Set-Theoretic Aspects *of* Almost Free Boolean Algebras

Habilitationsschrift

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「ナジャよ、この本は君には
わからないがおれはこれから読むのだ」

— 西脇 順三郎 『壌歌』 (1969)

Contents

1 Introduction

In the following, we give a survey on the study of almost free Boolean algebras. L. Heindorf and L. Shapiro recently wrote an excellent exposition (Heindorf–Shapiro [25]) which covers some of our topics. While Heindorf and Shapiro in [25] are mainly interested in problems of universal algebraic nature, we put here more stress on set-theoretic methods and independence results. Also, we shall look at several classes of Boolean algebras ($L_{\infty\kappa}$-free Boolean algebras, κ-free Boolean algebras, potentially free Boolean algebras, etc.) which were not discussed thoroughly in [25].

The word "almost free" is used here as a generic term. In general, an algebra A in a variety $\mathcal{V}$ is thought to be almost free if

(∗) *"most" of subalgebras of A of small cardinality are free.*

For example, in the variety $\mathcal{AB}$ of abelian groups, $G \in \mathcal{AB}$ is said to be almost free (κ-free respectively) if every subalgebra of G of cardinality less than $|B|$ (less than κ respectively) is free (e.g. Eklof [6]). Since a subalgebra of a free algebra need not to be free in the variety of Boolean algebras (see Proposition 2.4), we cannot define the almost freeness of Boolean algebras just in this way. One of the possible definitions would be the following: a Boolean algebra B is *almost free* (*κ-free* respectively) if the set of free subalgebras of B contains a club subset of $[B]^{<|B|}$ ($[B]^{<\kappa}$ respectively).

We can also consider an enhanced version of (∗) by saying that an algebra A is almost free (in a stronger sense) if

(†) *there is a class $\mathcal{C}$ of "most" of subalgebras of small cardinality such that elements of $\mathcal{C}$ are all free and are embedded "nicely" in A.*

Examples of this kind of notions of almost freeness are openly generated Boolean algebras and projective Boolean algebras. Another class of almost free Boolean algebras in this vein is $L_{\infty\kappa}$-free Boolean algebras. For the definition of these classes of Boolean algebras, see Chapter 2. We look at these classes of Boolean algebras more closely in Chapters 4, 5 and 6.

In Chapter 7 we give some method of construction of non-free almost free Boolean algebras. For the construction of almost free Boolean algebras by the method given in 7.2 for cardinality greater than $\aleph_1$, we need consequences of $V = L$. The results in Section 8.2 shows that ZFC is really not enough for the construction of Boolean algebras of cardinality grater than $\aleph_1$ with some of the properties of those almost free Boolean algebras obtained in 7.2. In 7.3, we describe another construction method (PC+) due to S. Shelah which provides (already in ZFC) $L_{\infty\kappa}$-free non-free Boolean algebras in κ for each cardinal κ such that there are κ-free non-free abelian groups in κ.

Using the language of forcing, we can introduce notions of "potentially free" Boolean algebras. For a class $\mathcal{C}$ of partial orderings, let us call a Boolean algebra B *potentially free with respect to* $\mathcal{C}$ (or $\mathcal{C}$*-potentially free*) if

(‡) there exists $P \in \mathcal{C}$ such that $\Vdash_P$ "B is free".

In Chapter 9, we study the $\mathcal{C}$-potentially freeness for some classes $\mathcal{C}$ of partial orderings.

Free Boolean algebras themselves play also a very important roll in the theory of forcing. The completions of free Boolean algebras provide namely the generic extensions used by P. Cohen when he first proved the consistency of the negation of the continuum hypothesis. These complete Boolean algebras are therefore often called *Cohen algebras* in the literature. By replacing relatively complete subalgebras in the definition of classes of almost free Boolean algebras by regular subalgebras, we obtain corresponding classes of almost Cohen algebras. Almost Cohen algebras B can be also considered as being "almost Cohen" because of the similarity of the forcing extensions by them to those by Cohen algebras. Some properties of notions of almost Cohen algebras are studied in Balcar–Jech–Zapletal [1].

For notions and elementary facts about Boolean algebras which remain unexplained, the reader may consult Koppelberg [31]. We shall though try hard to make the following as self-contained as possible. If not otherwise mentioned, we follow the notation and definitions in [31]. In particular we regard a Boolean algebra as an algebraic structure $B = (B, +, \cdot, -, 0, 1)$. $B^+ = B \setminus \{ 0 \}$ is the positive elements of B. Two positive elements a, b of a Boolean algebra B are said to be incompatible if $a \cdot b = 0$. Thinking in Stone representation of B, we shall also say that such a and b are disjoint. 2 is the two element Boolean algebra $\{ 0, 1 \}$. Our set-theoretic notation is standard. We refer Jech [27], Kunen [35] and/or Jech [28]. By κ, λ, μ etc., we always denote infinite cardinals. l, m, n etc. are reserved for natural numbers. (Not necessarily finite) ordinal numbers are denoted by α, β, γ,... ν, μ, ξ etc.

We shall use frequently the method of elementary substructures which allows us to treat various classes of almost free Boolean algebras in a uniform way. For a cardinal χ, $\mathcal{H}(\chi)$ denotes the sets of hereditarily of cardinality less than κ. For uncountable regular κ, $\mathcal{H}(\chi)$ is a model of ZFC possibly except powerset axiom. For an algebraic structure A, let χ be such that $A \in \mathcal{H}(\chi)$. For an elementary submodel M of $(\mathcal{H}(\chi), \in)$ such that $A \in M$, $A \cap M$ is then a subalgebra of A by the elementarity of M. This fact is used to characterize some classes of almost free/Cohen Boolean algebras. It is often convenient to add a well-ordering $\leq^*$ of $\mathcal{H}(\chi)$ to the structure of $\mathcal{H}(\chi)$. In $\mathcal{H}(\chi) = (\mathcal{H}(\chi), \in, \leq^*)$, there are canonical built-in Skolem functions, i.e. the functions f_φ for each existential formula $\varphi(x_1, \ldots, x_n)$ mapping each $(a_1, \ldots, a_n) \in \mathcal{H}(\chi)^n$ to the least element (with respect to $\leq^*$) exemplifying the existential assertion $\varphi(a_1, \ldots, a_n)$. We denote with $\tilde{h}_{\leq^*}(X)$ the Skolem hull of $X \subseteq \mathcal{H}(\chi)$ with respect to these Skolem functions. The following facts are used frequently without mention.

Lemma 1.1 *Let χ be a regular uncountable cardinal and M an elementary submodel of $\mathcal{H}(\chi)$. For $x \in M$, if $\kappa = |x|$ and $\kappa + 1 \subseteq M$ then $x \subseteq M$.*

Proof. By $\mathcal{H}(\chi) \models$ "$\exists \underline{f}(\underline{f}$ is a surjection from κ to $x)$", $\kappa \in M$ and by elementarity of M, we have $M \models$ "$\exists \underline{f}(\underline{f}$ is a surjection from κ to $x)$". Hence there is a surjection $f \in M$ from κ to x. Since κ is a subset of M, so is $x = \{\, f(\alpha) : \alpha \in \kappa \,\}$. ❏ (Lemma 1.1)

Lemma 1.2 *For a structure $A \in \mathcal{H}(\chi)$, if $M \prec \mathcal{H}(\chi)$ and $A \in M$, then $A \cap M$ is a substructure of A.*

Proof. If f is a function in the structure A, then we have $f \in M$. Hence, by elementarity of M it follows that $A \cap M$ is closed with respect to f. ❏ (Lemma 1.2)

At the beginning of a proof using elementary submodels, we often declare that χ be *sufficiently large*. This simply means that the regular cardinal χ should be chosen so that all objects we are going to consider in the proof are contained as elements in $\mathcal{H}(\chi)$ and all relevant assertions concerning these objects become absolute over $\mathcal{H}(\chi)$. Usually such "sufficiently large" χ need not to be that large. For example, in the proof of Theorem 5.1, it is enough to take any regular χ such that $\mathcal{P}(B)$, $\kappa \in \mathcal{H}(\chi)$.

Acknowledgments

I would like to thank Sabine Koppelberg who suggested me the study of $L_{\infty\kappa}$-free Boolean algebras. Following her suggestion I wrote [19] with her and Makoto Takahashi who then visited Berlin. And this was the beginning of my research in the field.

The present text grew out of [16] which I wrote as an appendix to Heindorf–Shapiro [25]. I would like to thank Lutz Heindorf and Leonid Shapiro for giving me the opportunity to write the appendix.

Thanks are also due to Ingo Bandlow, Aleksander Błaszczyk and Thomas Jech for stimulating discussions on the subject at various occaisions.

Last but not least, I have to thank Saharon Shelah who gave me the permission to present some of his results which are going to appear in [20] and [21]. I am afraid that every thing I wrote in the following is just trivial for him. Only, I hope that through my presentation which I tried to make as self-contained as possible, the reader may find an easier access to some of the results.

2 Free Boolean algebras and some other classes of Boolean algebras

In this chapter, we introduce several classes of Boolean algebras which share some of the properties of the class of free Boolean algebras.

A Boolean algebra B is said to be *free* if B is generated 'freely' from some $U \subset B$, i.e. $B = \langle U \rangle_B$ and

$$(*) \quad x_0 \cdot \cdots \cdot x_n \cdot -y_0 \cdot \cdots \cdot -y_m \neq 0 \text{ for any distinct } x_0,\ldots,x_n,y_1,\ldots,y_m \in U.$$

A subset U of a Boolean algebra B with the property $(*)$ above is said to be *independent*. If $U \subseteq B$ is independent and $B = \langle U \rangle_B$, we say that U is a *free generator of* B and B a *free Boolean algebra over* U. We write then $B = \mathrm{Fr}\, U$. This notation it is justified by the fact that any free Boolean algebras over given set U are isomorphic to each other over U. For each U, we consider one of the free Boolean algebras over U as $\mathrm{Fr}\, U$ depending on the context. $\mathrm{Fr}\, U$ may be also taken as the algebra of formulas of propositional logic over the set U of propositional variables modulo logical equivalence, or as the clopen algebra over the generalized Cantor space ${}^U 2$. Free Boolean algebras can be characterized by the following universal algebraic property. The uniqueness of the free Boolean algebra over a set U up to isomorphism over U mentioned above follows also immediately from this lemma.

Lemma 2.1 *Suppose that F is a free Boolean algebra with its free generator U. For any Boolean algebra A and mapping $f : U \rightarrow A$, there is a unique Boolean homomorphism $g : F \rightarrow A$.*

Proof. By Sikorski's Extension Theorem. The uniqueness is clear since $\langle U \rangle_F = F$. ❑ (Lemma 2.1)

If $X \subseteq Y$, we often assume that $\mathrm{Fr}\, X \leq \mathrm{Fr}\, Y$ via canonical embedding[1] , where $A \leq B$ reads "A is a subalgebra of B". Actually $\mathrm{Fr}\, X$ is a very nice subalgebra of $\mathrm{Fr}\, Y$ in the following sense. Let A, B be Boolean algebras such that $A \leq B$. A is said to be a *relatively complete subalgebra of* B if for every $b \in B$, $p^B_A(b) = \max_A\{\, a \in A \,:\, a \leq b \,\}$ exists. If A is a relatively complete subalgebra of B, we denote this by $A \leq_{\mathrm{rc}} B$. Otherwise $A \leq_{\neg\mathrm{rc}} B$. By duality, $A \leq_{\mathrm{rc}} B$ holds if and only if for every $b \in B$, $q^B_A(b) = \min_A\{\, a \in A \,:\, b \leq a \,\}$ exists. We shall refer $p^B_A(b)$ and $q^B_A(b)$ as *lower and upper projections of b onto A* respectively.

Lemma 2.2 *Suppose $X \subseteq Y$. Then* $\mathrm{Fr}\, X \leq_{\mathrm{rc}} \mathrm{Fr}\, Y$.

[1] i.e. by identifying $\mathrm{Fr}\, X$ with $\langle X \rangle_{\mathrm{Fr}\, Y}$.

Proof. Let $b \in \mathrm{Fr}\,Y$, say $b = \sum_{i<m} \prod_{j<n_i} y_{i,j}$ where $y_{i,j} \in Y$ or $-y_{i,j} \in Y$ for all $i < m$, $j < n_i$. Let $b' = \sum_{i<m} \prod_{j<n_i} x_{i,j}$ where

$$x_{i,j} = \begin{cases} 1, & \text{if } y_{i,j} \notin X \text{ and } -y_{i,j} \notin X, \\ y_{i,j}, & \text{otherwise.} \end{cases}$$

We claim that $b' = q^{\mathrm{Fr}\,Y}_{\mathrm{Fr}\,X}(b)$. Clearly $b' \in \mathrm{Fr}\,X$ and $b \leq b'$. If b' were not the upper projection of b onto $\mathrm{Fr}\,X$, there would be $c \in (\mathrm{Fr}\,X)^+$ such that $c \leq b'$ and $c \cdot b = 0$. Without loss of generality we may assume that $c \leq \prod_{j<n_i} x_{i,j}$ for some $i < m$ and c is of the form $u_1 \cdot \cdots \cdot u_k \cdot -v_1 \cdot \cdots \cdot v_l$ for some distinct $u_1, \ldots, u_l, v_1, \ldots, v_l \in X$. Since $c\dot{b} = 0$ and Y is independent, there must be some $j_0 < n_i$ such that $y_{i,j_0} \notin X$, $-y_{i,j_0} \notin X$ and $c \cdot y_{i,j_0} = 0$. Again by independence of Y it follows that $c = 0$. But this is a contradiction. ❑ (Lemma 2.2)

A Boolean algebra B is said to be *openly generated* if $\{\, A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B \,\}$ contains a club subset[2] of $[B]^{\aleph_0}$. The following easy lemma shows that the notion of openly generated Boolean algebras generalizes that of free Boolean algebras.

Lemma 2.3 *Every free Boolean algebra is openly generated.*

Proof. Let $B = \mathrm{Fr}\,Y$. Put

$$\mathcal{C} = \{\, \mathrm{Fr}\,X \;:\; X \in [Y]^{\aleph_0} \,\}.$$

Clearly $\mathcal{C}$ is club in $[B]^{\aleph_0}$. By Lemma 2.2 every element of $\mathcal{C}$ is relatively complete subalgebra of B. ❑ (Lemma 2.3)

Historically, the topological dual of the notion of openly generated Boolean algebras (*openly generated spaces*) was first introduced by E. V. Ščepin who proved in [43] that a topological space is openly generated if and only if it is κ-metrizable — a notion also introduced by himself. The name 'openly generated' is associated with the fact that surjective open mappings are the topological dual to the embeddings of Boolean algebras as relatively complete subalgebras. In Heindorf–Shapiro [25], openly generated Boolean algebras are called *rc-filtered*. Ščepin's study of openly generated spaces was motivated by topological questions from his investigation on uncountable products of metrizable spaces, inverse limit of topological spaces etc. In contrast, we consider here openly generated Boolean algebras in the context

[2] recall that $[X]^\kappa = \{\, u : u \subseteq X, |u| = \kappa \,\}$. $\mathcal{C} \subseteq [X]^\kappa$ is said to be *closed unbounded* (*club*) subset of $[X]^\kappa$ (or *club* in $[X]^\kappa$) if $\mathcal{C}$ is closed with respect to unions of chains of length $\leq \kappa$ and cofinal in $[X]^\kappa$ with respect to $\subseteq$. $\mathcal{S} \subseteq [X]^\kappa$ is said to be *stationary* (in $[X]^\kappa$) if $\mathcal{S} \cap \mathcal{C} \neq \emptyset$ for every club $\mathcal{C} \subseteq [X]^\kappa$. $[X]^{<\kappa}$ or $[X]^{\leq\kappa}$, club and stationary subsets of $[X]^{<\kappa}$ are defined similarly. Note that $\mathcal{C} \subseteq [X]^{<\kappa^+}$ is club (stationary resp.) in $[X]^{<\kappa^+}$ if $\mathcal{C} \cap [X]^\kappa$ is club (stationary resp.) in $[X]^\kappa$. $C \subseteq \kappa$ is said to be club (stationary resp.) if C is club (stationary resp.) in $[\kappa]^{<\kappa}$ in the sense above.

of almost freeness and our setting will be purely Boolean algebraic. In particular the topological translation of our results will be always bound to zero dimensional spaces. In practice, however, this restriction will not be very essential in many cases. For more detailed historical remarks on openly generated spaces and their roll in topology, the reader may consult Heindorf–Shapiro [25].

The notion of relatively complete subalgebras can be generalized to that of κ-substructures defined below. For an ordered structure $B = (B, <, \ldots)$, a substructure A of B and $b \in B$, let

$$A \restriction b = \{ a \in A : a \leq b \},$$
$$A \uparrow b = \{ a \in A : a \geq b \}.$$

For a regular cardinal κ, A is said to be *κ-substructure* (or κ-subalgebra in case of Boolean algebras) of B if for every $b \in B$, the set $A \restriction b$ has a cofinal subset of cardinality less than κ and the set $A \uparrow b$ has a coinitial subset of cardinality less than κ. If A is a κ-substructure of B we denote this by $A \leq_\kappa B$. Thus for Boolean algebras A, B such that $A \leq B$, we have $A \leq_{\mathrm{rc}} B$ if and only if $A \leq_{\aleph_0} B$. We shall write also "$A \leq_\sigma B$" in place of "$A \leq_{\aleph_1} B$". If $A \leq_\sigma B$ we say that A is a σ-substructure (or σ-subalgebra in case of Boolean algebras). For Boolean algebras A, B to show that $A \leq_\kappa B$, it is enough, by duality, to check that every ideal over A of the form $A \restriction b$, $b \in B$ is generated by a subset of cardinality less than κ or that every filter over A of the form $A \uparrow b$, $b \in B$ is generated by a subset of cardinality less than κ.

In analogy to openly generated Boolean algebras, let us say that an ordered structure B is *$\leq_\kappa$-generated* if the set $\{ A \in [B]^\kappa : A \leq_\kappa B \}$ contains a club subset of $[B]^\kappa$. Thus $\leq_{\aleph_0}$-generated Boolean algebras are just openly generated Boolean algebras. As before we shall also say $\leq_\sigma$-generated instead of $\leq_{\aleph_1}$-generated. In Heindorf–Shapiro [25], $\leq_\sigma$-generated Boolean algebras are called "σ-filtered" and characterized as Boolean algebras with the so-called weak Freese-Nation property. We shall prove this and some other characterizations of $\leq_\kappa$-generated Boolean algebras in Chapter 5. From this characterization, it follows that if B is $\leq_\kappa$-generated and $\kappa \leq \lambda$ then B is $\leq_\lambda$-generated (Proposition 5.8). In particular, for any κ, the class of κ-generated Boolean algebras include the class of openly generated Boolean algebras. For $\kappa > \aleph_0$, $\leq_\kappa$-generatedness is too general to be considered still as a notion of almost freeness — e.g. it follows immediately from the definition that every Boolean algebra of cardinality less or equal to κ is $\leq_\kappa$-generated. However we shall see later that this notion still enjoy several nice characterizations similar to those for openly generatedness (see Theorem 5.1, Proposition 5.11, Theorem 5.5) and is also quite useful in our context (see e.g. the proof of Theorem 8.19).

In contrast to the situation in groups or abelian groups, a subalgebra of a free Boolean algebra need not to be free. More generally, the following holds.

Proposition 2.4 (Fuchino–Koppelberg–Shelah [17]) *For any regular cardinal κ,* *there is a subalgebra of* $\mathrm{Fr}\,\kappa^+$ *which is not* $\leq_\kappa$*-generated.*

Proof. The topological dual to the Boolean algebra B below is used in Engelking [9] to show that there exists a dyadic space which in not Dugundji. In the language of Boolean algebras this means that there exists a subalgebra of a free Boolean algebra which is not projective. This assertion follows from the present proposition since every projective Boolean algebra is openly generated (see the remark before Theorem 2.7 below).

Let X be a set of cardinality κ^+. We shall show that there is a subalgebra of $\mathrm{Fr}\,X$ which is not $\leq_\kappa$-generated. Let U_1 and U_2 be the ultrafilters of $\mathrm{Fr}\,X$ generated by X and $\{\,-x\,:\,x\in X\,\}$ respectively. Let

$$B=\{\,b\in\mathrm{Fr}\,X\;:\;b\in U_1\;\Leftrightarrow\;b\in U_2\,\}.$$

Clearly B is a subalgebra of $\mathrm{Fr}\,X$. We claim that B is not $\leq_\kappa$-generated. For $Y\subseteq X$, let $B_Y=B\cap\mathrm{Fr}\,Y$.

