collapse of M and let $\nu = \mathrm{On} \cap M_0$. Note that we have $m \restriction \mu \cup \{a\} = id_{\,\mu \cup \{a\}}$.

Let $\mathfrak{M} := \langle \nu + \mu, E, f \rangle$ be the structure in the language $\mathcal{L} := \{\underset{\sqcup}{E}, \underset{\sqcup}{f}\}$ such that there is an isomorphism

(ℵ1.1) $\quad i : \langle M_0, \in, rank \rangle \xrightarrow{\cong} \langle \nu + \mu, E, f \rangle$ x-theintro-0-0-a
such that $i \restriction \nu = id_{\,\nu}$, $i(a) = \nu$, and $i(m(b)) = \nu + 1$

where $rank$ is the rank function restricted to M_0. Clearly we have $\langle \nu + \mu, E \rangle \models \psi^*(\nu, \nu + 1))$, where ψ^* is the formula obtained from ψ by replacing the symbol $\underset{\sqcup}{\in}$ by $\underset{\sqcup}{E}$.

Let $\underset{\sim}{\mathfrak{M}}$, $\underset{\sim}{E}$, $\underset{\sim}{f} \in \mathsf{V}$ be $\mathbb{P}$-names of $\mathfrak{M}$, E and f respectively. By replacing $\mathbb{P}$ with $\mathbb{P} \restriction \mathbb{p}$ for some $\mathbb{p} \in \mathbb{P}$ if necessary, we may assume that

(ℵ1.2) all the properties of $\langle \nu + \mu, E, f \rangle$ used below are forced (as a statement on $\langle \nu + \mu, \underset{\sim}{E}, \underset{\sim}{f} \rangle$) by $\mathbb{1}_{\mathbb{P}}$. x-theintro-0-0-0

In V, let $\mathcal{D}$ be the family of maximal antichains (each of size $\leq \mu < \kappa$) in $\mathbb{P}$ consisting of the following;

(ℵ1.3) $\quad \{[\![\, \underset{\sim}{f}(\alpha) = \beta \,]\!]_{\mathbb{B}} \,:\, \beta < \nu\} \setminus \{\mathbb{0}_{\mathbb{B}}\}, \qquad$ for all $\alpha \in \nu + \mu$. x-theintro-0-1

(ℵ1.4) $\quad \{[\![\, \underset{\sim}{\mathfrak{M}} \models \theta(a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}}, [\![\, \underset{\sim}{\mathfrak{M}} \models \neg\theta(a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}}\} \setminus \{\mathbb{0}_{\mathbb{B}}\},$ x-theintro-0-2
for all Σ_0-formulas θ in $\mathcal{L}$ and $a_0, ..., a_{k-1} \in \nu + \mu$.

(ℵ1.5) $\quad \{[\![\, \underset{\sim}{\mathfrak{M}} \models \eta \wedge \theta(a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}},$ x-theintro-0-3
$[\![\, \underset{\sim}{\mathfrak{M}} \models \neg\eta(a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}}, [\![\, \underset{\sim}{\mathfrak{M}} \models \neg\theta(a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}}\} \setminus \{\mathbb{0}_{\mathbb{B}}\},$
for all Σ_0-formulas η, θ in $\mathcal{L}$ and $a_0, ..., a_{k-1} \in \nu + \mu$.

(ℵ1.6) $\quad \big(\{[\![\, \neg(\exists x \,\underset{\sqcup}{E}\, c)\, \eta(x, a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}}\} \;\cup$ x-theintro-0-4
$\{[\![\, d \,\underset{\sqcup}{E}\, c \wedge \eta(d, a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}} \,:\, d \in \nu + \mu\}\big) \setminus \{\mathbb{0}_{\mathbb{B}}\},$
for all Σ_1-formulas $\eta = \eta(x, x_0, ..., x_{k-1})$ in $\mathcal{L}$ and c, $a_0, ..., a_{k-1} \in \nu + \mu$.

To see that each of the sets in (ℵ1.3) is a maximal antichain in $\mathbb{P}$ of size $\leq \mu < \kappa$, suppose that $\alpha \in \nu + \mu$ and $\mathbb{p} \in \mathbb{P}$. Then there is $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ which decides $\underset{\sim}{f}(\alpha)$. Since $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\alpha) \in \nu$" by (ℵ1.2), if follows that $\mathbb{q} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\alpha) = \beta$" for some $\beta \in \nu$.

It is clear that elements of each of the sets in ($\aleph$1.3) are pairwise incompatible and these sets are of size $\leq \mu < \kappa$.

It is also proved similarly that sets in ($\aleph$1.6) are maximal antichains in $\mathbb{P}$ of size $\leq \mu < \kappa$.

Now, in V, let $\mathbb{G}$ be $\mathcal{D}$-generic filter. $\mathbb{G}$ exists by $\mathsf{BFA}_{<\kappa}(\mathcal{P})$, and since $\mathcal{D}$ is a family of maximal antichains of size $<\kappa$ with $|\,\mathcal{D}\,| < \kappa$.

Let

$$\underset{\sim}{\mathfrak{M}}[\mathbb{G}] := \langle \nu+\mu, \underset{\sim}{E}[\mathbb{G}], \underset{\sim}{f}[\mathbb{G}]\rangle.$$

where

$$(\aleph 1.7)\quad \underset{\sim}{E}[\mathbb{G}] := \{\langle \xi,\eta\rangle \,:\, \xi,\eta \in \nu+\mu,\ \mathbb{p} \Vdash_{\mathbb{P}} \text{“}\langle \xi,\eta\rangle \in \underset{\sim}{E}\text{”} \text{ for some } \mathbb{p} \in \mathbb{G}\}, \quad \text{and}$$ x-theintro-0-3-0

$$(\aleph 1.8)\quad \underset{\sim}{f}[\mathbb{G}] := \{\langle \xi,\eta\rangle \,:\, \xi,\eta \in \nu+\mu,\ \mathbb{p} \Vdash_{\mathbb{P}} \text{“}\langle \xi,\eta\rangle \in \underset{\sim}{f}\text{”} \text{ for some } \mathbb{p} \in \mathbb{G}\}.$$ x-theintro-0-3-1

Claim 1.2.1 (1) *$\underset{\sim}{\mathfrak{M}}[\mathbb{G}]$ is an $\mathcal{L}$-structure.*

(2) *For each Σ_1-formula $\theta = \theta(x_0, ..., x_{k-1})$ in $\mathcal{L}$ and $a_0, ..., a_{k-1} \in \nu+\mu$,*

$$(\aleph 1.9)\quad [\![\, \underset{\sim}{\mathfrak{M}} \models \theta(a_0, ..., a_{k-1})\,]\!]_{\mathbb{B}} \in \mathbb{G} \ \text{ if and only if } \ \underset{\sim}{\mathfrak{M}}[\mathbb{G}] \models \theta(a_0, ..., a_{k-1}).$$ *x-theintro-0-4-a*

(3) *$\underset{\sim}{E}[\mathbb{G}]$ is extensional and well-founded. $\underset{\sim}{E}[\mathbb{G}]$ on $\nu+\mu$ coincides with the canonical ordering on $\nu+\mu$.*

$\vdash$ (1): Since the maximal antichains in ($\aleph$1.3) are in $\mathcal{D}$, we have $\underset{\sim}{f}[\mathbb{G}] : \nu+\mu \to \nu$.

(2): By induction on the construction of the formula θ using ($\aleph$1.4), ($\aleph$1.5), and ($\aleph$1.6).

Suppose that a_0, $a_1 \in \nu+\mu$. Then

$$\begin{aligned}&[\![\, \underset{\sim}{\mathfrak{M}} \models a_0 \,\underline{E}\, a_1 \,]\!]_{\mathbb{B}} \in \mathbb{G} \;\Leftrightarrow\; [\![\, a_0 \,\underset{\sim}{E}\, a_1 \,]\!]_{\mathbb{B}} \in \mathbb{G} \;\Leftrightarrow\; \exists \mathbb{p} \in \mathbb{G}\ \mathbb{p} \Vdash_{\mathbb{P}} \text{“}a_0 \,\underset{\sim}{E}\, a_1\text{”}\\ &\underbrace{\Leftrightarrow}_{\text{by } (\aleph 1.7)} \; a_0 \,\underset{\sim}{E}[\mathbb{G}]\, a_1 \;\Leftrightarrow\; \underset{\sim}{\mathfrak{M}}[\mathbb{G}] \models a_0 \,\underline{E}\, a_1.\end{aligned}$$

The proof for “$a_0 = \underset{\sim}{f}(a_1)$” can be done similarly.

The induction steps for $\neg$ and $\vee$ go through since $\mathbb{G}$ is $\mathcal{D}$-generic and the antichains in ($\aleph$1.4) are in $\mathcal{D}$.

Suppose now that the equivalence ($\aleph$1.9) holds for a Σ_1-formula $\theta = \theta(x, x_0, ..., x_{k-1})$ and all other Σ_1-formulas with the quantifier rank (with respect to bounded existential quantification) less than or equal to that of θ.

If $[\![\, \underset{\sim}{\mathfrak{M}} \models (\exists x \,\underline{E}\, b)\, \theta(x, a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}} \in \mathbb{G}$, then, by ($\aleph$1.6), there is $[\![\, \underset{\sim}{\mathfrak{M}} \models d \in \nu_\mu \wedge \theta(d, a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}} \in \mathbb{G}$. By the induction hypothesis, it follows that $\underset{\sim}{\mathfrak{M}}[\mathbb{G}] \models d \,\underline{E}\, b \wedge \theta(d, a_0, ..., a_{k-1})$. Thus $\underset{\sim}{\mathfrak{M}}[\mathbb{G}] \models (\exists x \,\underline{E}\, b)\, \theta(d, a_0, ..., a_{k-1})$.

If $\underset{\sim}{\mathfrak{M}}[\mathbb{G}] \models (\exists x \,\underset{\sim}{E}\, b)\,\theta(d, a_0, ..., a_{k-1})$, then $\underset{\sim}{\mathfrak{M}}[\mathbb{G}] \models d \,\underset{\sim}{E}\, b \wedge \theta(d, a_0, ..., a_{k-1})$ for some $d \in \nu + \mu$. By induction hypothesis, it follows that $[\![\, d \,\underset{\sim}{E}\, b \wedge \theta(d, a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}} \in \mathbb{G}$. Since $[\![\, d \,\underset{\sim}{E}\, b \wedge \theta(d, a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}} \leq_{\mathbb{B}} [\![\, (\exists x \,\underset{\sim}{E}\, b)\,\theta(x, a_0, ..., a_{kp-1}) \,]\!]_{\mathbb{B}}$, we have $[\![\, (\exists x \,\underset{\sim}{E}\, b)\,\theta(x, a_0, ..., a_{kp-1}) \,]\!]_{\mathbb{B}} \in \mathbb{G}$ since $\mathbb{G}$ is a filter.

(3): By ($\aleph$1.2), we have $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathfrak{M}} \models$ Axiom of Extensionality", and

$$(\aleph 1.10) \quad \Vdash_{\mathbb{P}} \text{“} \underset{\sim}{\mathfrak{M}} \models \forall x \forall y\, (x \underset{\sim}{E} y \rightarrow \underset{\sim}{f}(x) < \underset{\sim}{f}(y)) \text{”}.$$ x-theintro-0-5

By (2), it follows that $\underset{\sim}{E}[\mathbb{G}]$ is extensional and the statement on the structure $\underset{\sim}{\mathfrak{M}}[\mathbb{G}]$ corresponding to ($\aleph$1.10) holds. A similar argument shows that the canonical ordering on ν coincides with $\underset{\sim}{E}[\mathbb{G}] \restriction \nu^2$. This and the property of $\underset{\sim}{\mathfrak{M}}[\mathbb{G}]$ corresponding to ($\aleph$1.10) implies that $\underset{\sim}{E}[\mathbb{G}]$ is well-founded. $\dashv$ (Claim 1.2.1)

By Claim 1.2.1, (3), we can take the Mostowski collapse of the structure $\underset{\sim}{\mathfrak{M}}[\mathbb{G}]$ $m^* : \langle \nu + \mu, \underset{\sim}{E}[\mathbb{G}] \rangle \overset{\cong}{\rightarrow} \langle M_2, \in \rangle$. Since $\langle \nu + \mu, \underset{\sim}{E}[\mathbb{G}] \rangle \models$ "$\psi^*(\nu, \nu + 1)$" by ($\aleph$1.1), ($\aleph$1.2) and Claim 1.2.1, (2), we have $m^*(\nu) = a$, and $M_2 \models \psi(a, m^*(\mu + 1))$. Thus $M_2 \models \varphi(a)$. Since φ is Σ_1, it follows that $V \models \varphi(a)$.

(b) $\Rightarrow$ (a): Suppose that $\mathbb{P} \in \mathcal{P}$ is complete Boolean and $\mathcal{D}$ is a set of antichains each of size $< \kappa$ with $|\, \mathcal{D} \,| < \kappa$.

Let $X = \bigcup \mathcal{D}$ then $|\, X \,| < \kappa$. Say, $\mu := |\, X \,|$. Let λ be sufficiently large with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$ for sufficiently large n. Let $M \prec V_\lambda$ be such that $|\, M \,| = \mu$, $\mathbb{P}$, $\mathcal{D}$, $X \in M$, and $\mu + 1 \subseteq M$. Note that $\mathcal{D} \subseteq M$ and $I \subseteq M$ for each $I \in \mathcal{D}$.

Let $m : M \overset{\cong}{\rightarrow} M_0$ be the Mostowski collapse and $\langle \mathbb{P}_0, \leq_{\mathbb{P}_0} \rangle := m(\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle)$.

Since $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$ generates an $(M_0, \mathbb{P}_0)$-generic filter, we have

$$\Vdash_{\mathbb{P}} \text{“} \mathcal{H}(\kappa^{(+)}) \models \text{there is a } (M_0, \mathbb{P}_0)\text{-generic filter”}.$$

By assumption it follows that $\mathcal{H}(\kappa) \models$ "there is a $(M_0, \mathbb{P}_0)$-generic filter" in $\mathsf{V}[\mathbb{G}]$. Let $\mathbb{G}_0$ be such a filter. Then $m^{-1}{}''\mathbb{G}_0$ generates a $\mathcal{D}$-generic filter on $\mathbb{P}$.

❏ (Theorem 1.2)

The following is one of many nice applications of Theorem 1.2:

Corollary 1.3 *If $\mathcal{P}$ contains a poset adding a new real then* $\mathsf{BFA}_{<\kappa}(\mathcal{P})$ *for $\kappa > \aleph_1$ implies* $\neg\mathsf{CH}$.

Proof. Assume that $\mathsf{BFA}_{<\kappa}(\mathcal{P})$ holds for $\kappa > \aleph_1$, but also CH holds in V. Let $a = \mathcal{P}(\omega)$. We have $a \in \mathcal{H}(\kappa)^{\mathsf{V}}$ by CH and $\mathcal{H}(\kappa) \models$ "a is $\mathcal{P}(\omega)$". The statement can be formulated as a Π_1-formula with the parameter a. But if $\mathbb{P} \in \mathcal{P}$ adds a real, $\mathcal{H}(\kappa)^{\mathsf{V}[\mathbb{G}]} \models$ "a is not $\mathcal{P}(\omega)$". This is a contradiction to Theorem 1.2, (c).

❏ (Corollary 1.3)

Suppose that $\mathcal{R}$ is a definable class (proper or set). We shall say that a class $\mathcal{P}$ of posets is *provably correct for $\mathcal{R}$* if the following is provable in ZFC:

(1.5) for any $\mathbb{P} \in \mathcal{P}$ and a $(\in \mathsf{V})$, $a \in \mathcal{R} \Leftrightarrow \Vdash_{\mathbb{P}}$ "$a \in \mathcal{R}$". p-theintro-2

Thus if $\mathcal{P}$ is provably correct for $\mathcal{R}$ and $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic for a $\mathbb{P} \in \mathcal{P}$ then $(\mathsf{V}, \in, \mathcal{R}^{\mathsf{V}})$ is a (class) substructure of $(\mathsf{V}[\mathbb{G}], \in, \mathcal{R}^{\mathsf{V}[\mathbb{G}]})$.

Let I_{NS} denote the non stationary ideal over ω_1. Thus

$$I_{\mathsf{NS}} := \{X \subseteq \omega_1 \,:\, X \text{ is non stationary}\}.$$

Lemma 1.4 *If (we can prove that) all $\mathbb{P} \in \mathcal{P}$ are stationary preserving then $\mathcal{P}$ is provably correct for I_{NS}.* p-theintro-3

Proof. If $s \in I_{\mathsf{NS}}$ then there is a club $c \subseteq \omega_1$ such that $s \cap c = \emptyset$. Since $\Vdash_{\mathbb{P}}$ "c is a club in ω_1 and $s \cap c = \emptyset$", we have $\Vdash_{\mathbb{P}}$ "$s \in I_{\mathsf{NS}}$".

If $s \notin I_{\mathsf{NS}}$ then $\Vdash_{\mathbb{P}}$ "$s \notin I_{\mathsf{NS}}$" since $\mathbb{P}$ is stationary preserving. ❑ (Lemma 1.4)

Let $\mathcal{R}$ be (the $\mathcal{L}_\in$-definition of) a class. Let $\mathcal{L}_{\in,\underset{\smile}{\mathcal{R}}}$ be the language which extends $\mathcal{L}_\in$ with a new unary predicate symbol $\underset{\smile}{\mathcal{R}}$ where $M \models \underset{\smile}{\mathcal{R}}(a)$ is interpreted as $a \in \mathcal{R}^M$ in an $\in$-structure M. In the following, we shall often identify the (definition) of the class $\mathcal{R}$ with the symbol $\underset{\smile}{\mathcal{R}}$ of $\mathcal{R}$, and simply write $\mathcal{R}$ and $\mathcal{L}_{\in,\mathcal{R}}$ instead of $\underset{\smile}{\mathcal{R}}$ and $\mathcal{L}_{\in,\underset{\smile}{\mathcal{R}}}$. This also applies when we are talking about I_{NS} and $\mathcal{L}_{\in,I_{\mathsf{NS}}}$.

The following Lemma 1.5 can be proved in the same way as with the corresponding lemma for $\mathcal{L}_\in$ formulas:

Lemma 1.5 *For transitive (sets or classes) M, N with $M \subseteq N$ and a class $\mathcal{R}$ (i.e. an $\mathcal{L}_\in$-formula with one single free variable) such that $\mathcal{R}^M = \mathcal{R}^N \cap M$, we have:* p-theintro-4

(1) $\langle M, \in, \mathcal{R}^M \rangle \models \varphi(\overline{a}) \Leftrightarrow \langle N, \in, \mathcal{R}^N \rangle \models \varphi(\overline{a})$ for all Σ_0-formula $\varphi = \varphi(\overline{x})$ in $\mathcal{L}_{\in,\mathcal{R}}$ and $\overline{a} \in M$.

(2) $\langle M, \in, \mathcal{R}^M \rangle \models \varphi(\overline{a}) \Rightarrow \langle N, \in, \mathcal{R}^N \rangle \models \varphi(\overline{a})$ for all Σ_1-formula $\varphi = \varphi(\overline{x})$ in $\mathcal{L}_{\in,\mathcal{R}}$ and $\overline{a} \in M$. ❑

Lemma 1.6 *For a Σ_1-formula φ in $\mathcal{L}_{\in,I_{\mathsf{NS}}}$ we can find a Σ_2-formula in $\mathcal{L}_\in$ with the parameter ω_1 equivalent to φ.* p-theintro-4-0

Proof. "$x \in I_{\mathsf{NS}}$" can be expressed by a Σ_1-formula in $\mathcal{L}_\in$:

$$\exists y\, (y \subseteq \omega_1 \ \wedge\ x \subseteq \omega_1 \ \wedge\ y \text{ is a club in } \omega_1 \ \wedge\ x \cap y = \emptyset)$$

Thus "$x \notin I_{\mathsf{NS}}$" can be expressed by a Π_1-formula in $\mathcal{L}_\in$. ❑ (Lemma 1.6)

Lemma 1.7 (A special case of Lemma 6.3 in Venturi and Viale [38]) *For a cardinal $\lambda \geq 2^{\aleph_1}$, we have $\langle \mathcal{H}(\lambda), \in, I_{\mathsf{NS}} \rangle \prec_{\Sigma_1} \langle \mathsf{V}, \in, I_{\mathsf{NS}} \rangle$.* p-theintro-4-1

Proof. If $\langle \mathcal{H}(\lambda), \in, I_{\mathsf{NS}} \rangle \models \varphi(a)$ for a Σ_1-formula in $\mathcal{L}_{\in, I_{\mathsf{NS}}}$, and $a \in \mathcal{H}(\lambda)$, then $\langle \mathsf{V}, \in, I_{\mathsf{NS}} \rangle \models \varphi(a)$ by Lemma 1.5, (2).

Suppose now that $\langle \mathsf{V}, \in, I_{\mathsf{NS}} \rangle \models \varphi(a)$ for φ and a as above. Suppose that $\varphi = \exists y\, \psi(x, y)$ for a Σ_0-formula ψ in $\mathcal{L}_{\in, I_{\mathsf{NS}}}$ and let b be such that $\langle \mathsf{V}, \in, I_{\mathsf{NS}} \rangle \models \psi(a, b)$.

Let $\alpha \in \mathrm{On}$ be sufficiently large such that $\langle V_\alpha, \in, I_{\mathsf{NS}} \rangle \prec_{\Sigma_n} \langle \mathsf{V}, \in, I_{\mathsf{NS}} \rangle$ for sufficiently large $n \in \omega$.

Let $\mu := \sup\{|\, trcl^+(a)\, |, 2^{\aleph_1}\}$ and[3] $\langle M, \in I_{\mathsf{NS}} \rangle \prec \langle \mathsf{V}, \in, I_{\mathsf{NS}} \rangle$ be such that $|\, M\, | = \mu$, $trcl^+(a)$, $\mu + 1$, $I_{\mathsf{NS}} \subseteq M$, and $b \in M$. Then we have $\langle M, \in, I_{\mathsf{NS}} \rangle \models \psi(a, b)$.

Let $m : M \overset{\cong}{\to} M_0$ be the Mostowski collapse. Then we have $m \restriction trcl^+(a) = id_{\,trcl^+(a)}$ and $m \restriction \mathcal{P}(\omega_1) = id_{\mathcal{P}(\omega_1)}$. It follows that $\langle M_0, \in, I_{\mathsf{NS}} \rangle \models \psi(a, m(b))$ and hence $\langle M_0, \in, I_{\mathsf{NS}} \rangle \models \varphi(a)$. By Lemma 1.5, (2), it follows that $\langle \mathcal{H}(\lambda), \in, I_{\mathsf{NS}} \rangle \models \varphi(a)$. ❑ (Lemma 1.7)

The following Theorem 1.8 is an extension of Bagaria's Absoluteness Theorem 1.2. A special case of this theorem (the case where $\mathcal{P}$ = the stationary preserving posets) is also attributed to Bagaria in [40]. Though Theorem 1.8 in its generality must have been known, we included it here since we could not find any proof in the literature.

We consider the following "plus"-version of Bounded Forcing Axioms: For a (normal) class of posets $\mathcal{P}$,

($\mathsf{BFA}^{+<\kappa}_{<\kappa}(\mathcal{P})$): For any complete Boolean $\mathbb{P} \in \mathcal{P}$, a family $\mathcal{D}$ of maximal antichains in $\mathbb{P}$ such that $|\, \mathcal{D}\, | < \kappa$ and $|\, I\, | < \kappa$ for all $I \in \mathcal{D}$, and for a set $\mathcal{S}$ of $\mathcal{P}$-names of cardinality $< \kappa$ such that each $\underset{\sim}{S} \in \mathcal{S}$ is a $\mathbb{P}$-name of a stationary subset of ω_1, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ on $\mathbb{P}$ such that $\underset{\sim}{S}[\mathbb{G}]$ for all $\underset{\sim}{S} \in \mathcal{S}$ are stationary subsets of ω_1.

Theorem 1.8 *Suppose that $\mathcal{P}$ is a class of posets closed under forcing equivalence, and restriction (in the sense of (1.2)) such that all elements of $\mathcal{P}$ are stationary preserving and $\kappa = 2^{\aleph_0} = 2^{\aleph_1}$. Then the following are equivalent:* p-theintro-5

(a) $\mathsf{BFA}^{+<\kappa}_{<\kappa}(\mathcal{P})$.

(b) *For any Σ_1-formula $\varphi = \varphi(x)$ in $\mathcal{L}_{\in, I_{\mathsf{NS}}}$, $a \in \mathcal{H}(\kappa)$, and $\mathbb{P} \in \mathcal{P}$, we have*

$$\Vdash_{\mathbb{P}} \text{“}\varphi(a)\text{”} \Leftrightarrow \varphi(a).$$

[3] We denote with $trcl^+(a)$ the variant of transitive closure which is the minimal transitive set T with $a \cup \{a\} \subseteq T$.