Claim 2.4.1 *For every* $Y\in[X]^\kappa$*,* B_Y *is not a* κ*-subalgebra of* B*.*

$\vdash$ Let $x_0\in Y$ and let x_1, x_2 be two distinct elements of $X\setminus Y$. Let

$$b=x_0+x_1+-x_2.$$

Since $b\in U_1$ and $b\in U_2$, we have $b\in B$. Let $I=B_Y\restriction b$. We show that I cannot be generated by any subset of size $<\kappa$. let J be an arbitrary subset of I of cardinality less than κ. For $c\in J$, we have $c\leq x_0$. Since x_0 is not an element of B, c is strictly less than x_0. Let Y' be a subset of Y of cardinality less than κ such that $J\subseteq B_{Y'}$. Let y_1, y_2 be two distinct elements of $Y\setminus Y'$. Let

$$d=x_0\cdot y_1\cdot -y_2.$$

Then we have $d\notin U_1$, $d\notin U_2$ and $d\leq x_0$. Hence $d\in B_Y\restriction b$. But d is incomparable with every non-zero element of J. $\dashv$ (Claim 2.4.1)

Since there are club many $C\in[B]^\kappa$ of the form $C=B_Y$ for $Y\in[X]^\kappa$, it follows that B is not $\leq_\kappa$-generated. ❑ (Proposition 2.4)

Adopting the terminology in topology, let us call a Boolean algebra B *dyadic* if B is embeddable in a free Boolean algebra. The dyadic Boolean algebras have a very important subclass namely that of projective Boolean algebras. A Boolean algebra B is said to be *projective* if $B\oplus\mathrm{Fr}\,\kappa$ is free for some κ, where $A\oplus B$ denotes the free product[3] of Boolean algebras A and B. This definition differs from the original one which states that a Boolean algebra is projective if and only if it is projective

[3] see Chapter 4 for more about free products.

in the sense of universal algebra. The equivalence of the present definition to the original one was stated implicitly in Koppelberg [32]. A proof of the equivalence will be given in Chapter 4. An advantage of our definition of projectivity is that we can formulate the same notion in arbitrary variety. In Mekler–Shelah [37], the projective algebras in this general setting is called *essentially free*. Topological duals of projective Boolean algebras are also studied in topology where these topological spaces are called *Dugundji spaces*. From our definition of projective Boolean algebras, it is seen immediately that every projective Boolean algebra is dyadic. For a structure B, let us call $\mathcal{C} \subseteq \mathcal{P}(B)$ *algebraically tight* if every $A \in \mathcal{C}$ is subalgebra of B and $\langle A \cup A' \rangle \in \mathcal{C}$ holds for every A, $A' \in \mathcal{C}$. The following theorem will be proved in the next chapter. For Boolean algebras A, B, such that $A \leq B$, B is said to be *countably generated over* A if there is a countable $X \subseteq B$ such that $B = A[X]$.

Theorem 2.5 *For a Boolean algebra B the following are equivalent.*

a) *B is projective;*

b) *$B \oplus \mathrm{Fr}\, X$ is free for some κ;*

c) (Ščepin [44], see also Bandlow [2]) $\{ A \in [B]^{\aleph_0} : A \leq_{\mathrm{rc}} B \}$ *contains an algebraically tight club subset of* $[B]^{\aleph_0}$.

d) (Haydon [24]) *There exists a continuously increasing sequence $(B_\alpha)_{\alpha<\kappa}$ of subalgebras of B such that $B_0 = 2$, $\bigcup_{\alpha<\kappa} B_\alpha = B$, $B_\alpha \leq_{\mathrm{rc}} B$ and $B_{\alpha+1}$ is countably generated over B_α for every $\alpha < \kappa$;*

e) (Koppelberg [32]) *There exists a continuously increasing sequence $(B_\alpha)_{\alpha<\kappa}$ of subalgebras of B and a sequence $(a_\alpha)_{\alpha<\kappa}$ of elements of B such that $B_0 = 2$, $\bigcup_{\alpha<\kappa} B_\alpha = B$, $B_\alpha \leq_{\mathrm{rc}} B$ and $B_{\alpha+1} = B(a_\alpha)$ for every $\alpha < \kappa$.* ❑

Lemma 2.6 *For a Boolean algebra B,*

1) *if B is free then B is projective;*

2) *if B is projective then B is openly generated;*

3) *if B is openly generated and of cardinality $\leq \aleph_1$ then B is projective.* ❑

Proof. *1*) follows directly from the definition of projective Boolean algebras. *2*) by *c*) and *3*) by *d*) of the characterization above. ❑ (Lemma 2.6)

Openly generated Boolean algebras can be regarded as "almost free" in the sense along the line with (†) on page 1. In the following we introduce yet two other kinds of almost freeness. To be general, let $\mathcal{E}$ be a property of Boolean algebras (e.g. free, projective, openly generated etc.). For a regular cardinal κ a Boolean algebra B is said to be κ-$\mathcal{E}$ (e.g. κ-free, κ-projective, κ-openly generated etc.) if the set $\{ A \in [B]^{<\kappa} : A \leq B$ and A satisfies $\mathcal{E} \}$ contains a club subset of $[B]^{<\kappa}$. Note that for a Boolean algebra B of cardinality less than κ this simply means that B has the property $\mathcal{E}$. For a regular κ, a Boolean algebra B is κ-$\mathcal{E}$ if and only if there exists an upward directed partial ordering $I = (I, \leq)$ and an indexed family $(B_i)_{i \in I}$

of subalgebras of B such that

0) $|B_i| < \kappa$ and B_i satisfies the property $\mathcal{E}$ for every $i \in I$;

1) $B_i \leq B_j$ for all $i, j \in I$ with $i \leq j$;

2) for every increasing chain $(i_\alpha)_{\alpha<\delta}$ in I of length $\delta < \kappa$, $i^* = \sup\{\, i_\alpha : \alpha < \delta \,\}$ exists and $B_{i^*} = \bigcup_{\alpha<\delta} B_{i_\alpha}$;

3) $B = \bigcup_{i\in I} B_i$.

This is just because of the following purely set-theoretic fact.

Proposition 2.7 *Suppose that $\mathcal{C} \subseteq [A]^{<\kappa}$ for a regular cardinal κ and a set A. Then the following are equivalent:*

a) $\mathcal{C}$ contains a club subset of $[A]^{<\kappa}$;

b) there exists an upward directed partial ordering $I = (I, \leq)$ and an indexed family $(C_i)_{i\in I}$ of elements of $\mathcal{C}$ such that

> *0) $|C_i| < \kappa$ for every $i \in I$;*
>
> *1) $C_i \subseteq C_j$ for all $i, j \in I$ with $i \leq j$;*
>
> *2) for every increasing chain $(i_\alpha)_{\alpha<\delta}$ in I of length $\delta < \kappa$, $i^* = \sup\{\, i_\alpha : \alpha < \delta \,\}$ exists and $C_{i^*} = \bigcup_{\alpha<\delta} C_{i_\alpha}$;*
>
> *3) $A = \bigcup_{i\in I} C_i$.*

Remark. In *b*) above, since I is upward directed, we may replace *2*) by

> 2′) for every upward directed subset J of I of cardinality less than κ, $i^* = \sup I$ exists and $C_{i^*} = \bigcup_{i\in J} C_i$.

Proof of Proposition 2.7 *a*) $\Rightarrow$ *b*): Let $\mathcal{C}$ be club in $[A]^{<\kappa}$. Then $I = (\mathcal{C}, \subseteq)$ and $(C_i)_{i\in\mathcal{C}}$, where $C_i = i$ for $i \in \mathcal{C}$ satisfies *0*) – *3*).

b) $\Rightarrow$ *a*): Let I and $(C_i)_{i\in I}$ be as in *b*). Fix a sufficiently large χ and a well-ordering $\leq^*$ on $\mathcal{H}(\chi)$. Let $\tilde{h}_{\leq^*}$ be the Skolem-hull operator corresponding to the canonical built-in Skolem functions in $\mathcal{M} = (\mathcal{H}(\chi), \in, \leq^*)$. Let us call $i \in I$ good if $\sup(\tilde{h}_{\leq^*}(C_i) \cap I) \leq i$. Note that $|\tilde{h}_{\leq^*}(C_i) \cap I| < \kappa$ by *0*) and $\tilde{h}_{\leq^*}(C_i) \cap I$ is upward directed. Hence, by *2*), $\sup(\tilde{h}_{\leq^*}(C_i) \cap I)$ exists. Let $\mathcal{C}' = \{\, C_i : i \text{ is good} \,\}$. Then $\mathcal{C}' \subseteq \mathcal{C}$ and $\mathcal{C}'$ is club in $[A]^{<\kappa}$. ❑ (Proposition 2.7)

Proposition 2.7 provides a short proof of the following well-known theorem.

Theorem 2.8 *Suppose that $A \subseteq B$ and $\mathcal{C} \subseteq [B]^{<\kappa}$ is club. Then $\mathcal{C}' = \{\, C \cap A : C \in \mathcal{C} \,\}$ contains a club in $[A]^{<\kappa}$.*

Proof. Let $I = \mathcal{C}$ be the partial ordering with $D \leq D' \Leftrightarrow D \subseteq D'$. For each $D \in I$, let $C_D = D \cap A$. Then $(C_D)_{D\in I}$ satisfies the conditions *0*) – *3*) in Proposition 2.7, b). Hence $\mathcal{C}'$ contains a club in $[A]^{<\kappa}$. ❑ (Theorem 2.8)

The relation of openly generated Boolean algebras and κ-projective Boolean algebras will be one of the main subjects in Sections 7.2 and 8.2. We show in Section 7.2 that under $V = L$, there are κ-free and $L_{\infty\kappa}$-free Boolean algebras which are not openly generated for any regular non-weakly-compact cardinal κ (Theorem 7.15). In contrast to this, we show in Section 8.2 that it follows from Fleissner's Axiom R that every Boolean algebra B is openly generated if and only if B is $\aleph_2$-projective (Theorem 8.19).

We need the infinitary logic $L_{\infty\kappa}$ to define another kind of almost freeness. $L_{\infty\kappa}$ is the logic whose formulas are constructed recursively just like in the first order logic with the difference that conjunction and disjunction of any set of formulas as well as quantification over a block (or sequence) of variables of length less than κ are allowed. The semantics of $L_{\infty\kappa}$ is defined in a canonical way. We say that two structures A, B of the same signature are *$L_{\infty\kappa}$-elementarily equivalent* (or *elementarily equivalent in $L_{\infty\kappa}$*) and denote this by $A \equiv_{L_{\infty\kappa}} B$, if $A \models \varphi$ holds if and only if $B \models \varphi$ holds for all $L_{\infty\kappa}$-sentence φ. Let us call a Boolean algebra B $L_{\infty\kappa}$-$\mathcal{E}$ if there is a Boolean algebra C with the property $\mathcal{E}$ such that B is elementarily equivalent to C in $L_{\infty\kappa}$, i.e. $B \models \varphi$ iff $C \models \varphi$ for every $L_{\infty\kappa}$-sentence φ. The following characterizations of $L_{\infty\kappa}$-free/projective Boolean algebras to be proved in Chapter 6 show that $L_{\infty\kappa}$-free/projective Boolean algebras may also be regarded as notions of almost freeness along the line of (†) on page 1. A Boolean algebra B *is free over* A ($A \leq_{\mathrm{free}} B$) if $A \leq B$ and B is isomorphic to $A \oplus \mathrm{Fr}\,\kappa$ over A for some $\kappa \leq |B|$. B *is projective over* A ($A \leq_{\mathrm{proj}} B$) if $B \oplus \mathrm{Fr}\,\kappa$ is free over A for $\kappa = |B| + \aleph_0$. The restriction to the $L_{\infty\kappa}$-free Boolean algebras for cardinals κ of uncountable cofinality in the next theorem is not essential: by the back-and-forth argument (see Theorem 6.2), it is easy to show that atomless Boolean algebras are $L_{\infty\aleph_0}$-equivalent to each other. Hence $L_{\infty\aleph_0}$-free Boolean algebras are just atomless Boolean algebras. For singular κ, every $L_{\infty\kappa}$-free Boolean algebra is κ^+-free (Theorem 6.23).

Theorem 2.9 (see Corollary 6.15) *Let κ be a cardinal of uncountable cofinality. A Boolean algebra B is $L_{\infty\kappa}$-free if and only if there exists a family $\mathcal{F}$ of subalgebras of B of cardinality less than κ such that every $F \in \mathcal{F}$ is free; $\bigcup \mathcal{F} = B$ and for every $\mathcal{F}' \subseteq \mathcal{F}$ of cardinality less than κ there exists an $F_0 \in \mathcal{F}$ such that $F \leq_{\mathrm{free}} F_0$ holds for every $F \in \mathcal{F}$.* ❑

Theorem 2.10 (see Proposition 6.18) *Let κ be a cardinal such that $\mathrm{cf}\,\kappa > \omega$. For a Boolean algebra B we have the implication $a) \Rightarrow b) \Rightarrow c)$ for the following statements:*

a) *B is $L_{\infty\kappa}$-projective;*

b) *there is a family $\mathcal{F}$ of subalgebras of B of cardinality less than κ such that every $F \in \mathcal{F}$ is projective; $\bigcup \mathcal{F} = B$ and for every $\mathcal{F}' \subseteq \mathcal{F}$ of cardinality less than*

κ, *there is* $F_0 \in \mathcal{F}$ *such that* $F \leq_{\text{proj}} F_0$ *for every* $F \in \mathcal{F}'$;

c) $B \oplus \operatorname{Fr} \kappa$ *is* $L_{\infty\kappa}$*-free.* ❑

Under $V = L$, there exists $L_{\infty\kappa}$-projective Boolean algebra which is not openly generated for any κ (Theorem 7.15). On the other hand under Fleissner's Axiom R, every $L_{\infty\aleph_2}$-projective Boolean algebra is openly generated (Corollary 8.21).

3 Embeddings of Boolean algebras

We put together here some of the basic lemmas on the relations $\leq_{\rm rc}$, $\leq_\kappa$, $\leq_{\rm proj}$, $\leq_{free}$ introduced in the last chapter. Remember that $\leq_{\rm rc}$ is equal to $\leq_\kappa$ for $\kappa = \aleph_0$ and $\leq_\sigma$ to $\leq_\kappa$ for $\kappa = \aleph_1$.

Lemma 3.1 *Let A, B, C be Boolean algebras.*

1) If $A \leq_\kappa B$ and $B \leq_\kappa C$ then $A \leq_\kappa C$.

2) If $A \leq_\kappa C$ and $A \leq B \leq C$ then $A \leq_\kappa B$.

3) Suppose that κ is regular. If $A \leq_\kappa B$ and $X \in [B]^{<\kappa}$ then $A[X] \leq_\kappa B$. In particular, if $A \leq_{\rm rc} B$ and $b_1, \ldots, b_n \in B$ then $A(b_1, \ldots, b_n) \leq_{\rm rc} B$.

Proof. *1*) and *2*) are trivial. For *3*), since κ is regular, it is enough to show that $A(y) \leq_\kappa B$ holds for every $y \in B$.

Suppose $b \in B$. for $b' \in A(y)$ let $b' = a_0 \cdot y + a_1 \cdot -y$ for some a_0, $a_1 \in A$. Then we have

$$(*) \quad b' \leq b \;\Leftrightarrow\; a_0 \cdot y \leq b \text{ and } a_1 \cdot -y \leq b \;\Leftrightarrow\; a_0 \leq b + -y \text{ and } a_1 \leq b + y.$$

Let U_0 and U_1 be generators of ideals $A \restriction (b + -y)$ and $A \restriction (b + y)$ respectively of cardinality less than κ. Then $U = \{\, u_0 \cdot y + u_1 \cdot -y \;:\; u_0 \in U_0 \text{ and } u_1 \in U_1 \,\}$ has cardinality less than κ and generates $A(y) \restriction b$ by $(*)$ above. ❑ (Lemma 3.1)

Suppose that A, B, C are Boolean algebras such that $A \leq C$ and $B \leq C$. A and B are said to be *independent* (*in* C) if $a \cdot b \neq 0$ for all $a \in A^+$ and $b \in B^+$. A subset X of C is said to be independent over A if

$$a \cdot x_0 \cdot \cdots \cdot x_n \cdot -y_0 \cdot \cdots \cdot -y_m \neq 0$$

for any $a \in A^+$ and distinct $x_0, \ldots, x_n$, $y_1, \ldots$ $y_m \in X$. Thus $X \subseteq C$ is independent over A iff A and $\langle X \rangle_C$ are independent and $\langle X \rangle_C$ is a free Boolean algebra with free generator X. For $c \in C$ we say that c is independent over B if $\{\, c \,\}$ is. Note that c is independent over B iff B and $\langle c \rangle_C$ are independent if and only if $b \cdot c \neq 0$ and $b \cdot -c \neq 0$ for every $b \in B^+$.

If A and B are independent in C and $C = \langle A \cup B \rangle_C$, we say that C is *the free product of A and B* and write $C = A \oplus B$. We may talk here about *the* free product, since, if C is as above and C' is such that A, $B \leq C'$, A and B are independent in C' and $C' = \langle A \cup B \rangle_{C'}$, then C and C' are isomorphic over $A \cup B$. If Boolean algebras A, B are given, we take isomorphic copies A', B' of A and B such that $A' \cap B' = 2$. and define $C = A \oplus B$ to be the Boolean algebra generated from the set $A' \cup B'$ freely except equations holding already in A' and B'. Identifying A and B with A' and B' respectively, C is then the free product of A and B in the sense above. In the following we always regard A and B as subalgebras of $A \oplus B$ by this identification. For $A \leq A'$ and $B \leq B'$ we suppose that $A \oplus B$ is a subalgebra of $A' \oplus B'$ just by identifying $A \oplus B$ with $\langle A \cup B \rangle_{A' \oplus B'}$. Free products are characterized by the following universal algebraic property. Note that the uniqueness of $A \oplus B$ up to isomorphism over $A \cup B$ mentioned above is an immediate consequence of this characterization.

Lemma 3.2 *For any Boolean algebras A, B, C (such that $A \cap B = 2$) and Boolean homomorphisms $f : A \to C$, $g : B \to C$, there is a unique Boolean homomorphism $h : A \oplus B \to C$ such that $f, g \subseteq h$.*

Proof. By Sikorski's Extension Theorem. Uniqueness is clear since $\langle A \cup B \rangle_{A \oplus B} = A \oplus B$. ❏ (Lemma 3.2)

Let us denote the homomorphism h as above with $f \oplus g$. As we defined in the last chapter, $A \leq_{\rm free} B$ means that B is isomorphic to $A \oplus F$ over A for some free F. In the terminology above this is equivalent to the assertion that there is $X \subseteq B$ such that X is independent over A and $B = A[X]$. We shall call such X a *free generator of B over A*. We write $A \leq_{\neg{\rm free}} B$ if A is a subalgebra of B but there is no free F such that $B \cong_A A \oplus F$.

Lemma 3.3 *Let A, A', B, B' be Boolean algebras.*

1) $A \leq_{\rm rc} A \oplus B$ and $B \leq_{\rm rc} A \oplus B$.

2) If $A \leq_{\rm rc} A'$ and $B \leq_{\rm rc} B'$ then $A \oplus B \leq_{\rm rc} A' \oplus B'$.

3) If $A \leq_{\rm free} A'$ and $B \leq_{\rm free} B'$ then $A \oplus B \leq_{\rm free} A' \oplus B'$.

4) If $A \leq_{\rm proj} A'$ and $B \leq_{\rm proj} B'$ then $A \oplus B \leq_{\rm proj} A' \oplus B'$.

5) If $A \leq_{\rm free} B$ then $A \leq_{\rm proj} B$. If $A \leq_{\rm proj} B$ then $A \leq_{\rm rc} B$.

Proof. *1*): We show $A \leq_{\rm rc} A \oplus B$. Let $c \in A \oplus B$, say $c = a_0 \cdot b_0 + \cdots + a_{n-1} \cdot b_{n-1}$ for some $a_i \in A^+$, $b_i \in B^+$ for $i < n$. Let $c^* = a_0 + \cdots + a_{n-1}$. We claim that $c^* = q_A^{A \oplus B}(c)$. Clearly $c^* \in A$ and $c^* \geq c$. Hence if c^* were not the upper projection of c onto A, there would be $d \in A^+$ such that $d \leq c^*$ and $d \cdot c = 0$. For such d, there is an $i < n$ such that $d \cdot a_i \neq 0$. But we have $(d \cdot a_i) \cdot b_i \leq d \cdot c = 0$. This is a contradiction since $d \cdot a_i \in A^+$ and $b_i \in B^+$. $B \leq_{\rm rc} A \oplus B$ can be proved similarly.

2): Let $c \in A' \oplus B'$, say $c = a_0 \cdot b_0 + \cdots + a_{n-1} \cdot b_{n-1}$ for some $a_i \in (A')^+$, $b_i \in (B')^+$ for $i < n$. Let $c^* = q_A^{A'}(a_0) \cdot q_B^{B'}(b_0) + \cdots + q_A^{A'}(a_{n-1}) \cdot q_B^{B'}(b_{n-1})$. We claim that c^* is the upper projection of c onto $A \oplus B$. Clearly $c^* \in A \oplus B$ and $c^* \geq c$. Hence if c^* were not the upper projection of c onto $A \oplus B$, there would be $d \in (A \oplus B)^+$ such that $d \leq c^*$ and $d \cdot c = 0$. Without loss of generality we may assume that d is of the form $a \cdot b$ for some $a \in A^+$ and $b \in B^+$. Let $i < n$ be such that $d \cdot q_A^{A'}(a_i) \cdot q_B^{B'}(b_i) \neq 0$. We have $d \cdot a_i \cdot b_i \leq d \cdot c = 0$. Hence either $a \cdot a_i = 0$ or $b \cdot b_i = 0$. If $a \cdot a_i = 0$, then $a_i \leq q_A^{A'}(a_i) \cdot -a < q_A^{A'}(a_i)$. This is a contradiction since $q_A^{A'}(a_i) \cdot -a \in A$. Similarly, $b \cdot b_i = 0$ also leads to a contradiction.

3): Suppose that U is free over A in A' such that $A' = A[U]$ and V is free over

B in B' such that $B' = B[U]$. Then $U \cup V$ is free over $A \oplus B$ in $A' \oplus B'$ and $A' \oplus B' = A \oplus B[U \cup V]$.

4): Let X, Y be two disjoint sets such that $|X| = |A'| + \aleph_0$ and $|Y| = |B'| + \aleph_0$. Considering the all of the following Boolean algebras as subalgebras of $A' \oplus B' \oplus \mathrm{Fr}\,(X \cup Y)$, we have $A \leq_{\mathrm{free}} A' \oplus \mathrm{Fr}\,X$ and $B \leq_{\mathrm{free}} B' \oplus \mathrm{Fr}\,Y$. Hence by *3*), $A \oplus B \leq_{\mathrm{free}} (A' \oplus \mathrm{Fr}\,X) \oplus (B' \oplus \mathrm{Fr}\,Y) = (A' \oplus B') \oplus \mathrm{Fr}\,(X \cup Y)$. Thus we have $A \oplus B \leq_{\mathrm{proj}} A' \oplus B'$.

5): The first assertion follows directly from the definitions. For the second, suppose that $B \oplus F$ is isomorphic to $A \oplus F$ over A for a free Boolean algebra F. By *1*), we have $A \leq_{\mathrm{rc}} B \oplus F$. Hence $A \leq_{\mathrm{rc}} B$ by Lemma 3.1, *2*). ❑ (Lemma 3.3)

2) of the lemma above holds, more generally, for "$\leq_\kappa$" for an arbitrary infinite cardinal κ in place of "$\leq_{\mathrm{rc}}$". The proof is a bit more intricate and uses the following lemma from Koppelberg [31]:

Lemma 3.4 (Lemma 11.7 in [31]) *For Boolean algebras A, B, let a_i, $a'_j \in A$ and b_i, $b'_j \in B$ for $i < n$, $j < m$. Then, in $A \oplus B$,*

$$a_0 \cdot b_0 + \cdots + a_{n-1} \cdot b_{n-1} \leq a'_0 \cdot b'_0 + \cdots + a'_{m-1} \cdot b'_{m-1}$$

if and only if, for all $i \in n$ and $J \subseteq m$,

$$a_i \leq \Sigma\{\, a'_j \,:\, j \in J \,\} \quad \text{or} \quad b_i \leq \Sigma\{\, b'_j \,:\, j \notin J \,\}.$$

❑

Lemma 3.5 *For any κ, if $A \leq_\kappa A'$ and $B \leq_\kappa B'$, then $A \oplus B \leq_\kappa A' \oplus B'$.*

Proof. Let $c \in A' \oplus B'$, say $c = a'_0 \cdot b'_0 + \cdots + a'_{m-1} \cdot b'_{m-1}$ for some $m \in \omega$ and $a'_i \in A'$, $b'_i \in B'$ for $i < m$. We show that $(A \oplus B) \restriction c$ has a cofinal subset of cardinality less than κ. For each $J \subseteq J$, let $s_J = \sum\{\, a'_j \,:\, j \in J \,\}$ and $t_J = \sum\{\, b'_j \,:\, j \notin J \,\}$. By the assumption there are cofinal subsets X_J and Y_J of $A \restriction s_J$ and $B \restriction t_J$ respectively of cardinality less than κ. Let $A'' = [\bigcup_{J \in \mathcal{P}(m)} X_J]_A$ and $B'' = [\bigcup_{J \in \mathcal{P}(m)} Y_J]_B$. Let $Z = (A'' \oplus B'') \restriction c$. Then $|Z| < \kappa$, and $Z \subseteq (A \oplus B) \restriction c$. We show that Z is cofinal in $Z \subseteq (A \oplus B) \restriction c$. Let $d \in A \oplus B$, say $d = a_0 \cdot b_0 + \cdots + a_{n-1} \cdot b_{n-1}$ for some $n \in \omega$ and $a_i \in A$, $b_i \in B$ for $i < n$. For each $i < n$ and $J \subseteq m$, let $a_{i,J} \in X_J \cup \{\, 1 \,\}$ and $b_{i.J} \in Y_J \cup \{\, 1 \,\}$ be such that $a_i \leq a_{i,J} \leq s_J$ if $a_i \leq s_J$, or $a_{i,J} = 1$ otherwise; and $b_i \leq b_{i,J} \leq t_J$ if $b_i \leq t_J$, or $b_{i,J} = 1$ otherwise. For $i < n$, let $a''_i = \prod_{J \in \mathcal{P}(m)} a_{i,J}$ and $b''_i = \prod_{J \in \mathcal{P}(m)} b_{i,J}$. Let $d'' = a''_0 \cdot b''_0 + \cdots + a''_{n-1} \cdot b''_{n-1}$. Since $a_i \leq a''_i$ and $b_i \leq b''_i$ for all $i < n$, we have $d \leq d''$. Clearly $d'' \in A'' \oplus B''$. By Lemma 3.4, we have also $d'' \leq c$. Hence $d'' \in Z$. ❑ (Lemma 3.5)

Lemma 3.6 *Let A, B, C be Boolean algebras such that $A \leq B \leq C$.*

1) *If $A \leq_{\mathrm{rc}} B$ and $B \leq_{\mathrm{rc}} C$ then $A \leq_{\mathrm{rc}} C$.*

2) *If $A \leq_{\mathrm{free}} B$ and $B \leq_{\mathrm{free}} C$ then $A \leq_{\mathrm{free}} C$.*

3) *If B is free and $B \leq_{\rm free} C$, then C is free.*
4) *If $A \leq_{\rm proj} B$ and $B \leq_{\rm proj} C$ then $A \leq_{\rm proj} C$.*
5) *If B is projective and $B \leq_{\rm proj} C$, then C is projective.*

Proof. *1)*: $p^B_A \circ p^C_B$ is a lower projection on C onto A.

2): Let $U \subseteq B$ be a free generator of B over A and $V \subseteq C$ be the free generator of C over B. Then $U \cup V$ is a free generator of C over A.

3) follows from *2)* for $A = 2$.

4): Let $\kappa = |B|$ and $\lambda = |C|$. Then we have $A \leq_{\rm free} B \oplus {\rm Fr}\,\kappa$ and $B \leq_{\rm free} C \oplus {\rm Fr}\,\lambda$. By Lemma 3.3, *3)*, it follows that $B \oplus {\rm Fr}\,\kappa \leq_{\rm free} (C \oplus {\rm Fr}\,\lambda) \oplus {\rm Fr}\,\kappa$. Hence by *2)*, $A \leq_{\rm free} (C \oplus {\rm Fr}\,\lambda) \oplus {\rm Fr}\,\kappa$. But $(C \oplus {\rm Fr}\,\lambda) \oplus {\rm Fr}\,\kappa$ is isomorphic to $C \oplus {\rm Fr}\,\lambda$ over C. Hence $A \leq_{\rm free} C \oplus \lambda$ and $A \leq_{\rm proj} C$.

5) follows from *4)* for $A = 2$. ❑ (Lemma 3.6)

Lemma 3.7 *Suppose that $(B_\alpha)_{\alpha<\delta}$ is continuously increasing chain of Boolean algebras and $B = \bigcup_{\alpha<\delta} B_\alpha$.*

1) *If $B_\alpha \leq_{\rm free} B_{\alpha+1}$ for every $\alpha < \delta$ then $B_0 \leq_{\rm free} B$.*
2) *If $B_\alpha \leq_{\rm proj} B_{\alpha+1}$ for every $\alpha < \delta$ then $B_0 \leq_{\rm proj} B$.*
3) *If $B_\alpha \leq_{\rm rc} B_{\alpha+1}$ for every $\alpha < \delta$ then $B_0 \leq_{\rm rc} B$.*
4) *For a regular κ, if $B_\alpha \leq_\kappa B_{\alpha+1}$ for every $\alpha < \delta$ then $B_0 \leq_\kappa B$.*

Proof. *1)*: For each $\alpha < \delta$ let $U_\alpha \subseteq B_{\alpha+1}$ be independent over B_α such that $B_{\alpha+1} = B_\alpha[U_\alpha]$. Then $U = \bigcup_{\alpha<\delta} U_\alpha$ is independent over B_0 and $B = B_0[U]$.

2): Let $(X_\alpha)_{\alpha<\delta}$ be a sequence of pairwise disjoint sets such that $B_\alpha \leq_{\rm free} B_{\alpha+1} \oplus {\rm Fr}\,X_\alpha$. Put $X = \bigcup_{\alpha<\delta} X_\alpha$. We show that $B_0 \leq_{\rm free} B \oplus {\rm Fr}\,X$. For $\alpha < \delta$ let $C_\alpha = \langle B \cup \bigcup_{\beta<\alpha} X_\beta \rangle_{B \oplus {\rm Fr}\,X}$ $(= B_\alpha \oplus {\rm Fr}\,(\bigcup_{\beta<\alpha} X_\beta))$. Then $C_0 = B_0$, $\bigcup_{\alpha<\delta} C_\alpha = B \oplus {\rm Fr}\,X$ and $(C_\alpha)_{\alpha<\delta}$ is continuously increasing. $C_\alpha = B_\alpha \oplus {\rm Fr}\,(\bigcup_{\beta<\alpha} X_\alpha) \leq_{\rm free} B_{\alpha+1} \oplus {\rm Fr}\,X_\alpha \oplus {\rm Fr}\,(\bigcup_{\beta<\alpha} X_\alpha) = C_{\alpha+1}$ by Lemma 3.3, *3)*. Hence by *1)* it follows that $B \oplus {\rm Fr}\,X$ is free over B_0.

3) is just *4)* for $\kappa = \aleph_0$.

4): By induction on $\alpha \leq \delta$, we show that $B_0 \leq_\kappa B_\alpha$. For $\alpha = 0$ this is clear. Suppose that α is a limit and we have shown that $B_0 \leq_\kappa B_\beta$ for every $\beta < \alpha$. Let $b \in B_\alpha$. By the continuity there is some $\beta_0 < \alpha$ such that $b \in B_{\beta_0}$. By the assumption $B_0 \restriction b$ has a cofinal subset of cardinality less than κ. If $\alpha = \beta + 1$ and we have shown that $B_0 \leq_\kappa B_\beta$ then for any $b \in B_{\beta+1}$, if X is cofinal in $B_{\beta+1} \restriction b$ and of cardinality $< \kappa$ and Y_x is cofinal in $B_0 \restriction x$ and of cardinality $< \kappa$ for each $x \in X$, then $\bigcup_{x\in X} Y_x$ is cofinal in $B_0 \restriction b$ and of cardinality $< \kappa$. ❑ (Lemma 3.7)

Lemma 3.8 *Suppose that κ is a regular cardinal and $2^{<\kappa} = \kappa$. If $(B_\alpha)_{\alpha<\kappa^+}$ is a continuously increasing sequence of subalgebras of B such that $B_\alpha \leq_\kappa B$ and $|B_\alpha| \leq \kappa$ for every $\alpha < \kappa^+$, then, for $B' = \bigcup_{\alpha<\kappa^+} B_\alpha$, we have $B' \leq_\kappa B$.*

Proof. Suppose not. Then there is a $b \in B$ such that $B' \restriction b$ does not have any cofinal subset of cardinality less than κ. Let χ be sufficiently large and M be an elementary submodel of $\mathcal{H}(\chi)$ of cardinality κ such that B, $(B_\alpha)_{\alpha<\kappa^+}$, b, $\kappa \in M$ and $[\kappa]^{<\kappa} \in M$. Note that there is such M by our assumption $2^{<\kappa} = \kappa$. Now, let $\alpha^* = \sup \kappa^+ \cap M$. As $\kappa \subseteq [\kappa]^{<\kappa} \subseteq M$, for $\alpha \in \kappa^+ \cap M$, we have $B_\alpha \subseteq M$. Hence $B' \cap M = B_{\alpha^*}$. Since

$$M \models \text{“} B' \restriction b \text{ does not have any cofinal subset of size } < \kappa \text{”},$$

$B_{\alpha^*} \restriction b$ does not have any cofinal subset of cardinality less than κ. But this is a contradiction to $B_{\alpha^*} \leq_\kappa B$. ❑ (Lemma 3.8)

An instance of Lemma 3.8 is that, if $(B_\alpha)_{\alpha<\omega_1}$ is a continuously increasing sequence of relatively complete subalgebras of B, then $B' = \bigcup_{\alpha<\omega_1} B_\alpha$ is also a relatively complete subalgebra of B. If B satisfies the ccc, we can obtain a stronger result.

Lemma 3.9 *Suppose that B is a κ-cc Boolean algebra and δ an ordinal of cofinality $\geq \kappa$. If $(B_\alpha)_{\alpha<\delta}$ is an increasing sequence of relatively complete subalgebras of B, then, for $B' = \bigcup_{\alpha<\delta} B_\alpha$, we have $B' \leq_{\mathrm{rc}} B$.*

Proof. Otherwise, there is a $b \in B$ without the lower projection onto B'. Then $(p^B_{B_\alpha}(b))_{\alpha<\delta}$ is a non eventually constant increasing sequence of elements of B. But this is impossible since B satisfies the κ-cc. ❑ (Lemma 3.9)

Note that $(B_\alpha)_{\alpha<\delta}$ need not to be continuous.

Lemma 3.10 *Suppose that $A \leq_{\mathrm{rc}} B$ and $b \in B$. If there is $u \in B$ such that u is independent over $A(b)$ then there is $v \in B$ such that v is independent over A and $A(b) \leq A(v)$.*

Proof. Put $a_0 = p^B_A(b)$, $a_1 = p^B_A(-b)$ and $a_2 = 1 \cdot -(a_0 + a_1)$. We show in the following claims that $v = b \cdot u + a_2 \cdot b + a_1 \cdot u$ is as desired.

Claim 3.10.1 *v is independent over A.*

$\vdash$ Suppose $a \in A^+$. Then $a \cdot v = a \cdot b \cdot u + a \cdot a_2 \cdot b + a \cdot a_1 \cdot u$. If $a \cdot b \neq 0$ then $a \cdot v \geq a \cdot b \cdot u > 0$. The last inequality is because u is independent over $A(b)$. If $a \cdot b = 0$ then $a \leq -b$ and hence $a \leq a_1$ by the definition of a_1. It follows that $a \cdot v \geq a \cdot a_1 \cdot u = a \cdot u > 0$. Similarly we can prove $-v \cdot a \neq 0$ noting that $-v = -b \cdot -u + a_2 \cdot -b + a_0 \cdot -u$. $\dashv$ (Claim 3.10.1)

Claim 3.10.2 *$A(b) \leq A(v)$.*

$\vdash$ This is clear since $b = -a_1 \cdot v + a_0$. $\dashv$ (Claim 3.10.2)

❑ (Lemma 3.10)

In general, the converse of the second assertion of Lemma 3.3, *5*) is not true. However we have the following.

Lemma 3.11 *1) Suppose that $A \leq_{\mathrm{rc}} B$ and B is countably generated over A. Then $A \leq_{\mathrm{free}} B \oplus \mathrm{Fr}\,\omega$. Hence, for Boolean algebras A, B such that $A \leq B$ and B is countably generated over A, we have $A \leq_{\mathrm{rc}} B$ if and only if $A \leq_{\mathrm{proj}} B$.*

2) Suppose that $A \leq_{\mathrm{rc}} B$, B is countably generated over A and, for any $b_0, \ldots, b_{n-1} \in B$, there is $b \in B$ independent over $A(b_0, \ldots, b_{n-1})$. Then we have $A \leq_{\mathrm{free}} B$.

Proof. *1*): Let $X \subseteq B$ be countable such that $B = A[X]$. Say $X = \{\, x_m : m \in \omega \,\}$. Let $(y_n)_{n\in\omega}$ be a sequence in $B \oplus \mathrm{Fr}\,\omega$ such that[4]

$$y_n = \begin{cases} x_m, & \text{if } n \text{ is even and } n = 2m \text{ for some } m \in \omega; \\ m, & \text{if } n \text{ is odd and } n = 2m+1 \text{ for some } m \in \omega. \end{cases}$$

We construct inductively a sequence $(u_n)_{n\in\omega}$ of elements of $B \oplus \mathrm{Fr}\,\omega$ such that, for every $n \in \omega$,

i) u_n is independent over $A(u_0, \ldots, u_{n-1})$;

ii) $y_0, \ldots, y_{n-1} \in A(u_0, \ldots, u_{n-1})$.

Suppose that we have chosen $u_0, \ldots, u_{n-1}$. By Lemma 3.3, *1*),*2*) and Lemma 3.1, *3*) we have $A(u_0, \ldots, u_{n-1}) \leq_{\mathrm{rc}} B \oplus \mathrm{Fr}\,\omega$. Let $l \in \omega$ be such that $A(u_0, \ldots, u_{n-1}) \leq B \oplus \mathrm{Fr}\,l$. Then l (as an element of the free generator ω of $\mathrm{Fr}\,\omega$) is independent over $B \oplus \mathrm{Fr}\,l$ and hence also over $A(u_0, \ldots, u_{n-1})$. By Lemma 3.10 there is $u_n \in B \oplus \mathrm{Fr}\,\omega$ such that $y_n \in A(u_0, \ldots, u_{n-1}, u_n)$ and u_n is independent over $A(u_0, \ldots, u_{n-1})$. Now, let $U = \{\, u_n : n \in \omega \,\}$. By *i*), U is independent over B and $A[U] = B \oplus \mathrm{Fr}\,\omega$ by *ii*).

2): Let $X = \{\, x_n : n \in \omega \,\}$ be as in *1*). We construct inductively a sequence $(u_n)_{n\in\omega}$ of elements of B such that, for every $n \in \omega$,

i) u_n is independent over $A(u_0, \ldots, u_{n-1})$;

ii′) $x_0, \ldots, x_{n-1} \in A(u_0, \ldots, u_{n-1})$.

The construction is possible since, if $u_0, \ldots, u_{n-1}$ have been chosen, then we have $A(u_0, \ldots, u_{n-1}) \leq_{\mathrm{rc}} B$ by Lemma 3.1, *3*). Hence, by our assumption and Lemma 3.10, there is $u_n \in B$ independent over $A(u_0, \ldots, u_{n-1})$ such that $x_n \in A(u_0, \ldots, u_{n-1}, u_n)$. By *i*) and *ii′*), $U = \{\, u_n : n \in \omega \,\}$ is a free generator of B over A. ❏ (Lemma 3.11)

[4] here we regard m as an element of $\omega \subseteq \mathrm{Fr}\,\omega \leq B \oplus \mathrm{Fr}\,\omega$.

4 Projective Boolean algebras

In this chapter, we give a proof of Theorem 2.5. Most of the results here are (at least implicitly) stated in Koppelberg [32]. The main theorem in this chapter is the following Theorem 4.1 which is a slight generalization of Theorem 2.5: note that a Boolean algebra B is projective if and only if $2 \leq_{\text{proj}} B$. We then show the equivalence of our definition of projectivity with the usual one in Proposition 4.5. In Theorem 4.6, we give a kind of prototype of the arguments with elementary submodels which play an important role in the later chapters.

Theorem 4.1 *For Boolean algebras B, C such that $C \leq B$ the following are equivalent:*

a) *B is projective over C;*

b) *$B \oplus \mathrm{Fr}\,\kappa$ is isomorphic to $C \oplus \mathrm{Fr}\,\kappa$ over C for some κ;*

c) *$C \leq_{\mathrm{rc}} B$ and $\{ A \in [B]^{\aleph_0} : C[A] \leq_{\mathrm{rc}} B \}$ contains an algebraically tight club subset of $[B]^{\aleph_0}$.*

d) *There exists a continuously increasing sequence $(B_\alpha)_{\alpha<\kappa}$ of subalgebras of B such that $B_0 = C$, $\bigcup_{\alpha<\kappa} B_\alpha = B$, $B_\alpha \leq_{\mathrm{rc}} B$ and $B_{\alpha+1}$ is countably generated over B_α for every $\alpha < \kappa$;*

e) *There exists a continuously increasing sequence $(B_\alpha)_{\alpha<\kappa}$ of subalgebras of B and a sequence $(a_\alpha)_{\alpha<\kappa}$ of elements of B such that $B_0 = C$, $\bigcup_{\alpha<\kappa} B_\alpha = B$, $B_\alpha \leq_{\mathrm{rc}} B$ and $B_{\alpha+1} = B(a_\alpha)$ for every $\alpha < \kappa$.*

Proof. $a) \Rightarrow b)$ is trivial.