(c) *For any* $(\mathsf{V},\mathbb{P})$*-generic* $\mathbb{G}$*, we have*

$$\langle \mathcal{H}(2^{\aleph_0})^{\mathsf{V}}, \in, I_{\mathsf{NS}}^{\mathsf{V}} \rangle \prec_{\Sigma_1} \langle \mathcal{H}((2^{\aleph_0})^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]}, \in, I_{\mathsf{NS}}^{\mathsf{V}[\mathbb{G}]} \rangle.$$

Proof. The equivalence of (b) and (c) follows from Lemma 1.7. (a) $\Rightarrow$ (b): Let $\mathbb{P} \in \mathcal{P}$. Without loss of generality, $\mathbb{P}$ is completely Boolean with $\mathbb{P} = \mathbb{B} \setminus \{\mathbb{0}_{\mathbb{B}}\}$. Suppose $a \in \mathcal{H}(\kappa)$ and φ is a Σ_1-formula in $\mathcal{L}_{\in, I_{\mathsf{NS}}}$. If $\varphi(a)$ holds in V, then we also have $\Vdash_{\mathbb{P}}$ "$\varphi(a)$" by Lemma 1.5.

Suppose now that $\varphi = \exists y\, \psi(x, y)$ for a bounded formula ψ in $\mathcal{L}_{\in, I_{\mathsf{NS}}}$, and $\Vdash_{\mathbb{P}}$ "$\varphi(a)$". Without loss of generality, we may assume that $a \subseteq \mu$ for some cardinal $\mu < \kappa$ (this is because a can be reconstructed from $trcl^+(a)$, and $trcl^+(a)$ can be coded by a subset a^* of $|\, trcl^+(a)\,|$). The formula $\varphi(a)$ can be replaced by the formula saying:

$$\exists x\,(\, x \text{ is the set “}a\text{” reconstructed from the transitive set coded by } a^* \text{ and } \varphi(x) \text{ holds}\,).$$

Note that this formula is Σ_1 in $\mathcal{L}_{\in, I_{\mathsf{NS}}}$ with the parameter a^* if φ is Σ_1 in $\mathcal{L}_{\in, I_{\mathsf{NS}}}$. We may also assume that a is not an ordinal (if necessary, we can replace a with a subset of μ with some redundant complexity to make $a \notin \mathrm{On}$).

Let $\underset{\sim}{b}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\psi(a, \underset{\sim}{b})$". Let $\mathbb{G}$ be a $(\mathsf{V},\mathbb{P})$-generic filter and we work in $\mathsf{V}[\mathbb{G}]$. Letting $b = \underset{\sim}{b}[\mathbb{G}]$, we have $\psi(a, b)$.

Working further in $\mathsf{V}[\mathbb{G}]$, let λ be large enough such that V_λ satisfies a large enough fragment of ZFC, a, $b \in V_\lambda$, and $V_\lambda \models \psi(a, b)$. Let $M \prec V_\lambda$ be such that $\mu \subseteq M$, a, $b \in M$, and $|\, M\,| = \mu$. Note that we have $\langle M, \in, I_{\mathsf{NS}} \cap M \rangle \prec \langle V_\lambda, \in, I_{\mathsf{NS}} \rangle$ since I_{NS} is definable in $\langle V_\lambda, \in \rangle$. Let $m : M \xrightarrow{\cong} M_0$ be the Mostowski collapse of M and let $\nu = \mathrm{On} \cap M_0$. Note that we have $m \restriction \mu \cup \{a\} \cup (I_{\mathsf{NS}} \cap M) = id_{\,\mu \cup \{a\} \cup (I_{\mathsf{NS}} \cap M)}$ and hence $I_{\mathsf{NS}} \cap M = I_{\mathsf{NS}} \cap M_0$.

Let $\mathfrak{M} := \langle \nu + \mu, E, I, f, g \rangle$ be the structure in the language $\mathcal{L} := \{\underset{\sqcup}{E}, \underset{\sqcup}{I}, \underset{\sqcup}{f}, \underset{\sqcup}{g}\}$ such that there is an isomorphism

$$\begin{aligned} &i : \langle M_0, \in, I_{\mathsf{NS}} \cap M_0, rank, g_0 \rangle \xrightarrow{\cong} \langle \nu + \mu, E, I, f, g \rangle \\ &\text{such that } i \restriction \nu = id_{\,\nu},\ i(a) = \nu, \text{ and } i(m(b)) = \nu + 1 \end{aligned} \tag{1.6}$$

where $rank$ is the rank function restricted to M_0 and $g_0 : M_0 \to M_0$ is a mapping such that $f \restriction \mu$ is an enumeration of $(\mathcal{P}(\omega_1) \cap M_0) \setminus I_{\mathsf{NS}}$ (= (the set of all stationary subsets of $\omega_1)^{M_0}$) and $g''(M_0 \setminus \mu) = \{\emptyset\}$.

Clearly, $\langle \nu + \mu, E, I \rangle \models \psi^*(\nu, \nu + 1))$ where ψ^* is the formula obtained from ψ by replacing symbols $\underset{\sqcup}{\in}$ and $\underset{\sqcup}{I_{\mathsf{NS}}}$ in ψ by $\underset{\sqcup}{E}$ and $\underset{\sqcup}{I}$.

Note that we have

(1.7) $\quad \langle \nu+\mu, E, I, f, g\rangle \models \forall x \subseteq \omega_1\ (I(x) \vee \exists \alpha < \kappa\,(g(\alpha) = x)).$ a:x-theintro-1-0

Let $\underset{\sim}{\mathfrak{M}}$, $\underset{\sim}{E}$, $\underset{\sim}{I}$, $\underset{\sim}{f}$ $\underset{\sim}{g} \in \mathsf{V}$ be $\mathbb{P}$-names of $\mathfrak{M}$, E, I, f and g respectively. By replacing $\mathbb{P}$ with $\mathbb{P} \restriction \mathbb{p}$ for some $\mathbb{p} \in \mathbb{P}$ if necessary, we may assume that

(1.8) all the properties of $\langle \nu+\mu, E, I, f, g\rangle$ used below are forced (as a statement on $\langle \nu+\mu, \underset{\sim}{E}, \underset{\sim}{I}, \underset{\sim}{f}, \underset{\sim}{g}\rangle$) by $\mathbb{1}_{\mathbb{P}}$. a:x-theintro-0-0-0

In V, let $\mathcal{D}$ be the family of maximal antichains (each of size $\leq \mu < \kappa$) in $\mathbb{P}$ consisting of the following;

(1.9) $\{[\![\, \underset{\sim}{f}(\alpha) = \beta \,]\!]_{\mathbb{B}} : \beta < \nu\} \setminus \{\mathbb{0}_{\mathbb{B}}\}$, and $\{[\![\, \underset{\sim}{g}(\alpha) = \beta \,]\!]_{\mathbb{B}} : \beta < \nu\} \setminus \{\mathbb{0}_{\mathbb{B}}\}$, for all $\alpha \in \nu+\mu$. a:x-theintro-0-1

(1.10) $\{[\![\, \underset{\sim}{\mathfrak{M}} \models \theta(a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}}, [\![\, \underset{\sim}{\mathfrak{M}} \models \neg\theta(a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}}\} \setminus \{\mathbb{0}_{\mathbb{B}}\}$, a:x-theintro-0-2
for all Σ_0-formulas θ in $\mathcal{L}$ and $a_0, ..., a_{k-1} \in \nu+\mu$.

(1.11) $\{[\![\, \underset{\sim}{\mathfrak{M}} \models \eta \wedge \theta(a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}},$ a:x-theintro-0-3
$[\![\, \underset{\sim}{\mathfrak{M}} \models \neg\eta(a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}}, [\![\, \underset{\sim}{\mathfrak{M}} \models \neg\theta(a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}}\} \setminus \{\mathbb{0}_{\mathbb{B}}\}$,
for all Σ_0-formulas η, θ in $\mathcal{L}$ and $a_0, ..., a_{k-1} \in \nu+\mu$.

(1.12) $\big(\{[\![\, \neg(\exists x \mathrel{\underset{\smile}{E}} c)\, \eta(x, a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}}\} \cup$ a:x-theintro-0-4
$\{[\![\, d \mathrel{\underset{\smile}{E}} c \wedge \eta(d, a_0, ..., a_{k-1}) \,]\!]_{\mathbb{B}} : d \in \nu+\mu\}\big) \setminus \{\mathbb{0}_{\mathbb{B}}\}$,
for all Σ_1-formulas $\eta = \eta(x, x_0, ..., x_{k-1})$ in $\mathcal{L}$ and c, $a_0, ..., a_{k-1} \in \nu+\mu$.

To see that each of the sets in (1.9) is a maximal antichain in $\mathbb{P}$ of size $\leq \mu$, suppose that $\alpha \in \nu+\mu$ and $\mathbb{p} \in \mathbb{P}$. Then there is $\mathbb{q} \leq_{\mathbb{P}} \mathbb{p}$ which decides $\underset{\sim}{f}(\alpha)$. Since $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\alpha) \in \nu$" by (1.8), if follows that $\mathbb{q} \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\alpha) = \beta$" for some $\beta \in \nu$. It is clear that elements of each of the sets in (1.9) are pairwise incompatible and these sets are of size $\leq \mu < \kappa$.

It is also proved similarly that sets in (1.12) are maximal antichains in $\mathbb{P}$ of size $\leq \mu < \kappa$.

By (1.8), we have that

(1.13) $\quad \Vdash_{\mathbb{P}}$ "$\{\xi \in \omega_1 : \xi \mathrel{\underset{\sim}{E}} \underset{\sim}{g}(\alpha)\}$ is a stationary subset of ω_1". a:x-theintro-0-4-a

Now, in V, let $\mathbb{G}$ be $\mathcal{D}$-generic filter such that

(1.14) $\underset{\sim}{g}(\alpha)[\mathbb{G}] := \{\xi \in \omega : \mathbb{p} \Vdash_{\mathbb{P}}$ "$\xi \mathrel{\underset{\sim}{E}} \underset{\sim}{g}(\alpha)$" for some $\mathbb{p} \in \mathbb{G}\}$ is a stationary subset of ω_1 for all $\alpha < \mu$. a:x-theintro-0-4-a-0

$\mathbb{G}$ exists by $\mathsf{BFA}^{+<\kappa}_{<\kappa}(\mathcal{P})$, by (1.13), and since $\mathcal{D}$ is a family of maximal antichains of size $<\kappa$ with $|\,\mathcal{D}\,| < \kappa$.

Let

$$\underset{\sim}{\mathfrak{M}}[\mathbb{G}] := \langle \nu+\mu, \underset{\sim}{E}[\mathbb{G}], \underset{\sim}{I}[\mathbb{G}], \underset{\sim}{f}[\mathbb{G}], \underset{\sim}{g}[\mathbb{G}]\rangle.$$

where

$$\begin{aligned}
&\underset{\sim}{E}[\mathbb{G}] := \{\langle\xi,\eta\rangle \,:\, \xi,\eta\in\nu+\mu,\ \mathbb{p}\Vdash_{\mathbb{P}} \text{“}\langle\xi,\eta\rangle\in\underset{\sim}{E}\text{” for some } \mathbb{p}\in\mathbb{G}\},\\
&\underset{\sim}{I}[\mathbb{G}] := \{\xi \,:\, \xi\in\nu+\mu,\ \mathbb{p}\Vdash_{\mathbb{P}} \text{“}\xi\in\underset{\sim}{I}\text{” for some } \mathbb{p}\in\mathbb{G}\},\\
&\underset{\sim}{f}[\mathbb{G}] := \{\langle\xi,\eta\rangle \,:\, \xi,\eta\in\nu+\mu,\ \mathbb{p}\Vdash_{\mathbb{P}} \text{“}\langle\xi,\eta\rangle\in\underset{\sim}{f}\text{” for some } \mathbb{p}\in\mathbb{G}\}, \quad\text{and}\\
&\underset{\sim}{g}[\mathbb{G}] := \{\langle\xi,\eta\rangle \,:\, \xi,\eta\in\nu+\mu,\ \mathbb{p}\Vdash_{\mathbb{P}} \text{“}\langle\xi,\eta\rangle\in\underset{\sim}{g}\text{” for some } \mathbb{p}\in\mathbb{G}\}.
\end{aligned}$$

Claim 1.8.1 (1) *$\underset{\sim}{\mathfrak{M}}[\mathbb{G}]$ is an $\mathcal{L}$-structure.*

(2) *For each Σ_1-formula $\theta=\theta(x_0,\dots,x_{k-1})$ in $\mathcal{L}$ and $a_0,\dots,a_{k-1}\in\nu+\mu$,*

$$(1.15)\quad [\![\, \underset{\sim}{\mathfrak{M}} \models \theta(a_0,\dots,a_{k-1})\,]\!]_{\mathbb{B}} \in \mathbb{G} \ \text{ if and only if } \ \underset{\sim}{\mathfrak{M}}[\mathbb{G}] \models \theta(a_0,\dots,a_{k-1}).$$ *a:x-theintro-0-4-0*

(3) *$\underset{\sim}{E}[\mathbb{G}]$ is extensional and well-founded. $\underset{\sim}{E}[\mathbb{G}]$ on $\nu+\mu$ coincides with the canonical ordering on $\nu+\mu$.*

$\vdash$ (1): Since the maximal antichains in (1.9) are in $\mathcal{D}$, we have $\underset{\sim}{f}[\mathbb{G}]:\nu+\mu\to\nu$ and $\underset{\sim}{g}[\mathbb{G}]:\nu+\mu\to\nu+\mu$.

(2): By induction on the construction of the formula θ using (1.10), (1.11), and (1.12).

(3): By (1.8), we have $\Vdash_{\mathbb{P}}$ “$\underset{\sim}{\mathfrak{M}}\models$ Axiom of Extensionality”, and

$$(\aleph 1.11)\quad \Vdash_{\mathbb{P}} \text{“}\underset{\sim}{\mathfrak{M}} \models \forall x\forall y\,(x\underset{\sim}{E}y \to \underset{\sim}{f}(x) < \underset{\sim}{f}(y))\text{”}.$$ a:x-theintro-0-5

By (2), it follows that $\underset{\sim}{E}[\mathbb{G}]$ is extensional and the statement on the structure $\underset{\sim}{\mathfrak{M}}[\mathbb{G}]$ corresponding to ($\aleph$1.11) holds. A similar argument shows that the canonical ordering on ν coincides with $\underset{\sim}{E}[\mathbb{G}]\restriction\nu^2$. This and the property of $\underset{\sim}{\mathfrak{M}}[\mathbb{G}]$ corresponding to ($\aleph$1.11) implies that $\underset{\sim}{E}[\mathbb{G}]$ is well-founded. $\dashv$ (Claim 1.8.1)

By Claim 1.8.1, (3), we can take the Mostowski collapse of the structure $\underset{\sim}{\mathfrak{M}}[\mathbb{G}]$ $m^*:\langle\nu+\mu,\underset{\sim}{E}[\mathbb{G}],\underset{\sim}{I}[\mathbb{G}]\rangle \xrightarrow{\cong} \langle M_2,\in,I\rangle$. Since $\langle\nu+\mu,\underset{\sim}{E}[\mathbb{G}],\underset{\sim}{I}[\mathbb{G}]\rangle\models$ “$\psi^*(\nu,\nu+1)$” by (1.6), (1.8) and Claim 1.8.1, (2), we have $m^*(\nu)=a$ and hence $\langle M_2,\in,I\rangle\models\psi(a,m^*(\nu{+}1))$ where the predicate I_{NS} is interpreted as I. Thus $\langle M_2,\in,I\rangle\models\varphi(a)$. By (1.7), (1.8), (1.14) and Claim 1.8.1, (2), we have $I=I_{\mathsf{NS}}\cap M_2$.

Since φ is Σ_1, it follows that $V\models\varphi(a)$ by Lemma 1.5, (2).

(b) $\Rightarrow$ (a): Suppose that $\mathbb{P}\in\mathcal{P}$ is complete Boolean, $\mathcal{D}$ is a set of antichains each of size $<\kappa$ with $|\,\mathcal{D}\,|<\kappa$, and $\mathcal{S}$ is a set of $\mathbb{P}$-names of stationary subsets of ω_1 with $|\,\mathcal{S}\,|<\kappa$.

By replacing elements of $\mathcal{S}$ by equivalent $\mathbb{P}$-names which are sufficiently nice, we may assume that each element of $\mathcal{S}$ is nice $\mathbb{P}$-name of size $\aleph_1$ (this is possible since we assumed that $\mathbb{P}$ is completely Boolean).

Let $X = \bigcup \mathcal{D}$ then $|X| < \kappa$. Let $\mu := \max\{|X|, |\mathcal{S}|\}$. Let λ be sufficiently large with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$ for sufficiently large n. Let $M \prec V_\lambda$ be such that $|M| = \mu$, (1.16): $\mathbb{P}$, $\mathcal{D}$, X, $\mathcal{S} \in M$, and $\mu + 1 \subseteq M$. Note that (1.16) implies $\mathcal{D}, \mathcal{S} \subseteq M$ and $I, \underset{\sim}{S} \subseteq M$ for each $I \in \mathcal{D}$ and $\underset{\sim}{S} \in \mathcal{S}$. x-theintro-0-6

Let $m : M \overset{\cong}{\to} M_0$ be the Mostowski collapse and $\langle \mathbb{P}_0, \leq_{\mathbb{P}_0} \rangle := m(\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle)$. Let $\mathcal{S}_0 := \{m(\underset{\sim}{S}) : \underset{\sim}{S} \in \mathcal{S}\}$.

Since $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$ generates an $(M_0, \mathbb{P}_0)$-generic filter, we have

$$\Vdash_{\mathbb{P}} \text{“ there is a } (M_0, \mathbb{P}_0)\text{-generic filter which realizes each element of } \mathcal{S}_0 \text{ to be a stationary subset of } \omega_1 \text{ ”.}$$

By the assumption (b), it follows that

$$\mathsf{V} \models \text{“ there is a } (M_0, \mathbb{P}_0)\text{-generic filter which realizes each element of } \mathcal{S}_0 \text{ to be a stationary subset of } \omega_1 \text{ ”.}$$

Let $\mathbb{G}_0$ be such a filter. Then $m^{-1}{}''\mathbb{G}_0$ generates a $\mathcal{D}$-generic filter $\mathbb{G}_1$ on $\mathbb{P}$ which realizes each element of $\mathcal{S}$ to be a stationary subset of ω_1. ❑ (Theorem 1.8)

2 Recurrence Axioms and the Ground Axiom

2.1 Hierarchies of Recurrence and Maximality

rec-GA

The term “Recurrence Axiom” was coined in Fuchino and Usuba [20] (see also Fuchino [13]). The Recurrence Axiom for a (normal) class $\mathcal{P}$ of posets, a set A of parameters, and a set Γ of $\mathcal{L}_\in$-formulas ($(\mathcal{P}, A)_\Gamma$-RcA, for short) is the statement (2.1) below. hierarchies-rec-max

A *ground of a* (transitive set or class) *model* W (of some set theory) is an inner model W_0 of W such that there is a poset $\mathbb{P} \in \mathsf{W}_0$ such that W is a $\mathbb{P}$-generic extension of W_0. For a class $\mathcal{P}$ of posets, a ground W_0 of W is a *$\mathcal{P}$-ground of* W if there is a poset $\mathbb{P} \in \mathsf{W}_0$ such that $\mathsf{W}_0 \models$ “ $\mathbb{P} \in \mathcal{P}$ ” and W is a $\mathbb{P}$-generic extension of W_0.

The Recurrence Axiom $(\mathcal{P}, A)_\Gamma$-RcA is the following statement formulated in an axiom scheme in $\mathcal{L}_\in$ (that this axiom is not formalizable in a single formula is discussed in [12]):

(2.1) For any $\varphi(\overline{x}) \in \Gamma$ and $\overline{a} \in A$, if $\Vdash_{\mathbb{P}}$ “ $\varphi(\overline{a}^{\checkmark})$ ”, then there is a ground W of V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$. x-theintro-1

The definition of a stronger variant $(\mathcal{P}, A)_\Gamma$-RcA^+ of $(\mathcal{P}, A)_\Gamma$-RcA is obtained when we replace "ground" in (2.1) by "$\mathcal{P}$-ground". If $\Gamma = \mathcal{L}_\in$, we simply drop Γ and talk about $(\mathcal{P}, A)$-$\mathsf{RcA}(^+)$.

In the following, we often identify check names of sets with the sets themselves and drop the symbol "$^\checkmark$". Also we shall often replace tuples $\overline{a}$ of parameters by a single parameter a for simplicity (actually without loss of generality in most of the cases).

Recurrence Axioms are almost identical with Maximality Principles introduced in [25] with the same parameters. For $\mathcal{P}$, A as above, the Maximality Principle for $\mathcal{P}$ and A ($\mathsf{MP}(\mathcal{P}, A)$ for short) is defined as below.

For a class $\mathcal{P}$ of posets, an $\mathcal{L}_\in$-formula $\varphi(\overline{a})$ with parameters $\overline{a}$ ($\in \mathsf{V}$) is said to be a *$\mathcal{P}$-button* if there is $\mathbb{P} \in \mathcal{P}$ such that for any $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of poset with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", we have $\Vdash_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}}$ "$\varphi(\overline{a}^\checkmark)$".

If $\varphi(\overline{a})$ is a $\mathcal{P}$-button then we call $\mathbb{P}$ as above a *push of the $\mathcal{P}$-button $\varphi(\overline{a})$*.

For a class $\mathcal{P}$ of posets and a set A (of parameters), the *Maximality Principle for $\mathcal{P}$ and A* ($\mathsf{MP}(\mathcal{P}, A)$, for short) is the following assertion which is formulated as an axiom scheme in $\mathcal{L}_\in$:

$\mathsf{MP}(\mathcal{P}, A)$: For any $\mathcal{L}_\in$-formula $\varphi(\overline{x})$ and $\overline{a} \in A$, if $\varphi(\overline{a})$ is a $\mathcal{P}$-button then $\varphi(\overline{a})$ holds.

Similarly to the restricted versions of Recurrence Axiom, we define, for a set Γ of $\mathcal{L}_\in$-formulas:

$\mathsf{MP}(\mathcal{P}, A)_\Gamma$: For any $\varphi(\overline{x}) \in \Gamma$ and $\overline{a} \in A$, if $\varphi(\overline{a})$ is a $\mathcal{P}$-button then $\varphi(\overline{a})$ holds.

Proposition 2.1 (Barton et al. [7], see also Proposition 2.2 in [20]) *Suppose that $\mathcal{P}$ is a class of posets and A a set (of parameters).* p-intro-1

(1) $(\mathcal{P}, A)$-RcA^+ *is equivalent to* $\mathsf{MP}(\mathcal{P}, A)$.

(2) $(\mathcal{P}, A)$-RcA *is equivalent to the following assertion:*

(2.2) *For any $\mathcal{L}_\in$-formula $\varphi(\overline{x})$ and $\overline{a} \in A$, if $\varphi(\overline{a})$ is a $\mathcal{P}$-button then $\varphi(\overline{a})$ holds in a ground of* V. x-intro-5-0 ❑

See Lemma 3.4 in Section 3 below and its proof.

Recurrence Axiom ($\Leftrightarrow$ Maximality Principle) can be also characterized as the ZFC version of Sy-David Friedman's Inner Model Hypothesis [10] (see Barton et al [7], see also Fuchino, and Usuba [20] or Fuchino [13]).

In contrast to the proposition above, $(\mathcal{P}, A)_\Gamma$-RcA^+ is not necessarily equivalent to $\mathsf{MP}(\mathcal{P}, A)_\Gamma$ for some set Γ of formulas. In the next section, we prove that (under the consistency of certain large cardinal axioms) $\mathsf{MP}(\mathcal{P},A)_{\Sigma_2}$ does not imply $(\mathcal{P},A)_{\Sigma_2}$-$\mathsf{RcA}$ and $\mathsf{MP}(\mathcal{P},A)_{\Pi_2}$ does not imply $(\mathcal{P},A)_{\Pi_2}$-RcA (see Corollary 3.12).

Later we shall also consider a restricted form of (2.2) which we will call $\mathsf{MP}^-(\mathcal{P},A)_\Gamma$:

$\mathsf{MP}^-(\mathcal{P},A)_\Gamma$: For any $\varphi(\overline{x}) \in \Gamma$ and $\overline{a} \in A$, if $\varphi(\overline{a})$ is a $\mathcal{P}$-button then $\varphi(\overline{a})$ holds in a ground of V.

Writing $\mathsf{MP}^-(\mathcal{P},A)$ for $\mathsf{MP}^-(\mathcal{P},A)_{\mathcal{L}_\in}$, the assertion of Proposition 2.1, (2) is reformulated as $(\mathcal{P},A)$-$\mathsf{RcA} \;\Leftrightarrow\; \mathsf{MP}^-(\mathcal{P},A)$.

While Recurrence Axioms are assertions about the richness of the grounds of the universe V, their characterizations as Maximality Principles may be seen as a variation of generic absoluteness. This is best seen in their further characterization as the principle $\mathsf{MP}^*(\mathcal{P},A)$ defined around (3.21), see also Subsection 6.3.

The following is an immediate consequence of Bagaria's Absoluteness Theorem 1.2:

Theorem 2.2 (Ikegami-Trang (reformulated for our hierarchy of restricted Recurrence Axioms) [27]) *For a (normal) class $\mathcal{P}$ of posets and a cardinal κ, the following are equivalent:* p-theintro-1

(a) $(\mathcal{P},\mathcal{H}(\kappa))_{\Sigma_1}$-$\mathsf{RcA}^+$.

(b) $(\mathcal{P},\mathcal{H}(\kappa))_{\Sigma_1}$-$\mathsf{RcA}$.

(c) $\mathsf{BFA}_{<\kappa}(\mathcal{P})$.

Proof. (a) $\Rightarrow$ (b): is trivial.