$b) \Rightarrow c)$: Let F be the free Boolean algebra $\mathrm{Fr}\,|B|$. By assumption $B \oplus F$ is isomorphic to $C \oplus F$ over C. Let $U \subseteq B \oplus F$ be independent over C such that $C[U] = B \oplus F$. By Lemma 3.2, there exists a homomorphism $\varphi : B \oplus F \to B$ such that $\varphi \restriction B = id_B$. For each $b \in F \oplus B$ let $u(b) \in [U]^{<\aleph_0}$ be minimal such that $b \in C[u(b)]$. Let us call $A \leq B$ good if, for $V_A = \bigcup u[A]$, we have

$$A[V_A] \cap B \subseteq A \quad \text{and} \quad \varphi[A[V_A]] \subseteq A.$$

Actually, we have then $A[V_A] \cap B = A$ and $\varphi[A[V_A]] = A$ since the inclusion in other direction holds anyway. Let

$$\mathcal{C} = \{ A \in [B]^{\aleph_0} : A \text{ is good} \}.$$

Clearly $\mathcal{C}$ is a club subset of $[B]^{\aleph_0}$.

Claim 4.1.1 *$\mathcal{C}$ is algebraically tight.*

$\vdash$ Suppose A, $A' \in \mathcal{C}$. Let $A'' = \langle A \cup A' \rangle_B$. Let $V_A = \bigcup u[A]$, $V_{A'} = \bigcup u[A']$ and $V_{A''} = \bigcup u[A'']$. We have $V_{A''} = \bigcup u[\langle A \cup A' \rangle_B]$. By the definition of u this is equal to $\bigcup u[A] \cup \bigcup u[A'] = V_A \cup V_{A'}$. Hence

$$\begin{aligned} A''[V_{A''}] \cap B &= A''[V_A \cup V_{A'}] \cap B \\ &\subseteq \langle (A[V_A] \cap B) \cup (A'[V_{A'}] \cap B) \rangle_B \\ &\subseteq \langle A \cup A' \rangle_B = A'' \end{aligned}$$

and

$$\begin{aligned} \varphi[A''[V_{A''}]] &= \varphi[\langle A[V_A] \cup A'[V_{A'}] \rangle_B] \\ &= \langle \varphi[A[V_A] \cup A'[V_{A'}]] \rangle_{B \oplus F} \\ &\subseteq \langle A \cup A' \rangle_{B \oplus F} = A''. \end{aligned}$$

$\dashv$ (Claim 4.1.1)

Claim 4.1.2 $C[A] \leq_{\mathrm{rc}} B$ *for any good* $A \subseteq B$.

$\vdash$ Let $V_A = \bigcup u[A]$. Note that $C[V_A] \leq_{\mathrm{free}} B \oplus F$ and hence $C[V_A] \leq_{\mathrm{rc}} B \oplus F$ by Lemma 3.3, *5*). For $b \in B \oplus F$, let $c = q^{B \oplus F}_{C[V_A]}(b)$. Then $\varphi(c) \geq \varphi(b) = b$. Since A is good, $\varphi(c) \in A$. We show that $\varphi(c) = q^B_{C[A]}(b)$. Otherwise there would be some $d \in (C[A])^+$ such that $d \leq \varphi(c)$ and $d \cdot b = 0$. Let $d' = d \cdot c$. Then $d' \in C[V_A]$. Since $\varphi(d') = \varphi(d) \cdot \varphi(c) = d \cdot \varphi(c) = d \neq 0$, we have $d' \neq 0$. Also $d' \leq c$ and $d' \cdot b \leq d \cdot b = 0$. This is a contradiction to $c = q^{B \oplus F}_{C[V_A]}(b)$. $\dashv$ (Claim 4.1.2)

c) $\Rightarrow$ *d*): Assume that $A \leq_{\mathrm{rc}} B$ and $\mathcal{C} \subseteq [B]^{\aleph_0}$ is algebraically tight club subset of $[B]^{\aleph_0}$ such that $C[A] \leq_{\mathrm{rc}} B$ for every $A \in \mathcal{C}$.

Claim 4.1.3 $C[\bigcup \mathcal{D}] \leq_{\mathrm{rc}} B$ *for every* $\mathcal{D} \subseteq \mathcal{C}$.

$\vdash$ If $\mathcal{D} = \emptyset$ then $C[\bigcup \mathcal{D}] = C \leq_{\mathrm{rc}} B$. Let $\mathcal{D} \neq \emptyset$. Toward a contradiction, let us assume that there is $b \in B$ such that $p^B_{C[\bigcup \mathcal{D}]}(b)$ does not exist. We can choose $A_n \in \mathcal{D}$, $n \in \omega$ such that

$$(*) \quad p^B_{C[A_0]}(b) < p^B_{C[A_0 \cup A_1]}(b) < p^B_{C[A_0 \cup A_1 \cup A_2]}(b) < \cdots$$

Note that $p^B_{C[A_0 \cup \cdots \cup A_n]}(b)$ exists since $C[A_0 \cup \cdots \cup A_n] = C[\langle A_0 \cup \cdots \cup A_n \rangle_B]$ and $\langle A_0 \cup \cdots \cup A_n \rangle_B \in \mathcal{C}$ by algebraic tightness of $\mathcal{C}$. Let $A = \langle \bigcup_{n \in \omega} A_n \rangle_B$. We have $A = \bigcup_{m \in \omega} \langle \bigcup_{n < m} A_n \rangle_B$. $\langle \bigcup_{n < m} A_n \rangle_B \in \mathcal{C}$ for all $n \in \omega$ by algebraic tightness of $\mathcal{C}$. Since $\mathcal{C}$ is is club, it follows that $A \in \mathcal{C}$. On the other hand, $p^B_{C[A]}(b)$ cannot exist because of $(*)$. This is a contradiction. $\dashv$ (Claim 4.1.3)

Now let $\mathcal{C} = \{ A_\alpha : \alpha < \kappa \}$ for some cardinal κ. For $\alpha < \kappa$, let $\mathcal{D}_\alpha = \{ A_\beta : \beta < \alpha \}$ and $B_\alpha = C[\bigcup \mathcal{D}_\alpha]$. Then $B_0 = C$, $\bigcup_{\alpha < \kappa} B_\alpha = B$ and $(B_\alpha)_{\alpha < \kappa}$ is continuously increasing. $B_\alpha \leq_{\mathrm{rc}} B$ by the claim above and $B_{\alpha+1} = B_\alpha[A_\alpha]$ for every $\alpha < \kappa$.

d) $\Rightarrow$ *e*): By Lemma 3.1, *3*).

e) $\Rightarrow$ *d*): Trivial.

d) ⇒ *a*): Let $(B_\alpha)_{\alpha<\kappa}$ be as in *c*). By Lemma 3.11, *1*) we have $B_\alpha \leq_{\mathrm{proj}} B_{\alpha+1}$ for every $\alpha < \kappa$. By Lemma 3.7, *2*), it follows that $C = B_0 \leq_{\mathrm{proj}} B$. ❑ (Theorem 4.1)

Theorem 4.2 *Suppose that B, C are Boolean algebras such that $C \leq_{\mathrm{proj}} B$. Let $\kappa = \min\{ X \subseteq B : C[X] = B \}$. If, for every $X \in [B]^{<\kappa}$, there is $b \in B$ independent over $C[X]$, then we have $C \leq_{\mathrm{free}} B$.*

Proof. If $\kappa = \aleph_0$, the theorem follows from Lemma 3.11, *2*). Hence we may assume that κ is uncountable. By Theorem 4.1, there is an algebraically tight club subset $\mathcal{C}$ of $[B]^{\aleph_0}$ such that $C[A] \leq_{\mathrm{rc}} B$ for every $A \in \mathcal{C}$. By induction we can construct a continuously increasing sequence $(\mathcal{C}_\alpha)_{\alpha<\kappa}$ of subsets of $\mathcal{C}$ such that

(0) $\mathcal{C}_0 = \emptyset$;

(1) $|\mathcal{C}_{\alpha+1} \setminus \mathcal{C}_\alpha| = \aleph_0$;

(2) for any $b_0, \ldots, b_{n-1} \in C[\bigcup \mathcal{C}_{\alpha+1}]$, there is $b \in C[\bigcup \mathcal{C}_{\alpha+1}]$ independent over $(C[\bigcup \mathcal{C}_\alpha])(B_0, \ldots, b_{n-1})$;

(3) $C[\bigcup(\bigcup_{\alpha<\kappa} \mathcal{C}_\alpha)] = B$.

For $\alpha < \kappa$, let $B_\alpha = C[\bigcup \mathcal{C}_\alpha]$. By (0), $B_0 = C$. $(B_\alpha)_{\alpha<\kappa}$ is continuously increasing and $\bigcup_{\alpha<\kappa} B_\alpha = B$ by (3). By Claim 4.1.3, we have $B_\alpha \leq_{\mathrm{rc}} B$ for every $\alpha < \kappa$. By (1), $B_{\alpha+1}$ is countably generated over B_α for every $\alpha < \kappa$. Hence, by (2) and Lemma 3.11, *2*), we have $B_\alpha \leq_{\mathrm{free}} B_{\alpha+1}$ for ever α. By Lemma 3.7, *1*), it follows that $C \leq_{\mathrm{free}} B$. ❑ (Theorem 4.2)

Corollary 4.3 *An infinite projective Boolean algebra B of cardinality κ is free if and only if, for every $A \in \mathrm{Sub}^{<\kappa}(B)$, there is $b \in B$ such that b is independent over A.*

Proof. Clearly infinite free Boolean algebras have the property as above. Conversely "if" direction is just Theorem 4.2 for $C = 2$. ❑ (Corollary 4.3)

As mentioned in the last chapter, the original definition of projective Boolean algebras was different from ours (see Halmos [22], [23]). We shall call the Boolean algebras which are projective in the original sense in the following as algebraically projective. Thus Boolean algebra B is *algebraically projective* if for any Boolean algebras A, C, homomorphism $f : B \to A$ and epimorphism $g : C \to A$, there exists a homomorphism $h : B \to C$ such that $g \circ h = f$. Let us call such h *a lifting of f over g*. Below, we show the equivalence of algebraic projectivity with our definition of projective Boolean algebras.

A Boolean algebra B is a *retract* of a Boolean algebra C if there are homomorphisms $i : B \to C$ and $j : C \to B$ such that $j \circ i = id_B$.

Lemma 4.4

1) *Free Boolean algebras are algebraically projective.*

2) *If B and B' are algebraically projective then so is $B \oplus B'$.*

Proof. *1)*: Let F be a free Boolean algebra and U a free generator of F. Suppose that $f : F \to A$ is a homomorphism and $g : C \to A$ an epimorphism for some Boolean algebras A, C. For each $u \in U$ choose $h'(u) \in C$ so that $g \circ h'(u) = f(u)$. This is possible since g is surjective. By Lemma 2.1, there is a homomorphism $h : F \to C$ extending h'. Since $g \circ h \restriction U = g \circ h' \restriction U = f \restriction U$ and since U generates F, it follows that $g \circ h = f$.

2): Let $f : B \oplus B' \to A$ be a homomorphism and $g : C \to A$ an epimorphism for some Boolean algebras A, C. Let h be a lifting of $f \restriction B$ over g and h' a lifting of $f \restriction B'$. Then $f \oplus f'$ is the lifting of f over g. ❏ (Lemma 4.4)

Proposition 4.5 *For a Boolean algebra B, the following are equivalent.*

a) *B is projective;*
b) *B is algebraically projective;*
c) *B is a retract of a free Boolean algebra.*

Proof. *a)* ⇒ *b)*: Suppose that B is projective. Let F be a free Boolean algebra such that $B \oplus F$ is free. For a homomorphism $f : B \to A$ and a epimorphism $g : C \to A$ let $\tilde{f} : B \oplus F \to A$ be such that $f \subseteq \tilde{f}$. By Lemma 4.4, *1)*, there exits a homomorphism $\tilde{h} : B \oplus F \to C$ such that $g \circ \tilde{h} = \tilde{f}$. Let $h = \tilde{h} \restriction B$. Then we have $g \circ h = f$.

b) ⇒ *c)*: Let F be a free Boolean algebra such that $|F| = |B|$. There is an epimorphism $g : F \to B$. For $C = B$ and $f = id_B$, there is $h : B \to F$ such that $g \circ h = f = id_B$. This shows that B is a retract of the free Boolean algebra F.

c) ⇒ *a)*: Suppose that F is a free Boolean algebra and homomorphisms $f : B \to F$, $g : F \to B$ are such that $g \circ f = id_B$. We show that B satisfies the assertion of Theorem 4.1, *c)*. Let U be a free generator of F. For each $a \in F$, let $u(a) \in [U]^{<\aleph_0}$ be minimal such that $a \in \langle u(a) \rangle_F$. Let us call $A \leq B$ good if $g[\bigcup u[f[A]]] \subseteq A$.

Claim 4.5.1 *If $A \leq B$ is good then $A \leq_{\mathrm{rc}} B$.*

⊢ Suppose that $A \leq B$ is good and $V = \bigcup u[f[A]]$. For $b \in B$ let $d = q^F_{\langle V \rangle_F}(f(b))$. Since A is good, we have $g(d) \in A$. We show that $g(d) = q^B_A(b)$. By $f(b) \leq d$ we have $b = g \circ f(b) \leq g(d)$. Hence if $g(d)$ were not an upper projection of b onto A, there would be $a \in A^+$ such that $a \leq g(d)$ and $a \cdot b = 0$. Let $b' = f(a) \cdot d$. Since $g(b') = (g \circ f(a)) \cdot g(d) = a \cdot g(d) = a > 0$, we have $b' \neq 0$. $b' \leq d$ by definition, $b' \in \langle V \rangle_F$ and $b' \leq d$. But $b' \cdot f(b) \leq f(a) \cdot f(b) = f(a \cdot b) = 0$. This is a contradiction as d is the upper projection of $f(b)$ onto $\langle V \rangle_F$. ⊣ (Claim 4.5.1)

Now let

$$\mathcal{C} = \{ A \in [B]^{\aleph_0} \ : \ A \text{ is good} \}.$$

Clearly $\mathcal{C}$ club subset of $[B]^{\aleph_0}$.

Claim 4.5.2 *$\mathcal{C}$ is algebraically tight.*

$\vdash$ Suppose that A, $A' \in \mathcal{C}$. Then

$$\begin{aligned} g[\bigcup u[f[\langle A \cup A' \rangle_B]]] &= g[\langle \bigcup u[f[A]] \cup \bigcup u[f[A']]\rangle_{\langle V\rangle_F}] \\ &= \langle g[\bigcup u[f[A]]] \cup g[\bigcup u[f[A']]]\rangle_B \\ &\subseteq \langle A \cup A'\rangle_B. \end{aligned}$$

Hence $\langle A \cup A'\rangle_B \in \mathcal{C}$. $\dashv$ (Claim 4.5.2)

By Theorem 4.1, it follows that B is projective. ❑ (Proposition 4.5)

The following rather trivial theorem gives the first example of the phenomena we are going to observe frequently in the following chapters: for a Boolean algebra B, if M is an elementary submodel of $\mathcal{H}(\chi)$ for sufficiently large χ and $B, \ldots,$ $\in M$, then $B \cap M$ is a "nice" subalgebra of B and some properties of B can be characterized in term of this.

Theorem 4.6 *Let B, B' be Boolean algebras and χ a sufficiently large cardinal. Suppose that M is an elementary submodel of $\mathcal{H}(\chi)$ such that B, $B' \in M$*

1) If B is free, then so is $B \cap M$ and $B \cap M \leq_{\text{free}} B$. Further, if $M' \prec \mathcal{H}(\chi)$ is such that $M \subseteq M'$ and $B \leq_{\text{free}} B'$, then $B \cap M \leq_{\text{free}} B' \cap M'$.

2) If B is projective, then so is $B \cap M$ and $B \cap M \leq_{\text{proj}} B$. Further, if $M' \prec \mathcal{H}(\chi)$ is such that $M \subseteq M'$ and $B \leq_{\text{proj}} B'$, then $B \cap M \leq_{\text{proj}} B' \cap M'$.

Proof. *1*): Since $M \models$ "B is free", there is a free generator $U \in M$ of B. Then $U \cap M$ is a free generator of $B \cap M$. Hence $B \cap M$ is free. Let $U' = U \setminus M$. Then $B = (B \cap M)[U']$ and U' is independent over $B \cap M = [U \cap M]_B$. Thus $B \cap M \leq_{\text{free}} B$. If $M' \prec \mathcal{H}(\chi)$ is such that $M \subseteq M'$ and $B \leq_{\text{free}} B'$, let $U' \in M$ be a free generator of B' extending U. Then $U' \cap M'$ is a free generator of $B' \cap M'$. Let $U'' = (U' \cap M') \setminus (U \cap M)$. Then $B' \cap M' = (B \cap M)[U'']$ and U'' is independent over $B \cap M$. Hence $B \cap M \leq_{\text{free}} B' \cap M'$.

2): Let $\kappa = |B|$. As $B \in M$, we have also $\kappa \in M$. Hence $B \oplus \mathrm{Fr}\,\kappa \in M$. Since B is projective, $B \oplus \mathrm{Fr}\,\kappa$ is free. By *1*), it follows that $(B \oplus \mathrm{Fr}\,\kappa) \cap M$ is free. But $(B \oplus \mathrm{Fr}\,\kappa) \cap M = (B \cap M) \oplus \mathrm{Fr}\,(\kappa \cap M)$. Hence $B \cap M$ is projective. Also from *1*), it follows that $(B \oplus \mathrm{Fr}\,\kappa) \cap M \leq_{\text{free}} B \oplus \mathrm{Fr}\,\kappa$. On the other hand, by Lemma 3.1, *1*), we have $B \cap M \leq_{\text{free}} (B \cap M) \oplus \mathrm{Fr}\,(\kappa \cap M) = (B \oplus \mathrm{Fr}\,\kappa) \cap M$. It follows that $B \cap M \leq_{\text{free}} B \oplus \mathrm{Fr}\,\kappa$. Hence $B \cap M \leq_{\text{proj}} B$. If $M' \prec \mathcal{H}(\chi)$ is such that $M \subseteq M'$ and $B \leq_{\text{free}} B'$, let $\kappa' = |B'|$. By *1*), $B \cap M \leq_{\text{free}} (B' \oplus \mathrm{Fr}\,\kappa') \cap M' \leq_{\text{free}}=$ $(B' \cap M) \oplus \mathrm{Fr}\,(\kappa' \cap M')$. It follows that $B \cap M \leq_{\text{proj}} B' \cap M'$. ❑ (Theorem 4.6)

5 $\leq_\kappa$-generated Boolean algebras

In this chapter, we prove theorems on openly generated Boolean algebras. We formulate and prove most of the assertions not only for openly generated Boolean algebras but, more generally, for $\leq_\kappa$-generated Boolean algebras for arbitrary regular κ. Remember that a Boolean algebra is openly generated if and only if it is $\leq_{\aleph_0}$-generated.

5.1 Characterizations of $\leq_\kappa$-generated Boolean algebras

Let us begin with the following characterization of $\leq_\kappa$-generated Boolean algebras.

Theorem 5.1 (Fuchino–Koppelberg–Shelah [17]) *For a regular cardinal κ and a Boolean algebra B, the following are equivalent.*

a) B is $\leq_\kappa$-generated.

b) There exists an upward directed partial ordering $I = (I, \leq)$ and an indexed family $(B_i)_{i\in I}$ of subalgebras of B such that

> *0) $|B_i| \leq \kappa$ for every $i \in I$;*
> *1) $B_i \leq_\kappa B_j$ for all i, $j \in I$ such that $i \leq j$;*
> *2) for every increasing chain $(i_\alpha)_{\alpha<\delta}$ in I of length $\delta < \kappa^+$, $i^* = \sup\{ i_\alpha : \alpha < \delta \}$ exists and $B_{i^*} = \bigcup_{\alpha<\delta} B_{i_\alpha}$;*
> *3) $B = \bigcup_{i\in I} B_i$.*

c) For some, or equivalently any sufficiently large[5] χ and elementary submodel M of $\mathcal{H}(\chi)$, if $B \in M$ and $\kappa + 1 \subseteq M$, then $B \cap M \leq_\kappa B$.

d) For some, or equivalently any sufficiently large[5] χ and elementary submodel M of $\mathcal{H}(\chi)$, if $B \in M$, $\kappa + 1 \subseteq M$ and $|M| = \kappa$, then $B \cap M \leq_\kappa B$.

Remark. Note that, for $\kappa = \aleph_0$, the condition $\kappa+1 \subseteq M$ holds for any elementary submodel M of $\mathcal{H}(\chi)$.

Proof. $a) \Leftrightarrow b)$ follows immediately from Proposition 2.7.

$a) \Rightarrow c)$: Suppose that B is $\leq_\kappa$-generated and χ, M are as in c). Then there is $\mathcal{C} \in M$ such that $\mathcal{C}$ is club subset of $[B]^\kappa$ and $C \leq_\kappa B$ for every $C \in \mathcal{C}$. For $C \in \mathcal{C} \cap M$, we have $C \subseteq M$ by $|C| = \kappa$ and $\kappa + 1 \subseteq M$. It follows that $B \cap M = \bigcup(\mathcal{C} \cap M)$. But the right side of the equation is an element of $\mathcal{C}$ since $\mathcal{C} \cap M$ is directed, $|\mathcal{C} \cap M| \leq \kappa$ and $\mathcal{C}$ is club in $[B]^\kappa$. It follows that $B \cap M \leq_\kappa B$.

$c) \Rightarrow d)$: Trivial.

$d) \Rightarrow b)$: Suppose that B satisfies d). Let $I = \{ M \prec \mathcal{H}(\chi) : B \in M, \kappa + 1 \subseteq$

[5] see the remark on p.3.

M, $|B| = \kappa\}$. Let $\leq$ be the partial ordering on I defined by $M \leq N \Leftrightarrow M \subseteq N$ for M, $N \in I$ and $B_M = B \cap M$ for each $M \in I$. Then it is easy to check that $(B_M)_{M\in I}$ satisfies the conditions *0) – 3)* in *b*). ❑ (Theorem 5.1)

The characterization of the $\leq_\kappa$-generated Boolean algebras above often enables us to give quite elegant proofs to assertions on $\leq_\kappa$-generated Boolean algebras. The proof of the following lemmas are such examples.

Lemma 5.2 *Suppose that κ is regular and A, B are $\leq_\kappa$-generated Boolean algebras. Then $A \oplus B$ is also $\leq_\kappa$-generated.*

Proof. Let χ be sufficiently large and M be an elementary submodel of $\mathcal{H}(\chi)$ such that $A \oplus B \in M$ and $\kappa + 1 \subseteq M$. By Theorem 5.1, it is enough to show that $(A \oplus B) \cap M \leq_\kappa A \oplus B$. Since $M \models$ " there are A', B' such that $A \oplus B$ is the free product of A' and B'", we may assume without loss of generality that A, $B \in M$. By Theorem 5.1, we have $A\cap M \leq_\kappa A$ and $B\cap M \leq_\kappa B$. Hence, by the elementarity of M and Lemma 3.5, it follows $(A \oplus B) \cap M = (A \cap M) \oplus (B \cap M) \leq_\kappa A \oplus B$.
❑ (Lemma 5.2)

Lemma 5.3 *Suppose that κ is regular and A, B are Boolean algebras. If $A \leq_{\kappa^+} B$ and B is $\leq_\kappa$-generated, then A is also $\leq_\kappa$-generated. In particular, if $A \leq_\sigma B$ (or even $A \leq_{\rm rc} B$) and B is openly generated, then so is A.*

Proof. Let χ be sufficiently large. By Theorem 5.1, *c*), it is enough to show that $A \cap M \leq_\kappa A$ for every elementary submodel M of $\mathcal{H}(\chi)$ such that $A \in M$ and $\kappa + 1 \subseteq M$. Since the statement "there exists a $\leq_\kappa$-generated Boolean algebra $\underline{B}$ such that $A \leq_\kappa \underline{B}$" holds in $\mathcal{H}(\chi)$ and hence also in M, we may assume without loss of generality that $B \in M$.

Let $a \in A$. By Theorem 5.1, $B \cap M \leq_\kappa B$. Hence there is a cofinal subset X of $(B\cap M)\restriction a$ of cardinality less than κ. For each $x \in X$ the filter $A \uparrow x$ is an element of M. By $A \leq_{\kappa^+} B$, there is a coinitial subset Y_x of $A \uparrow x$ of cardinality less or equal to κ. Since $A \uparrow x \in M$ we may take Y_x as an element of M. By $\kappa + 1 \subseteq M$, we have $Y_x \subseteq M$.

Now, for $x \in X$, since $a \in A \uparrow x$, there is some $y_x \in Y_x$ such that $x \leq y_x \leq a$. Put $Z = \{\, y_x : x \in X \,\}$. Then $Z \subseteq A \cap M$ and $|Z| \leq |X| < \kappa$. As Z dominates X, it is cofinal in $(B \cap M)\restriction a$. But since $Z \subseteq A \cap M$, Z is also cofinal in $(A \cap M)\restriction a$.
❑ (Lemma 5.3)

For a regular κ, a mapping on B is called *κ-Freese-Nation mapping* if $f : B \to [B]^{<\kappa}$ and for all a, $b \in B$, if $a \leq b$ then there is $c \in f(a) \cap f(b)$ such that $a \leq c \leq b$. A Boolean algebra B has the *κ-Freese-Nation property* if there is a κ-Freese-Nation mapping on B. The $\aleph_0$-Freese-Nation property was introduced in Freese–Nation [11] in connection with projective lattices. L. Heindorf called this *the Freese-Nation property* and proved that a Boolean algebra B has $\aleph_0$-Freese-Nation property if and only if B is openly generated (see Heindorf–Shapiro [25]). L. Heindorf and L. Shapiro [25] then generalized the notion to $\aleph_1$-Freese-Nation property and proved some elementary facts about it. Ordered structures with κ-Freese-Nation property are further studied in Fuchino–Koppelberg–Shelah [17]. Many of the following lemmas can be formulated more generally for partially ordered sets or even for quasi orderings.

Lemma 5.4 *Let B be a Boolean algebra and κ a regular cardinal.*

1) If $|B| \leq \kappa$ then B has the κ-Freese-Nation property.

2) If B is the union of a continuously increasing sequence of Boolean algebras $(B_\alpha)_{\alpha<\delta}$ for some ordinal δ such that B_α has the κ-Freese-Nation property and $B_\alpha \leq_\kappa B_{\alpha+1}$ for every $\alpha < \delta$, then B has also the κ-Freese-Nation property.

3) If f is a κ-Freese-Nation mapping on B and $A \leq B$ is closed with respect to f then $A \leq_\kappa B$.

Proof. *1)*: Let $\sqsubseteq$ be a well-ordering on B of order type less or equal to κ. For $b \in B$, let $f(b) = \{\, c \in B : c \sqsubseteq b \,\}$. Then f is a κ-Freese-Nation mapping on B.

2): For each $\alpha < \delta$, let $f_\alpha : B_\alpha \to [B_\alpha]^{<\kappa}$ be a κ-Freese-Nation mapping on B_α. By induction on $\alpha < \delta$, we define an increasing sequence $(g_\alpha)_{\alpha<\delta}$ such that $g_\alpha : B_\alpha \to [B_\alpha]^{<\kappa}$ is a κ-Freese-Nation mapping on B_α for every $\alpha < \delta$. $g = \bigcup_{\alpha<\delta} g_\alpha$ is then a κ-Freese-Nation mapping on B. Let $g_0 = f_0$. Now, suppose that g_α has been defined. For each $b \in B_{\alpha+1} \setminus B_\alpha$ fix $X_b \subseteq B_\alpha \restriction b$ and $Y_b \subseteq B_\alpha \uparrow b$ such that $|X_b|$, $|Y_b| < \kappa$, X_b is cofinal in $B_\alpha \restriction b$ and Y_b is coinitial in $B_\alpha \uparrow b$. Let

$$g_{\alpha+1}(b) = \begin{cases} g_\alpha(b), & \text{if } b \in B_\alpha \\ \bigcup\{\, g_\alpha(x) : x \in X_b \cup Y_b \,\} \cup f_{\alpha+1}(b), & \text{otherwise.} \end{cases}$$

Clearly $g_\alpha \subseteq g_{\alpha+1}$. Since κ is regular, $|g_{\alpha+1}(b)| < \kappa$ for every $b \in B_{\alpha+1}$. We show that $g_{\alpha+1}$ is a κ-Freese-Nation mapping on $B_{\alpha+1}$. Suppose that $b, b' \in B_{\alpha+1}$ are such that $b \leq b'$. If $b, b' \in B_{\alpha+1} \setminus B_\alpha$, then there is $c \in f_{\alpha+1}(b) \cap f_{\alpha+1}(b') \subseteq g_{\alpha+1}(b) \cap g_{\alpha+1}(b')$ such that $b \leq c \leq b'$. If $b, b' \in B_\alpha$, then there is $c \in g_\alpha(b) \cap g_\alpha(b') = g_{\alpha+1}(b) \cap g_{\alpha+1}(b')$ such that $b \leq c \leq b'$. If $b \in B_\alpha$ and $b' \in B_{\alpha+1} \setminus B_\alpha$, then there is $x \in X_{b'}$ such that $b \leq x \leq b'$. Hence there is $c \in g_\alpha(b) \cap g_\alpha(x) \subseteq g_{\alpha+1}(b) \cap g_{\alpha+1}(b')$ such that $b \leq c \leq x \leq b'$. The case that $b \in B_{\alpha+1} \setminus B_\alpha$ and $b' \in B_\alpha$ can be treated similarly. For limit $\alpha < \delta$, if g_β, $\beta < \alpha$ have been defined, then $g_\alpha = \bigcup_{\beta<\alpha} g_\beta$ is a κ-Freese-Nation mapping on B_α.

3): Let $b \in B$. For $a \in A$ such that $a \leq b$, we have that $f(a) \subseteq A$ and there is $c \in f(a) \cap f(b)$ such that $a \leq c \leq b$. Since A is closed under f, we have $c \in A$. It follows that $X = f(b) \cap (A \restriction b)$ is cofinal in $A \restriction b$. But $|X| \leq |f(b)| < \kappa$. ❑ (Lemma 5.4)

Theorem 5.5 (Heindorf–Shapiro [25]) *For a regular κ and a Boolean algebra B, the following are equivalent.*

a) *B is $\leq_\kappa$-generated.*

b) *B is the union of a continuously increasing chain of subalgebras $(B_\alpha)_{\alpha<\lambda}$ of length $\lambda = |B|$ such that $|B_\alpha| \leq |\alpha+\kappa|$, $B_\alpha \leq_\kappa B$ and B_α is $\leq_\kappa$-generated for every $\alpha < \lambda$.*

c) *B has the κ-Freese-Nation property.*

Proof. *a)* $\Rightarrow$ *b)*: Let χ be sufficiently large and let $M \prec \mathcal{H}(\chi)$ be such that $|M| = |B| = \lambda$, $B \in M$ and $\kappa+1 \subseteq M$. Let $(M_\alpha)_{\alpha<\lambda}$ be a continuously increasing sequence of elementary submodels of M such that $B \in M_0$, $\kappa+1 \subseteq M_0$, $|M_\alpha| \leq |\alpha+\kappa|$ for all $\alpha < \lambda$ and $M = \bigcup_{\alpha<\lambda} M_\alpha$. For $\alpha < \lambda$, let $B_\alpha = B \cap M_\alpha$. Then $(B_\alpha)_{\alpha<\lambda}$ is continuously increasing, $|B_\alpha| \leq |M_\alpha| \leq |\alpha+\kappa|$ for every $\alpha < \lambda$ and $B = \bigcup_{\alpha<\lambda} B_\alpha$. Hence, by Theorem 5.1, $B_\alpha \leq_\kappa B$ for every $\alpha < \lambda$. By Lemma 5.3, each B_α is $\leq_\kappa$-generated.

b) $\Rightarrow$ *c)*: By induction on $\lambda = |B|$, we show that, if there is $(B_\alpha)_{\alpha<\lambda}$ as in *b)*, then B has the κ-Freese-Nation property. For $\lambda \leq \kappa$, B has the κ-Freese-Nation property by Lemma 5.4, *1)*. Suppose that we have shown the assertion for every $\mu < \lambda$. If $|B| = \lambda$ and $(B_\alpha)_{\alpha<\lambda}$ is as in *b)*, then by *a)* $\Rightarrow$ *b)*, each B_α can be represented as the union of a continuously increasing sequence $(B^\beta_\alpha)_{\beta<\mu_\alpha}$ for $\mu_\alpha = |B_\alpha| < \lambda$ such that $B^\beta_\alpha \leq_\kappa B_\alpha$, $|B^\beta_\alpha| \leq |\beta+\kappa|$ and $B^\beta_\alpha \leq_\kappa B$ for every $\beta < \mu_\alpha$. By induction hypothesis, it follows that B_α has the κ-Freese-Nation property for every $\alpha < \lambda$. Hence by Lemma 5.4, *2)*, B has the κ-Freese-Nation property.

c) $\Rightarrow$ *a)*: Suppose that B has the κ-Freese-Nation property. Let χ be sufficiently large and M an elementary submodel of $\mathcal{H}(\chi)$ such that $B \in M$ and $\kappa+1 \subseteq M$. By Theorem 5.1, it is enough to show that $B \cap M \leq_\kappa B$. Since $M \models$ "B has the κ-Freese-Nation property", there is $f \in M$ such that f is a κ-Freese-Nation mapping on B. For each $b \in B \cap M$, $f(b) \in M$ and $|f(b)| < \kappa$. Hence by $\kappa \subseteq M$, we have $f(b) \subseteq M$, i.e., $B \cap M$ is closed with respect to f. By Lemma 5.4, *3)*, if follows that $B \cap M \leq_\kappa B$. ❏ (Theorem 5.5)

One of the advantages of the characterization of $\leq_\kappa$-generated Boolean algebras as such having κ-Freese-Nation property, is that we can obtain immediately the following theorem while the original proof by E.V. Ščepin for the case $\kappa = \aleph_0$ was already fairly complicated.

Theorem 5.6 ([25], Ščepin [44] for $\kappa = \aleph_0$, see also Fuchino [14]) *Suppose that $(B_\alpha)_{\alpha<\delta}$ is a continuously increasing sequence of $\leq_\kappa$-generated Boolean algebras for some limit ordinal δ such that $B_\alpha \leq_\kappa B_{\alpha+1}$ for every $\alpha < \delta$. Then $B = \bigcup_{\alpha<\delta} B_\alpha$ is also $\leq_\kappa$-generated.*

Proof. By Theorem 5.5 and Lemma 5.4, *2)*. ❏ (Theorem 5.6)

For a short sequence of Boolean algebras Theorem 5.6 can be still improved.

Theorem 5.7 (Bandlow [**?**] for $\kappa = \aleph_0$) *Suppose that $(B_\alpha)_{\alpha<\delta}$ is a continuously increasing sequence of $\leq_\kappa$-generated Boolean algebras for some $\delta \leq \kappa$ such that $B_\alpha \leq_{\kappa^+} B_{\alpha+1}$ holds for every $\alpha < \kappa$. Then $B = \bigcup_{\alpha<\kappa} B_\alpha$ is also $\leq_\kappa$-generated.*

Proof. Let χ be sufficiently large. By Theorem 5.1, it is enough to show $B \cap M \leq_\kappa B$ for any elementary submodel M of $\mathcal{H}(\chi)$ such that $\kappa + 1 \subseteq M$ and $B \in M$. By $\delta \in M$, we may assume without loss of generality that $(B_\alpha)_{\alpha<\delta} \in M$. Since $\delta \subseteq M$ we have $B_\alpha \in M$ for every $\alpha < \delta$. Let $b \in B$. Then there is $\alpha < \delta$ such that $b \in B_\alpha$. Since B_α is $\leq_\kappa$-generated, we have $B_\alpha \cap M \leq_\kappa B_\alpha$. Hence there is a cofinal subset X of $(B_\alpha \cap M) \restriction b$ of cardinality less than κ. We show that X is also cofinal in $(B \cap M) \restriction b$. Suppose that $b' \in (B \cap M) \restriction b$. By Lemma 3.7, *4*), we have $B_\alpha \leq_{\kappa^+} B$. Hence there is a coinitial subset Y of the filter $B_\alpha \uparrow b'$ of cardinality less or equal to κ. In particular there is such Y in M. Since $\kappa \subseteq M$, we have $Y \subseteq M$. Let $y \in Y$ be such that $b' \leq y \leq b$. Since $y \in B_\alpha \cap M$, there is $x \in X$ such that $y \leq x \leq b$. Hence $b' \leq x$. ❑ (Theorem 5.7)

The following is also an immediate consequence of the characterization of $\leq_\kappa$-generated Boolean algebras.

Proposition 5.8 *Suppose that κ and λ are regular cardinals such that $\kappa \leq \lambda$. If a Boolean algebra B is $\leq_\kappa$-generated then B is $\leq_\lambda$-generated.*

Proof. If $f : B \to [B]^{<\kappa}$ is a κ-Freese-Nation mapping on B, then f is clearly also a λ-Freese-Nation mapping on B. ❑ (Proposition 5.8)

For uncountable κ, Theorem 5.1, *c*), *d*) can be still slightly improved. Weakenings of the following notion are needed to formulate the result.

Definition 5.9 (S. Shelah) *An elementary submodel M of $\mathcal{H}(\chi)$ for a regular χ is said to be* internally approachable *if M is the union of continuously increasing chain $(M_\alpha)_{\alpha<\lambda}$ for $\lambda = |M|$ such that $|M_\alpha| < |M|$ and $(M_\beta)_{\beta\leq\alpha} \in M_{\alpha+1}$ for every $\alpha < \lambda$.*

We shall call an elementary submodel M of $\mathcal{H}(\chi) = (\mathcal{H}(\chi), \in)$ $\mathcal{H}(\kappa)$*-like*, if $\kappa = \omega$, or $\kappa > \omega$ and for every $X \in [M]^{<\kappa}$ there is $N \in M$ such that $N \prec M$, $|N| < \kappa$ and $X \subseteq N$. M is V_κ*-like* if M is $\mathcal{H}(\kappa)$-like and $|M| = \kappa$. Hence M is V_κ-like if and only if M is countable or M is the union of an increasing sequence $(M_\alpha)_{\alpha<\kappa}$ of elementary submodels of $\mathcal{H}(\chi)$ such that $|M_\alpha| < \kappa$ and $M_\alpha \in M_{\alpha+1}$ for all $\alpha < \kappa$. We shall call such $(M_\alpha)_{\alpha<\kappa}$ a *witness* of V_κ-likeness of M.

An advantage of V_κ-likeness is the following Lemma 5.10, *3*) which is not true in general for internally approachable elementary submodels.

Lemma 5.10 *Suppose that κ and χ are regular and $\kappa < \chi$. Then:*

0) *If $M \prec \mathcal{H}(\chi)$ is internally approachable and $|M| = \kappa$, then M is V_κ-like.*

1) *If $M \prec \mathcal{H}(\chi)$ is V_κ-like and $\kappa \in M$ then $\kappa \subseteq M$.*

2) *For any $X \in [\mathcal{H}(\chi)]^{\leq\kappa}$ there is an internally approachable $M \prec \mathcal{H}(\chi)$ such that $X \subseteq M$ and $|M| = \kappa$.*

3) *If $(M^\gamma)_{\gamma<\kappa}$ is a continuously increasing sequence of V_κ-like elementary submodels of $\mathcal{H}(\chi)$ then $\bigcup_{\gamma<\kappa} M^\gamma$ is also V_κ-like.*

Proof. *0)*: By definition.

1): Suppose not. Let M be a V_κ-like elementary submodel of $\mathcal{H}(\chi)$ such that $\kappa \not\subseteq M$. Let $\alpha^* < \kappa$ be minimal such that $\alpha \notin M$. Let $(M_\alpha)_{\alpha<\kappa}$ be a witness of V_κ-likeness of M. Without loss of generality we may assume that $\kappa \in M_0$. Let $\beta < \kappa$ be such that $\alpha \subseteq M_\beta$. Let $\gamma \leq \kappa$ be minimal such that $\alpha \leq \gamma$ and $\gamma \in M_\beta$. Then α^* is definable in $M_{\beta+1}$ as $\gamma \cap M_\beta$. Hence $\alpha^* \in M_{\beta+1} \subseteq M$. This is a contradiction.

2): Let $X = \{\, x_\alpha \;:\; \alpha < \kappa \,\}$. By induction we can construct a continuously increasing sequence $(M_\alpha)_{\alpha<\kappa}$ of elementary submodels of $\mathcal{H}(\chi)$ such that $|M_\alpha| \leq |\alpha + \omega|$, $x_\alpha \in M_\alpha$ and $(M_\beta)_{\beta\leq\alpha} \in M_{\alpha+1}$ for all $\alpha < \kappa$. $M = \bigcup_{\alpha<\kappa} M_\alpha$ is then internally approachable and $X \subseteq M$.