(b) $\Rightarrow$ (c): Assume that $(\mathcal{P},\mathcal{H}(\kappa))_{\Sigma_1}$-$\mathsf{RcA}$ holds, and suppose that $\mathbb{P} \in \mathcal{P}$, φ is a Σ_1-formula in $\mathcal{L}_\in$ and $a \in \mathcal{H}(\kappa)$. By Bagaria's Absoluteness Theorem 1.2, it is enough to show that $\Vdash_{\mathbb{P}}$ "$\varphi(a)$" $\Leftrightarrow \varphi(a)$ holds.

$\Vdash_{\mathbb{P}}$ "$\varphi(a)$" $\Leftarrow \varphi(a)$: is clear since φ is Σ_1.

$\Vdash_{\mathbb{P}}$ "$\varphi(a)$" $\Rightarrow \varphi(a)$: Assume that $\Vdash_{\mathbb{P}}$ "$\varphi(a)$". By $(\mathcal{P},\mathcal{H}(\kappa))_{\Sigma_1}$-$\mathsf{RcA}$, there is a ground W of V such that $a \in \mathsf{W}$ and $\mathsf{W} \models \varphi(a)$. Since φ is Σ_1 it follows that $\mathsf{V} \models \varphi(a)$.

(c) $\Rightarrow$ (a): Assume that $\mathsf{BFA}_{<\kappa}(\mathcal{P})$ holds. Suppose that $\Vdash_{\mathbb{P}}$ "$\varphi(a)$" for $\mathbb{P}$, φ, a as above. Then, by Bagaria's Absoluteness Theorem 1.2, we have $\varphi(a)$. In particular, since $\{\mathbb{1}\} \in \mathcal{P}$ (remember the convention set at (1.1)), $\varphi(a)$ holds in a $\mathcal{P}$-ground of V (namely V itself). ❑ (Theorem 2.2)

According to Joel Hamkins [25] it is an observation of his former PhD student George Leibman that MA follows from Maximality Principle for $\mathcal{P}$ = ccc posets, and the set of parameters $\mathcal{H}(2^{\aleph_0})$. This observation is now a part of Theorem 2.2, since MA is equivalent to $\mathsf{BFA}_{<2^{\aleph_0}}(\mathcal{P})$ for this $\mathcal{P}$.

Strictly speaking, Theorem 2.2 is different from the original theorem in Ikegami-Trang [27] (Theorem 1.13 there) in that Ikegami and Trang are talking about the Σ_n, Π_n-hierarchy $\mathsf{MP}(\cdots)_\Gamma$ for $\Gamma = \Sigma_n$, Π_n etc., which is shown to be different from $(\cdots)_\Gamma$-RcA^+ hierarchy (see Corollary 3.12 and the remark before the corollary). The proof above together with the proof of Theorem 1.13 in [27] actually shows that for a normal class of posets, $(\mathcal{P}, \mathcal{H}(\aleph_2))_{\Sigma_1}$-$\mathsf{RcA}^+$ coincides with $\mathsf{MP}(\mathcal{P}, \mathcal{H}(\aleph_2))_{\Sigma_1}$ (see Theorem 3.1 and Corollary 3.2).

$(\mathcal{P}, \mathcal{H}(\aleph_2))_\Gamma$-$\mathsf{RcA}^+$ and $\mathsf{MP}(\mathcal{P}, \mathcal{H}(\aleph_2))_\Gamma$ in general can be different principles. We will address to this subtle difference in the next Section 3, and show that these two hierarchies can split up drastically on the Π_2 and Σ_2 levels (see Corollary 3.12).

2.2 (In)compatibility of Recurrence and Maximality with Ground Axiom

The *Ground Axiom* (abbreviation: GA) is the axiom asserting that there is no proper ground of the universe V. The axiom is introduced by Joel Hamkins and Jonas Reitz. Its basic properties including the formalizability of the axiom in $\mathcal{L}_\in$ are proved in Reitz [33]. ground-axiom

The relative consistency of GA with PFA is proved in [33] (see also the proof of Theorem 3.8 below; actually GA is even consistent with MM^{++}, see Theorem 6.3). In particular, this and Ikegami-Trang Theorem 2.2 imply:

Theorem 2.3 GA *is relatively consistent with* $(\mathcal{P}, \mathcal{H}(\aleph_2))_{\Sigma_1}$-$\mathsf{RcA}^+$ *for a class of posets* $\mathcal{P}$ *whose elements are proper.* ❑ *p-rec-GA-0*

Since the Recurrence Axiom implies that there are "many" different grounds, it is clear that Proposition 2.3 cannot be generalized for $(\cdots)_\Gamma$-RcA for arbitrary Γ. In particular, since Ground Axiom itself is formalizable in a Π_3-sentence in $\mathcal{L}_\in$ (see the remark after Lemma 3.4), and it is not true in any non-trivial generic extension of the ground model, we obtain:

Theorem 2.4 *Suppose* $(\mathcal{P}, \emptyset)_{\Sigma_3}$-$\mathsf{RcA}$ *holds for a non-trivial class* $\mathcal{P}$ *of posets. Then* GA *does not hold.* *p-rec-GA-0-a*
$\mathsf{MP}^-(\mathcal{P}, \emptyset)_{\Sigma_3}$ *for a non-trivial* $\mathcal{P}$ *also implies* $\neg\mathsf{GA}$.

Proof. Assume toward a contradiction that $(\mathcal{P}, \emptyset)_{\Sigma_3}$-$\mathsf{RcA}$ holds for a non-trivial class $\mathcal{P}$ of posets, and GA also holds. Let ψ be the Π_3-sentence expressing that the universe does not have a non-trivial ground. Let $\mathbb{P} \in \mathcal{P}$ be non-trivial forcing. Then $\Vdash_{\mathbb{P}}$ "$\neg\psi$". Since $\neg\psi$ is a Σ_3-sentence, $(\mathcal{P}, \emptyset)_{\Sigma_3}$-$\mathsf{RcA}$ implies that there is a ground W of V such that $\mathsf{W} \models \neg\psi$. Since we also assumed GA, W must be identical with V. Thus $\mathsf{V} \models \neg\psi$. This is a contradiction.

Since ψ above is a $\mathcal{P}$-button, the same proof leads to a contradiction under $\mathsf{MP}^-(\mathcal{P},\emptyset)_{\Sigma_3}$. □ (Proposition 2.4)

Actually we also have the following delimitation, which show that Theorem 2.3 is optimal in many instances of $\mathcal{P}$.

In the following, we use a variant of the cardinal invariant $\mathfrak{b}^*$ introduced in Eda-Kada-Yuasa [9]:

$$\begin{aligned}\mathfrak{b}^{**} := \min\{\kappa \in \mathit{Card} \ :\ & \text{for any } B \subseteq {}^{\omega}\omega, \text{ if } B \text{ is unbounded in } {}^{\omega}\omega \text{ with} \\ & \text{respect to } \leq^* \text{, then there is } B' \subseteq B \text{ with} \\ & |\,B'\,| \leq \kappa \text{ such that } B' \text{ is unbounded in } B\}.\end{aligned}$$

Lemma 2.5 (1) *$\mathfrak{b} = \aleph_1$ can be formulated as a Σ_2-sentence φ in $\mathcal{L}_\in$.* p-rec-GA-0-0

(2) *$\mathfrak{b}^{**} = \aleph_1$ can be formulated as a Π_2-sentence ψ in $\mathcal{L}_\in$.*

(3) *$\mathfrak{b} < \mathfrak{d}$ can be formulated as a Π_2-sentence η in $\mathcal{L}_\in$.*

Proof. (1): The following formula φ will do:

$$\begin{aligned}\exists B\, \exists R\, \exists F\, (\, & B \subseteq {}^{\omega}\omega \wedge \text{ ``} R \text{ is an } \omega_1\text{-like linear ordering on } B \text{ which is} \\ & \text{witnessed by } F\text{''} \ \wedge\ \forall f\, (f \in {}^{\omega}\omega \ \rightarrow\ (\exists g \in B)\, (g \not\leq^* f)))\,.\end{aligned}$$

(2): The following formula ψ will do:

$$\begin{aligned}\forall B\, (\, & \overbrace{B \subseteq {}^{\omega}\omega}^{\Sigma_0} \wedge \overbrace{B \text{ is unbounded in } {}^{\omega}\omega}^{\Pi_1 \text{ and hence its negation is } \Sigma_1} \ \rightarrow \\ & \exists B'\, \exists R\, \exists F\, (\, \overbrace{B' \subseteq B}^{\Sigma_0} \wedge \overbrace{B' \text{ is unbounded in } B}^{\Sigma_0} \\ & \qquad \wedge \underbrace{R \text{ is } \omega_1\text{-like order on } B' \text{ which is witnessed by } F}_{\Sigma_0}))\,.\end{aligned}$$

(3): "$\mathfrak{b} = \mathfrak{d}$" is characterized by the existence of a bounding family $\subseteq {}^{\omega}\omega$ which is well ordered with respect to $\leq^*$. Similarly to above, this can be formulated by a Σ_2-sentence. Hence, the negation of the equality ($\Leftrightarrow \mathfrak{b} < \mathfrak{d}$) is Π_2 in $\mathcal{L}_\in$. □ (Lemma 2.5)

Lemma 2.6 $\mathfrak{b} \leq \mathfrak{b}^{**} \leq \mathfrak{d}$. p-rec-GA-0-1

Proof. Let $\langle f_\alpha : \alpha < \mathfrak{b}\rangle$ be such that $f_\alpha \leq^* f_{\alpha'}$ for all $\alpha < \alpha' < \mathfrak{b}$, and $\{f_\alpha : \alpha < \mathfrak{b}\}$ is unbounded in ${}^{\omega}\omega$ (this can be done by letting $\{g_\alpha : \alpha < \mathfrak{b}\}$ be a unbounded subset of ${}^{\omega}\omega$, and defining f_α, $\alpha < \mathfrak{b}$ inductively such that we have $g_\alpha \leq^* f_\alpha$ for all $\alpha < \mathfrak{b}$). Let $B = \{f_\alpha : \alpha < \mathfrak{b}\}$. Then no $B' \subseteq B$ with $|\,B'\,| < |\,B\,| = \mathfrak{b}$ is unbounded in $\mathcal{B}$.

Note that the sequence $\langle f_\alpha : \alpha < \mathfrak{b}\rangle$ as above also shows that $\mathfrak{b}$ is a regular cardinal.

This proves that $\mathfrak{b} \leq \mathfrak{b}^{**}$.

To show that $\mathfrak{b}^{**} \leq \mathfrak{d}$, suppose that $D \subseteq {}^\omega\omega$ is dominating in ${}^\omega\omega$ (with respect to $\leq^*$), and $|\,D\,| = \mathfrak{d}$.

For any unbounded $B \subseteq {}^\omega\omega$, and for each $d \in D$ let $b_d \in B$ be such that $b_d \not\leq^* d$. Then $B' = \{b_d : d \in D\} \subseteq B$ is unbounded in ${}^\omega\omega$ and hence also unbounded in B and is of cardinality $\leq \mathfrak{d}$. This shows that $\mathfrak{b}^{**} \leq \mathfrak{d}$. □ (Lemma 2.6)

In the following, we denote with $\mathbb{C}_\kappa$ the finite support κ-product of Cohen forcing, and with $\mathbb{D}$ the finite support iteration of Hechler forcing of length ω_1. It is easy to see that (over an arbitrary ground model V) we have $\Vdash_{\mathbb{C}_\kappa}$ "$\mathfrak{b} = \aleph_1, \mathfrak{d} \geq \kappa$" for any regular $\kappa \geq \aleph_1$, and $\Vdash_{\mathbb{D}}$ "$\mathfrak{d} = \aleph_1$". More generally, letting $\mathbb{D}_\kappa$ be the FS-iteration of Hechler forcing of length κ for regular κ, we have $\Vdash_{\mathbb{D}_\kappa}$ "$\mathfrak{b} = \mathfrak{d} = \kappa$".

Proposition 2.7 *Suppose $\mathcal{P}$ is a class of posets with $\mathbb{D} \in \mathcal{P}$ and (2.3): $\mathfrak{b} > \aleph_1$ holds.* p-rec-GA-1 x-rec-GA-a-0

(1) *If $(\mathcal{P}, \emptyset)_{\Sigma_2}$-$\mathsf{RcA}$ holds, then GA does not hold.*

(2) *If $(\mathcal{P}, \emptyset)_{\Pi_2}$-$\mathsf{RcA}$ holds, then GA does not hold.*

Proof. Suppose that $\mathcal{P}$ is as above, and (2.3) holds.

(1): Suppose that $(\mathcal{P}, \emptyset)_{\Sigma_2}$-$\mathsf{RcA}$ and GA hold.

Since we have $\Vdash_{\mathbb{D}}$ "$\mathfrak{b} = \aleph_1$", and since "$\mathfrak{b} = \aleph_1$" is expressible in a Σ_2-sentence in $\mathcal{L}_\in$ by Lemma 2.5, (1), it follows by $(\mathcal{P}, \emptyset)_{\Sigma_2}$-$\mathsf{RcA}$, that there is a ground W_0 with $\mathsf{W}_0 \models$ "$\mathfrak{b} = \aleph_1$". Since $\mathsf{V} = \mathsf{W}_0$ by GA, this is a contradiction to (2.3).

(2): Suppose that $(\mathcal{P}, \emptyset)_{\Pi_2}$-$\mathsf{RcA}$ and GA hold.

Note that by (2.3) and Lemma 2.6, we have (2.4): $\mathsf{V} \models \mathfrak{b}^{**} > \aleph_1$. x-rec-GA-0

Since we have $\Vdash_{\mathbb{D}}$ "$\mathfrak{d} = \aleph_1$", we have $\Vdash_{\mathbb{D}}$ "$\mathfrak{b}^{**} = \aleph_1$" by Lemma 2.6. Since "$\mathfrak{b}^{**} = \aleph_1$" is expressible in a Π_2-sentence in $\mathcal{L}_\in$ by Lemma 2.5, (2), it follows by $(\mathcal{P}, \emptyset)_{\Pi_2}$-$\mathsf{RcA}$, that there is a ground W_0 with $\mathsf{W}_0 \models$ "$\mathfrak{b}^{**} = \aleph_1$". Since $\mathsf{V} = \mathsf{W}_0$ by GA, it follows that $\mathsf{V} \models$ "$\mathfrak{b}^{**} = \aleph_1$". This is a contradiction to (2.4). □ (Proposition 2.7)

The following can be proved similarly to Proposition 2.7, (1).

Proposition 2.8 *Suppose that $\mathcal{P}$ is a class of posets with $\mathbb{C}_{\aleph_1} \in \mathcal{P}$ and $\mathfrak{b} \geq \aleph_2$ holds. Then $(\mathcal{P}, \emptyset)_{\Sigma_2}$-$\mathsf{RcA}$ implies that GA does not hold.* □ p-rec-GA-1-0

Note:

Lemma 2.9 CH *can be formulated both as Σ_2-sentence and Π_2-sentence in $\mathcal{L}_\in$ without parameters.* p-rec-GA-1-1

Proof. Consider

$$\exists F\ \exists R\ (\ \overbrace{R \text{ is } \omega_1\text{-like ordering on } {}^{\omega}\omega \text{ and } R \text{ witnesses the } \omega_1\text{-likeness}}^{\Pi_1\text{: note that we need here the quantification of the sort } \forall f\ (f \in {}^{\omega}\omega\ \rightarrow\ ...)}\ )$$

and

$$\forall F\,\forall S\,\forall\, A\,(\,(\overbrace{S = \text{“}{}^{\omega}\omega\text{”}\ \wedge\ A \text{ is an ordinal}\ \wedge\ F : S \to A \text{ is a surjection}}^{\Pi_1\text{, so the negation is }\Sigma_1}) \rightarrow\ \underbrace{A < \omega_2}_{\Sigma_1}\ ).$$

❑ (Lemma 2.9)

Proposition 2.10 (1) *Suppose that* $\neg$CH *holds and* $\mathcal{P}$ *contains a poset collapsing* $2^{\aleph_0}$ *to* $\aleph_1$ *without adding reals. Then each of* $(\mathcal{P},\emptyset)_{\Sigma_2}$-RcA *and* $(\mathcal{P},\emptyset)_{\Pi_2}$-RcA *implies* $\neg$GA. p-rec-GA-1-1-0

(2) *Suppose that* CH *holds and* $\mathcal{P}$ *contains a poset* $\mathbb{Q}$ *adding* $\geq \aleph_2$ *reals without collapsing cardinals* $\leq \aleph_2$*. Then each of* $(\mathcal{P},\emptyset)_{\Sigma_2}$-RcA *and* $(\mathcal{P},\emptyset)_{\Pi_2}$-RcA *implies* $\neg$GA.

(3) *Suppose that* $\mathcal{P}$ *contains sufficiently many ccc posets (containing enough CS-iterations of Cohen and Hechler posets would suffice), then each of* $(\mathcal{P},\emptyset)_{\Sigma_2}$-RcA *and* $(\mathcal{P},\emptyset)_{\Pi_2}$-RcA *implies* $\neg$GA.

Proof. Similarly to the proof of Proposition 2.7 using Lemma 2.9. For (3), we consider cases where (a) $\aleph_1 = \mathfrak{b} = \mathfrak{d}$, (b) $\aleph_1 < \mathfrak{b} = \mathfrak{d}$, or (c) $\aleph_1 < \mathfrak{b} < \mathfrak{d}$, and apply Lemma 2.5 in all of the cases. ❑ (Proposition 2.10)

2.3 Incompatibility of Laver genericity with Ground Axiom

In the following we want to discuss the impact of the results we obtained above on axioms stating that there is a Laver-generic large cardinal (Laver-gen. large cardinal axioms).

The strongest Laver-generic large cardinal axiom which has been considered so far, is the tightly super-$C^{(\infty)}$ $\mathcal{P}$-Laver-generically hyperhuge cardinal (see Fuchino and Usuba [20]). Here, a cardinal κ is said to be (*tightly*, resp.) *super* $C^{(\infty)}$ $\mathcal{P}$-*Laver-generically hyperhuge* if for all $n \in \mathbb{N}$ and for any $\lambda_0 > \kappa$ there are $\lambda \geq \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, and j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$, $j(\kappa) > \lambda$, ${}^{j(\lambda)}M \subseteq M$, $V_{j(\lambda)} \prec_{\Sigma_n} \mathsf{V}$ (and $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is of size $\leq j(\kappa)$, resp.).[4)]

We can also define (tightly) super $C^{(\infty)}$ $\mathcal{P}$-Laver gen. large cardinal analogously for notions of large cardinal other than hyperhugeness (see [20] or [13]). For an

[4)] When we say “a poset $\mathbb{P}$ is of cardinality $\leq \mu$” we actually mean that there is a poset $\mathbb{Q}$ forcing equivalent to $\mathbb{P}$ such that $|\,\mathbb{Q}\,| \leq \mu$.

iterable class $\mathcal{P}$ of posets which also permits transfinite iteration with some suitable support, we can prove that the existence of the tightly super $C^{(\infty)}$ $\mathcal{P}$-Laver gen. hyperhuge cardinal is consistent under a 2-huge cardinal ([20]).

Note that we cannot formulate the (genuine) large cardinal property corresponding to (tightly) super $C^{(\infty)}$ $\mathcal{P}$-Laver-generically large cardinal in $\mathcal{L}_\in$. However, for a natural class $\mathcal{P}$ of posets like proper posets, semiproper posets, ccc posets, etc. we can formulate the notion of (tightly) super $C^{(\infty)}$ $\mathcal{P}$-Laver-generically hyperhugeness in an axiom scheme in $\mathcal{L}_\in$. This is because $\mathcal{P}$-Laver-generically hyperhugeness of a cardinal κ implies $\kappa = \kappa_{\mathfrak{refl}}$ $(:= \max\{\aleph_2, 2^{\aleph_0}\})$ for these classes $\mathcal{P}$ of posets, and hence we can formulate the (tightly) super $C^{(\infty)}$ $\mathcal{P}$-Laver-generically hyperhugeness of such κ in infinitely many formulas without introducing a new constant symbol for the cardinal.

In Fuchino and Usuba [20], it is proved that if κ is tightly super $C^{(\infty)}$ $\mathcal{P}$-Laver-generically ultrahuge, then $(\mathcal{P}, \mathcal{H}(\kappa))$-$\mathsf{RcA}^+$ holds. Here the tightly super $C^{(\infty)}$ $\mathcal{P}$-Laver-generically ultrahugeness is apparently much weaker than tightly super $C^{(\infty)}$ $\mathcal{P}$-Laver-generically hyperhugeness.

Note that $(\mathcal{P}, \mathcal{H}(\kappa))$-$\mathsf{RcA}^+$ is also an assertion formalizable only in infinitely many formulas. In contrast, it is proved in [12], that, in a sense, Laver-genericity without "super $C^{(\infty)}$" details never implies the full $(\mathcal{P}, \mathcal{H}(\kappa))$-$\mathsf{RcA}^+$.

By the result mentioned above and by Theorem 2.4, it follows immediately that:

Proposition 2.11 *For any iterable class $\mathcal{P}$ of posets, if κ is tightly super $C^{(\infty)}$ $\mathcal{P}$-Laver-generically ultrahuge, then* GA *does not hold.* ❏ p-rec-GA-1-1-1

In [20], it is proved that if κ is tightly $\mathcal{P}$-generically hyperhuge (not necessarily Laver-generic) then there is the bedrock (i.e. the ground satisfying GA) and κ his hyperhuge in the bedrock.

On the other hand, it is shown in [13] that a tightly $\mathcal{P}$-Laver-generically ultrahuge cardinal for nice iterable $\mathcal{P}$ is $\leq \kappa_{\mathfrak{refl}}$. Here an iterable class $\mathcal{P}$ of posets is said to be *nice* if either $\mathcal{P}$ preserves ω_1 and $\mathrm{Col}(\omega_1, \{\omega_2\}) \in \mathcal{P}$, or $\mathcal{P}$ contains a poset which adds a new real. Actually the following lemma is one of the main rationales of the definition of the cardinal $\kappa_{\mathfrak{refl}}$.

Lemma 2.12 *Suppose that $\mathcal{P}$ is a nice iterable class of posets. If κ is $\mathcal{P}$-Laver-gen. supercompact, then $\kappa \leq \kappa_{\mathfrak{refl}}$.* p-rec-GA-1-1-2

Proof. By Lemma 6.,(2) and (3) in [13]. ❏ (Lemma 2.12)

Thus we obtain:

Proposition 2.13 *For a nice iterable class $\mathcal{P}$ of posets, suppose that there is a tightly $\mathcal{P}$-Laver-generically hyperhuge cardinal. Then the bedrock exists and it is different from* V*. In particular,* GA *does not hold.* p-rec-GA-1-2

Proof. By Lemma 2.12, the tightly $\mathcal{P}$-Laver-generically hyperhuge cardinal is $\leq \kappa_{\mathfrak{refl}}$ (in V) while κ is hyperhuge in the bedrock $\overline{\mathsf{W}}$. This implies that $\mathsf{V} \neq \overline{\mathsf{W}}$. In particular, GA does not hold. ❑ (Proposition 2.13)

At the moment we do not know if the existence of a tightly $\mathcal{P}$-generically hyperhuge in theorem in [20] mentioned above can be weakened to the existence of some tight generic large cardinal of lower consistency strength. However, in [13], it is proved that for an iterable class $\mathcal{P}$ of posets, if κ is tightly $\mathcal{P}$-Laver-gen. ultrahuge then $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_2}$-$\mathsf{RcA}^+$ holds (Theorem 21 in [13]). Note that ultrahuge cardinal is apparently much weaker than hyperhuge cardinal.

Theorem 2.14 *Suppose that $\mathcal{P}$ is an iterable class of posets satisfying one of the conditions in Proposition 2.10.*
If $\kappa_{\mathfrak{refl}}$ is tightly $\mathcal{P}$-Laver-gen. ultrahuge then GA *does not hold.* p-rec-GA-2

Proof. If κ is tightly $\mathcal{P}$-Laver-gen. ultrahuge then $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_2}$-$\mathsf{RcA}^+$ holds by Theorem 21 in [13]. Thus if $\mathbb{D} \in \mathcal{P}$ then (by Proposition 5.5 below and) by Proposition 2.7,(1), it follows that GA does not hold.

Other cases can treated similarly by applying other assertions of Proposition 2.10.

❑ (Theorem 2.14)

3 Hierarchies of restricted Recurrence Axioms and Maximality Principles

Ikegami and Trang [27] formulated Maximality Principle slightly different from our notation. Their Maximality Principle in restricted form is defined for a class $\mathcal{P}$ of posets, cardinal κ and set Γ of formulas (in $\mathcal{L}_\in$) as: hierarchies

(3.1) For any formula $\varphi \in \Gamma$ and $A \subseteq \kappa$, if $\varphi(A)$ is a $\mathcal{P}$-button then $\varphi(A)$ holds in V. x-hierarchies-a

Since (tuples of) elements of $\mathcal{H}(\kappa^+)$ can be coded by subsets of κ we have:

Lemma A 3.1 *For any class of posets $\mathcal{P}$, cardinal κ, and a set Γ of $\mathcal{L}_\in$-formulas, we have:*
The Maximality Principle of Ikegami and Trang (3.1) $\Leftrightarrow$ $\mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa^+))_\Gamma$. ❑ p-hierarchies-0

Thus, Ikegami and Trang's Theorem (Theorem 1.13 in [27]) is reformulated as:

(3.2) $\mathsf{MP}(\mathcal{P},\mathcal{H}(\aleph_2))_{\Sigma_1} \;\Leftrightarrow\; \mathsf{BFA}_{<\aleph_2}(\mathcal{P})$ for any (normal) class $\mathcal{P}$ of posets. x-hierarchies-0

Actually, almost the same argument as the proof of Theorem 2.2 given above, we can show the following more general theorem:

Theorem 3.1 (Generalization of the original Ikegami-Trang Theorem) *For a (normal) class $\mathcal{P}$ of posets, and a cardinal κ the following are equivalent:* p-hierarchies-1

(a) $\mathsf{MP}(\mathcal{P},\mathcal{H}(\kappa))_{\Sigma_1}$.