3): For $\gamma < \kappa$, let $(M^\gamma_\xi)_{\xi<\kappa}$ be a witness of V_κ-likeness of M^γ. Let $\{\, (\eta_\alpha, \nu_\alpha) \;:\; \alpha < \kappa \,\}$ be an enumeration of $\{\, (\gamma, \beta) \;:\; \gamma,\, \beta < \kappa \,\}$. Let $(\gamma_\alpha)_{\alpha<\kappa}$ and $(\xi_\alpha)_{\alpha<\kappa}$ be defined inductively such that *i)* $\gamma_\alpha, \xi_\alpha < \kappa$, *ii)* $M^{\gamma_\alpha}_{\xi_\alpha} \in M^{\gamma_{\alpha+1}}_{\xi_{\alpha+1}}$, *iii)* $M^{\gamma_\beta}_{\xi_\beta} \subseteq M^{\gamma_\alpha}_{\xi_\alpha}$ for all $\beta < \alpha$ and *iv)* $M^{\eta_\alpha}_{\nu_\alpha} \subseteq M^{\gamma_\alpha}_{\xi_\alpha}$ for all $\alpha < \kappa$. Let $M_\alpha = M^{\gamma_\alpha}_{\xi_\alpha}$ for $\alpha < \kappa$. Then $(M_\alpha)_{\alpha<\kappa}$ is a witness of V_κ-likeness of M. ❑ (Lemma 5.10)

Proposition 5.11 (essentially in Fuchino–Koppelberg–Shelah [17]) *For regular κ, and a Boolean algebra B of cardinality $\leq \kappa^+$ the following are equivalent:*

a) *B is $\leq_\kappa$-generated;*

b) *For some, or equivalently any sufficiently large χ and V_κ-like elementary submodel M of $\mathcal{H}(\chi)$, if B, $\kappa \in M$ then $B \cap M \leq_\kappa B$.*

Proof. The assertions *a)*, and *b)* both hold if $|B| \leq \kappa$. Hence we may assume that $|B| = \kappa^+$. *a)* ⇒ *b)* follows from Theorem 5.1. For *b)* ⇒ *a)*, let $(M_\alpha)_{\alpha<\kappa^+}$ be an increasing sequecne of V_κ-like elementary submodel of $\mathcal{H}(\chi)$ such that B, $\kappa \in M_0$ and $B \subseteq \bigcup_{\alpha<\kappa^+} M_\alpha$. For $\alpha < \kappa^+$, let $B_\alpha = B \cap (\bigcup_{\beta<\alpha} M_\beta)$. Then we have $|B_\alpha| \leq \kappa$ for every $\alpha < \kappa^+$. Hence each B_α is $\leq_\kappa$-generated. Also we have:

Claim 5.11.1 *$B_\alpha \leq_\kappa B$ for every $\alpha < \kappa^+$.*

$\vdash$ If $\alpha = \beta + 1$ then $B_\alpha = B \cap M_\beta$. Hence by assumption, we have $B_\alpha \leq_\kappa B$. If α is a limit and $\mathrm{cf}(\alpha) < \kappa$ then it follows from this that we also have $B_\alpha \leq_\kappa B$. If α is a limit and $\mathrm{cf}(\alpha) = \kappa$, then $N = \bigcup_{\beta<\alpha} M_\alpha$ is V_κ-like by Lemma 5.10, *3)*. Hence, again by assumption, it follows that $B_\alpha = B \cap N =\leq_\kappa B$. $\dashv$ (Claim 5.11.1)

By Lemma 5.4, *2*), it follows that B is $\leq_\kappa$-generated. ❑ (Proposition 5.11)

For B with $|B| > \kappa^+$ we still have the equivalence of *a*) and *b*) in the theorem above under some additional assumptions: Recall that $\square_\mu$ means that there is a sequence (a $\square_\mu$-sequence) $(C_\alpha)_{\alpha\in Lim(\mu^+)}$ such that, for every $\alpha < \mu^+$,

α) C_α is club in α;
β) $\beta \in (C_\alpha)'$ implies $C_\beta = C_\alpha \cap \beta$;
γ) $\mathrm{cf}(\gamma) < \mu$ implies $otp(C_\gamma) < \mu$.

From γ), it follows that for singular μ, we have $|C_\alpha| < \mu$ for every $\alpha \in Lim(\mu^+)$. Without loss of generality we may also assume:

δ) if $\delta \in (C_\alpha)' \cap \alpha$, then $\delta + 1 \in C_\alpha$

(just replace each C_α by $C_\alpha \cup \{\,\delta+1 : \delta \in (C_\alpha)' \cap \alpha\,\}$).

Theorem 5.12 (Fuchino, Soukup [?]) *Let κ be a regular uncountable cardinal and $\kappa \leq \lambda$. Suppose that*

i) *($[\mu]^{<\kappa}, \subseteq$) has a cofinal subset of cardinality μ for every μ such that $\kappa < \mu < \lambda$ and* $\mathrm{cf}(\mu) \geq \kappa$*; and*

ii) *$\square_\mu$ holds for every μ such that $\kappa \leq \mu < \lambda$ and* $\mathrm{cf}(\mu) < \kappa$*.*

Then, for any Boolean algebra B of cardinality $\leq \lambda$, the following are equivalent:

1) *B is $\leq_\kappa$-generated;*

2) *for some, or equivalently any sufficiently large χ, and any V_κ-like $M \prec \mathcal{H}(\chi)$ with B, $\kappa \in M$, we have $B \cap M \leq_\kappa B$;*

3) *for some, or equivalently any sufficiently large χ, there is $x_0 \in \mathcal{H}(\chi)$ such that for any V_κ-like $M \prec \mathcal{H}(\chi)$ with B, $x_0 \in M$, we have $B \cap M \leq_\kappa B$.*

Note that $\neg 0^\#$ implies the conditions *i*) and *ii*).
Proof. This will be written later. ❑ (Theorem 5.12)

5.2 Cardinal invariants of $\leq_\kappa$-generated Boolean algebras

Theorem 5.13 (S. Shelah, [17]) *Suppose that κ is a regular cardinal and B a $\leq_\kappa$-generated Boolean algebra. For any cardinal λ such that $\lambda = \lambda^{<\lambda}$, if $X \subseteq B$ is of cardinality greater than λ, then there is an independent subset Y of X of cardinality greater than λ.*

Proof. Let χ be sufficiently large and $(M_\alpha)_{\alpha<\lambda^+}$ be a continuously increasing sequence of elementary submodels of $\mathcal{H}(\chi)$ such that $\kappa{+}1 \subseteq M_0$, $B \in M_0$, $|M_\alpha| \leq \lambda$ and $X \cap (M_{\alpha+1} \setminus M_\alpha) \neq \emptyset$ for every $\alpha < \lambda^+$. For $\alpha < \lambda^+$, let $B_\alpha = B \cap M_\alpha$. $(B_\alpha)_{\alpha<\lambda^+}$ is then a continuously increasing sequence of subalgebras of B. By Theorem 5.1, $B_\alpha \leq_\kappa B$ for every $\alpha < \lambda^+$. For $\alpha < \lambda^+$, let $a_\alpha \in B_{\alpha+1} \setminus B_\alpha$. Let $S = \{\,\alpha <$

$\lambda^+ : \mathrm{cf}(\alpha) \geq \kappa \}$. For each $\alpha \in S$, let I_α and J_α be cofinal subsets of $B_\alpha \restriction a_\alpha$ and $B_\alpha \restriction -a_\alpha$ respectively, both of cardinality less than κ. Let $h(\alpha) = \langle I_\alpha, J_\alpha \rangle$. By $\lambda = \lambda^{<\kappa}$ we can apply Fodor's lemma and find a stationary $T \subseteq S$ such that $h \restriction T$ is constant, say $h(\alpha) = \langle I, J \rangle$ for all $\alpha \in T$. Let $\alpha^* = \min\{ \alpha < \lambda^+ : I, J \subseteq B_\alpha \}$. Without loss of generality we may assume that $\alpha^* < \alpha$ holds for every $\alpha \in T$. Let $L = \{ b \in B_{\alpha^*} : b \not\leq i + j \text{ for all } i \in I, j \in J \}$. Then we have

(1) $1 \in L$ (since, by $a_\alpha \notin B_{\alpha^*}$ for any $\alpha \in T$, $I \cup J$ generates a proper ideal of B_{α^*}). In particular we have $L \neq \emptyset$;

(2) If $b \in L$ and $k \in I \cup J$ then $b \cdot -k \in L$.

Now, by *1*) above, the following claim shows that $\{ a_\alpha : \alpha \in T \}$ is independent. Since $|T| = \lambda^+$, this proves the theorem.

Claim 5.13.1 *If $b \in L$ and p is an elementary product over $a_{\alpha_0},\ldots,a_{\alpha_{n-1}}$ for $\alpha_i \in T$ such that $\alpha_0 < \cdots < \alpha_{n-1}$ (i.e. p is of the form*[6] *$(a_{\alpha_0})^{\tau_0} \cdot \cdots \cdot (a_{\alpha_{n-1}})^{\tau_{n-1}}$ for some $\tau_i \in 2$, $i < n$) then $b \cdot p \neq 0$.*

$\vdash$ We prove the claim by induction on n. For $n = 0$, this is trivial since $0 \notin L$. Assume that the claim holds for n. Let $\alpha_0,\ldots,\alpha_n \in T$ be such that $\alpha_0 < \cdots \alpha_{n-1} < \alpha_n$ and let p be an arbitrary elementary product over $a_{\alpha_0},\ldots,a_{\alpha_{n-1}}$. Let $b \in L$. By the induction hypothesis, we have $b \cdot p \neq 0$. We have to show that $b \cdot p \cdot a_{\alpha_n} \neq 0$ and $b \cdot p \cdot -a_{\alpha_n} \neq 0$. Toward a contradiction, assume that $b \cdot p \cdot a_{\alpha_n} = 0$ holds. Then $b \cdot p \leq -a_{\alpha_n}$. Since $b \cdot p \in B_{\alpha_n}$, we can find $j \in J$ such that $b \cdot p \leq j$. Hence $(b \cdot -j) \cdot p = 0$. Since $b \cdot -j \in L$ by *2*) above, this is a contradiction to the induction hypothesis. Similarly, from $b \cdot p \cdot -a_{\alpha_n} = 0$, we obtain $(b \cdot -i) \cdot p = 0$ for some $i \in I$ which again is a contradiction to *2*). $\dashv$ (Claim 5.13.1)

❏ (Theorem 5.13)

Theorem 5.13 implies the following inequalities about cardinal invariants for $\leq_\kappa$-generated Boolean algebras. Let us first review definitions of some of cardinal invariants on Boolean algebras introduced in Monk [38]. For a Boolean algebra B

$$\mathrm{Ind}(B) = \sup\{ |X| : X \text{ is an independent subset of } B \},$$

$$c(B) = \sup\{ |X| : X \text{ is a disjoint subset of } B \},$$

$$\mathrm{Length}(B) = \sup\{ |X| : X \text{ is a linear ordered subset of } B \}.$$

Corollary 5.14 (Fuchino–Koppelberg–Shelah [17]) *For a regular cardinal κ, if a Boolean algebra B is $\leq_\kappa$-generated, then $|B| \leq \mathrm{Ind}(B)^{<\kappa}$. In particular, for an openly generated Boolean algebra B, $|B| = \mathrm{Ind}(B)$.*

[6] for a Boolean algebra B, $b \in B$ and $i \in 2$, we define $(b)^i$ by:

$$(b)^i = \begin{cases} b, & \text{if } i = 1, \\ -b, & \text{if } i = 0. \end{cases}$$

Proof. Suppose that B is $\leq_\kappa$-generated but $|B| > \text{Ind}(B)^{<\kappa}$ holds. Then $\text{Ind}(B)^{<\kappa} = (\text{Ind}(B)^{<\kappa})^{<\kappa}$. Hence we have $\text{Ind}(B) > \text{Ind}(B)^{<\kappa}$ by Theorem 5.13. This is a contradiction. ❑ (Corollary 5.14)

In Corollary 5.14 the equality is attained for every regular κ. For the case of $\kappa = \aleph_1$, the simplest example to see this would be $Intalg(\mathbb{R})$.

Corollary 5.15 (Fuchino–Koppelberg–Shelah [17]; Heindorf–Shapiro [25] for $\kappa \leq \aleph_1$) *Let κ be a regular cardinal. If a Boolean algebra B is $\leq_\kappa$-generated, then $c(B) \leq 2^{<\kappa}$ and* $\text{Length}(B) \leq 2^{<\kappa}$. *In particular, any openly generated Boolean algebra B has the ccc and* $\text{Length}(B) = \aleph_0$. ❑

5.3 Bockstein separation property

For a Boolean algebra B and $X \subseteq B$, let $X^\perp = \{\, c \in B : b{\cdot}c = 0 \text{ for every } b \in X \,\}$. An ideal I on B is said to be regular if $(I^\perp)^\perp = I$. Note that $(X^\perp)^\perp \supseteq X$ holds for any $X \subseteq B$ (see below). In the following we shall also simply write e.g. $I^{\perp\perp}$ in place of $(I^\perp)^\perp$. A Boolean algebra B is said to have the Bockstein separation property if every regular ideal I of B is countably generated, i.e. if there is a countable cofinal subset of such I. We shall consider the following generalization of this notion: For a cardinal κ, a Boolean algebra B has the *κ-Bockstein separation property*, if every regular ideal I on B has a cofinal subset of cardinality less than κ. Thus the usual Bockstein separation property is just the $\aleph_1$-Bockstein separation propertyin our terminology. For $\kappa \leq \lambda$, if B has the κ-Bockstein separation property, then B clearly has also the λ-Bockstein separation property. The following fact on the $\perp$-operation is quite useful.

Lemma 5.16 *For a Boolean algebra B and $X, Y \subseteq B$,*

1) *if $X \subseteq Y$, then $X^\perp \supseteq Y^\perp$;*
2) $X \subseteq X^{\perp\perp}$;
3) $X^{\perp\perp\perp} = X^\perp$

Proof. *1)*: Suppose $X \subseteq Y$. Then we have

$$\begin{aligned} z \in Y^\perp &\Leftrightarrow y \cdot z = 0 \text{ for every } y \in Y \\ &\Rightarrow x \cdot z = 0 \text{ for every } x \in X \\ &\Leftrightarrow z \in X^\perp. \end{aligned}$$

2): If $x \in X$, then $x \cdot y = 0$ for any $y \in X^\perp$. Hence $x \in X^{\perp\perp}$.
3): By *2)*, we have $X^{\perp\perp} \supseteq X$. Hence $X^{\perp\perp\perp} = (X^{\perp\perp})^\perp \subseteq X^\perp$ by *1)*. On the other hand, by *2)*, we have $X^\perp \subseteq (X^\perp)^{\perp\perp} = X^{\perp\perp\perp}$. ❑ (Lemma 5.16)

Lemma 5.17 *Suppose that I is an ideal on a Boolean algebra B. I is regular if and only if there exists a pairwise disjoint $X \subseteq B^+$ with $I = X^\perp$.*

Proof. If $I = X^\perp$ then $I^{\perp\perp} = X^{\perp\perp\perp} = X^\perp = I$ by Lemma 5.16, *3*). Hence I is regular. Suppose now that I is regular ideal on B. Let $X \subseteq B^+$ be maximal pairwise disjoint such that $X \subseteq I^\perp$, i.e. each $x \in X$ satisfies $b\cdot x = 0$ for every $b \in I$. We show that $I = X^\perp$. If $a \in I$ then $a \cdot x = 0$ for every $x \in X$. Hence $a \in X^\perp$. If $a \in X^\perp$ and yet $a \notin I$, then by regularity of I, we would have $a \notin (I^\perp)^\perp$. Hence there is some $c \in I^\perp$ such that $a \cdot c \neq 0$. Note that $a \cdot c \in I^\perp$ since $a \cdot c \leq c$. Hence by maximality of X there is $x \in X$ such that $a\cdot c\cdot x \neq 0$. But this is a contradiction as $a \cdot c \cdot x \leq a \cdot x = 0$. ❑ (Lemma 5.17)

Proposition 5.18 *Suppose that B is a κ-cc Boolean algebra and $\{ A \in [B]^{<\kappa} : A \leq_{\rm rc} B \}$ is unbounded in $[B]^{<\kappa}$. Then B has the κ-Bockstein separation property. In particular, every openly generated Boolean algebra has the Bockstein separation property.*

Proof. Suppose that I is a regular ideal on B. By Lemma 5.17 there is a pairwise disjoint subset X of B^+ such that $I = X^\perp$. By the κ-cc of B, we have $|X| < \kappa$. By the assumption there is $A \leq_{\rm rc} B$ such that $|A| < \kappa$ and $X \subseteq A$. For $a \in I$, $q^B_A(a)$ is disjoint from each element of X: if there were $x \in X$ such that $q^B_A(a) \cdot x \neq 0$ then we would have $a \cdot x \neq 0$ which contradicts the choice of X. Hence $a \leq q^B_A(a) \in I$. As $q^B_A(a) \in A$, it follows that $I \cap A$ is cofinal in I.

If B is openly generated, then B has the ccc by Corollary 5.15. Hence, by what we proved above, B has the Bockstein separation property. ❑ (Proposition 5.18)

For a Boolean algebra B, $X \subseteq B^+$ is *dense in* B if, for every $b \in B^+$, there is $c \in X$ such that $c \leq b$. A subalgebra A of B is said to be *dense in* B if A^+ is dense in B in the sense above. If A is dense in B we denote this by $A \leq_{\rm dense} B$.

Lemma 5.19 *Suppose that A, B are Boolean algebras such that $A \leq_{\rm dense} B$. Then for any $b \in B$, $A \restriction b$ is a regular ideal in A.*

Proof. Otherwise there would be $a \in (A \restriction b)^{\perp\perp}$ such that $a \not\leq b$. Let $c = a \cdot -b$. Then $c \in B^+$. Since A is dense in B, there is $d \in A^+$ such that $d \leq c$. Then $d \leq a$. Hence $d \in (A \restriction b)^{\perp\perp}$. On the other hand, since $d \leq -b$, we have $d \in (A \restriction b)^\perp$. As $d \neq 0$, this is a contradiction. ❑ (Lemma 5.19)

Proposition 5.20 *Suppose that κ is regular and A, B are Boolean algebras such that $A \leq B$. If B is $\leq_\kappa$-generated and A has the κ-Bockstein separation property, then A is $\leq_\kappa$-generated.*

Proof. Let J be an ideal on B maximal with the property $A \cap J = \{ 0 \}$. Let $\varphi : B \to B/J$ be the canonical epimorphism. Then, by $A \cap J = \{ 0 \}$, φ is injective on A. Identifying A with $\varphi[A]$, we may assume that $\varphi \restriction A = id_A$ and $A \leq B/J$.

Claim 5.20.1 $A \leq_{\rm dense} B/J$.

$\vdash$ Let $\tilde{b} \in (B/J)^+$, say $\tilde{b} = \varphi(b)$ for some $b \in B$. Then $b \notin J$. Hence, by maximality of J, there is some $b' \in J$ and $a \in A^+$ such that $a \leq b + b'$. But this just means $a \leq \tilde{b}$. $\dashv$ (Claim 5.20.1)

Let χ be sufficiently large and M an elementary submodel of $\mathcal{H}(\chi)$ such that $A \in M$ and $\kappa + 1 \subseteq M$. To show that A is $\leq_\kappa$-generated, it is enough to show that $A \cap M \leq_\kappa A$ holds by Theorem 5.1. By elementarity of M, there are B', J', $\varphi' \in M$ such that $A \leq B'$, B' is $\leq_\kappa$-generated, J' is an ideal on B' maximal with the property $A \cap J' = \{0\}$ and $\varphi' : B' \to B'/J$ is the canonical mapping. Hence, without loss of generality, we may assume that B, φ, $J \in M$. Now, let $a \in A$. Since B is $\leq_\kappa$-generated, we have $B \cap M \leq_\kappa B$ by Theorem 5.1. Hence there is $X \subseteq (B \cap M) \restriction a$ such that $|X| < \kappa$ and X is cofinal in $(B \cap M) \restriction a$. For each $x \in X$, $A \restriction \varphi(x)$ is regular ideal on A by Claim 5.20.1 and Lemma 5.19. Hence, by κ-Bockstein separation property of A and $\varphi(x) \in M$, there is $Y_x \in M$ such that $Y_x \subseteq A \restriction \varphi(x)$, $|Y_x| < \kappa$ and Y_x is cofinal in $A \restriction \varphi(x)$. By $\kappa + 1 \subseteq M$, we have $Y_x \subseteq A \cap M$. Let $Z = \bigcup_{x \in X} Y_x$. For $x \in X$, we have $\varphi(x) \leq \varphi(a) = a$. Hence $Z \subseteq (A \cap M) \restriction a$. As κ is regular, we have $|Z| < \kappa$.

Claim 5.20.2 *Z is cofinal in $(A \cap M) \restriction a$.*

$\vdash$ Suppose that $a' \in (A \cap M) \restriction a$. Then $a' \in (B \cap M) \restriction a$. Hence there is $x \in X$ such that $a' \leq x$. It follows that $a' = \varphi(a') \leq \varphi(x)$. Hence $a' \in A \restriction \varphi(x)$ and so there is some $u \in Y_x \subseteq Z$ such that $a' \leq u$. $\dashv$ (Claim 5.20.2)

This shows that $A \cap M \leq_\kappa A$. ❑ (Proposition 5.20)

Theorem 5.21 *Suppose that κ is a regular cardinal such that $2^{<\kappa} = \kappa$ and A, B are Boolean algebras such that $A \leq B$, B is $\leq_\kappa$-generated and $\{C : C \leq_{\mathrm{rc}} B, |C| < \kappa\}$ is unbounded in $[B]^{<\kappa}$. Then A is $\leq_\kappa$-generated if and only if A has the κ-Bockstein separation property.*

Proof. If A is $\leq_\kappa$-generated then A has the κ-cc by Corollary 5.15. Hence, by Proposition 5.18, A has the κ-Bockstein separation property. Conversely, if A has the κ-Bockstein separation property, then it follows form Proposition 5.20, that A is $\leq_\kappa$-generated. ❑ (Theorem 5.21)

Corollary 5.22 (L. Heindorf, see Heindorf–Shapiro [25]) *Suppose that A, B are Boolean algebras such that $A \leq B$ and B is openly generated. Then A is openly generated if and only if A has the Bockstein separation property.* ❑

6 $L_{\infty\kappa}$-free Boolean algebras

$L_{\infty\kappa}$-freeness was first studied in connection with groups and abelian groups. This motivated the study of $L_{\infty\kappa}$-freeness in arbitrary variety. Eklof [6] contains a compact survey of this subject. $L_{\infty\kappa}$-free algebras are also called strongly κ free in the literature (see e.g. Eklof–Mekler [8] where strongly κ free algebras are defined by the property d) of Proposition 6.15 below). In Fuchino–Koppelberg–Takahashi [19] we began the study of $L_{\infty\kappa}$-free algebras in the variety of Boolean algebras which is continued in Fuchino–Shelah [20]. In this chapter, we shall present some of basic results of this field.