(b) $\mathsf{MP}^-(\mathcal{P},\mathcal{H}(\kappa))_{\Sigma_1}$.

(c) $\mathsf{BFA}_{<\kappa}(\mathcal{P})$.

Proof. (a) $\Rightarrow$ (b): is trivial,

(b) $\Rightarrow$ (c): Assume $\mathsf{MP}^-(\mathcal{P},\mathcal{H}(\kappa))_{\Sigma_1}$. By Bagaria's Absoluteness Theorem 1.2, it is enough to show that $\varphi(\overline{a}) \Leftrightarrow\ \Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$" for all $\mathbb{P}\in\mathcal{P}$, Σ_1-formula φ in $\mathcal{L}_\in$ and $\overline{a}\in\mathcal{H}(\kappa)$.

If $\varphi(\overline{a})$ holds then, since φ is Σ_1, $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$" also holds.

Suppose that $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$". Then, for any $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of poset $\Vdash_{\mathbb{P}}$ " $\Vdash_{\underset{\sim}{\mathbb{Q}}}$ "$\varphi(\overline{a})$" " since φ is Σ_1. In particular $\varphi(\overline{a})$ is a $\mathcal{P}$-button with the push $\mathbb{P}$ of the button. By $\mathsf{MP}^-(\mathcal{P},\mathcal{H}(\kappa))_{\Sigma_1}$, there is a ground W_0 of V such that $\overline{a}\in\mathsf{W}_0$ and $\mathsf{W}_0\models\varphi(\overline{a})$. Since φ is Σ_1, it follows that $\mathsf{V}\models\varphi(\overline{a})$.

(b) $\Rightarrow$ (c): Assume $\mathsf{BFA}_{<\kappa}(\mathcal{P})$. By Bagaria's Absoluteness Theorem 1.2, this means that $\varphi(\overline{a}) \Leftrightarrow\ \Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$" for all $\mathbb{P}\in\mathcal{P}$, Σ_1-formula φ in $\mathcal{L}_\in$ and $\overline{a}\in\mathcal{H}(\kappa)$.

Suppose that $\Vdash_{\mathbb{P}}$ " $\Vdash_{Q}$ "$\varphi(\overline{a})$" for all $Q\in\mathcal{P}$" for $\mathbb{P}$, φ, $\overline{a}$ as above. Since $\{\mathbb{1}\}\in\mathcal{P}$ it follows that $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$". By assumption, it follow that $\varphi(\overline{a})$ holds.

❑ (Theorem 3.1)

Corollary 3.2 *For a class $\mathcal{P}$ of posets and for an infinite cardinal κ, we have* p-hierarchies-2

$$\begin{aligned}&\mathsf{MP}(\mathcal{P},\mathcal{H}(\kappa))_{\Sigma_1}, \;\Leftrightarrow\; \mathsf{MP}^-(\mathcal{P},\mathcal{H}(\kappa))_{\Sigma_1}, \;\Leftrightarrow\; \mathsf{BFA}_{<\kappa}(\mathcal{P}),\\ &\Leftrightarrow\; (\mathcal{P},\mathcal{H}(\kappa))_{\Sigma_1}\text{-}\mathsf{RcA}, \;\Leftrightarrow\; (\mathcal{P},\mathcal{H}(\kappa))_{\Sigma_1}\text{-}\mathsf{RcA}^+.\end{aligned}$$

Proof. By Theorem 2.2 and Theorem 3.1. ❑ (Corollary 3.2)

The following lemma holds since Π_1-formulas are downward absolute.

Lemma 3.3 *For any class $\mathcal{P}$ of posets, and any set A, $(\mathcal{P},A)_{\Pi_1}$-RcA^+ and $\mathsf{MP}(\mathcal{P},A)_{\Pi_1}$ hold (in ZFC). In particular, we have* p-hierarchies-3

$$(\mathcal{P},A)_{\Pi_1}\text{-}\mathsf{RcA} \;\Leftrightarrow\; (\mathcal{P},A)_{\Pi_1}\text{-}\mathsf{RcA}^+ \;\Leftrightarrow\; \mathsf{MP}^-(\mathcal{P},A)_{\Pi_1} \;\Leftrightarrow\; \mathsf{MP}(\mathcal{P},A)_{\Pi_1}\,.$$

❑

In the following, we show that the equivalence in Lemma 3.3 does not hold for Π_2.

Nevertheless, we have the following implications.

Lemma 3.4 *Suppose that $\mathcal{P}$ is a (normal) class of posets defined by a Σ_m-formula without parameters for some number m, and A a set.*[5)] p-hierarchies-4

(1) $(\mathcal{P}, A)_{\Pi_n}$-$\mathsf{RcA}^+ \;\Rightarrow\; \mathsf{MP}(\mathcal{P}, A)_{\Pi_n}$, *for all* $n \geq \max\{m, 1\}$.

(2) $\mathsf{MP}(\mathcal{P}, A)_{\Sigma_n} \;\Rightarrow\; (\mathcal{P}, A)_{\Sigma_n}$-$\mathsf{RcA}^+$, *for all* $n \geq \max\{m, 3\}$.

Proof. The following proofs are just re-examinations of the easy proof of Proposition 2.1, (1) (e.g. the one given in Fuchino and Usuba [20]).

(1): Note that, for $n = 1$, the claim also follows from Lemma 3.3.

Assume that $(\mathcal{P}, A)_{\Pi_n}$-RcA^+ holds for $n \geq \max\{m, 2\}$. To show that $\mathsf{MP}(\mathcal{P}, A)_{\Pi_n}$ holds, suppose that $\varphi = \varphi(\overline{x})$ is a Π_n-formula, $\overline{a} \in A$, and $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}}$ "$\forall P \in \mathcal{P}\,(\Vdash_P$ "$\varphi(\overline{a})$")" holds in V.

"$\forall P \in \mathcal{P}\,(\Vdash_P$ "$\varphi(\overline{x})$")" is Π_n by the choice of n. Let us denote this formula by φ^*. Thus, we have $\Vdash_{\mathbb{P}}$ "$\varphi^*(\overline{a})$".

By $(\mathcal{P}, A)_{\Pi_n}$-RcA^+, it follows that there is a $\mathcal{P}$-ground W of V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi^*(\overline{a})$. By the definition of φ^*, and since W is a $\mathcal{P}$-ground, it follows that $\mathsf{V} \models \varphi(\overline{a})$.

(2): Assume that $\mathsf{MP}(\mathcal{P}, A)_{\Sigma_n}$ holds. Suppose that φ is Σ_n-formula, $\overline{a} \in A$, and $\mathbb{P} \in \mathcal{P}$ is such that

(3.3) $\quad \Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$". x-hierarchies-1-0

Then we have $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$ holds in a $\mathcal{P}$-ground".

The assertion

(3.4) $\quad$ "$\varphi(\overline{x})$ holds in a $\mathcal{P}$-ground" x-hierarchies-2

can be expressed in a Σ_n-formula $\varphi^{**} = \varphi^{**}(\overline{x})$ (see the remark after the proof of the present lemma). By (3.3) and by the definition (3.4) of φ^{**} we have $\Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}}$ "$\varphi^{**}(\overline{a})$" for all $\mathbb{P}$-names $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$". Thus, by $\mathsf{MP}(\mathcal{P}, A)_{\Sigma_n}$, it follows that $\mathsf{V} \models \varphi^{**}(\overline{a})$. By the definition of φ^{**}, it follows that there is a $\mathcal{P}$-ground W_0 such that $\mathsf{W}_0 \models \varphi(\overline{a})$. ❑ (Lemma 3.4)

The fact that (3.4) can be formulated in a Σ_n-formula for $n \geq \max\{m, 3\}$, can be seen as follows: First, let us recall the following fact.

---

5) Note that "x is c.c.c. poset", "x is proper posets", "x is semi-proper poset" are all Σ_2-statements. In case of "x is (semi-)proper poset", this can be seen in the formulation:

$$\text{“}\exists \underline{\kappa}\, \exists F\, (\underline{\kappa} \text{ is a cardinal } \wedge F \text{ “codes” the fact “}\underline{\kappa} \geq (\beth_\omega)^+(|\,x\,|)\text{”} \ \wedge\ \cdots).\text{”}$$

Here, the underline to κ is added to suggest that the symbol does not denote a constant symbol but rather a variable in the language $\mathcal{L}_\in$. We shall keep this convention in the following.

Lemma 3.5 *Suppose that $\psi = \psi(x, \overline{y})$ is a Σ_m-formula and φ is Σ_n-formula (Π_n-formula, resp.). Then $\varphi^{\psi(x;\overline{y})}$ is a $\Sigma_{max\{m,n\}}$-formula (a $\Pi_{max\{m,n\}}$-formula, resp.).*[6)] p-hierarchies-4-0

Proof. For quantifier free formula φ, the claim of the Lemma is true since $\varphi^{\psi} = \varphi$.

Suppose that, for a Σ_n-formula $\varphi_0 = \varphi_0(x_0, \overline{x})$ (Π_n-formula $\varphi_1 = \varphi_1(x_0, \overline{x})$), ${\varphi_0}^{\psi(x;\overline{y})}$ is Σ_k (${\varphi_1}^{\psi(x;\overline{y})}$ is Π_k resp.).

Then

$$(\forall x_0)(\psi(x_0, \overline{y}) \rightarrow {\varphi_0}^{\psi(x;\overline{y})}) \text{ is } \Pi_{\max\{m,k+1\}}, \text{ and}$$
$$(\exists x_0)(\psi(x_0, \overline{y}) \wedge {\varphi_1}^{\psi(x;\overline{y})}) \text{ is } \Sigma_{\max\{m,k+1\}}.$$

Using this fact, the claim of the Lemma can be proved now by induction on n.

❏ (Lemma 3.5)

An examination of [6] and [33] reveals a construction of a Π_2-formula $\Phi(x, \underline{P}, \underline{\delta}, r, \underline{G})$ which says that

> $\underline{P}$ is a poset, δ is a regular cardinal in V, there is a uniquely determined inner model $\mathcal{M}$ with δ-cover and δ-approximation properties such that $r = ({}^{\delta>}2)^{\mathcal{M}}$, $\underline{P} \in \mathcal{M}$, $\mathcal{M} \neq \mathsf{V}$, $\underline{G}$ is an $(\mathcal{M}, \underline{P})$-generic set such that $\mathsf{V} = \mathcal{M}[\underline{G}]$, and $x \in \mathcal{M}$.

Let $\psi = \psi(x)$ be a Σ_m-formula expressing "$x \in \mathcal{P}$". Then, for a Σ_n-formula $\varphi = \varphi(\overline{x})$ for $n \geq \max\{m, 3\}$, the formula $\varphi^{**}(\overline{x})$ defined as

$$\exists\, \underline{P} \exists\, \underline{\delta} \,\exists\, r \,\exists\, \underline{G}\, (\Phi(\emptyset, \underline{P}, \underline{\delta}, r, \underline{G}) \wedge \psi^{\Phi(x;\cdots)}(\underline{P}) \wedge \varphi^{\Phi(x;\cdots)}(\overline{x}))$$

is Σ_n by Lemma 3.5, and $\varphi^{**}(\overline{a})$ expresses "$\varphi(\overline{a})$ holds in a $\mathcal{P}$-ground".

We shall also use the following variant of Maximality Principle. Let $\mathcal{P}$, A, Γ be as before.

$\mathsf{MP}^+(\mathcal{P}, A)_\Gamma$: For $\varphi \in \Gamma$, and $\overline{a} \in A$, if $\varphi(\overline{a})$ is a $\mathcal{P}$-button, then $\{\mathbb{1}\}$ is a push of the $\mathcal{P}$-button $\varphi(\overline{a})$.

As before, we drop the subscript Γ from $\mathsf{MP}^+(\mathcal{P}, A)_\Gamma$ if $\Gamma = \mathcal{L}_\in$.

Lemma 3.6 (1) $\mathsf{MP}^+(\mathcal{P}, A)_\Gamma \;\Rightarrow\; \mathsf{MP}(\mathcal{P}, A)_\Gamma$. p-hierarchies-5

(2) (Hamkins [25]) $\mathsf{MP}^+(\mathcal{P}, A) \;\Leftrightarrow\; \mathsf{MP}(\mathcal{P}, A)$. *More precisely, if $\mathcal{P}$ is Σ_m-*

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6) Here, we denote by $\varphi^{\psi(x;\overline{y})}$ the formula φ restricted to $\psi(x, \overline{y})$ where $\psi(x, \overline{y})$ is thought to be the definition of the class $\mathcal{A}_{\overline{y}} = \{x : \psi(x, \overline{y})\}$ with parameters (or, more precisely, place holders for parameters) $\overline{y}$. The semi-colon in "$\varphi^{\psi(x;\overline{y})}$" should remind this allocation of roles among the free variables of ψ.

Thus $\varphi^{\psi(x,\overline{y})}$ corresponds to the informal statement: $\mathcal{A}_{\overline{y}} \models \varphi$.

definable, then for any $n \geq \max\{m,1\}$, we have $\mathsf{MP}^+(\mathcal{P},A)_{\Pi_n} \;\Leftrightarrow\; \mathsf{MP}(\mathcal{P},A)_{\Pi_n}$.

(3) *For an iterable $\mathcal{P}$, if $\mathsf{MP}^+(\mathcal{P},A)_\Gamma$, then for any $\mathbb{P} \in \mathcal{P}$, we have*

$$\Vdash_{\mathbb{P}} \text{“}\,\mathsf{MP}^+(\mathcal{P},A)_\Gamma\,\text{”}.$$

Proof. (1): is clear by definition.

(2): By (1), it is enough to show “$\Leftarrow$”, Assume that $\mathsf{MP}(\mathcal{P},A)$ holds, and suppose that $\varphi(\overline{a})$ is a $\mathcal{P}$-button for an $\mathcal{L}_\in$-formula φ and $\overline{a} \in A$. Then $\varphi^* := \forall Q\,(Q \in \mathcal{P} \;\rightarrow\; \Vdash_Q \text{“}\,\varphi(\overline{a})\,\text{”})$ is a $\mathcal{P}$-button. Hence, by $\mathsf{MP}(\mathcal{P},A)$, φ^* holds in V. But this means that $\{\mathbb{1}\}$ is a push for the button φ.

(3): Suppose that $\mathsf{MP}^+(\mathcal{P},A)_\Gamma$ holds (in V). For $\varphi \in \Gamma$, and $\overline{a} \in A$, let $\mathbb{P} \in \mathcal{P}$ be such that it forces that $\varphi(\overline{a})$ is a $\mathcal{P}$-button. By Maximal Principle (of forcing), there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of a poset such that

$$\Vdash_{\mathbb{P}} \text{“}\,\underset{\sim}{\mathbb{Q}} \in \mathcal{P} \;\wedge\; \Vdash_{\underset{\sim}{\mathbb{Q}}} \text{“}\,\forall R\,(R \in \mathcal{P} \;\rightarrow\; \Vdash_R \text{“}\,\varphi(\overline{a})\,\text{”})\,\text{”}\,\text{”}.$$

Since $\mathcal{P}$ is iterable, it follows that $\varphi(\overline{a})$ is a $\mathcal{P}$-button over V. Thus, by $\mathsf{MP}^+(\mathcal{P},A)_\Gamma$, $\{\mathbb{1}\}$ is a push of the $\mathcal{P}$-button $\varphi(\overline{a})$ (in V).

Again since $\mathcal{P}$ is iterable, it follows that $\Vdash_{\mathbb{P}} \text{“}\,\{\mathbb{1}\}$ is a push of the $\mathcal{P}$-button $\varphi(\overline{a})\,\text{”}$.

❑ (Lemma 3.6)

The following should be folklore:

Lemma 3.7 (1) *If α is a limit ordinal and V_α satisfies a sufficiently large finite fragment of* ZFC, *then for any $\mathbb{P} \in V_\alpha$ and $(\mathsf{V},\mathbb{P})$-generic $\mathbb{G}$, we have $V_\alpha[\mathbb{G}] = {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$.* p-hierarchies-5-0

(2) *If α is a limit ordinal and V_α satisfies a sufficiently large finite fragment of* ZFC, *then for any direct limit $\mathbb{P}$ of an iteration of length On^{V_α} in $\mathbb{P} \in V_\alpha$ definable in V_α and preserving cardinals in V_α, if $\mathbb{G}$ is $(\mathsf{V},\mathbb{P})$-generic, then we have*

$$V_\alpha[\mathbb{G}] = {V_\alpha}^{\mathsf{V}[\mathbb{G}]}.$$

(3) *For each natural number k, there is a sufficiently large $k' > k$ such that for any $\alpha \in \mathrm{On}$ if $V_\alpha \prec_{\Sigma_{k'}} \mathsf{V}$ (i.e. α is $\Sigma_{k'}$-correct), then for any poset $\mathbb{P} \in V_\alpha$ and $(\mathsf{V},\mathbb{P})$-generic $\mathbb{G}$, ${V_\alpha}^{\mathsf{V}[\mathbb{G}]} \prec_{\Sigma_k} \mathsf{V}[\mathbb{G}]$.*

(4) *Suppose that $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha \in \mathrm{On}\rangle$ is an Easton support class iteration of increasingly directed closed posets and $\mathfrak{P}$ is the class direct limit of the iteration. If k is a natural number and κ is a regular cardinal which is $\Sigma_{k'}$-correct for a sufficiently large $k' > k$, then we have ${V_\kappa}^{\mathsf{V}[\mathbb{G}_\kappa]} \prec_{\Sigma_k} \mathsf{V}[\mathfrak{G}]$ for any $(\mathsf{V},\mathfrak{P})$-generic $\mathfrak{G}$ and $\mathbb{G}_\kappa = \mathfrak{G} \cap \mathbb{P}_\kappa$.*

Proof. (1): See e.g. Lemma 3.2 in [20]. (2): follows from (1).

(3): see the proof of Lemma 4.8, (1) in the extended version of [20].

(4): Let $\Phi = \Phi(x)$ be an $\mathcal{L}_\in$-formula which defines $\mathfrak{P}$. Then by the choice of κ, we have $\mathbb{P}_\kappa = \Phi^{V_\kappa{}^{\mathsf{V}}}$. The claim (3) follows from this fact with an argument practically identical to that for (3). □ (Lemma 3.7)

By Theorem 2.4, we cannot replace Π_2 in the next theorem by Σ_3.

Theorem 3.8 *Suppose that $\mathcal{P}$ is a Σ_2-definable iterable class of posets containing all σ-closed posets, and that $\mathsf{MP}^+(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Pi_2}$ holds. Suppose further that there is a proper class $\mathcal{K}$ of supercompact cardinals.* p-hierarchies-6

If $\mathfrak{P}$ is the class poset for Laver preparation for $\mathcal{K}$ (see the proof below for more details), then we have

$$\Vdash_{\mathfrak{P}} \text{“}\,\mathsf{GA}\ +\ \mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Pi_2} \\ +\ \textit{there are class many supercompact cardinals}\text{”}.$$

Proof. Let $\mathcal{P}$, $\mathcal{K}$ be as above.

Let $f : \mathrm{On} \to \mathsf{V}$ be a universal Laver function for $\mathcal{K}$. I.e., a class function f such that

(3.5) for any $\kappa \in \mathcal{K}$, we have $f \restriction \kappa : \kappa \to V_\kappa$, and for any $x \in \mathsf{V}$ and any $\lambda \geq \max\{\kappa, |\,trcl(x)\,|\}$, there is a normal ultrafilter $\mathcal{U}_{\kappa,\lambda,x}$ over $\mathcal{P}_\kappa(\lambda)$ and associated elementary embedding $f_{\kappa,\lambda,x} : \mathsf{V} \xrightarrow{\preceq}_\kappa M$ with $j_{\kappa,\lambda,x}(f)(\kappa) = x$. x-hierarchies-2-0

We may also assume that

(3.6) $f(\alpha) = 0$ for all $\alpha < \kappa_{\mathfrak{refl}}$. x-hierarchies-2-1

Note that $f \restriction \kappa$, $\kappa \in \mathcal{K}$ are uniformly definable across V_κ ($= \mathcal{H}(\kappa)$) for all $\kappa \in \mathcal{K}$.

A universal Laver function exists (see e.g. Apter [1], Lemma 1).

Let $\langle \mathbb{P}_\alpha : \alpha \in \mathrm{On}\rangle$ be the Laver preparation along with f making supercompactness of all $\kappa \in \mathcal{K}$ indestructible by κ-directed closed forcing.[7] I.e., $\langle \mathbb{P}_\alpha : \alpha \in \mathrm{On}\rangle$ is defined as the iterative part of the Easton support[8] iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha \in \mathrm{On}\rangle$ with a control sequence $\langle \lambda_\alpha : \alpha \in \mathrm{On}\rangle$ of cardinals defined recursively by

(3.7) If $\alpha \in \mathrm{On}$ is a limit and closed with respect to $\langle \lambda_\beta : \beta < \alpha\rangle$, $f(\alpha) = \langle \underset{\sim}{\mathbb{Q}}, \lambda\rangle$ with $\Vdash_{\mathbb{P}_\alpha}$ “$\underset{\sim}{\mathbb{Q}}$ is $<\alpha$-directed closed poset”, then $\underset{\sim}{\mathbb{Q}}_\alpha = \underset{\sim}{\mathbb{Q}}$ and $\lambda_\alpha = \lambda$; x-hierarchies-3

(3.8) Otherwise $\lambda_\alpha = \sup\{\lambda_\beta : \beta < \alpha\}$ and $\Vdash_{\mathbb{P}_\alpha}$ “$\mathbb{Q}_\alpha = \{\mathbb{1}\}$”. x-hierarchies-4

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7) Note that κ-directed closed means $\leq\kappa$-directed closed.

8) I.e. direct limit at $\mathbb{P}_\alpha$ for regular α and inverse ($\approx$ full support) limit at singular α.

Let $\mathfrak{P}$ be the class forcing which is the direct limit of $\langle \mathbb{P}_\alpha : \alpha \in \mathrm{On} \rangle$. For each $\kappa \in \mathcal{K}$, let $\mathfrak{P}_{>\kappa}$ be (the class $\mathcal{P}_\kappa$-name of) the κ-directed closed tail part of the iteration. Thus $\mathfrak{P} \sim \mathbb{P}_\kappa * \mathfrak{P}_{>\kappa}$ and $\Vdash_{\mathbb{P}_\kappa}$ "$\mathfrak{P}_{>\kappa}$ is a κ-directed closed class poset".

Let $\mathfrak{G}$ be $(\mathsf{V}, \mathfrak{P})$-generic. For each $\kappa \in \mathcal{K}$, let $\mathbb{G}_\kappa = \mathfrak{G} \cap \mathbb{P}_\kappa$. Each $\kappa \in \mathcal{K}$ is made indestructible under κ-directed closed forcing by $\mathbb{P}_\kappa$ (see e.g. [31]). In particular, κ remains supercompact in $\mathsf{V}[\mathbb{G}_\lambda]$ for all $\lambda \in \mathcal{K}$. It follows that κ remains supercompact also in $\mathsf{V}[\mathfrak{G}]$.

$\mathsf{V}[\mathfrak{G}] \models \mathsf{GA}$. This is because $\mathsf{V}[\mathfrak{G}]$ satisfies Continuum Coding Axiom (CCA), and GA follows from it (see [33] Theorem 3.2). That $\mathsf{V}[\mathfrak{G}]$ satisfies CCA follows from the fact that in $\mathsf{V}[\mathfrak{G}]$ there are cofinally many indestructible supercompact cardinals. ⟦The first author learned the following from G. Goldberg: Suppose that $a \subseteq \alpha$. Let $\kappa > \alpha$ be an indestructible supercompact cardinal. Let $\mathbb{P}$ be κ-directed closed poset such that $\Vdash_{\mathbb{P}}$ "$\forall \beta < \alpha\, (2^{\aleph_{\kappa+\beta+1}} = (\aleph_{\kappa+\beta+1})^+ \;\leftrightarrow\; \beta \in a)$". Since $\Vdash_{\mathbb{P}}$ "κ is supercompact" by assumption and a supercompact cardinal is Σ_2-correct. It follows that

$$\Vdash_{\mathbb{P}} \text{“} V_\kappa \models \exists \delta\, \forall \beta < \alpha\, (2^{\aleph_{\delta+\beta+1}} = (\aleph_{\delta+\beta+1})^+ \;\leftrightarrow\; \beta \in a) \text{”}$$

Since $\Vdash_{\mathbb{P}}$ "$V_\kappa = (V_\kappa)^{\mathsf{V}}$" by κ-directed closedness of $\mathbb{P}$, it follows that

$$\mathsf{V} \models \text{“} V_\kappa \models \exists \delta\, \forall \beta < \alpha\, (2^{\aleph_{\delta+\beta+1}} = (\aleph_{\delta+\beta+1})^+ \;\leftrightarrow\; \beta \in a) \text{”}.$$

Hence the assertion holds in V. ⟧

Thus, it is enough to show that $\mathsf{V}[\mathfrak{G}] \models \mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Pi_2}$. Note that we have

$$\mathcal{H}(\kappa_{\mathfrak{refl}})^{\mathsf{V}} = \mathcal{H}(\kappa_{\mathfrak{refl}})^{\mathsf{V}[\mathbb{G}_\kappa]} = \mathcal{H}(\kappa_{\mathfrak{refl}})^{\mathsf{V}[\mathfrak{G}]} \tag{3.9}$$ x-hierarchies-5

by $\min(\mathcal{K})$-directed closedness of $\mathfrak{P}$, and (3.6). Also

$$V_\kappa^{\mathsf{V}[\mathbb{G}_\kappa]} = V_\kappa^{\mathsf{V}[\mathfrak{G}]} \tag{3.10}$$ x-hierarchies-6

for all $\kappa \in \mathcal{K}$ by κ-directed closedness of $\mathfrak{P}_{>\kappa}$ and Lemma 3.7, (2).

Working in $\mathsf{V}[\mathfrak{G}]$, suppose that $\varphi = \varphi(\overline{x})$ is a Π_2-formula and $\overline{a} \in \mathcal{H}(\kappa_{\mathfrak{refl}})$ $(= \mathcal{H}(\kappa_{\mathfrak{refl}})^{\mathsf{V}})$.

Further in $\mathsf{V}[\mathfrak{G}]$, suppose that $\mathbb{S} \in \mathcal{P}$ is such that

$$\mathsf{V}[\mathfrak{G}] \models \;\Vdash_{\mathbb{S}} \text{“} \forall T \in \mathcal{P}\, (\Vdash_T \text{“} \varphi(\overline{a}) \text{”}) \text{”}. \tag{3.11}$$ x-hierarchies-7-a-a

We want to show that $\varphi(\overline{a})$ holds (in $\mathsf{V}[\mathfrak{G}]$).

By replacing V by $\mathsf{V}[\mathbb{G}_{\kappa_0}]$, and $\mathcal{K}$ by $\mathcal{K} \setminus \kappa_0 + 1$ for a large enough $\kappa_0 \in \mathcal{K}$ with $\mathbb{S} \in \mathsf{V}[\mathbb{G}_{\kappa_0}]$, we may assume that $\mathbb{S} \in {V_{\kappa_0}}^{\mathsf{V}}$ for a $\kappa_0 < \min(\mathcal{K})$. Let $\mathfrak{g}$ be $(\mathsf{V}[\mathfrak{G}], \mathbb{S})$-generic. Since $\mathbb{S} \in \mathcal{P}$ and since $\mathcal{P}$ is iterable, we have

(3.12) $\quad \mathsf{V}[\mathfrak{G}][\mathbb{g}] \models \forall T \in \mathcal{P}\,(\,\Vdash_T \text{“}\,\varphi(\overline{a})\,\text{”}).$ x-hierarchies-7-a

By (3.10) (and Lemma 3.7, (2)), we have

(3.13) $\quad {V_\kappa}^{\mathsf{V}[\mathbb{G}_\kappa][\mathbb{g}]} = {V_\kappa}^{\mathsf{V}[\mathfrak{G}][\mathbb{g}]}$ x-hierarchies-6-0

for all $\kappa \in \mathcal{K}$.

Note that each $\kappa \in \mathcal{K}$ remains supercompact in all of $V[\mathbb{G}_\kappa]$, $\mathsf{V}[\mathbb{G}_\kappa][\mathbb{g}]$, $V[\mathfrak{G}]$, and $\mathsf{V}[\mathfrak{G}][\mathbb{g}]$. Thus,

(3.14) $\quad {V_\kappa}^{\mathsf{V}[\mathfrak{G}]} \prec_{\Sigma_2} \mathsf{V}[\mathfrak{G}],$ x-hierarchies-7-0-a

(3.15) $\quad {V_\kappa}^{\mathsf{V}[\mathfrak{G}][\mathbb{g}]} \prec_{\Sigma_2} \mathsf{V}[\mathfrak{G}][\mathbb{g}],$ x-hierarchies-7-0

(3.16) $\quad {V_\kappa}^{\mathsf{V}[\mathbb{G}_\kappa]} \prec_{\Sigma_2} \mathsf{V}[\mathbb{G}_\kappa], \quad$ and x-hierarchies-7-a-0-0

(3.17) $\quad {V_\kappa}^{\mathsf{V}[\mathbb{G}_\kappa][\mathbb{g}]} \prec_{\Sigma_2} \mathsf{V}[\mathbb{G}_\kappa][\mathbb{g}]$ x-hierarchies-7-a-1

for all $\kappa \in \mathcal{K}$. By (3.10), and (3.14), we have

(3.18) $\quad {V_\kappa}^{\mathsf{V}[\mathbb{G}_\kappa]} \prec_{\Sigma_2} \mathsf{V}[\mathfrak{G}]$ x-hierarchies-7-a-2

for all $\kappa \in \mathcal{K}$. Similarly

(3.19) $\quad {V_\kappa}^{\mathsf{V}[\mathbb{G}_\kappa][\mathbb{g}]} \prec_{\Sigma_2} \mathsf{V}[\mathfrak{G}][\mathbb{g}]$ x-hierarchies-7-0-0

holds for all $\kappa \in \mathcal{K}$ by (3.13) and (3.15),

“$\forall T \in \mathcal{P}\,(\,\Vdash_T \text{“}\,\varphi(\overline{a})\,\text{”})$” is Π_2 (note that we need Σ_2-definability of $\mathcal{P}$ for this). Hence ${V_\kappa}^{\mathsf{V}[\mathbb{G}_\kappa][\mathbb{g}]} \models \forall T \in \mathcal{P}\,(\,\Vdash_T \text{“}\,\varphi(\overline{a})\,\text{”})$ by (3.12) and (3.19). By (3.17), it follows that $\mathsf{V}[\mathbb{G}_\kappa][\mathbb{g}] \models \forall T \in \mathcal{P}\,(\,\Vdash_T \text{“}\,\varphi(\overline{a})\,\text{”})$. This implies that $\varphi(\overline{a})$ is a $\mathcal{P}$-button in $\mathsf{V}[\mathbb{G}_\kappa]$.

Similarly, since $\mathsf{V}[\mathfrak{G}] \models \mathbb{S} \in \mathcal{P}$, and $\mathcal{P}$ is Σ_2, we have $\mathsf{V}[\mathbb{G}_\kappa] \models \mathbb{S} \in \mathcal{P}$ by (3.18) and (3.16).

Since we have $\mathsf{V}[\mathbb{G}_\kappa] \models \mathsf{MP}^+(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Pi_2}$ by Lemma 3.6, (3), it follows that $V[\mathbb{G}_\kappa] \models \varphi(\overline{a})$ for all $\kappa \in \mathcal{K}$.

Since φ is Π_2 and $\mathcal{K}$ is cofinal in On, it follows that $\mathsf{V}[\mathfrak{G}] \models \varphi(\overline{a})$ by (3.18).

This shows that $\mathsf{V}[\mathfrak{G}] \models \mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Pi_2}$ holds. ❑ (Theorem 3.8)

If we start from a ground model with a proper class $\mathcal{K}$ of $C^{(n)}$-supercompact cardinals (see Bagaria [5]) for sufficiently large n, we can improve the condition “$\mathcal{P}$ is Σ_2-definable” in Theorem 3.8 by “$\mathcal{P}$ is Σ_3-definable” (see the remark after the proof of Proposition 3.10).

“Π_2” and “$\mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Pi_2}$” in Theorem 3.8 can be also replaced by “Δ_3” and “$\mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Delta_3}$” which is not covered by $\mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Pi_2}$ (Proposition 3.10).

In the following, we quickly review the definition and some needed facts about the variation $\mathsf{MP}^*(\mathcal{P}, A)_\Gamma$ of the Maximality Principle which will be further studied in Fuchino, Gappo, Lietz, and Parente [14].

For an iterable class $\mathcal{P}$ of posets, a set A of parameters, and a set Γ of $\mathcal{L}_\in$-formulas, let $(\Gamma)^*_\mathcal{P}$ be the set of provably $\mathcal{P}$-persistent formulas in Γ. That is, the collection of all formulas $\varphi \in \Gamma$, $\varphi = \varphi(\overline{x})$ such that the $\mathcal{L}_\in$-sentence

$$(3.20) \quad (\varphi)^*_\mathcal{P} :\equiv \forall \overline{x}\, (\varphi(\overline{x}) \rightarrow \forall \underline{\mathbb{P}} \in \mathcal{P}\, (\Vdash_{\underline{\mathbb{P}}} \text{“}\varphi(\overline{x})\text{”})).$$ x-YAH-a

is provable in ZFC.

Note that if Γ is closed with respect to equivalence (which is provable in ZFC) and has a recursive representatives (modulo the equivalence), then the same holds for $(\Gamma)^*_\mathcal{P}$ (as far as $\mathcal{P}$ is a definable class but this is always assumed).

Now $\mathsf{MP}^*(\mathcal{P}, A)_\Gamma$ is defined as the axiom scheme consisting of formulas of the form

$$(3.21)_\varphi \quad (\forall \underline{\mathbb{P}} \in \mathcal{P})\, (\forall \overline{x} \in A)\, (\Vdash_{\underline{\mathbb{P}}} \text{“}\varphi(\overline{x})\text{”} \;\rightarrow\; \varphi(\overline{x}))$$ x-YAH-0

for each $\varphi \in (\Gamma)^*_\mathcal{P}$.

Similarly to the $\mathsf{MP}(\cdots)_\Gamma$ and $(\cdots)_\Gamma$-RcA^+ hierarchies, we write $\mathsf{MP}^*(\mathcal{P}, A)$ for $\mathsf{MP}^*(\mathcal{P}, A)_{\mathcal{L}_\in}$.

The main point of the definition of $\mathsf{MP}^*(\mathcal{P}, A)_\Gamma$ is that the persistence is coded in the collection $(\Gamma)^*_\mathcal{P}$ so that each formula $(3.21)_\varphi$ in $\mathsf{MP}^*(\mathcal{P}, A)_\Gamma$ remains at about the same complexity of φ.

Lemma 3.9 *For an iterable class $\mathcal{P}$ of posets, arbitrary set A of parameters and set Γ of formulas, the following are equivalent:* p-YAH-0

(a) $\mathsf{MP}(\mathcal{P}, A)_{(\Gamma)^*_\mathcal{P}}$, (b) $\mathsf{MP}^+(\mathcal{P}, A)_{(\Gamma)^*_\mathcal{P}}$, (c) $(\mathcal{P}, A)_{(\Gamma)^*_\mathcal{P}}$-$\mathsf{RcA}^+$,
(d) $\mathsf{MP}^*(\mathcal{P}, A)_\Gamma$.

Proof. We show (a) $\Leftrightarrow$ (d). Other equivalences can be proved similarly.

(a) $\Rightarrow$ (d): Assume that $\mathsf{MP}(\mathcal{P}, A)_{(\Gamma)^*_\mathcal{P}}$ holds. Suppose that $\varphi \in (\Gamma)^*_\mathcal{P}$, $\overline{a} \in A$ and $\Vdash_\mathbb{P} \text{“}\varphi(\overline{a})\text{”}$ holds for a $\mathbb{P} \in \mathcal{P}$. Then, since $\varphi \in (\Gamma)^*_\mathcal{P}$, $\varphi(\overline{a})$ is a $\mathcal{P}$-button and $\mathbb{P}$ is its push. By $\mathsf{MP}(\mathcal{P}, A)_{(\Gamma)^*_\mathcal{P}}$ it follows that $\mathsf{V} \models \varphi(\overline{a})$. This shows that $\mathsf{MP}^*(\mathcal{P}, A)_\Gamma$ holds.

(d) $\Rightarrow$ (a): Assume that $\mathsf{MP}^*(\mathcal{P}, A)_\Gamma$ holds and suppose that $\varphi \in (\Gamma)^*_\mathcal{P}$, $\overline{a} \in A$, $\varphi(\overline{a})$ is a $\mathcal{P}$-button and $\mathbb{P} \in \mathcal{P}$ is its push. Then, since $\{\mathbb{1}\} \in \mathcal{P}$, $\Vdash_\mathbb{P} \text{“}\varphi(\overline{a})\text{”}$. By $\mathsf{MP}^*(\mathcal{P}, A)_\Gamma$ it follows that $\mathsf{V} \models \varphi(\overline{a})$. This shows that $\mathsf{MP}(\mathcal{P}, A)_{(\Gamma)^*_\mathcal{P}}$ holds.

❑ (Lemma 3.9)

The hierarchy of this type of restricted Maximality Principles appears in Goodman [23]. Our $\mathsf{MP}^*(\mathcal{P},A)_\Gamma$ is called "Γ-$\mathsf{MP}_{\mathcal{P}}(A)$" in [23]. Though the choice of symbols in [23] is so that letter Γ is used to denote the class of posets and Φ to denote the class of formulas.

The hierarchy of MP^* is actually a special case of the hierarchy of "$\mathsf{BFA}(\mathcal{P},\Gamma)_{\kappa,\lambda}$" in Asperó [2] (see [14]: see also Lemma 6.10).[9)]

The proof of Lemma 3.4, (2), shows also the implication:

(2') $\mathsf{MP}^*(\mathcal{P},A)_{\Sigma_n} \;\Rightarrow\; (\mathcal{P},A)_{\Sigma_n}$-$\mathsf{RcA}^+$, for all $n \geq \max\{m,3\}$ where $\mathcal{P}$ is Σ_m.

Thus, by Theorem 2.4, $\mathsf{MP}^*(\mathcal{P},A)_{\Sigma_n}$ for $n \geq \max\{m,3\}$ for m as above implies $\neg\mathsf{GA}$. In particular, for Σ_3-definable $\mathcal{P}$, $\mathsf{MP}^*(\mathcal{P},A)_{\Sigma_3}$ implies $\neg\mathsf{GA}$. This shows that the condition Δ_3 in the following proposition is (almost) optimal.

Proposition 3.10 *Suppose that $\mathcal{P}$ is a Σ_3-definable iterable class of posets containing all σ-closed posets, and that $\mathsf{MP}^+(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Delta_3}$ holds. Suppose further that there is a proper class $\mathcal{K}$ of $C^{(n)}$-supercompact cardinals for a sufficiently large n and $\mathfrak{P}$ is the class poset defined as in the proof of Theorem 3.8 for this $\mathcal{K}$.* p-hierarchies-6-1
Then we have

$$\Vdash_{\mathfrak{P}} \text{“}\mathsf{GA} \;+\; \mathsf{MP}^*(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Delta_3}\text{”}.$$

Proof. Suppose that $\mathcal{K}$ and $\mathfrak{P}$ are as above and $\mathsf{MP}^+(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Delta_3}$ holds.

Let $\mathfrak{G}$ be a $(\mathsf{V},\mathfrak{P})$-generic filter.

As it has been already shown in the proof of Theorem 3.8, we have $\mathsf{V}[\mathfrak{G}] \models \mathsf{GA}$.

So we prove $\mathsf{V}[\mathfrak{G}] \models \mathsf{MP}^*(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Delta_3}$. Working in $\mathsf{V}[\mathfrak{G}]$, suppose that $\varphi = \varphi(\overline{x})$ is a $(\Delta_3)^*_{\mathcal{P}}$-formula and $\overline{a} \in \mathcal{H}(\kappa_{\mathfrak{refl}})$ $(= \mathcal{H}(\kappa_{\mathfrak{refl}})^{\mathsf{V}})$.

Further in $\mathsf{V}[\mathfrak{G}]$, suppose that $\mathbb{S} \in \mathcal{P}$ is such that

(3.22) $\mathsf{V}[\mathfrak{G}] \models \;\Vdash_{\mathbb{S}} \text{“}\varphi(\overline{a})\text{”}.$ x-hierarchies-8-0

We want to show that $\varphi(\overline{a})$ holds (in $\mathsf{V}[\mathfrak{G}]$).

Similarly to the proof of Theorem 3.8, we may assume that $\mathbb{S} \in {V_{\kappa_0}}^{\mathsf{V}}$ for a $\kappa_0 < \min(\mathcal{K})$. Let $\mathfrak{g}$ be $(\mathsf{V}[\mathfrak{G}],\mathbb{S})$-generic. By the choice (3.22) of $\mathbb{S}$, we have

(3.23) $\mathsf{V}[\mathfrak{G}][\mathfrak{g}] \models \varphi(\overline{a})$. x-hierarchies-8-1

By the choice of the "sufficiently large" n (in terms of Lemma 3.7, (4) and (3)), and by (3.10),

(3.24) ${V_\kappa}^{\mathsf{V}[\mathbb{G}_\kappa]} \prec_{\Sigma_3} \mathsf{V}[\mathfrak{G}]$, x-hierarchies-8-2

9) These principles are related but different from the Bounded Forcing Axioms $\mathsf{BFA}_{<\kappa}(\mathcal{P})$.

$$(3.25)\quad V_\kappa^{\mathsf{V}[\mathbb{G}_\kappa][\mathbb{g}]} \prec_{\Sigma_3} \mathsf{V}[\mathfrak{G}][\mathbb{g}],$$ x-hierarchies-9

and, since κ remains supercompact in $\mathsf{V}[\mathbb{G}_\kappa]$ and $\mathsf{V}[\mathbb{G}_\kappa][\mathbb{g}]$,

$$(3.26)\quad V_\kappa^{\mathsf{V}[\mathbb{G}_\kappa]} \prec_{\Sigma_2} \mathsf{V}[\mathbb{G}_\kappa],$$ x-hierarchies-9-0

$$(3.27)\quad V_\kappa^{\mathsf{V}[\mathbb{G}_\kappa][\mathbb{g}]} \prec_{\Sigma_2} \mathsf{V}[\mathbb{G}_\kappa][\mathbb{g}]$$ x-hierarchies-10

for all $\kappa \in \mathcal{K}$.

By (3.24), $\mathbb{S} \in \mathcal{P}$ (in $\mathsf{V}[\mathfrak{G}]$), and since $\mathcal{P}$ is Σ_3, $V_\kappa^{\mathsf{V}[\mathbb{G}_\kappa]} \models \mathbb{S} \in \mathcal{P}$. Hence, by (3.26), $\mathsf{V}[\mathbb{G}_\kappa] \models \mathbb{S} \in \mathcal{P}$.

Since φ is Δ_3, (3.23) and (3.25) implies $V_\kappa^{\mathsf{V}[\mathbb{G}_\kappa][\mathbb{g}]} \models \varphi(\overline{a})$. This and (3.27) imply $\mathsf{V}[\mathbb{G}_\kappa][\mathbb{g}] \models \varphi(\overline{a})$.

By $\varphi \in (\Delta_3)^*_{\mathcal{P}}$, it follows that $\mathsf{V}[\mathbb{G}_\kappa][\mathbb{g}] \models \forall \underset{\sim}{\mathbb{Q}} \in \mathcal{P}\,(\Vdash_{\underset{\sim}{\mathbb{Q}}} \text{“}\varphi(\overline{a})\text{”})$. Thus $\mathsf{V}[\mathbb{G}_\kappa] \models \exists \underset{\sim}{\mathbb{S}} \in \mathcal{P}\,(\forall \underset{\sim}{\mathbb{Q}}\,(\Vdash_{\underset{\sim}{\mathbb{S}}} \text{“}\underset{\sim}{\mathbb{Q}} \in \mathcal{P} \rightarrow \varphi(\overline{a})\text{”}))$.

Since $\mathsf{V}[\mathbb{G}_\kappa] \models \mathsf{MP}^+(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Delta_3}$ by Lemma 3.6, (3), it follows that

$$(3.28)\quad \mathsf{V}[\mathbb{G}_\kappa] \models \varphi(\overline{a}).$$ x-hierarchies-11

Since φ is Δ_3, and hence Π_3 in particular, $V_\kappa^{\mathsf{V}[\mathbb{G}_\kappa]} \models \varphi(\overline{a})$ by (3.26). Thus, by (3.28), it follows that $\mathsf{V}[\mathfrak{G}] \models \varphi(\overline{a})$.

This shows that $\mathsf{V}[\mathfrak{G}] \models \mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Delta_3}$ holds. ❏ (Proposition 3.10)

The first half of the proof of Proposition 3.10 can be applied to the proof of Theorem 3.8 to obtain:

(A variant of Theorem 3.8) *Suppose that $\mathcal{P}$ is a Σ_3-definable iterable class of posets containing all σ-closed posets, and that $\mathsf{MP}^+(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Pi_2}$ holds. Suppose further that there is a proper class $\mathcal{K}$ of $C^{(n)}$-supercompact cardinals for a sufficiently large n.*

If $\mathfrak{P}$ is the class poset for Laver preparation for $\mathcal{K}$ (see the proof below for more details), then we have

$$\Vdash_{\mathfrak{P}} \text{“}\mathsf{GA} \;+\; \mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Pi_2}$$
$$+\ \textit{there are class many supercompact cardinals}\text{”}.$$

The following theorem is also obtained by combining the proofs of Theorem 3.8 and Proposition 3.10 taking Lemma 3.9 into account.

Theorem 3.11 (1) *Suppose that $\mathcal{P}$ is a Σ_2-definable iterable class of posets containing all σ-closed posets such that $\mathsf{MP}^+(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_\Gamma$ holds where Γ denotes here the set of all formulas representable as the conjunction of a Σ_2-formula and a $(\Delta_3)^*_{\mathcal{P}}$-formula. Suppose further that there is a proper class $\mathcal{K}$ of supercompact cardinals and $\mathfrak{P}$ is the class poset defined as in the proof of Theorem 3.8 for this $\mathcal{K}$.* p-hierarchies-6-2

Then we have

$$\Vdash_{\mathfrak{P}} \text{“}\mathsf{GA}\ +\ \mathsf{MP}^*(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_\Gamma\text{”}.$$

(2) *Suppose that $\mathcal{P}$ is a Σ_3-definable iterable class of posets containing all σ-closed posets such that $\mathsf{MP}^+(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_\Gamma$ holds where Γ is the set of formulas defined as in (1). Suppose further that there is a proper class $\mathcal{K}$ of $C^{(n)}$-supercompact cardinals for a sufficiently large n, and $\mathfrak{P}$ is the class poset defined as in the proof of Theorem 3.8 for this $\mathcal{K}$.*
Then we have

$$\Vdash_{\mathfrak{P}} \text{“}\mathsf{GA}\ +\ \mathsf{MP}^*(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_\Gamma\text{”}. \qquad \square$$

The following Corollary shows in particular that the implication in Lemma 3.4, (1) for $n = 2$ cannot be reversed.

Corollary 3.12 (1) *Suppose that $\mathcal{P}$ is a Σ_2-definable iterable class of posets containing all σ-closed posets, and also a poset adding a real. Assume further that there is a proper class $\mathcal{K}$ of supercompact cardinals, and $\mathsf{MP}^+(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_\Gamma$ holds where Γ is defined just as in Theorem 3.11. Then, there is a class poset $\mathfrak{P}$ such that we have* p-hierarchies-7

$$\Vdash_{\mathfrak{P}} \text{“}\neg\,(\mathcal{P},\emptyset)_{\Pi_2}\text{-}\mathsf{RcA},\ \neg\,(\mathcal{P},\emptyset)_{\Sigma_2}\text{-}\mathsf{RcA}\ \textit{and}\ \mathsf{MP}(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_\Gamma\text{”}.$$

(2) *Suppose that $\mathcal{P}$ is a Σ_3-definable iterable class of posets containing all σ-closed posets, and a poset adding a real. Assume further that there is a stationary proper class $\mathcal{K}$ of $C^{(n)}$-supercompact cardinals for sufficiently large n, and $\mathsf{MP}^+(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_\Gamma$ holds for the set of formulas Γ as defined in Theorem 3.11. Then, there is a class poset $\mathfrak{P}$ such that we have*

$$\Vdash_{\mathfrak{P}} \text{“}\neg\,(\mathcal{P},\emptyset)_{\Pi_2}\text{-}\mathsf{RcA},\ \neg\,(\mathcal{P},\emptyset)_{\Sigma_2}\text{-}\mathsf{RcA}\ \textit{and}\ \mathsf{MP}^*(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_\Gamma\text{”}.$$

Proof. Note that $\mathsf{MP}^+(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$ for $\mathcal{P}$ as here implies $\neg\mathsf{CH}$ (see Fuchino and Usuba [20], Theorem 3.3).

(1): By Proposition 2.10, (1) and Proposition 3.11, (1).

(2): By Proposition 2.10, (1) and Theorem 3.11, (2). $\square$ (Corollary 3.12)

Typical instances of $\mathcal{P}$ in Corollary 3.12 are when $\mathcal{P}$ is the class of all proper posets, the class of all semi-proper posets, or the class of all stationary preserving posets.