Let us first introduce some notation used in the following. For a cardinal κ and (the underlying set of) a structure A, let ${}^{\kappa>}A$ be the set of sequences of elements of A of length less than κ, i.e.

$$ {}^{\kappa>}A = \{\, \overline{a} : \ \overline{a} : \alpha \to A \text{ for some } \alpha < \kappa \,\}. $$

Abusing the notation, for $\overline{a} \in {}^{\kappa>}A$, we shall often write simply $\overline{a}$ to denote $\mathrm{rng}\,(\overline{a})$. Note that this coincides with the usual convention for finite sequences. For $\overline{a} \in {}^{\kappa>}A$, $l(\overline{a})$, *the length of* $\overline{a}$, is just $\mathrm{dom}\,(\overline{a})$. For $\overline{a} \in {}^{\kappa>}A$ and $\overline{b} \in {}^{\kappa>}B$ with $l(\overline{a}) = l(\overline{b})$, let $\overline{a} \mapsto \overline{b}$ be the mapping $\{\, (\overline{a}(\beta), \overline{b}(\beta)) \ : \ \beta < l(\overline{a}) \,\}$. Thus $\overline{a} \mapsto \overline{b}$ is the mapping which sends the β'th element of $\overline{a}$ to the β'th element of $\overline{b}$. For $\overline{a}$, $\overline{a}' \in {}^{\kappa>}A$, $\overline{a}^\frown\overline{a}'$ denotes the concatenation of $\overline{a}$ and $\overline{a}'$, i.e. the element $\overline{b}$ of ${}^{\kappa>}A$ such that $l(\overline{b}) = l(\overline{a}) + l(\overline{a}')$, $\overline{b}(\gamma) = \overline{a}(\gamma)$ for all $\gamma < l(\overline{a})$ and $\overline{b}(l(\overline{a}) + \gamma) = \overline{a}'(\gamma)$ for all $\gamma < l(\overline{a}')$. For $\overline{a}_\alpha \in {}^{\kappa>}A$, $\alpha < \lambda$, let us denote by $\frown_{\alpha<\lambda} \overline{a}_\alpha$ the concatenation of $\overline{a}_\alpha$, $\alpha < \lambda$ one after another. Note that, if $\lambda < \mathrm{cf}(\kappa)$, then we have $\frown_{\alpha<\lambda} \overline{a}_\alpha \in {}^{\kappa>}A$. For $X \in [A]^{<\kappa}$, $\overline{a} \in {}^{\kappa>}A$ is an *enumeration* of X if $\mathrm{rng}\,(\overline{a}) = X$. For structures A, B, a mapping f is said to be a *partial isomorphism from* A *to* B if $\mathrm{dom}\,(f) \subseteq A$, $\mathrm{rng}\,(f) \subseteq B$, and if $\mathrm{dom}\,(f)$ and $\mathrm{rng}\,(f)$ generate substructures A' and B' of A and B respectively, then f generates an isomorphism from A' to B'. As for Boolean algebras, $B \le A$ means that B is a substructure of the structure A. We define

$$ \mathrm{Sub}^{<\kappa}(A) = \{\, B \in [A]^{<\kappa} \ : \ B \le A \,\}. $$

$\mathrm{Sub}^{\le\kappa}(B)$, $\mathrm{Sub}^{\kappa}(B)$ etc. are defined similarly. For $\overline{a} \in {}^{\kappa>}A$, $(A, \overline{a})$ is the expansion of the structure A obtained by adding $l(\overline{a})$ many new constant symbols to the signature of A whose interpretations are $\overline{a}(\alpha)$, $\alpha < l(\overline{a})$. For $\overline{a}_\alpha \in {}^{\kappa>}A$, $\alpha < \lambda$, we denote by $(A, \overline{a}_\alpha)_{\alpha<\lambda}$ the structure $(A, \overline{b})$ where $\overline{b} = \frown_{\alpha<\lambda} \overline{a}_\alpha$.

Most of the assertions about $L_{\infty\kappa}$-free/projective Boolean algebras in this chapter are valid for algebras in an arbitrary variety $\mathcal{V}$. An algebra $F \in \mathcal{V}$ is defined to be *free* if F is generated by a set U with the property as in Lemma 2.1 while *free products* of algebras in $\mathcal{V}$ are defined by the property in Lemma 3.2. Elementary properties of free products mentioned in Chapter 3. Projectivity, $\le_{\mathrm{free}}$, $\le_{\mathrm{proj}}$, $L_{\infty\kappa}$-freeness etc. in $\mathcal{V}$ can be defined then just as for Boolean algebras. In the following, "*A is an algebra*" means that A is an algebra in some fixed but arbitrary variety $\mathcal{V}$. Fr κ means, in this context, the free algebra in the variety $\mathcal{V}$ with the free generator κ.

A few of the assertions below are rather typical for Boolean algebras, e.g. Theorem 6.9 where completions of $L_{\infty\kappa}$-elementarily equivalent Boolean algebras are considered.

6.1 Back-and-forth theorem

Lemma 6.1 *Suppose that A, B are structures of the same signature such that $A \equiv_{L_{\infty\kappa}} B$ and $\overline{a} \in {}^{\kappa>}A$.*

1) There is an $L_{\infty\kappa}$-formula $\psi_{\overline{a}}(\overline{x})$ with $l(\overline{x}) = l(\overline{a})$ such that $A \models \psi_{\overline{a}}[\overline{a}]$ and for any $\overline{b} \in {}^{\kappa>}B$ with $l(\overline{b}) = l(\overline{a})$, we have $B \models \psi_{\overline{a}}[\overline{b}]$ if and only if $(A, \overline{a}) \equiv_{L_{\infty\kappa}} (B, \overline{b})$.

2) There is $\overline{b} \in B$ such that $(A, \overline{a}) \equiv_{L_{\infty\kappa}} (B, \overline{b})$.

Proof. *1*): For each $\overline{b} \in {}^{\kappa>}B$ with $l(\overline{b}) = l(\overline{a})$, let $\varphi_{\overline{b}}(\overline{x})$ be an $L_{\infty\kappa}$-formula such that $A \models \varphi_{\overline{b}}[\overline{a}]$ and $B \models \neg\varphi_{\overline{b}}[\overline{b}]$ if $(A, \overline{a}) \not\equiv_{L_{\infty\kappa}} (B, \overline{b})$. Otherwise, let $\varphi_{\overline{b}}(\overline{x})$ be any $L_{\infty\kappa}$-formula φ such that $A \models \varphi[\overline{a}]$. Then

$$\psi_{\overline{a}}(\overline{x}) = \bigwedge\{\, \varphi_{\overline{b}}(\overline{x}) \,:\, \overline{b} \in {}^{\kappa>}B, l(\overline{b}) = l(\overline{a}) \,\}$$

is as desired.

2): Let $\psi_{\overline{a}}(\overline{x})$ be as in *1*). Since $A \models \psi_{\overline{a}}[\overline{a}]$, we have $A \models \exists\overline{x}\,\psi_{\overline{a}}(\overline{x})$. Hence $B \models \exists\overline{x}\,\psi_{\overline{a}}(\overline{x})$. Let $\overline{b} \in {}^{\kappa>}B$ be such that $B \models \psi_{\overline{a}}[\overline{b}]$. Then $(A, \overline{a}) \equiv_{L_{\infty\kappa}} (B, \overline{b})$.
❑ (Lemma 6.1)

The following characterization of $L_{\infty\kappa}$-elementary equivalence is known as the "*Back-and-Forth Theorem*".

Theorem 6.2 *For structures A, B of the same signature, the following are equivalent:*

a) $A \equiv_{L_{\infty\kappa}} B$;

b) there is a non-empty family $\mathcal{F}$ of partial isomorphisms from A to B such that

- *0) $|f| < \kappa$ for every $f \in \mathcal{F}$,*
- *i) if $\overline{a} \mapsto \overline{b} \in \mathcal{F}$ and $\overline{a}' \in {}^{\kappa>}A$, then there is $\overline{b}' \in {}^{\kappa>}B$ such that $\overline{a}^\frown\overline{a}' \mapsto \overline{b}^\frown\overline{b}' \in \mathcal{F}$,*
- *ii) if $\overline{a} \mapsto \overline{b} \in \mathcal{F}$ and $\overline{b}' \in {}^{\kappa>}B$, then there is $\overline{a}' \in {}^{\kappa>}A$ such that $\overline{a}^\frown\overline{a}' \mapsto \overline{b}^\frown\overline{b}' \in \mathcal{F}$,*
- *iii) if $f \in \mathcal{F}$ and $g \subseteq f$, then $g \in \mathcal{F}$.*

We shall refer such a family $\mathcal{F}$ as above a *family of partial isomorphisms form A to B with back-and-forth property for $L_{\infty\kappa}$.*

Proof. *a)* ⇒ *b)*: Suppose that $A \equiv_{L_{\infty\kappa}} B$. Let

$$\mathcal{F} = \{\, \overline{a} \mapsto \overline{b} \,:\, \overline{a} \in {}^{\kappa>}A,\, \overline{b} \in {}^{\kappa>}B,\, (A, \overline{a}) \equiv_{L_{\infty\kappa}} (B, \overline{b}) \,\}.$$

We show that $\mathcal{F}$ is as in *b)*. By Lemma 6.1, *2)*, $\mathcal{F}$ is non-empty. By definition $\mathcal{F}$ satisfies *0)* in *b)*. It is also clear that $\mathcal{F}$ satisfies *iii)* in *b)*. Hence, by symmetry, it is enough to show that $\mathcal{F}$ satisfies *i)*. Suppose $\overline{a} \mapsto \overline{b} \in \mathcal{F}$ and $\overline{a}' \in {}^{\kappa>}A$. Let $\psi_{\overline{a}^\frown\overline{a}'}$ be as in Lemma 6.1, *1)* for $\overline{a}^\frown\overline{a}'$. Since $A \models \psi_{\overline{a}^\frown\overline{a}'}[\overline{a}, \overline{a}']$, we have $A \models \exists\overline{x}'\psi_{\overline{a}^\frown\overline{a}'}[\overline{a}, \overline{x}']$. By $(A, \overline{a}) \equiv_{L_{\infty\kappa}} (B, \overline{b})$ it follows that $B \models \exists\overline{x}'\psi_{\overline{a}^\frown\overline{a}'}[\overline{b}, \overline{x}']$. So there is some $\overline{b}' \in {}^{\kappa>}B$ such that $B \models \psi_{\overline{a}^\frown\overline{a}'}[\overline{b}^\frown\overline{b}']$. By definition of $\psi_{\overline{a}^\frown\overline{a}'}$, it follows that $(A, \overline{a}^\frown\overline{a}') \equiv_{L_{\infty\kappa}} (B, \overline{b}^\frown\overline{b}')$. Hence $\overline{a}^\frown\overline{a}' \mapsto \overline{b}^\frown\overline{b}' \in \mathcal{F}$.

b) ⇒ *a)*: Suppose that $\mathcal{F}$ is as in *b)*. By induction on (∈-rank[7] of) $L_{\infty\kappa}$-formula φ, we show that

$(*)_\varphi$: *for any* $\overline{a} \mapsto \overline{b} \in \mathcal{F}$, $A \models \varphi[\overline{a}]$ *if and only if* $B \models \varphi[\overline{b}]$.

For atomic φ this is clear as elements of $\mathcal{F}$ are partial isomorphisms. If $(*)_\varphi$ holds for every $\varphi \in \Phi$, then clearly $(*)_{\bigwedge\Phi}$ also holds. It is also clear that $(*)_{\neg\varphi}$ follows from $(*)_\varphi$. For a formula φ of the form $\exists\overline{y}\psi(\overline{x}, \overline{y})$, assume that $(*)_{\psi(\overline{x},\overline{y})}$ holds. If $\overline{a} \mapsto \overline{b} \in \mathcal{F}$ and $A \models \varphi[\overline{a}]$, then there is $\overline{a}' \in {}^{\kappa>}A$ such that $A \models \psi[\overline{a}^\frown\overline{a}']$. By *i)*, there is $\overline{b}' \in {}^{\kappa>}B$ such that $\overline{a}^\frown\overline{a}' \mapsto \overline{b}^\frown\overline{b}' \in \mathcal{F}$. By $(*)_{\psi(\overline{x},\overline{y})}$ we have $B \models \psi[\overline{b}^\frown\overline{b}']$. Hence $B \models \varphi[\overline{b}]$. Similarly, by *ii)*, $A \models \varphi[\overline{a}]$ follows from $B \models \varphi[\overline{b}]$ for $\overline{a} \in {}^{\kappa>}A$ and $\overline{b} \in {}^{\kappa>}B$ such that $\overline{a} \mapsto \overline{b} \in \mathcal{F}$. Thus $(*)_\varphi$ holds. ❑ (Theorem 6.2)

From the proof of *b)* ⇒ *a)* above, we additionally obtained the following:

Corollary 6.3 *For structures A, B, if $\mathcal{F}$ is a family of partial isomorphisms from A to B with the back-and-forth property for $L_{\infty\kappa}$ and $f \in \mathcal{F}$ with $f = \overline{a} \mapsto \overline{b}$, then $(A, \overline{a}) \equiv_{L_{\infty\kappa}} (B, \overline{b})$.* ❑

Lemma 6.4 *Let κ, λ be such that $\kappa \leq \lambda$. Then we have* $\mathrm{Fr}\,\kappa \equiv_{L_{\infty\kappa}} \mathrm{Fr}\,\lambda$.

Proof. Let

$$\mathcal{F} = \{\, f \,: f \subseteq \text{the isomorphism from } \mathrm{Fr}\,\overline{a} \text{ to } \mathrm{Fr}\,\overline{b} \text{ induced by } \overline{a} \mapsto \overline{b} \text{ for some } \overline{a} \in {}^{\kappa>}\kappa \text{ and } \overline{b} \in {}^{\kappa>}\lambda \text{ of the same length } \}.$$

Then it is easily seen that $\mathcal{F}$ is a family of partial isomorphisms from $\mathrm{Fr}\,\kappa$ to $\mathrm{Fr}\,\lambda$ with the back-and-forth property for $L_{\infty\kappa}$. By Theorem 6.2, it follows that $\mathrm{Fr}\,\kappa \equiv_{L_{\infty\kappa}} \mathrm{Fr}\,\lambda$. ❑ (Lemma 6.4)

Lemma 6.5 *For algebras A, B, A', B', such that $A \equiv_{L_{\infty\kappa}} A'$, $B \equiv_{L_{\infty\kappa}} B'$, we have $A \oplus B \equiv_{L_{\infty\kappa}} A' \oplus B'$.*

[7] $L_{\infty\kappa}$-formulas are of course sets constructed inside set theory. It is not relevant how formulas are "coded" in sets. Here, we use merely the fact that, in a natural coding, subformulas of a given $L_{\infty\kappa}$-formula φ have strictly smaller ∈-rank than φ itself.

Proof. Let $\mathcal{F}$ ($\mathcal{F}'$ resp.) be a family of partial isomorphisms from A to A' (from B to B' resp.) with the back-and-forth property for $L_{\infty\kappa}$. Let

$$\mathcal{F}^* = \{\, g : g \subseteq f \oplus f', \text{ for some } f \in \mathcal{F},\ f' \in \mathcal{F}' \text{ such that } \operatorname{dom}(f) \leq A \text{ and } \operatorname{dom} f' \leq B \,\}.$$

Then it is easy to check that $\mathcal{F}^*$ is a family of partial isomorphisms from $A \oplus B$ to $A' \oplus B'$ with back-and-forth property for $L_{\infty\kappa}$. By Theorem 6.2, it follows that $A \oplus B \equiv_{L_{\infty\kappa}} A' \oplus B'$. ❑ (Lemma 6.5)

Proposition 6.6 *If an algebra B is $L_{\infty\kappa}$-projective, then $B \oplus \mathrm{Fr}\,\kappa$ is $L_{\infty\kappa}$-free.*

Proof. Let B be $L_{\infty\kappa}$-projective. By the definition of $L_{\infty\kappa}$-projectivity, there is a projective algebra C such that $B \equiv_{L_{\infty\kappa}} C$. Let $\lambda = |C|$. Then $C \oplus \mathrm{Fr}\,\lambda$ is free. By Lemma 6.4 and Lemma 6.5, we have $B \oplus \mathrm{Fr}\,\kappa \equiv_{L_{\infty\kappa}} C \oplus \mathrm{Fr}\,\lambda$. Hence $B \oplus \mathrm{Fr}\,\kappa$ is $L_{\infty\kappa}$-free. ❑ (Proposition 6.6)

For a Boolean algebra B, we denote by B^{cm} the the completion of B. For uncountable κ, $B^{\kappa\text{-cm}}$ — the κ-completion of B — is the smallest subalgebra C of B^{cm} such that $B \leq C$ and C is closed with respect to sum and product of less than κ many elements of C. If B is a κ-complete Boolean algebra and $X \subseteq A$, we denote by $[X]^\kappa_B$ the smallest κ-complete subalgebra of B containing X.

Lemma 6.7 *Let κ be a regular uncountable cardinal. For Boolean algebras A, B, and $\overline{a} \in {}^{\kappa>}A$, $\overline{b} \in {}^{\kappa>}B$ of the same length, if $(A, \overline{a}) \equiv_{L_{\infty\kappa}} (B, \overline{b})$, then the mapping $\overline{a} \mapsto \overline{b}$ can be extended uniquely to an isomorphism f from $[\overline{a}]^\kappa_{A^{\mathrm{cm}}}$ to $[\overline{b}]^\kappa_{B^{\mathrm{cm}}}$.*

Proof. For $c \in [\overline{a}]^\kappa_{A^{\kappa\text{-cm}}}$, there is an infinitary term $t(\overline{x})$ constructed from $+$, $\cdot$, $-$ and $\sum$, $\prod$ (infinitary sum and product over less than κ many elements) such that $t(\overline{a}) = c$. We define the mapping f by $f(c) = t(\overline{b})$. Since f should be κ-complete (i.e. preserving infinitary sums and products over less than κ many elements), this is the only possible definition of f. To see that this is well-defined and gives an isomorphism from $[\overline{a}]^\kappa_{A^{\mathrm{cm}}}$ to $[\overline{a}]^\kappa_{A^{\mathrm{cm}}}$, let $t_0(\overline{x})$,..., $t_{n-1}(\overline{x})$ be infinitary terms in the sense as above. Then there is an $L_{\infty\kappa}$-formula[8] $\varphi(\overline{x})$ saying that "$t_0(\overline{x}) \cdot \cdots \cdot t_{n-1} = 0$". Then we have

$$\begin{aligned} t_0(\overline{a}) \cdot \cdots \cdot t_{n-1}(\overline{a}) = 0 &\Leftrightarrow A \models \varphi[\overline{a}] \\ &\Leftrightarrow B \models \varphi[\overline{b}] \\ &\Leftrightarrow t_0(\overline{b}) \cdot \cdots \cdot t_{n-1}(\overline{b}) = 0. \end{aligned}$$

❑ (Lemma 6.7)

[8] note that we need here the regularity of κ so that we can formulate assertions like "$\overline{x}(\alpha) \leq t_i(\overline{x})$" using less than κ many bounded variables and hence remaining in $L_{\infty\kappa}$.

Lemma 6.8 *If a Boolean algebra B is $L_{\infty\kappa}$-free for some $\kappa > \aleph_1$ then B has the ccc.*

Proof. Let φ be the $L_{\infty\aleph_2}$-formula

$$\forall \overline{x}\,(\bigvee\{\,\overline{x}(\alpha)\cdot\overline{x}(\beta) \neq 0 \,:\, \alpha,\beta < \omega_1,\ \alpha \neq \beta\,\}),$$

where $\overline{x}$ is an injective sequence of variables of length ω_1. Then $B \models \varphi$ if and only if B has the ccc. Since $\operatorname{Fr}\kappa$ has the ccc, $\operatorname{Fr}\kappa \models \varphi$. Hence for any $L_{\infty\kappa}$-free Boolean algebra B, we have $B \models \varphi$ and thus B has the ccc. ❑ (Lemma 6.8)

Theorem 6.9 *Let κ be regular and A, B Boolean algebras.*

1) *If $A \equiv_{L_{\infty\kappa}} B$, then $A^{\kappa\text{-cm}} \equiv_{L_{\infty\kappa}} B^{\kappa\text{-cm}}$.*
2) *If A, B satisfy the κ-cc and $A \equiv_{L_{\infty\kappa}} B$, then $A^{\mathrm{cm}} \equiv_{L_{\infty\kappa}} B^{\mathrm{cm}}$.*

Proof. *1*): Let $\mathcal{F}$ be a family of partial isomorphisms from A to B with back-and-forth property for $L_{\infty\kappa}$. For $f \in \mathcal{F}$ with $f = \overline{a} \mapsto \overline{b}$, we have $(A,\overline{a}) \equiv_{L_{\infty\kappa}} (B,\overline{b})$ by Corollary 6.3. Hence, by Lemma 6.7, each $f \in \mathcal{F}$ with $f = \overline{a} \mapsto \overline{b}$ can be extended uniquely to an isomorphism $\tilde{f} : [\overline{a}]^{\kappa}_{A^{\mathrm{cm}}} \longrightarrow [\overline{b}]^{\kappa}_{B^{\mathrm{cm}}}$. Let

$$\tilde{\mathcal{F}} = \{\,g \,:\, |g| < \kappa,\ g \subseteq \tilde{f} \text{ for some } f \in \mathcal{F}\,\}.$$

Then it is easy to see that $\tilde{\mathcal{F}}$ is a family of partial isomorphisms from $A^{\kappa\text{-cm}}$ to $B^{\kappa\text{-cm}}$ with the back-and-forth property for $L_{\infty\kappa}$.