Problem 3.13 *Do some theorems hold which would imply certain non-implications similar to those in Corollary 3.12 for $\mathcal{P} = \sigma$-closed posets, or $\mathcal{P} =$ ccc posets?*

4 Generic absoluteness under Recurrence Axioms

The conclusion of the following Theorem 4.1 generalizes that of Viale's Theorem 1.1 genabs-rec (Theorem 1.4 in [39]). Note that the assumption in our Theorem 4.1, the Recurrence Axiom $(\mathcal{P}, \mathcal{H}(\kappa))$-$\mathsf{RcA}^+$ for an uncountable cardinal κ and an iterable class $\mathcal{P}$ of posets, namely, is of much lower consistency strength than the assumptions in Viale's Theorem for some instances of $\mathcal{P}$. Actually the assumption of Theorem 4.1 is even compatible with $V = L$ for many "natural" classes $\mathcal{P}$ of posets including the cases "$\mathcal{P}$ = all ccc posets" or "$\mathcal{P}$ = all proper posets" (see Theorems 5.6, 5.10 in [25]). For "$\mathcal{P}$ = all stationary preserving posets", Theorem 1.6 in Ikegami and Trang [27] proves that the existence of proper class many strongly compact cardinals plus a reflecting cardinal is an upper bound of the consistency strength of the Maximality Principle for the $\mathcal{P}$.

Known lower bound of this Recurrence Axiom is also large. By Ikegami-Trang Theorem 2.2, (stationary preserving, $\mathcal{H}(2^{\aleph_0})$)-$\mathsf{RcA}$ is equivalent with BMM. Schindler [34] shows that BMM implies that there is an inner model with a strong cardinal.

In contrast, the Maximality Principle for $\mathcal{P}$ = semi-proper posets, the consistency strength is much lower than this by Asperó [2]. Note that, in general, semi-proper and stationary preserving are not identical notions.

The existence of the tightly super-$C^{(\infty)}$ $\mathcal{P}$-Laver-gen. hyperhuge cardinal κ is the known Laver-generic large cardinal axiom which implies the full Recurrence Axiom for $\mathcal{P}$ and $\mathcal{H}(\kappa)$ (see Fuchino, and Usuba [20]).

There is practically no (consistent) generic large cardinal axiom formalizable in a single formula which also implies $(\mathcal{P}, \emptyset)$-RcA for any sufficiently general class $\mathcal{P}$ of posets ([12]).

In [20], it is proved that for an iterable class $\mathcal{P}$ of posets, the existence of the tightly super-$C^{(\infty)}$ $\mathcal{P}$-Laver-gen. hyperhuge cardinal κ with $\kappa = \kappa_{\mathfrak{refl}}$ implies $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))$-$\mathsf{RcA}^+$ where $\kappa_{\mathfrak{refl}}$ is defined as $\kappa_{\mathfrak{refl}} := \max\{2^{\aleph_0}, \aleph_2\}$. Note that this does not contradict what we mentioned in the last paragraph since the tightly super-$C^{(\infty)}$ $\mathcal{P}$-Laver-gen. hyperhugeness of $\kappa_{\mathfrak{refl}}$ is only expressed by an axiom schema. Note that by [20] we know the exact consistency strength of this principle (as that of $\kappa_{\mathfrak{refl}}$ being super-$C^{(\infty)}$ hyperhuge in the bedrock).

In Section 5 we show that the generalization of the conclusion of Viale's Theorem 1.1 (like that of the following Theorem 4.1) already follows from tight $\mathcal{P}$-Laver-gen. hugeness. This assumption is still much stronger than that of Viale's Theorem 1.1 but the upper bound of the consistency strength of this Laver-genericity is far below the consistency strength of a tight super-$C^{(\infty)}$ $\mathcal{P}$-Laver-gen. hyperhuge cardinal.

Theorem 4.1 *Suppose that $\mathcal{P}$ is an iterable Σ_n-definable class of posets for $n \geq 2$ and $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_n \cup \Gamma}$-$\mathsf{RcA}^+$ holds for an uncountable cardinal κ where Γ is the set of all formulas which are conjunction of a Σ_2-formula and a Π_2-formula. Then, for any $\mathbb{P} \in \mathcal{P}$ such that $\Vdash_{\mathbb{P}}$ "$\mathsf{BFA}_{<\kappa}(\mathcal{P})$",* p-genabs-rec-0

$$\mathcal{H}(\mu^+)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}(\mu^+)^{\mathsf{V}[\mathbb{G}]} \quad \textit{holds for all } \mu < \kappa \textit{ and for } (\mathsf{V}, \mathbb{P})\textit{-generic } \mathbb{G}.$$

Thus, we have $\mathcal{H}(\kappa)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}((\kappa^{(+)})^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]}$.

Proof. Suppose that $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}}$ "$\mathsf{BFA}_{<\kappa}(\mathcal{P})$" and $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic filter. Let $\varphi = \varphi(x)$ be a Σ_2-formula in $\mathcal{L}_\in$, and $\varphi(x) = \exists y\, \psi(x, y)$ for a Π_1-formula ψ in $\mathcal{L}_\in$. Let $\mu < \kappa$ and $a \in \mathcal{H}(\mu^+)$ ($\subseteq \mathcal{H}(\kappa)$). We have to show that $\mathcal{H}(\mu^+)^{\mathsf{V}} \models \varphi(a) \Leftrightarrow \mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \varphi(a)$.

Suppose first that $\mathcal{H}(\mu^+)^{\mathsf{V}} \models \varphi(a)$. Let $b \in \mathcal{H}(\mu^+)^{\mathsf{V}}$ be such that $\mathcal{H}((\mu^+)^{\mathsf{V}})^{\mathsf{V}} \models \psi(a, b)$. Since we have $\mathsf{V} \models \mathsf{BFA}_{<\kappa}(\mathcal{P})$ by Ikegami-Trang Theorem 2.2, it follows that $\mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \psi(a, b)$ by Bagaria's Absoluteness Theorem 1.2, and thus $\mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \varphi(a)$.

Note that we did not use the assumption "$\Vdash_{\mathbb{P}}$ "$\mathsf{BFA}_{<\kappa}(\mathcal{P})$"" for this direction.

Suppose now $\mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \varphi(a)$. By $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_n \cup \Gamma}$-$\mathsf{RcA}^+$, there is a $\mathcal{P}$-ground W of V such that

(4.1) $\quad \mathsf{W} \models$ "$\mathsf{BFA}_{<\mu^+}(\mathcal{P}) \ \wedge\ \mathcal{H}(\mu^+) \models \varphi(a)$". x-genabs-rec-a-0

Note that the formula in (4.1) is Σ_n if $n \geq 3$ and Γ if $n = 2$.

Let $b \in \mathcal{H}((\mu^+)^{\mathsf{W}})^{\mathsf{W}}$ be such that $\mathsf{W} \models$ "$\mathcal{H}(\mu^+) \models \psi(a, b)$". By Bagaria's Absoluteness Theorem 1.2, and since V is a $\mathcal{P}$-generic extension of W, it follows that $\mathsf{V} \models$ "$\mathcal{H}(\mu^+) \models \psi(a, b)$" and hence $\mathcal{H}(\mu^+)^{\mathsf{V}} \models \varphi(a)$.

For the last statement of the present theorem, let φ be a Σ_2-formula, and $a \in \mathcal{H}(\kappa)$. If $\mathcal{H}(\kappa) \models \varphi(a)$, then, by (1.4), there is $\mu < \kappa$ such that $\mathcal{H}(\mu^+) \models \varphi(a)$. By the first part of the theorem, it follows that $\mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \varphi(a)$. Thus $\mathcal{H}((\kappa^{(+)})^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \varphi(a)$ by (1.4).

If $\mathcal{H}((\kappa^{(+)})^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \varphi(a)$, then there is $\mu < \kappa$ such that $\mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \varphi(a)$ (this is also shown using (1.4)). Hence $\mathcal{H}((\mu^+)^{\mathsf{V}}) \models \varphi(a)$ by the first part of the theorem. ❏ (Theorem 4.1)

Note that by Lemma 1.6, the conclusion (4.1) of Theorem 4.1 can be yet strengthened to

(4.2) $\quad (\mathcal{H}(\mu^+), \in, I_{\mathsf{NS}})^{\mathsf{V}} \prec_{\Sigma_2} (\mathcal{H}(\mu^+), \in, I_{\mathsf{NS}})^{\mathsf{V}[\mathbb{G}]}$ holds for all $\mu < \kappa$ and for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$. x-genabs-rec-a-1

5 Generic absoluteness under Laver-genericity

Laver-genericity also implies a conclusion similar to that of Viale's Theorem 1.1. genabs-Laver
Although this fact does not have an advantage in terms of consistency strength in comparison with Theorem 4.1, the Laver-generic large cardinal we need in Theorem 5.7 below is much weaker than the tight super $C^{(\infty)}$ $\mathcal{P}$-Laver-generic hyperhugeness, the generic large cardinal property which is known to imply the corresponding Recurrence Axiom used in Theorem 4.1 (see [12] and [20]).

In Viale [39], the absoluteness statement of his Theorem 1.1 is also discussed in connection with the Resurrection Axiom (see Theorem 5.2).

Adopting the generalized setting introduced in Fuchino [12], we define the Resurrection Axiom as follows: for an iterable class $\mathcal{P}$ of posets and a definition $\mu^\bullet$ of an uncountable cardinal (e.g. as $\aleph_1$, $\aleph_2$, $2^{\aleph_0}$, $(2^{\aleph_0})^+$, $\kappa_{\mathfrak{refl}}$ etc.),[10] *the Resurrection Axiom* for $\mathcal{P}$ and $\mu^\bullet$ is the statement:

$\mathsf{RA}(\mathcal{P},\mu^\bullet)$: For any $\mathbb{P}\in\mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of a poset such that $\Vdash_{\mathbb{P}}$ " $\underset{\sim}{\mathbb{Q}}\in\mathcal{P}$ " and $\mathcal{H}(\mu^\bullet)^{\mathsf{V}}\prec\mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}$ for any $(\mathsf{V},\mathbb{P}*\underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$.[11]

Lemma 5.1 (Hamkins, and Johnstone [26]) *Assume that $\mathcal{P}$ is an iterable class of posets, $\mu^\bullet$ is a definition of an uncountable cardinal, and $\mathsf{RA}(\mathcal{P},\mu^\bullet)$ holds. Then* p-genabs-Laver-a-0

(1) $\mathsf{BFA}_{<\mu^\bullet}(\mathcal{P})$ *holds.*

(2) *If all elements of $\mathcal{P}$ preserve stationarity of subsets of ω_1, $2^{\aleph_0}=2^{\aleph_1}$, and $\mu^\bullet=$ "$2^{\aleph_0}$", then $\mathsf{BFA}^{+<\mu^\bullet}_{<\mu^\bullet}(\mathcal{P})$ holds.*

Proof. (1): It is easy to check that (c) of Bagaria's Theorem 1.2 holds.[12]

Suppose that $\overline{a}\in\mathcal{H}(\mu^\bullet)$ and φ is a Σ_1-formula in $\mathcal{L}_\in$. For $\mathbb{P}\in\mathcal{P}$, let $\mathbb{G}$ be a $(V,\mathbb{P})$-generic filter.

If $\mathcal{H}(\mu^\bullet)^{\mathsf{V}}\models\varphi(\overline{a})$, then we have $\mathsf{V}\models\varphi(\overline{a})$ and hence $\mathsf{V}[\mathbb{G}]\models\varphi(\overline{a})$. Thus we have $\mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{G}]}\models\varphi(\overline{a})$ by (1.4).

Suppose now that $\mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{G}]}\models\varphi(\overline{a})$. Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name of a poset such that $\Vdash_{\mathbb{P}}$ " $\underset{\sim}{\mathbb{Q}}\in\mathcal{P}$ " such that for a $(\mathsf{V}[\mathbb{G}],\underset{\sim}{\mathbb{Q}}^{\mathbb{G}})$-generic $\mathbb{H}$,

(ℵ5.1) $\mathcal{H}(\mu^\bullet)^{\mathsf{V}}\prec\mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{G}*\mathbb{H}]}$. x-genabs-Laver-a-a

[10] More precisely, when we say "$\mu^\bullet$ is a definition of an uncountable cardinal" we mean that ZF or ZFC proves the statement "$\mu^\bullet$ uniquely exists and $\mu^\bullet$ is an uncountable cardinal".

[11] Here we mean with "$\mathcal{H}(\mu^\bullet)^{\mathsf{V}}\prec\mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}$" the elementarity $\mathcal{H}((\mu^\bullet)^{\mathsf{V}})^{\mathsf{V}}\prec\mathcal{H}((\mu^\bullet)^{\mathsf{V}[\mathbb{H}]})^{\mathsf{V}[\mathbb{H}]}$.

[12] Actually we do not need this condition in this part of the proof and hence we can obtain the Bounded Forcing Axiom under a weaker notion of Resurrection Axiom in which the second step $\underset{\sim}{\mathbb{Q}}$ may be anything.

Since $\mathsf{V}[\mathbb{G}] \models \varphi(\overline{a})$, we have $\mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{G}*\mathbb{H}]} \models \varphi(\overline{a})$ by the same argument as in the first part of this proof. Thus, by ($\aleph$5.1), it follows that $\mathcal{H}(\mu^\bullet)^{\mathsf{V}} \models \varphi(\overline{a})$.

(2): Similarly to (1) using (c) of Theorem 1.8.[13)] ❏ (Lemma 5.1)

Theorem 5.2 (A generalization of Theorem 5.1 in Viale [39]) *Suppose that $\mathcal{P}$ is an iterable class of posets, $\mu^\bullet$ is a definition of an uncountable cardinal and* $\mathsf{RA}(\mathcal{P}, \mu^\bullet)$ *holds. Then we have* p-genabs-Laver-a-1

$$\mathcal{H}(\mu^\bullet)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{G}]}$$

for any $\mathbb{P} \in \mathcal{P}$ such that $\Vdash_{\mathbb{P}}$ "$\mathsf{BFA}_{<\mu^\bullet}(\mathcal{P})$", and $(\mathsf{V}, \mathcal{P})$-generic $\mathbb{G}$.

Proof. An argument similar to that of Lemma 5.1 will do. ❏ (Theorem 5.2)

In [12], the boldface version of the following is proved:

Theorem 5.3 (Fuchino [12]) *For an iterable class $\mathcal{P}$ of posets, and a definition $\mu^\bullet$ of a cardinal, if $\mu^\bullet$ is tightly $\mathcal{P}$-Laver gen. superhuge, then* $\mathsf{RA}(\mathcal{P}, \mu^\bullet)$ *holds.* ❏ p-genabs-Laver-a-2

Corollary 5.4 *For an iterable class $\mathcal{P}$ of posets, , and a definition $\mu^\bullet$ of a cardinal, if $\mu^\bullet$ is tightly $\mathcal{P}$-Laver gen. superhuge, Then we have* p-genabs-Laver-a-3

$$\mathcal{H}(\mu^\bullet)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{G}]}$$

for any $\mathbb{P} \in \mathcal{P}$ such that $\Vdash_{\mathbb{P}}$ "$\mathsf{BFA}_{<\mu^\bullet}(\mathcal{P})$" and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$.

Proof. By Theorem 5.2 and Theorem 5.3. ❏ (Corollary 5.4)

Note that for many cases (with natural setting of $\mathcal{P}$ and the notion of large cardinal), if κ is $\mathcal{P}$-Laver-gen. large cardinal, then $\kappa = \kappa_{\mathfrak{refl}}$ (see Fuchino, Ottenbreit Maschio Rodrigues, and Sakai [15]).

In the following, we show in a direct proof, that Corollary 5.4 can be yet slightly improved (see Theorem 5.7 below).

It is known that Laver-gen. large cardinal axiom proves strong forms of double-plus forcing axioms (see Theorem 5.7 in [15]). In the Proposition 5.5 below we only recap a part of this result needed for the following argument.

For a class $\mathcal{P}$ of posets, and cardinals κ and λ,

($\mathsf{FA}^{+<\lambda}_{<\kappa}(\mathcal{P})$): For any $\mathbb{P} \in \mathcal{P}$, any family $\mathcal{D}$ of dense subsets of $\mathcal{P}$ with $|\mathcal{D}| < \kappa$, and any family $\mathcal{S}$ of $\mathbb{P}$-names of stationary subsets of ω_1 with $|\mathcal{S}| < \lambda$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ on $\mathbb{P}$ such that $\underset{\sim}{S}[\mathbb{G}]$ is a stationary subset of ω_1 for all $\underset{\sim}{S} \in \mathcal{S}$.

13) Note that for this proof, the weaker variant of $\mathsf{RA}(\mathcal{P})$ as in the proof of (1) is apparently not sufficient.

Note that MM^{++} is just $\mathsf{FA}^{+<\aleph_2}_{<\aleph_2}$(stationary preserving posets). $\mathsf{FA}_{<\kappa}(\mathcal{P})$ is the principle we obtain by dropping the mention about $\mathcal{S}$ from the definition of $\mathsf{FA}^{+<\lambda}_{<\kappa}(\mathcal{P})$.

Proposition 5.5 (1) *Suppose that κ is $\mathcal{P}$-Laver-gen. supercompact. Then $\mathsf{FA}_{<\kappa}(\mathcal{P})$ holds.* p-genabs-Laver-a

(2) *If all elements of the class $\mathcal{P}$ of posets are stationary preserving and κ is $\mathcal{P}$-Laver-gen. supercompact, then $\mathsf{FA}^{+<\kappa}_{<\kappa}(\mathcal{P})$ holds.*

Proof. We prove (2). (1) can be shown by a subset of this proof.

Assume that κ is a $\mathcal{P}$-Laver-gen. supercompact cardinal, and let $\mathcal{P}$, $\mathcal{D}$, $\mathcal{S}$ be as in the definition of $\mathsf{FA}^{+<\kappa}_{<\kappa}(\mathcal{P})$. Let D_α, $\alpha < \mu$ and $\underset{\sim}{S}_\alpha$, $\alpha < \mu'$ be enumerations of $\mathcal{D}$ and $\mathcal{S}$ respectively.

Let $\lambda = |\,\mathbb{P}\,|$. Without loss of generality, we may assume that $\lambda > \kappa$ and the underlying set of $\mathcal{P}$ is λ. Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and such that for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ such that $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, $j(\kappa) > \lambda$, $j''\lambda$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$.

Let $\mathbb{G}$ be the $\mathbb{P}$ part of $\mathbb{H}$. Then, since $j(\mu) = \mu$, $j(\mu') = \mu'$, $j(\mathcal{D}) = \{j(D_\alpha) : \alpha < \mu\}$, and $j(\mathcal{S}) = \{j(\underset{\sim}{S}_\alpha) : \alpha < \mu\}$, we have

(5.1) $\quad M \models$ " $j''\mathbb{G}$ generates a filter on $j(\mathbb{P})$ which is $j(\mathcal{D})$-generic, and realizes elements of $j(\mathcal{S})$ to be stationary". x-genabs-rec-0

Note that we need here the condition that $\mathcal{P}$ is stationary preserving since otherwise the stationary set $\underset{\sim}{S}[\mathbb{G}]$ in $\mathsf{V}[\mathbb{G}]$ might be no more stationary in $\mathsf{V}[\mathbb{H}]$.

(5.1) implies that

$M \models$ " there is a $j(\mathcal{D})$-generic filter on $j(\mathbb{P})$ which realizes all elements of $j(\mathcal{S})$ to be stationary".

By elementarity, it follows that

$\mathsf{V} \models$ " there is a $\mathcal{D}$-generic filter on $\mathbb{P}$ which realizes all elements of $\mathcal{S}$ to be stationary".

❑ (Proposition 5.5)

Lemma 5.6 *Suppose that κ is $\mathcal{P}$-Laver-gen. supercompact for an iterable $\mathcal{P}$. Then we have $\mathcal{H}(\kappa)^{\mathsf{V}} \prec_{\Sigma_1} \mathcal{H}((\kappa^{(+)})^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]}$ for any $\mathbb{P} \in \mathcal{P}$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$.* p-genabs-Laver-0

Proof. By Proposition 5.5, (1) and Bagaria's Absoluteness Theorem 1.2. ❑ (Lemma 5.6)

The following theorem improves Corollary 5.4.

Theorem 5.7 *For an iterable class $\mathcal{P}$ of posets, suppose that $\mathsf{BFA}_{<\kappa}(\mathcal{P})$ holds, and κ is tightly $\mathcal{P}$-Laver-gen. huge.*[14] *Then, for any $\mathbb{P} \in \mathcal{P}$ such that $\Vdash_{\mathbb{P}}$ "$\mathsf{BFA}_{<\kappa}(\mathcal{P})$",* p-genabs-Laver-1

$$\mathcal{H}(\mu^+)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}(\mu^+)^{\mathsf{V}[\mathbb{G}]} \quad \textit{holds for all } \mu < \kappa \textit{ and for } (\mathsf{V}, \mathbb{P})\textit{-generic } \mathbb{G}.$$

Thus, we have $\mathcal{H}(\kappa)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}((\kappa^{(+)})^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]}$.

Proof. Suppose that $\Vdash_{\mathbb{P}}$ "$\mathcal{H}(\mu^+) \models \varphi(\overline{a})$" for $\mathbb{P} \in \mathcal{P}$ with $\Vdash_{\mathbb{P}}$ "$\mathsf{BFA}_{<\kappa}(\mathcal{P})$", $\mu < \kappa$, Σ_2-formula φ and for $\overline{a} \in \mathcal{H}(\mu^+)$. Let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic filter. Then we have

(5.2) $\quad \mathsf{V}[\mathbb{G}] \models$ "$\mathsf{BFA}_{<\kappa}(\mathcal{P}) \wedge \mathcal{H}(\mu^+) \models \varphi(\overline{a})$". x-genabs-Laver-a-0

Let $\varphi = \exists y \psi(\overline{x}, y)$ where ψ is a Π_1-formula in $\mathcal{L}_{\in}$. Let $b \in \mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]}$. be such that $\mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \psi(\overline{a}, b)$.

Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" such that, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$ with

(5.3) $\quad \mathbb{G} \subseteq \mathbb{H}$ (under the identification $\mathbb{P} \lessdot \mathbb{P} * \underset{\sim}{\mathbb{Q}}$), x-genabs-Laver-a-1

there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$,

(5.4) $\quad |\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,| \leq j(\kappa) \qquad$ (by tightness), x-genabs-Laver-0

(5.5) $\quad \mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$ and x-genabs-Laver-1

(5.6) $\quad j''j(\kappa) \in M$. x-genabs-Laver-2

By (5.2), (5.3) and Bagaria's Absoluteness Theorem 1.2 (applied to $V[\mathbb{G}]$), we have $\mathsf{V}[\mathbb{H}] \models$ "$\psi(\overline{a}, b)$" and hence $\mathsf{V}[\mathbb{H}] \models$ "$\mathcal{H}(\mu^+) \models \psi(\overline{a}, b)$".

By (5.4), (5.5) and (5.6), there is a $\mathbb{P}$-name of b in M. By (5.5), it follows that $b \in M$. By similar argument, we have $\mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{H}]})^{\mathsf{V}[\mathbb{H}]} \subseteq M$ and hence $\mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{H}]})^{\mathsf{V}[\mathbb{H}]} = \mathcal{H}((\mu^+)^M)^M \in M$. Thus we have $M \models$ "$\mathcal{H}(\mu^+) \models \psi(\overline{a}, b)$".

By elementarity, it follows that $\mathsf{V} \models$ "$\mathcal{H}(\mu^+) \models \psi(\overline{a}, b)$", and hence $\mathsf{V} \models$ "$\mathcal{H}(\mu^+) \models \varphi(\overline{a})$" as desired.

Suppose now that $\mathbb{P}$, μ, φ, $\overline{a}$ are as above and assume that $\mathsf{V} \models$ "$\mathcal{H}(\mu^+) \models \varphi(\overline{a})$" holds. For a Π_1-formula ψ as above, let $b \in \mathcal{H}(\mu^+)^{\mathsf{V}}$ be such that $V \models$ "$\mathcal{H}(\mu^+) \models \psi(\overline{a}, b)$".

Since $\mathsf{V} \models \mathsf{BFA}_{<\kappa}(\mathcal{P})$ by assumption, it follows that $\mathsf{V}[\mathbb{G}] \models \psi(\overline{a}, b)$ by Bagaria's Absoluteness Theorem 1.2, and hence $\mathsf{V}[\mathbb{G}] \models \varphi(\overline{a})$.

The last assertion of the theorem follows by the same argument as that given at the end of the proof of Theorem 4.1. ❑ (Theorem 5.7)

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[14] Note that by Proposition 5.5, $\mathsf{BFA}_{<\kappa}(\mathcal{P})$ follows from the assumption that κ is $\mathcal{P}$-Laver-generic supercompact. Thus the conclusion of the theorem follows from the combination of the assumption κ being $\mathcal{P}$-Laver-generic supercompact and tightly $\mathcal{P}$-Laver-gen. huge. Note also that this combination follows from the tight $\mathcal{P}$-Laver-gen. superhugeness of κ. fn-genabs-Laver-0

6 Some more remarks and open questions

In this final section we collect some observations we could not put in the appropriate places in previous sections, and mention some open questions. misc

Since most of the claims in this section are easy consequences of known results, some of them may be already known.

6.1 Restricted Recurrence Axioms under Laver-genericity

As we already mentioned at the end of Section 2, the existence of a tightly $\mathcal{P}$-Laver gen. ultrahuge cardinal κ implies $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_2}$-$\mathsf{RcA}^+$ (Theorem 21 in [13]). This result can be slightly improved so that its conclusion stands in line with the assumptions of Theorem 4.1 for the case of $n = 2$. RRAL

Theorem 6.1 (A slightly improved version of Theorem 21 in Fuchino [13]) *Suppose that κ is tightly $\mathcal{P}$-Laver-generically ultrahuge for an iterable class $\mathcal{P}$ of posets. Then $(\mathcal{P}, \mathcal{H}(\kappa))_\Gamma$-$\mathsf{RcA}^+$ holds where Γ is the set of all formulas which are conjunctions of a Σ_2-formula and a Π_2-formula.* *p-Lg-RcA-0*

Proof. A slight modification of the proof given in [13] will do. Nevertheless, we present the proof here because of the subtlety of the modification of the proof around (6.8).

Assume that κ is tightly $\mathcal{P}$-Laver generically ultrahuge for an iterable class $\mathcal{P}$ of posets.

Suppose that $\varphi = \varphi(\overline{x})$ is Σ_2 formula (in $\mathcal{L}_\in$), $\psi = \psi(\overline{x})$ is Π_2 formula (in $\mathcal{L}_\in$), $\overline{a} \in \mathcal{H}(\kappa)$, and $\mathbb{P} \in \mathcal{P}$ is such that

(6.1) $\quad \mathsf{V} \models \Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}) \wedge \psi(\overline{a})$". x-Lg-RcA-a

Let $\lambda > \kappa$ be such that $\mathbb{P} \in \mathsf{V}_\lambda$ and

(6.2) $\quad V_\lambda \prec_{\Sigma_n} \mathsf{V}$ for a sufficiently large n. x-Lg-RcA-0

In particular, we may assume that we have chosen the n above so that a sufficiently large fragment of ZFC holds in V_λ ("sufficiently large" means here, in particular, in terms of Lemma 3.7, (1) and that the argument at the end of this proof is possible).

Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", and for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with

(6.3) $\quad j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M,$ x-Lg-RcA-1

(6.4) $\quad j(\kappa) > \lambda,$ x-Lg-RcA-1-0

(6.5) $\quad \mathbb{P} * \underset{\sim}{\mathbb{Q}},\ \mathbb{P},\ \mathbb{H},\ V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \in M,$ and x-Lg-RcA-1-1

(6.6) $\quad |\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,| \leq j(\kappa).$ x-Lg-RcA-1-2

By (6.6), we may assume that the underlying set of $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is $j(\kappa)$ and $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in V_{j(\lambda)}{}^{\mathsf{V}}$.

Let $\mathbb{G} := \mathbb{H} \cap \mathbb{P}$. Note that $\mathbb{G} \in M$ by (6.5) and we have

(6.7) $$V_{j(\lambda)}{}^{M} \underbrace{=}_{\text{by (6.5)}} V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \overbrace{=}^{\substack{\text{Since } V_{j(\lambda)}{}^{M}\ (= V_{j(\lambda)}^{\mathsf{V}[\mathbb{H}]}) \text{ satisfies a sufficiently large fragment of } \mathsf{ZFC} \\ \text{by elementarity of } j\text{, and hence the equality follows by Lemma 3.7, (1)}}} V_{j(\lambda)}{}^{\mathsf{V}}[\mathbb{H}].$$ x-Lg-RcA-2

Thus, by (6.5) and by the definability of grounds, we have $V_{j(\lambda)}{}^{\mathsf{V}} \in M$ and $V_{j(\lambda)}{}^{\mathsf{V}}[\mathbb{G}] \in M$. We may assume that $V_{j(\lambda)}^{\mathsf{V}}$ as a ground of $V_{j(\lambda)}^{M}$ satisfies a large enough fragment of ZFC.

Claim 6.1.1 $V_{j(\lambda)}{}^{\mathsf{V}}[\mathbb{G}] \models \varphi(\overline{a}) \wedge \psi(\overline{a})$. *cl-Lg-RcA-0*

$\vdash$ By Lemma 3.7, (1), $V_\lambda{}^{\mathsf{V}}[\mathbb{G}] = V_\lambda{}^{\mathsf{V}[\mathbb{G}]}$, and $V_{j(\lambda)}{}^{\mathsf{V}}[\mathbb{G}] = V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{G}]}$. By (6.2), both $V_\lambda{}^{\mathsf{V}}[\mathbb{G}]$ and $V_{j(\lambda)}^{\mathsf{V}}[\mathbb{G}]$ satisfy still large enough fragment of ZFC. Thus, by Lemma 6.2 below, it follows that

(6.8) $\quad V_\lambda{}^{\mathsf{V}}[\mathbb{G}] \prec_{\Sigma_1} V_{j(\lambda)}{}^{\mathsf{V}}[\mathbb{G}] \prec_{\Sigma_1} V[\mathbb{G}].$ x-Lg-RcA-2-0

By (6.1) and (6.2), we have $V_\lambda{}^{\mathsf{V}}[\mathbb{G}] \models \varphi(\overline{a})$ and $\mathsf{V}[\mathbb{G}] \models \psi(\overline{a})$. By (6.8) and since φ is Σ_2, and ψ is Π_2, it follows that $V_{j(\lambda)}{}^{\mathsf{V}}[\mathbb{G}] \models \varphi(\overline{a}) \wedge \psi(\overline{a})$. $\dashv$ (Claim 6.1.1)

Thus we have

(6.9) $\quad M \models$ "there is a $\mathcal{P}$-ground N of $V_{j(\lambda)}$ with $N \models \varphi(\overline{a}) \wedge \psi(\overline{a})$". x-Lg-RcA-3

By the elementarity (6.3), it follows that

(6.10) $\quad \mathsf{V} \models$ "there is a $\mathcal{P}$-ground N of V_λ with $N \models \varphi(\overline{a}) \wedge \psi(\overline{a})$". x-Lg-RcA-4

Now by (6.2), it follows that there is a $\mathcal{P}$-ground W of V such that $\mathsf{W} \models \varphi(\overline{a}) \wedge \psi(\overline{a})$. ❑ (Theorem 6.1)

We used the following variation of (1.4) in the proof of Theorem 6.1 to obtain (6.8):

Lemma 6.2 *Suppose that δ, $\delta' \in$ On, $\delta < \delta'$ and both V_δ and $V_{\delta'}$ satisfy a sufficiently large fragment of* ZFC. *Then we have $V_\delta \prec_{\Sigma_1} V_{\delta'} \prec_{\Sigma_1} \mathsf{V}$.* *p-Lg-RcA-0-0*

Proof. Suppose that $\overline{a} \in V_\delta$ and $\psi(\overline{x}, \overline{y})$ is a bounded formula in $\mathcal{L}_\in$.

If $V_\delta \models \exists \overline{y} \psi(\overline{a}, \overline{y})$, then there are $\overline{b} \in V_\delta$ such that $V_\delta \models \psi(\overline{a}, \overline{b})$. It follows that $V_{\delta'} \models \psi(\overline{a}, \overline{b})$ and hence $V_{\delta'} \models \exists \overline{y} \psi(\overline{a}, \overline{y})$.

Suppose now that $V_{\delta'} \models \exists \overline{y} \psi(\overline{a}, \overline{y})$. Since $V_{\delta'}$ satisfies a sufficiently large fragment of ZFC, there is $M \in V_{\delta'}$ such that

$$V_{\lambda'} \models \text{“}\,|\,trcl(\overline{a})\,| = |\,M\,|,\ M \text{ is transitive, } \overline{a} \in M, \text{ there are } \overline{b} \in M \text{ such that } M \models \psi(\overline{a}, \overline{b})\,\text{”}.$$

But then such M as above must be an element of V_λ and thus

$$V_{\lambda} \models \text{“}\,|\,trcl(\overline{a})\,| = |\,M\,|,\ M \text{ is transitive, } \overline{a} \in M, \text{ there are } \overline{b} \in M \text{ such that } M \models \psi(\overline{a}, \overline{b})\,\text{”}.$$

It follows that $V_\delta \models \exists \overline{y}\, \psi(\overline{a}, \overline{y})$.

The argument above shows that $V_\delta \prec_{\Sigma_1} V_{\delta'}$. $V_{\delta'} \prec_{\Sigma_1} \mathsf{V}$ can be shown with practically the same argument. ❑ (Lemma 6.2)

6.2 Separation of some other axioms and assertions

In Section 3, we separated some instances of $(\mathcal{P}, \mathcal{H}(\kappa))_\Gamma$-$\mathsf{RcA}$ and $\mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa))_\Gamma$ Sep by compatibility with the Ground Axiom (GA). The same idea can be also used to separate some other principles and axioms.

Theorem 6.3 *“MM^{++} + there are class many supercompact cardinals” (or even class many extendible cardinals) is consistent with* GA. *p-Lg-RcA-1-0*

Proof. Sean Cox [8] proved that MM^{++} is preserved by $<\omega_2$-directed closed forcing (Theorem 4.7 in [8]). Starting from a model with cofinally many supercompact cardinals, use the first supercompact to force MM^{++}. Then the class forcing just like that in the proof of Theorem 3.8 (or like the one in [21]) will produce a desired model. ❑ (Theorem 6.3)

Corollary 6.4 MM^{++} *or even* MM^{++} + *“there are class many super compact cardinals” does not imply that the continuum is a tightly $\mathcal{P}$-Laver gen. ultrahuge cardinal for any of the large enough subclass $\mathcal{P}$ of the class of all semiproper posets.* *p-Lg-RcA-2*

Proof. Let $\mathcal{P}$ = semiproper posets. Note that, if κ is $\mathcal{P}$-Laver generically supercompact, then $\kappa = 2^{\aleph_0}$ follows (see e.g. Theorem 5 and Lemma 6 in Fuchino [13]).

If κ is the tightly $\mathcal{P}$-Laver generically ultrahuge continuum, then Theorem 6.1 together with each one of the Propositions 2.7, 2.8, 2.10 implies that GA does not hold. Thus the model of MM^{++} + “there are class many super compact cardinals” + GA of Theorem 6.3 witnesses the desired non-implication. ❑ (Corollary 6.4)

Note that Corollary 6.4 with “tightly $\mathcal{P}$-Laver gen. ultrahuge” replaced by “tightly $\mathcal{P}$-Laver gen. hyperhuge” is trivial. This is because consistency strength of the existence of the tightly $\mathcal{P}$-Laver gen. hyperhuge cardinal is known to be that of the existence of a (genuinely) hyperhuge cardinal (see the remark right after Proposition 2.11).

Our Theorems 4.1 and 5.7 generalize Viale's Theorem 1.1 in terms of possible instances of the class $\mathcal{P}$ not covered Theorem 1.1 and also in terms of the cardinal κ in the conclusion of the theorems which can be strictly bigger than $\aleph_2$ (which can really happen if e.g. $\mathcal{P}$ is the class of ccc posets).

On the other hand, the premise of Viale's Theorem 1.1 is consistent with GA by Theorem 6.3 while this is not the case with Theorem 4.1 for many natural instances of $\mathcal{P}$ by Propositions 2.7, 2.8, 2.10 and unclear in case of $\mathcal{P}$ = stationary preserving posets with Theorem 5.7.

Viale's Theorem 1.1 implies, in particular:

Corollary 6.5 (to Viale's Theorem 1.1 and Theorem 6.3) *The assertion* p-misc-0

$$\mathcal{H}(\aleph_2)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}(\aleph_2)^{\mathsf{V}[\mathbb{G}]} \quad \textit{for any stationary preserving poset } \mathbb{P} \textit{ with } \Vdash_{\mathbb{P}} \text{“}\mathsf{BMM}\text{”}, \textit{ and } (\mathsf{V},\mathbb{P})\textit{-generic } \mathbb{G}.$$

is consistent with GA. ❑

Concerning Theorem 5.7, it is open at the moment if the existence of a tightly $\mathcal{P}$-Laver-gen. huge cardinal is inconsistent with GA. However some of its strengthenings do contradict GA for many instances of the class $\mathcal{P}$ of posets as we saw in Proposition 2.13 and Theorem 2.14.

The positive answer to the following question would give a clear separation of Laver-genericity from the corresponding forcing axiom with double plus:

Problem 6.6 *Does the (tightly) $\mathcal{P}$-Laver gen. supercompact cardinal axiom (i.e., the existential statement of such a cardinal, e.g. for $\mathcal{P}$ as in Corollary 3.12) imply the negation of* GA*?*

Though Corollary 6.4 makes the positive answer to the following problem rather unpromising, Theorem 2.53 of Woodin [40] and its variants (e.g. Theorem 4.5 in [8]) seem to suggest a positive answer:

Problem 6.7 *Is there any reasonable assumption under which* MM^{++} *and (tightly) $\mathcal{P}$-Laver gen. supercompact cardinal axiom are equivalent?*

The next subsection has been removed from the version for publication of the current paper as we have decided to move the material to the subsequent paper which is currently in preparation.

6.3 Yet another hierarchy of restricted Recurrence Axioms

The hierarchy $\mathsf{MP}(\cdot,\cdot)$. of Maximality Principles ($\Leftrightarrow$ Recurrence Axioms) introduced in Section 3 is suitable for the analysis of consistency strength and strictness of the hierarchy of restricted form of these principles. Yah

Lemma 6.8 *Suppose that $\mathcal{P}$ is an iterable class of posets.* p-Yah-0

(1) $(\mathcal{P}, A)_\Gamma$-$\mathsf{RcA}^+ \ \Rightarrow\ \mathsf{MP}^*(\mathcal{P}, A)_\Gamma$.

(2) *If* $A \subseteq A'$, *and* $\Gamma \subseteq \Gamma'$ *then* $(\mathcal{P}, A')_{\Gamma'}$-$\mathsf{RcA}^+ \ \Rightarrow\ (\mathcal{P}, A)_\Gamma$-$\mathsf{RcA}^+$, *and* $\mathsf{MA}^*(\mathcal{P}, A')_{\Gamma'} \ \Rightarrow\ \mathsf{MA}^*(\mathcal{P}, A)_\Gamma$.

(3) *If* $\mathcal{P} \subseteq \mathcal{P}'$, $A \subseteq A'$, *and* $\Gamma \subseteq \Gamma'$ *then* $(\mathcal{P}', A')_{\Gamma'}$-$\mathsf{RcA} \ \Rightarrow\ (\mathcal{P}, A)_\Gamma$-$\mathsf{RcA}$.

(4) $\mathsf{MP}^*(\mathcal{P}, A)_{\Pi_1}$ *holds (in* ZFC*).*

(5) $(\mathcal{P}, A)_{\Sigma_1}$-$\mathsf{RcA} \Leftrightarrow (\mathcal{P}, A)_{\Sigma_1}$-$\mathsf{RcA}^+ \Leftrightarrow \mathsf{MP}^*(\mathcal{P}, A)_{\Sigma_1}$.

Proof. (1): Suppose that $(\mathcal{P}, A)_\Gamma$-RcA^+ holds. Let φ be a provably $\mathcal{P}$-persistent formula in Γ and $\overline{a} \in A$. If $\varphi(\overline{a})$ is forced by $\mathbb{P} \in \mathcal{P}$, then $\mathbb{P}$ is a push of the $\mathcal{P}$-button $\forall \underline{\mathbb{P}}(\Vdash_{\underline{\mathbb{P}}} \text{“}\varphi(\overline{a})\text{”})$ by $(\varphi)^*_{\mathbb{P}}$ which is provable by assumption. By $(\mathcal{P}, A)_\Gamma$-RcA^+, it follows that $\varphi(\overline{a})$ holds (in V). This shows that $(6.8)_\varphi$ holds.

(2), (3): by definitions.

(4): $(\mathcal{P}, A)_{\Pi_1}$-RcA^+ holds by Lemma 3.3. Thus (1) implies $\mathsf{MP}^*(\mathcal{P}, A)_{\Pi_1}$.

(5): The first equivalence is a part of Theorem 2.2. The second equivalence is proved from this and argument similar to the proof of Theorem 2.2. ❑ (Lemma 6.8)

The monotonicity Lemma 6.8, (3) is used in Fuchino [13] in the argument to single out the maximal instance of Recurrence Axiom (stationary preserving, $\mathcal{H}(\kappa_{\mathfrak{refl}})$)-$\mathsf{RcA}$ under $2^{\aleph_0} = \aleph_2$ and the other maximal instance (all posets, $\mathcal{H}(2^{\aleph_0})$)-$\mathsf{RcA}$ under CH.

Lemma 6.8, (4) and (5) shows that the list of equivalent assertions in Corollary 3.2 and Lemma 3.3 can also include $\mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_1}$ and $\mathsf{MP}^*(\mathcal{P}, A)_{\Pi_1}$ respectively.

$\mathsf{MP}^*(\cdots)_\Gamma$ is almost identical with $(\cdots)_\Gamma$-RcA^+.

Proposition 6.9 *Suppose that $\mathcal{P}$ is an iterable class of posets and A is any set.* p-Yah-1

(1) *If $\mathcal{P}$ is Σ_m-definable then for any natural number n with* $\max\{4, m\} \leq n$, *we have* $(\mathcal{P}, A)_{\Sigma_n}$-$\mathsf{RcA}^+ \Leftrightarrow \mathsf{MP}^*(\mathcal{P}, A)_{\Sigma_n}$.

(2) $(\mathcal{P}, A)$-RcA^+ *(*$\Leftrightarrow \mathsf{MP}(\mathcal{P}, A)$*)* $\Leftrightarrow \mathsf{MP}^*(\mathcal{P}, A)$.

(3) *If $\mathcal{P}$ is Σ_4-definable, and one of the conditions in Proposition 2.10 holds, then* $\mathsf{MP}^*(\mathcal{P}, \emptyset)_{\Sigma_4}$ *implies* $\neg\mathsf{GA}$ *(cf. Theorem 3.10).*

Proof. (1): Suppose that $\mathcal{P}$ is Σ_n-definable iterable class of posets, and $n \geq \max\{3, m\}$. If $(\mathcal{P}, A)_{\Sigma_n}$-$\mathsf{RcA}^+$ holds, then we have $\mathsf{MP}^*(\mathcal{P}, A)_{\Sigma_n}$ by Lemma 6.8, (1).

Assume now that $\mathsf{MP}^*(\mathcal{P}, A)_{\Sigma_n}$ holds, and suppose that Σ_n formula $\varphi = \varphi(\overline{x})$ and $\overline{a} \in A$ are such that $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$" for a $\mathbb{P} \in \mathcal{P}$.

With the proof of definability of grounds in mind (see e.g. [33]), let $\varphi^*(\overline{x})$ be the formula claiming

$$
\begin{aligned}
\exists X \,(&\text{“} X \text{ is the parameter which codes a } \mathcal{P}\text{-ground”} \\
&\quad \wedge \text{“}\overline{x} \in \text{ the } \mathcal{P}\text{-ground coded by } X \\
&\quad \wedge \text{“the ground coded by } X \text{ satisfies } \varphi(\overline{x})\text{”}).
\end{aligned}
$$

By the choice of n, m, $\mathcal{P}$, φ^* can be expressed as Σ_n-formula and, it is provably $\mathcal{P}$-persistent since $\mathcal{P}$ is iterable. We also have $\Vdash_{\mathbb{P}}$ "$\varphi^*(\overline{a})$". By $\mathsf{MP}^*(\mathcal{P}, A)_{\Sigma_n}$, it follows that $\mathsf{V} \models \varphi^*(\overline{a})$. By definition of φ^*, this means that there is a $\mathcal{P}$-ground W of V such that $\mathsf{W} \models \varphi(\overline{a})$.

This shows that $(\mathcal{P}, A)_{\Sigma_n}$-$\mathsf{RcA}^+$ holds.

(2): follows from (1) (the first equivalence in parentheses is by Proposition 2.1, (1)).

(3): By (1) and Proposition 2.10. ❑ (Proposition 6.9)

For a class of posets $\mathcal{P}$, a set Γ of formulas, and infinite cardinals κ, λ with $\kappa \leq \lambda$, the principle $\mathsf{BFA}(\mathcal{P}, \Gamma)_{\kappa,\lambda}$ is defined by

$\mathsf{BFA}(\mathcal{P}, \Gamma)_{\kappa,\lambda}$: For any $\overline{a} \in \mathcal{H}(\lambda)$ and $\varphi = \varphi(\overline{x}) \in \Gamma$ with $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$" for some $\mathbb{P} \in \mathcal{P}$, there are stationarily many $X \in [\mathcal{H}(\lambda)]^{<\kappa}$ such that X (as an $\in$-model) satisfies the Axiom of Extensionality, $\overline{a} \in X$, and $\mathsf{V} \models \varphi(m_X(\overline{a}))$ where m_X denotes the Mostowski collapse of X.

Note that X as above should satisfy the Axiom of Extensionality so that the Mostowski collapse of X is defined (and injective). $\mathsf{BFA}(\mathcal{P}, \Gamma)_{\kappa,\lambda}$ was introduced in Asperó [2]. Note that this principle is not formulated as a generalization of the original definition of $\mathsf{BFA}_{<\kappa}(\mathcal{P})$ (see Section 1) but rather a statement which stands in analogy with the property in Bagaria's Absoluteness Theorem 1.2 characterizing $\mathsf{BFA}_{<\kappa}(\mathcal{P})$.

Lemma 6.10 (Goodman [23]) *Suppose that $\mathcal{P}$ is a class of posets and Γ a set of formulas $\subseteq \mathcal{L}_{\in}$.* (1) $\mathsf{BFA}(\mathcal{P}, \Gamma)_{\kappa,\kappa}$ *holds if and only if* p-Yah-2

(6.11) *For any $\overline{a} \in \mathcal{H}(\kappa)$ and $\varphi = \varphi(\overline{x}) \in \Gamma$ with $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$", $\mathsf{V} \models \varphi(\overline{a})$ holds.* x-Yah-1

(2) $\mathsf{BFA}(\mathcal{P}, (\Gamma)^*_{\mathcal{P}})_{\kappa,\kappa} \Leftrightarrow \mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa))_{\Gamma}$ *holds.*

Proof. (1): Suppose that $\mathsf{BFA}(\mathcal{P},\Gamma)_{\kappa,\kappa}$ holds. For $\overline{a}\in\mathcal{H}(\kappa)$ and $\varphi=\varphi(\overline{x})\in\Gamma$, suppose that $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$". Then there are stationarily many $X\in[\mathcal{H}(\kappa)]^{<\kappa}=\mathcal{H}(\kappa)$ such that $\overline{a}\in X$ and $\mathsf{V}\models\varphi(m_x(\overline{a}))$. In particular, there is such X that $trcl^+(\overline{a})\subseteq X$. Then $m_X(\overline{a})=\overline{a}$ and $\mathsf{V}\models\varphi(\overline{a})$. Thus (6.11) holds.

Conversely, if (6.11) holds, then $\mathcal{X}:=\{X\in[\mathcal{H}(\kappa)]^{<\kappa}\ :\ trcl^+(\overline{a})\subseteq X\}$ is stationary (actually club) in $[\mathcal{H}(\kappa)]^{<\kappa}$. For each $X\in\mathcal{X}$, we have $m_X(\overline{a})=\overline{a}$. This shows that $\mathsf{BFA}(\mathcal{P},\Gamma)_{\kappa,\kappa}$ holds.

(2): follows from (1). □ (Lemma 6.10)

The following argument leading to Proposition 6.15 and Proposition 6.19 is mostly a combination of ideas already utilized somewhere in Asperó [2], Goodman [23], and/or Hamkins [25]. We include the details of the proofs here for convenience of the reader.

Lemma 6.11 *Suppose that* $\mathsf{MP}^*(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_2}$ *holds and* $2^{\aleph_0}$ *is regular. If, either* (a) *there is a poset in* $\mathcal{P}$ *collapsing* $\kappa_{\mathfrak{refl}}$, *or* (b) *there is a poset in* $\mathcal{P}$ *adding* ${\kappa_{\mathfrak{refl}}}^+$ *reals without collapsing* ${\kappa_{\mathfrak{refl}}}^+$, *then* ${\kappa_{\mathfrak{refl}}}^{\mathsf{V}}$ *is inaccessible in* L. p-Yah-3

Proof. Assume that $\mathsf{MP}^*(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_2}$ holds.

Consider the sentence $\varphi(x)$ saying

$$\exists\underline{\mu}'\,(\mathsf{L}\models\text{"}\underline{\mu}'\text{ is a cardinal"}\ \wedge\ x<\underline{\mu}'<\kappa_{\mathfrak{refl}}).$$

$\varphi(x)$ is Σ_2 and it is provably $\mathcal{P}$-persistent.

In both of the cases, there is $\mathbb{P}\in\mathcal{P}$ such that $\Vdash_{\mathbb{P}}$ "$|\,{\kappa_{\mathfrak{refl}}}^{\mathsf{V}}\,|<\kappa_{\mathfrak{refl}}$". Suppose that $\mu<\kappa_{\mathfrak{refl}}$ is a cardinal in L.

Since $\kappa_{\mathfrak{refl}}$ is a regular cardinal in L, it follows that $\Vdash_{\mathbb{P}}$ "$\varphi(\mu)$".

By $\mathsf{MP}^*(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_2}$, it follows that $\mathsf{V}\models\varphi(\mu)$. I.e., $(\mu^+)^{\mathsf{L}}<\kappa_{\mathfrak{refl}}$. □ (Lemma 6.11)

Lemma 6.12 *Suppose that* $2^{\aleph_0}$ *is regular and* $\mathcal{P}$ *is a class of posets such that either* (a') *for any cardinal* $\lambda\geq\kappa_{\mathfrak{refl}}$, *there is a poset* $\mathbb{P}\in\mathcal{P}$ *which collapses* λ *to cardinality* $\aleph_1$, *or* (b') *for cofinally many cardinals* $\lambda>2^{\aleph_0}$, *there is* $\mathbb{P}\in\mathcal{P}$ *adding at least* λ *many reals without collapsing* λ. *Then, for any* $n\geq 2$, $\mathsf{MP}^*(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_n}$ *implies* $\mathsf{L}_{{\kappa_{\mathfrak{refl}}}^{\mathsf{V}}}\prec_{\Sigma_n}\mathsf{L}$. p-Yah-4

Proof. For $n=2$, $\mathsf{MP}^*(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_n}$ implies that $\kappa_{\mathfrak{refl}}$ is an inaccessible cardinal in L by Lemma 6.11. Thus $\mathsf{L}\models L_{{\kappa_{\mathfrak{refl}}}^{\mathsf{V}}}=\mathcal{H}({\kappa_{\mathfrak{refl}}}^{\mathsf{V}})$. By (1.4), it follows that

(6.12) $\mathsf{L}_{{\kappa_{\mathfrak{refl}}}^{\mathsf{V}}}\prec_{\Sigma_1}\mathsf{L}$. x-Yah-1-0

Thus it is enough to show that $\mathsf{L}_{{\kappa_{\mathfrak{refl}}}^{\mathsf{V}}}\prec_{\Sigma_n}\mathsf{L}$ holds for $n\geq 2$ assuming

(6.13) $\mathsf{MP}^*(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_n}$, and x-Yah-2

(6.14) $\quad \mathsf{L}_{\kappa_{\mathfrak{refl}}{}^{\mathsf{V}}} \prec_{\Sigma_{n-1}} \mathsf{L}.$ x-Yah-3

Note that for $n = 2$, (6.14) is just (6.12).

To prove this claim, assume (6.13) and (6.14), and let $\delta := \kappa_{\mathfrak{refl}}{}^{\mathsf{V}}$, $\psi = \psi(\overline{x}, \overline{y})$ a Π_{n-1}-formula, and $\overline{a} \in \mathsf{L}_\delta$. Since δ is a limit ordinal, there is $\delta_0 < \delta$ such that $\overline{a} \in L_{\delta_0}$.

If $\mathsf{L}_\delta \models \exists \overline{y} \psi(\overline{a}, \overline{y})$, then $\mathsf{L} \models \exists \overline{y} \psi(\overline{a}, \overline{y})$ by (6.14).

Assume now that $\mathsf{L} \models \exists \overline{y} \psi(\overline{a}, \overline{y})$. Let $\eta = \eta(u)$ be the Σ_n-formula

$$\exists v\,(\text{“}v = \mathsf{L}_{\kappa_{\mathfrak{refl}}}\text{”} \wedge (\forall x \in u)\,(\text{“}\mathsf{L} \models \exists \overline{y} \psi(\overline{x}, \overline{y})\text{”} \rightarrow (\exists \overline{y} \in v)\,(\text{“}\mathsf{L} \models \psi(\overline{x}, \overline{y})\text{”}))).$$

This formula is clearly provably $\mathcal{P}$-persistent. By the assumption (a') or (b'), and by an argument similar to the proof of Lemma 6.11, it follows (in both of the cases (a') and (b')) that there is $\mathbb{P} \in \mathcal{P}$ such that $\Vdash_{\mathbb{P}}$ “$\eta(\mathsf{L}_{\delta_0})$”. Thus by (6.13), $\mathsf{V} \models \eta(\mathsf{L}_{\delta_0})$.

In particular, there is $\overline{b} \in \mathsf{L}_\delta$ such that $\mathsf{L} \models \psi(\overline{a}, \overline{b})$. By (6.14), it follows that $\mathsf{L}_\delta \models \psi(\overline{a}, \overline{b})$ and thus $\mathsf{L}_\delta \models \varphi(\overline{a})$. ❑ (Lemma 6.12)

The following proposition can be seen as a subset of Theorem 2.6 in [2] and the proof given here is also more or less identical with the one in [2]:

Proposition 6.13 *For a natural number n, assume that λ is an infinite cardinal and $\kappa \geq \lambda$ is an inaccessible Σ_n-correct cardinal (i.e. a cardinal κ with the property $V_\kappa (= \mathcal{H}(\kappa)) \prec_{\Sigma_n} \mathsf{V}$).* p-Yah-5

Suppose that $\mathcal{P}$ is a Σ_n-definable iterable class of posets such that

(6.15) *all $\mathbb{P} \in \mathcal{P}$ preserve cardinals $\leq \lambda$, and if $\lambda < \kappa$, then the equation* x-Yah-4

$$\lambda = \max\{\mu \in \mathit{Card} : \textit{all } \mathbb{P} \in \mathcal{P} \textit{ preserves cardinals } \leq \mu\}$$

is provable (in ZFC*);*[15)]

(6.16) *$\mathcal{P}$ admits iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa\rangle$ of length κ with some appropriate kind of iteration (e.g. either FS-, CS-, or Easton-support iteration) such that* x-Yah-5

(6.16 a) *$\mathbb{P}_\kappa$ is the direct limit of $\langle \mathbb{P}_\alpha : \alpha < \kappa\rangle$.* {x-Yah-5}{a}

(6.16 b) *$\mathbb{P}_\kappa \in \mathcal{P}$;* {x-Yah-5}{b}

(6.16 c) *$\Vdash_{\mathbb{P}_\alpha}$ “$\mathbb{P}_\kappa / \underset{\sim}{\mathbb{G}}_\alpha \in \mathcal{P}$” for all $\alpha < \kappa$;* {x-Yah-5}{c}

(6.16 d) *$\mathbb{P}_\alpha \in V_\kappa$ for all $\alpha < \kappa$, and $\mathbb{P}_\kappa$ satisfies the κ-cc.* {x-Yah-5}{d}

Then there is an iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa\rangle$ satisfying (6.16) above such that $\Vdash_{\mathbb{P}_\kappa}$ “$\mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_n}$”.

Proof. Let $f : \kappa \to \kappa \times \omega \times \kappa$ be a surjection such that each $\langle \alpha_0, n\alpha_1 \rangle \in \kappa \times \omega \times \kappa$ appears κ times as a value of f. Let $\langle \varphi_m : m \in \omega \rangle$ be an enumeration of $(\Sigma_n)^*_{\mathcal{P}}$ (as a set in ZFC) corresponding to the recursive enumeration of $(\Sigma_n)^*_{\mathcal{P}}$ in metamathematics.

Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be an iteration in $\mathcal{P}$ satisfying (6.16) defined along with the sequences $\langle \underset{\sim}{\overline{a}}{}^\alpha_\xi : \xi < \kappa \rangle$ in V_κ for each $\alpha < \kappa$ such that

(6.17) $\Vdash_{\mathbb{P}_\alpha}$ "$\{\underset{\sim}{\overline{a}}{}^\alpha_\xi : \xi < \kappa\}^\bullet = \mathcal{H}(\kappa)$".[16] x-Yah-6

The successor step of the iteration is set by the following:

(6.18) If $f(\alpha) = \langle \alpha_0, m, \alpha_1 \rangle$, $\alpha_0 \leq \alpha$, and x-Yah-7

$$\Vdash_{\mathbb{P}_\alpha} \text{“}(\exists \underline{\mathbb{Q}} \in \mathcal{P})\,(\underline{\mathbb{Q}} \in \mathcal{H}(\kappa) \ \wedge\ \Vdash_{\underline{\mathbb{Q}}} \text{“}\varphi_m(\underset{\sim}{\overline{a}}{}^{\alpha_0}_{\alpha_1})\text{”})\text{”},^{17)}$$

then $\underset{\sim}{\mathbb{Q}}_\alpha \in V_\kappa$ is such a $\mathbb{P}_\alpha$-name as $\underline{\mathbb{Q}}$ as above;

otherwise $\underset{\sim}{\mathbb{Q}}_\alpha = \{\mathbb{1}\}^\bullet$.

We show that this iteration with $\mathbb{P}_\kappa$ is as desired. Let $\mathbb{G}_\kappa$ be a $(\mathsf{V}, \mathbb{P}_\kappa)$-generic filter.

Suppose that $\underset{\sim}{\overline{a}}$ is a tuple of $\mathbb{P}_\kappa$-names of elements of $\mathcal{H}(\kappa)^{\mathsf{V}[\mathbb{G}_\kappa]}$ $(= \mathcal{H}(\kappa)^{\mathsf{V}[\mathbb{G}_\kappa]})$, and $\underset{\sim}{\mathbb{Q}}$ be $\mathbb{P}_\kappa$-name of poset in $\mathcal{P}$ and $\Vdash_{\mathbb{P}_\kappa * \underset{\sim}{\mathbb{Q}}}$ "$\varphi(\underset{\sim}{\overline{a}})$" for a (concretely given) provably $\mathcal{P}$-persistent Σ_n-formula φ. By definition of the sequence $\langle \varphi_m : m \in \omega \rangle$ there is a number m^* such that $\varphi = \varphi_{m^*}$. By (6.16 d), there is $\gamma < \kappa$ such that $\underset{\sim}{\overline{a}}$ corresponds to $\mathbb{P}_\gamma$-names which we shall also denote with $\underset{\sim}{\overline{a}}$. Thus, there is $\beta^* < \kappa$ such that $\underset{\sim}{\overline{a}} = \underset{\sim}{\overline{a}}{}^\gamma_{\beta^*}$.

Note that for all $\alpha \in \kappa \setminus \gamma$ we have

(6.19) $\Vdash_{\mathbb{P}_\alpha}$ " $\Vdash_{(\mathbb{P}_\kappa/\underset{\sim}{\mathbb{G}}_\alpha) * \underset{\sim}{\mathbb{Q}}}$ "$\varphi(\underset{\sim}{\overline{a}})$" " and $\Vdash_{\mathbb{P}_\alpha}$ "$\mathbb{P}_\kappa/\underset{\sim}{\mathbb{G}}_\alpha * \underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" x-Yah-8

by (6.16 c) and by iterability of $\mathcal{P}$.

Let $\alpha^* \in \kappa \setminus \gamma$ be such that

(6.20) $f(\alpha^*) = \langle \gamma, n^*, \beta^* \rangle$. x-Yah-9

By (6.19) we have $\Vdash_{\mathbb{P}_{\alpha^*}}$ " $\Vdash_{(\mathbb{P}_\kappa/\underset{\sim}{\mathbb{G}}_{\alpha^*}) * \underset{\sim}{\mathbb{Q}}}$ "$\varphi_{m^*}(\underset{\sim}{\overline{a}}{}^\gamma_{\beta^*})$" ", and $\Vdash_{\mathbb{P}_{\alpha^*}}$ "$\mathbb{P}_\kappa/\underset{\sim}{\mathbb{G}}_{\alpha^*} * \underset{\sim}{\mathbb{Q}} \in \mathcal{P}$".

Thus we have

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15) In this case, we assume that λ is definable e.g. as $\aleph_1$ so that we can formulate the condition without introducing a new constant symbol.

16) With the superscript bullet "$\cdots^\bullet$" in connection with forcing with a poset $\mathbb{P}$, we denote the canonical $\mathcal{P}$-name corresponding to the object "$\cdots$" describes where we assume that $\mathbb{P}$-names are introduced as in Kunen [30].

17) Strictly speaking we mean with $\underset{\sim}{\overline{a}}{}^{\alpha_0}_{\alpha_1}$ the ($\underline{\mathbb{Q}}$-check names) of the $\mathbb{P}_\alpha$-names corresponding to the $\mathbb{P}_{\alpha_0}$-names.

$$\mathsf{V} \models \exists \underset{\sim}{\mathbb{Q}} \,(\,\Vdash_{\mathbb{P}_{\alpha^*}} \text{“}\, \underset{\sim}{\mathbb{Q}} \in \mathcal{P} \;\wedge\; \Vdash_{\underset{\sim}{\mathbb{Q}}} \text{“}\, \varphi_{m^*}(\underset{\sim}{\overline{a}}{}^{\gamma}_{\beta^*})\,\text{”}\,\text{”}). \tag{6.21}$$ x-Yah-10

Since the property in (6.21) is formalizable in Σ_n, and κ is Σ_n-correct, it follows that

$$V_\kappa \models \exists \underset{\sim}{\mathbb{Q}} \,(\,\Vdash_{\mathbb{P}_{\alpha^*}} \text{“}\, \underset{\sim}{\mathbb{Q}} \in \mathcal{P} \;\wedge\; \Vdash_{\underset{\sim}{\mathbb{Q}}} \text{“}\, \varphi_{m^*}(\underset{\sim}{\overline{a}}{}^{\gamma}_{\beta^*})\,\text{”}\,\text{”}).$$

By (6.18) it follows that $\Vdash_{\mathbb{P}_{\alpha^*}}$ “ $\Vdash_{\underset{\sim}{\mathbb{Q}}_{\alpha^*}}$ “ $\varphi(\underset{\sim}{\overline{a}})$ ” ”. Since φ is provably $\mathcal{P}$-persistent, it follows by (6.16 c) that $\Vdash_{\mathbb{P}_{\alpha^*}}$ “ $\Vdash_{\mathbb{P}_\kappa/\underset{\sim}{\mathbb{G}}_{\alpha^*}}$ “ $\varphi(\underset{\sim}{\overline{a}})$ ” ”. This is equivalent to $\Vdash_{\mathbb{P}_\kappa}$ “ $\varphi(\underset{\sim}{\overline{a}})$ ”.

❑ (Proposition 6.13)

Lemma 6.14 (Fact 2.2 in Asperó [2]) *If κ is inaccessible and Σ_{n+1}-correct for some $n \geq 1$ then there are unboundedly many inaccessible Σ_n-correct cardinals below κ.* ❑ p-Yah-5-0

Proposition 6.15 *Assume that $n \geq 2$. Suppose $\mathcal{P}$ is a Σ_n-definable iterable class of posets such that $\mathcal{P}$ satisfies the condition (a') or (b') of Lemma 6.12, and it is also provable that $\mathcal{P}$ satisfies (6.15) and (6.16) of Proposition 6.13 for any infinite cardinal λ and inaccessible $\kappa \geq \lambda$.* p-Yah-6
Then, assuming the consistency of the theory

$$\mathsf{ZFC} + \text{“}2^{\aleph_0} \textit{ is regular”} + \mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_n}, \tag{6.22}$$ x-Yah-10-0

this theory does not imply $\mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_{n+1}}$.

Proof. Suppose, toward a contradiction, that ZFC proves that $\mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_n}$ implies $\mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_{n+1}}$.

Working in the theory $\mathsf{ZFC} + \mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_n}$, we also have $\mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_{n+1}}$ by the assumption. By Lemma 6.12 applied to $n+1$, and then by Lemma 6.14, combined with Downward Löwenheim-Skolem Theorem and Mostowski Collapsing Lemma, we obtain a countable transitive model M of $\mathsf{ZFC} + \exists \kappa\,(\kappa$ is inaccessible $\wedge$ $V_\kappa \prec_{\Sigma_n} \mathsf{V})$.

By Proposition 6.13, there is a generic extension $M[\mathbb{G}]$ which is a model of $\mathsf{ZFC} +\ \mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_n}$.[18] By (a') and (b') we have $M[\mathbb{G}] \models \kappa = \kappa_{\mathfrak{refl}}$.

Thus we obtained a proof of *consis*$(\mathsf{ZFC} + \mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_n})$ in $\mathsf{ZFC} + \mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_n}$. This is a contradiction by The Second Incompleteness Theorem. ❑ (Proposition 6.15)

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[18] The proof of Proposition 6.13 is written as a proof of $\mathsf{V}[\mathbb{G}_\kappa] \models \mathsf{MP}^*(\cdots)_{\Sigma_n}$ for the axiom scheme $\mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_n}$ in the sense of meta-mathematics, but for the set model M this proof can be almost verbosely adopted to prove $M[\mathbb{G}_\kappa] \models \mathsf{MP}^*(\cdots)_{\Sigma_n}$ for the axiom scheme $\mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_n}$ in the sense of ZFC.

Corollary 6.16 *Under the same assumption on n and $\mathcal{P}$ as in Proposition 6.15, if* p-Yah-6-0

$$(6.22)' \quad \mathsf{ZFC} + \text{“}2^{\aleph_0} \textit{ is regular”} + (\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_n}\text{-}\mathsf{RcA}^+,$$

is consistent then this theory does not prove $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_{n+1}}\text{-}\mathsf{RcA}^+$,

Proof. By Proposition 6.9 and Proposition 6.15. ❑ (Corollary 6.16)

In case of $n \geq 3$ Proposition 6.15 can be further improved (see Proposition 6.19 below).

Lemma 6.17 *Suppose $\mathcal{P}$ is a class of posets satisfying (a') or (b') of Lemma 6.12.* p-Yah-7

(1) *If $n \geq 3$, $\mathsf{MP}^*(\mathcal{P}, \emptyset)_{\Pi_n}$ implies that there are unboundedly many Σ_{n-1}-correct inaccessible cardinals in L.*

(2) *Suppose that $\mathsf{MP}^*(\mathcal{P}, \emptyset)_{\Pi_2}$ holds. If there is at least one inaccessible cardinal, then there are cofinally many inaccessible cardinals.*

Proof. (1): Assume, toward a contradiction that

$$(6.23) \quad \mathsf{MP}^*(\mathcal{P}, \emptyset)_{\Pi_n} \text{ holds}$$ x-Yah-11

but the class $\mathcal{B}$ of all Σ_{n-1}-correct cardinals in L is bounded.

By (a') or (b') there is a poset $\mathbb{P} \in \mathcal{P}$ such that

$$\Vdash_{\mathbb{P}} \text{“}\kappa_{\mathfrak{refl}} > \mathcal{B}\text{”}.$$

The forced statement is Π_n and it is provably $\mathcal{P}$-persistent. Thus, by (6.23), $\mathsf{V} \models$ “$\kappa_{\mathfrak{refl}} > \mathcal{B}$”. This is a contradiction to Lemma 6.12 with n replaced by $n - 1$.

(2): Suppose that $\mathsf{MP}^*(\mathcal{P}, \emptyset)_{\Pi_2}$ holds but there are only boundedly many but at least one inaccessible cardinals. Then, by the condition on $\mathcal{P}$, there is $\mathbb{P} \in \mathcal{P}$ such that

$$(6.24) \quad \Vdash_{\mathbb{P}} \text{“there are no inaccessible cardinals”}.$$ x-Yah-11-0

Since the statement in (6.24) is Π_2 and provably $\mathcal{P}$-persistent. It follows that $\mathsf{V} \models$ “there are no inaccessible cardinals”. This is a contradiction. ❑ (Lemma 6.17)

Lemma 6.18 *For $n \geq 2$, if the theory* p-Yah-8

$$\mathsf{ZFC} + \text{“}\exists \underline{\kappa}\,(\underline{\kappa} \textit{ is } \Sigma_n\textit{-correct and inaccessible})\text{”}$$

is consistent, then this theory does not prove

$$\exists \underline{\kappa} \exists \underline{\lambda}\,(\underline{\kappa} < \underline{\lambda} \ \wedge\ \underline{\kappa} \textit{ is } \Sigma_n\textit{-correct and inaccessible} \\ \wedge\ \underline{\lambda} \textit{ is } \Sigma_{n-1}\textit{-correct and inaccessible}).$$

Proof. Suppose otherwise. Working in the theory ZFC + "$\exists\,\underline{\kappa}\,(\underline{\kappa}$ is Σ_n-correct and inaccessible$)$", assume that κ is Σ_n-correct and inaccessible and there is a Σ_{n-1}-correct inaccessible cardinal $\lambda > \kappa$.

Claim 6.18.1 $\mathsf{V}_\lambda \models$ "*κ is Σ_n-correct and inaccessible*". *Cl-Yah-0*

$\vdash$ Suppose that $\psi(\overline{x}, \overline{y})$ is Π_{n-1} and $\overline{a} \in V_\kappa$. If $V_\kappa \models \exists\overline{y}\psi(\overline{a}, \overline{y})$, then there are $\overline{b} \in V_\kappa$ such that $V_\kappa \models \psi(\overline{a}, \overline{b})$. Since ψ is Π_{n-1}, it follows that $V_\lambda \models \psi(\overline{a}, \overline{b})$. Thus $V_\lambda \models \exists\overline{y}\psi(\overline{a}, \overline{y})$.

If $V_\lambda \models \exists\overline{y}\psi(\overline{a}, \overline{y})$, then there are $\overline{y} \in V_\lambda$ such that $V_\lambda \models \psi(\overline{a}, \overline{b})$. Ti follows that $\mathsf{V} \models \psi(\overline{a}, \overline{b})$. Hence $\mathsf{V} \models \exists\overline{y}\psi(\overline{a}, \overline{y})$. Now since κ is Σ_n-correct, it follows that $V_\kappa \exists\overline{y}\psi(\overline{a}, \overline{y})$. $\dashv$ (Claim 6.18.1)

Thus we proved *consis*($\mathsf{ZFC} + \exists\underline{\kappa}\,(\underline{\kappa}$ is Σ_n-correct and inaccessible$)$). But this is a contradiction by The Second Incompleteness Theorem. ❑ (Proposition 6.18)

Proposition 6.18 can be still improved as follows:

Proposition 6.19 *Suppose $\mathcal{P}$ is a Σ_n-definable iterable class of posets such that $\mathcal{P}$ satisfies the condition (a') or (b') of Lemma 6.12, and it is also provable that $\mathcal{P}$ satisfies (6.15) and (6.16) of Proposition 6.13 for any infinite cardinal λ and inaccessible $\kappa \geq \lambda$.* *p-Yah-9*

(1) *For $n \geq 3$, assuming the consistency of the theory*

$$\mathsf{ZFC} + \text{“}2^{\aleph_0} \textit{ is regular”} + \mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_n}, \tag{6.25}$$ *x-Yah-12*

this theory does not prove $\mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Pi_n}$.

(1') *For $n \geq 3$, assuming the consistency of the theory*

$$\mathsf{ZFC} + \text{“}2^{\aleph_0} \textit{ is regular”} + (\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_n}\text{-}\mathsf{RcA}^+, \tag{6.26}$$ *x-Yah-12-a*

this theory does not prove $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Pi_n}$-$\mathsf{RcA}^+$.

(2) *Assume the consistency of the theory*

$$\mathsf{ZFC} + \text{“\textit{there is a supercompact cardinal and an inaccessible above it.}”} \tag{6.27}$$ *x-Yah-12-0*

Then $\mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_2}$ *does not imply* $\mathsf{MP}^*(\mathcal{P}, \emptyset)_{\Pi_2}$.

Proof. (1): Assume otherwise and suppose that the theory (6.25) proves $\mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Pi_n}$ for some $n \geq 3$.

By Lemma 6.12, in the theory (6.25), we have $\mathsf{L}_{\kappa_{\mathfrak{refl}}{}^V} \prec \mathsf{L}$. Thus from the assumption of the consistency of the theory (6.25), it follows the consistency of

$$\mathsf{ZFC} + \text{“}\kappa_{\mathfrak{refl}} \text{ is inaccessible in } \mathsf{L}\text{”} + \mathsf{L}_{\kappa_{\mathfrak{refl}}{}^V} \prec \mathsf{L}.$$

By Lemma 6.18, we obtain the consistency of the following theory:

$$(6.28)\quad \mathsf{ZFC} + \mathsf{V} = \mathsf{L} + \exists \underline{\kappa}\,(\text{“}\underline{\kappa}\text{ is inaccessible”} \wedge \mathsf{L}_{\underline{\kappa}} \prec_{\Sigma_n} \mathsf{L} \wedge \underbrace{\text{“there is no } \Sigma_{n-1}\text{-correct inaccessible cardinals above } \underline{\kappa}\text{”}}_{(6.29)}).$$ x-Yah-13 x-Yah-14

Working in this theory (6.28), we find a poset $\mathbb{P}_\kappa \in \mathcal{P}$ such that $\Vdash_{\mathbb{P}_\kappa}$ “$\mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_n}$” by Proposition 6.13. By (6.29) and Lemma 6.17, (1) $\Vdash_{\mathbb{P}_\kappa}$ “$\neg\mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Pi_n}$”. This is a contradiction to the assumption we set at the beginning of the proof.

(1'): By (1) and Proposition 6.9, (1).

(2): We work in the theory ZFC + “there is a supercompact κ and a single inaccessible μ above it”. The consistency of this theory follows form (6.27). In the following we shall use some notions and results from Goodman [23] (see also the paragraph right before Lemma 6.8 above).

The supercompact κ is also supercompact for $C^{(1)}$ by Lemma 2.2.6 in [23]. By Theorem 3.1.6 in [23], there is a poset $\mathbb{P} \in \mathcal{P}$ of size κ such that $\mathsf{V}[\mathbb{G}] \models \Sigma_2\text{-CFA}_{<\kappa}(\mathcal{P})$ for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ (for the definition of this principle, see Definition 3.1.2 in [23] — this Theorem 3.1.6 is proved similarly to our Proposition 6.13). By modifying the construction of $\mathbb{P}$ slightly if necessary, we also obtain $\mathsf{V}[\mathbb{G}] \models \kappa = \kappa_{\mathfrak{refl}}$.

By Theorem 3.1.4 in [23], this implies $\mathsf{V}[\mathbb{G}] \models$ “$\mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_2}$”.

On the other hand, since μ is the unique inaccessible cardinal in $\mathsf{V}[\mathbb{G}]$ above κ, Lemma 6.17, (2) implies $\mathsf{V}[\mathbb{G}] \models$ “$\neg\mathsf{MP}^*(\mathcal{P}, \emptyset)_{\Pi_2}$”. ❑ (Theorem 6.19)

Some possible non-implications still remain. For example:

Problem 6.20 *Is* $\mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Pi_2} + \neg\mathsf{MP}^*(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_2}$ *consistent?*

Cf. Proposition 6.9, (3).

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