2): By Lemma 6.8, A and B has the ccc. Hence $A^{\mathrm{cm}} = A^{\kappa\text{-cm}}$ and $B^{\mathrm{cm}} = B^{\kappa\text{-cm}}$. Hence the assertion follows from *1*). ❑ (Theorem 6.9)

Corollary 6.10 *Let B be a Boolean algebra.*

1) *If B is $L_{\infty\aleph_1}$-free then $2^{\aleph_1}$-cm is $L_{\infty\aleph_1}$-Cohen.*
2) *For a regular $\kappa > \aleph_1$, if B is $L_{\infty\kappa}$-free, then B^{cm} is $L_{\infty\kappa}$-Cohen.*

Proof. By Theorem 6.9 and Lemma 6.8. ❑ (Corollary 6.10)

6.2 Kueker's theorem

Next, we turn to a theorem by D. Kueker in [34] which asserts, roughly speaking, that $L_{\infty\kappa}$-elementary equivalence of structures A, B implies more resemblance of these structures than it appears at the first sight. We shall formulate the theorem in a more general form than the original one in [34]. Let us begin with the following easy lemma. For a structure A and its substructure A', let us denote by (A, A') the structure obtained by expanding A by adding a unary relation for underlying set of A'.

Lemma 6.11 *Let A, B be structures of the same signature and $A' \in \mathrm{Sub}^{<\kappa}(A)$, $B' \in \mathrm{Sub}^{<\kappa}(B)$. Then $(A, A') \equiv_{L_{\infty\kappa}} (B, B')$ if and only if there are enumerations $\overline{a} \in {}^{\kappa>}A$ and $\overline{b} \in {}^{\kappa>}B$ of A' and B' respectively such that $(A, \overline{a}) \equiv_{\infty\kappa} (B, \overline{b})$.*

Proof. Suppose that $(A, A') \equiv_{L_{\infty\kappa}} (B, B')$. By Theorem 6.2, there is a family $\mathcal{F}$ of partial isomorphisms from (A, A') to (B, B') with the back-and-forth property for $L_{\infty\kappa}$. By the back-and-forth property of $\mathcal{F}$, there is $f \in \mathcal{F}$ such that $\mathrm{dom}\,(f) = A'$. Let $\overline{a} \in {}^{\kappa>}A$ be an enumeration of A' and $\overline{b} \in {}^{\kappa>}B$ be such that $f = \overline{a} \mapsto \overline{b}$. Let $\varphi(\overline{x})$ be an $L_{\infty\kappa}$-formula such that $l(\overline{x}) = l(\overline{a})$ saying that "$\overline{x}$ enumerates R" where R is the unary relation symbol corresponding to A' in (A, A') (or B' in (B, B')). Since $(A, A') \models \varphi[\overline{a}]$, we also have $(B, B') \models \varphi[\overline{b}]$ Thus $\overline{b}$ is an enumeration of B'. By Corollary 6.3, we also have $(A, \overline{a}) \equiv_{L_{\infty\kappa}} (B, \overline{b})$.

Conversely, suppose that there are enumerations $\overline{a} \in {}^{\kappa>}A$ and $\overline{b} \in {}^{\kappa>}B$ of A' and B' respectively such that $(A, \overline{a}) \equiv_{L_{\infty\kappa}} (B, \overline{b})$. Then

$$\mathcal{F}' = \{\, \overline{a}' \mapsto \overline{b}' \;:\; \overline{a}' \in {}^{\kappa>}A,\ \overline{b}' \in {}^{\kappa>}B,\ \overline{a} \mapsto \overline{b} \subseteq \overline{a}' \mapsto \overline{b}',\ (A, \overline{a}') \equiv_{L_{\infty\kappa}} (B, \overline{b}') \,\}$$

is a family of partial isomorphisms from (A, A') to (B, B') with the back-and-forth property for $L_{\infty\kappa}$. Hence, by Theorem 6.2, we have $(A, A') \equiv_{L_{\infty\kappa}} (B, B')$.

❏ (Lemma 6.11)

Theorem 6.12 (D. Kueker [34]) *Suppose that A and B are structures of the same signature such that $A \equiv_{L_{\infty\kappa}} B$ and $\mathcal{C} \subseteq \mathrm{Sub}^{<\kappa}(A)$ is cofinal in $([A]^{<\kappa}, \subseteq)$ and closed with respect to unions of increasing chains of countable length. Let*

$$\mathcal{D} = \{\, D \in \mathrm{Sub}^{<\kappa}(B) \;:\; (A, C) \equiv_{L_{\infty\kappa}} (B, D) \text{ for some } C \in \mathcal{C} \,\}.$$

Suppose that $\lambda < \mathrm{cf}(\kappa)$ and $C_\alpha \in \mathcal{C}$, $D_\alpha \in \mathcal{D}$ are such that $(A, C_\alpha) \equiv_{L_{\infty\kappa}} (B, D_\alpha)$ for every $\alpha < \lambda$. By Lemma 6.11 there are enumerations $\overline{c}_\alpha$ and $\overline{d}_\alpha$ of C_α and D_α respectively for $\alpha < \lambda$ such that $(A, \overline{c}_\alpha) \equiv_{L_{\infty\kappa}} (B, \overline{d}_\alpha)$. Then there are $C \in \mathcal{C}$, $D \in \mathcal{D}$ and isomorphism $f_\alpha : C \to D$ for $\alpha < \lambda$ such that f_α extends $\overline{c}_\alpha \mapsto \overline{d}_\alpha$.

For the proof of Theorem 6.12, we use the following infinitary game. Let A be a structure and $\mathcal{C} \subseteq \mathrm{Sub}^{<\kappa}(B)$ be cofinal in $([A]^{<\kappa}, \subseteq)$. For A', $A'' \in \mathrm{Sub}^{<\kappa}(A)$ such that $A' \cong A''$, and an isomorphism $f : A' \to A''$, $\mathcal{G}^{\mathcal{C}}_{A',A'',f}$ is the following game of length ω played by Player I and Player II. In a play in $\mathcal{G}^{\mathcal{C}}_{A',A'',f}$, Player I and Player II choose $X_n \in [A]^{<\kappa}$ and $C_n \in \mathcal{C}$ alternately for $n \in \omega$ such that

$$A' \cup A'' \subseteq X_0 \subseteq C_0 \subseteq X_1 \subseteq C_1 \subseteq \cdots$$

where X_n, $n \in \omega$ are Player I's moves and C_n, $n \in \omega$ are Player II's moves. Player II wins in the play if, for $C = \bigcup_{n\in\omega} C_n$, there is an automorphism $\tilde{f} : C \to C$ extending f. Note that, if $\mathcal{C}$ is closed under unions of countable chains, then we have $C \in \mathcal{C}$.

Lemma 6.13 *Suppose that A, $\mathcal{C}$, A', A'', f are as above and $f = \overline{a}' \mapsto \overline{a}''$ for some enumerations $\overline{a}'$, $\overline{a}''$ of A' and A''. If $(A, \overline{a}') \equiv_{L_{\infty\kappa}} (A, \overline{a}'')$, then Player II has a winning strategy in $G^{\mathcal{C}}_{A',A'',f}$.*

Proof. Assume that $(A, \overline{a}') \equiv_{L_{\infty\kappa}} (A, \overline{a}'')$. Then, by Theorem 6.2, there is a family $\mathcal{F}$ of partial isomorphisms from $(A, \overline{a})$ to $(B, \overline{b})$ with the back-and-forth property for $L_{\infty\kappa}$. Player II wins by choosing $f_n \in \mathcal{F}$ parallel to her moves C_n so that, for every $n \in \omega$,

(1) $f_n \subseteq f_{n+1}$;
(2) $X_n \subseteq \mathrm{dom}\,(f), \mathrm{rng}\,(f)$;
(3) $\mathrm{dom}\,(f_n) \cup \mathrm{rng}\,(f_n) \subseteq C_n$.

(1) and (2) are possible by the back-and-forth property of $\mathcal{F}$ while (3) is possible since $\mathcal{C}$ is cofinal in $([A]^{<\kappa}, \subseteq)$. Now, if Player II have taken her moves as described above, let $\tilde{f} = \bigcup f_n$. Then, by (1), (2), (3), $\mathrm{dom}\,(\tilde{f}) = \mathrm{rng}\,(\tilde{f}) = C$ where $C = \bigcup_{n\in\omega} C_n$. Since $\tilde{f}$ is a union of partial isomorphisms from $(A, \overline{a}')$ to $(A, \overline{a}'')$, it follows that $\tilde{f}$ is an automorphism on C extending $f = \overline{a}' \mapsto \overline{a}''$. Thus the strategy as above is actually a winning strategy for Player II. ❑ (Lemma 6.13)

Lemma 6.14 *Suppose that A is a structure and $\mathcal{C} \subseteq \mathrm{Sub}^{<\kappa}(A)$ is cofinal in $[A]^{<\kappa}$ and closed with respect to unions of increasing chains of countable length. Further, suppose that $\lambda < \mathrm{cf}(\kappa)$ and $C_\alpha \in \mathcal{C}$, $D_\alpha \in \mathrm{Sub}^{<\kappa}(A)$ for $\alpha < \lambda$ are such that $(A, C_\alpha) \equiv_{L_{\infty\kappa}} (A, D_\alpha)$ for all $\alpha < \lambda$. If $\overline{c}_\alpha$, $\overline{d}_\alpha \in {}^{\kappa>}A$ are enumerations of C_α and D_α respectively such that $(A, \overline{c}_\alpha) \equiv_{L_{\infty\kappa}} (A, \overline{d}_\alpha)$ for every $\alpha < \lambda$, then there is $C \in \mathcal{C}$ such that, for each α, there is an automorphism $\tilde{f}_\alpha$ on C extending $f_\alpha = \overline{c}_\alpha \mapsto \overline{d}_\alpha$.*

Proof. By Lemma 6.13, Player II has a winning strategy σ^α in $\mathcal{G}^{\mathcal{C}}_{C_\alpha,D_\alpha,f_\alpha}$ for every $\alpha < \lambda$. Let $(X^\alpha_n, C^\alpha_n)_{n\in\omega,\alpha<\lambda}$ be matches in $\mathcal{G}^{\mathcal{C}}_{C_\alpha,D_\alpha,f_\alpha}$, $\alpha < \lambda$ played simultaneously in which Player II applies σ^α, $\alpha < \lambda$. More exactly,

(1) for each $\alpha < \lambda$, $(X^\alpha_n, C^\alpha_n)_{n\in\omega}$ is a play in $\mathcal{G}^{\mathcal{C}}_{C_\alpha,D_\alpha,f_\alpha}$ in which Player II's moves C^α_n, $n \in \omega$ have been chosen according to σ^α.

Additionally, we also require that

(2) $X^\alpha_{n+1} = \bigcup_{\beta<\lambda} C^\beta_n$ for every $n \in \omega$ and $\alpha < \lambda$.

(2) is possible since, by $\lambda < \mathrm{cf}(\kappa)$, we have $|\bigcup_{\beta<\lambda} C^\beta_n| < \kappa$. Now, let $C^\alpha = \bigcup_{n\in\omega} C^\alpha_n$. Then, by (2), C^α, $\alpha < \lambda$ are all the same. Let $C = C^\alpha$ for all $\alpha < \lambda$. By the closedness of $\mathcal{C}$, we have $C \in \mathcal{C}$. By (1), $C = C^\alpha$ is the result of a play in $\mathcal{G}^{\mathcal{C}}_{C_\alpha,D_\alpha,f_\alpha}$ in which Player II wins. Hence, for each $\alpha < \lambda$, there is an automorphism $\tilde{f}_\alpha$ on C extending f_α. Thus this C is as desired. ❑ (Lemma 6.14)

Now, we are ready to prove Kueker's theorem:

Proof of Theorem 6.12: Since $\lambda < \operatorname{cf}\kappa$, we have $|\bigcup_{\alpha<\lambda} \operatorname{rng}(\overline{d}_\alpha)| < \kappa$. Hence by Lemma 6.1, there are $\overline{a}_\alpha \in {}^{\kappa>}A$, $\alpha < \lambda$ such that

$$(A, \overline{a}_\alpha)_{\alpha<\lambda} \equiv_{L_{\infty\kappa}} (B, \overline{d}_\alpha)_{\alpha<\lambda}.$$

For each $\alpha < \lambda$, we have $(A, \overline{c}_\alpha) \equiv_{L_{\infty\kappa}} (A, \overline{a}_\alpha)$. Hence, by Lemma 6.14, there is $C \in \mathcal{C}$ such that, for each $\alpha < \lambda$, there is an automorphism g_α on C extending $\overline{c}_\alpha \mapsto \overline{a}_\alpha$. Let $\overline{c}$ be an enumeration of C. Then, by Lemma 6.1, there is $\overline{d} \in {}^{\kappa>}B$ such that

$$(A, \overline{a}_\alpha, \overline{c})_{\alpha<\lambda} \equiv_{L_{\infty\kappa}} (B, \overline{d}_\alpha, \overline{d})_{\alpha<\lambda}.$$

Let D be the substructure of B with the underlying set $\overline{d}$. Then $h = \overline{c} \mapsto \overline{d}$ is an isomorphism from C to D extending each $\overline{a}_\alpha \mapsto \overline{d}_\alpha$, $\alpha < \lambda$. For $\alpha < \lambda$, let $f_\alpha = h \circ g_\alpha$. f_α is an isomorphism from C to D. We have

$$f_\alpha \restriction C_\alpha = (\overline{a}_\alpha \mapsto \overline{d}_\alpha) \circ (\overline{c}_\alpha \mapsto \overline{a}_\alpha) = \overline{c}_\alpha \mapsto \overline{d}_\alpha.$$

Thus C, D, f_α, $\alpha < \lambda$ are as desired.

❑ (Theorem 6.12)

As corollaries of Theorem 6.12 we obtain the following characterizations of $L_{\infty\kappa}$-free/projective algebras which we already mentioned in Chapter 2.

Proposition 6.15 (Kueker [34] see also Eklof [6]) *For an algebra B the following are equivalent:*

a) *B is $L_{\infty\kappa}$-free;*

b) *there is $\mathcal{D} \subseteq \operatorname{Sub}^{<\kappa}(B)$ such that*

i) *every $F \in \mathcal{D}$ is free,*
ii) *$\mathcal{D}$ is cofinal in $([B]^{<\kappa}, \subseteq)$,*
iii) *for any $\mathcal{X} \in [\mathcal{D}]^{<\operatorname{cf}(\kappa)}$ there is $F \in \mathcal{D}$ such that $A \leq_{\text{free}} F$ for every $A \in \mathcal{X}$;*

c) *there is $\mathcal{D} \subseteq \operatorname{Sub}^{<\kappa}(B)$ such that i) and ii) as in b) and*

iii′) *for any $\mathcal{X} \in [\mathcal{D}]^{<\aleph_0}$ there is $F \in \mathcal{D}$ such that $A \leq_{\text{free}} F$ for every $A \in \mathcal{X}$;*

d) *there is $\mathcal{D} \subseteq \operatorname{Sub}^{<\kappa}(B)$ such that i) as in b) and*

ii′) *for any $A \in \mathcal{D}$, $\{ D \in \mathcal{D} : A \leq_{\text{free}} D \}$ is cofinal in $([B]^{<\kappa}, \subseteq)$.*

Proof. *a)* ⇒ *b)*: Let

$$\mathcal{C} = \{ C \in \operatorname{Sub}^{<\kappa}(\operatorname{Fr}\kappa) : C = \operatorname{Fr} X \text{ for some } X \in [\kappa]^{<\kappa} \}.$$

Note that for C, $C' \in \mathcal{C}$, if $C \leq C'$, then we have $C \leq_{\mathrm{free}} D$. Let

$$\mathcal{D} = \{\, D \in \mathrm{Sub}^{<\kappa}(B) \,:\, (B, D) \equiv_{L_{\infty\kappa}} (\mathrm{Fr}\,\kappa, C) \text{ for some } C \in \mathcal{C} \,\}.$$

Clearly, $\mathcal{D}$ satisfies *i*) and *ii*). By Theorem 6.12 and the remark above, $\mathcal{D}$ also satisfies *iii*).

b) $\Rightarrow$ *c*): Trivial.

c) $\Rightarrow$ *d*): Let $\mathcal{D}$ be as in *c*). Then it is easy to see that $\mathcal{D}$ satisfies *ii'*).

d) $\Rightarrow$ *a*): Let $\mathcal{C}$ be as above and $\mathcal{D}$ be as in *d*). Let

$$\mathcal{F} = \{\, f \,:\, f \subseteq g \text{ for some } C \in \mathcal{C},\ D \in \mathcal{D} \text{ and } g : C \overset{\cong}{\rightarrow} D \,\}.$$

Claim 6.15.1 *$\mathcal{F}$ is a family of partial isomorphisms from* $\mathrm{Fr}\,\kappa$ *to B with the back-and-forth property for $L_{\infty\kappa}$.*

$\vdash$ It is clear that elements of $\mathcal{F}$ are partial isomorphisms from $\mathrm{Fr}\,\kappa$ to B. So it is enough to show that $\mathcal{F}$ satisfies *0*) – *iii*) of Theorem 6.2, *b*). *0*) and *iii*) are clear from the definition of $\mathcal{F}$. For *i*), let $g \in \mathcal{F}$, say $g \subseteq$ for some $f : C \overset{\cong}{\rightarrow} D$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Let $X \in [\kappa]^{<\kappa}$ be such that $C = \mathrm{Fr}\,X$. For any $\bar{b} \in {}^{\kappa>}(\mathrm{Fr}\,\kappa)$, there is $X' \in [\kappa]^{<\kappa}$ such that $X \subseteq X'$ and $\bar{b} \subseteq \mathrm{Fr}\,X'$. By *ii'*), there is $D' \in \mathcal{D}$ such that $D \leq_{\mathrm{free}} D'$ and $D' = D[Y]$ for some $Y \subseteq D'$ independent over D such that $|Y| \geq |X' \setminus X|$. Let $X'' \in [\kappa]^{<\kappa}$ be such that $X' \subseteq X''$ and $|Y| = |X'' \setminus X|$. Let $f' : X'' \setminus X \rightarrow Y$ be a bijection. Then $f \cup f''$ can be extended to an isomorphism $h : C' \overset{\cong}{\rightarrow} D'$. Let $g' = h \restriction (\mathrm{dom}\,(g) \cup \bar{b})$. Then $g \subseteq g'$ and $g' \in \mathcal{F}$. *ii*) can be shown similarly. $\dashv$ (Claim 6.15.1)

By Theorem 6.2, it follows that B is $L_{\infty\kappa}$-free. ❑ (Proposition 6.15)

Corollary 6.16 *For a regular uncountable κ and an algebra B of cardinality κ the following are equivalent:*

a) *B is $L_{\infty\kappa}$-free;*

b) *B is the union of an increasing chain $(B_\alpha)_{\alpha<\kappa}$ of elements of* $\mathrm{Sub}^{<\kappa}(B)$ *such that B_α is free and $B_\alpha \leq_{\mathrm{free}} B_\beta$ for every $\alpha < \beta < \kappa$.*

Proof. *a*) $\Rightarrow$ *b*): If B is $L_{\infty\kappa}$-free and of cardinality κ, let $\mathcal{D} \subseteq \mathrm{Sub}^{<\kappa}(B)$ be as in Proposition 6.15, *c*). Then we can construct an increasing sequence $(B_\alpha)_{\alpha<\kappa}$ as in *b*) above such that each B_α is an element of $\mathcal{D}$.

b) $\Rightarrow$ *a*): If $(B_\alpha)_{\alpha<\kappa}$ is as in *b*), then $\mathcal{D} = \{\, B_\alpha \,:\, \alpha < \kappa \,\}$ satisfies the conditions *i*) – *iii*) in Proposition 6.15, *b*). ❑ (Corollary 6.16)

Corollary 6.17

1) *Suppose that B is an $L_{\infty\kappa}$-free algebra. If χ is sufficiently large and M is an elementary submodel of $\mathcal{H}(\chi)$ such that $B \in M$, $\kappa \subseteq M$ and* $\mathrm{Sub}^{<\kappa}(B) \cap M$ *is*

cofinal[9] *in* $\mathrm{Sub}^{<\kappa}(B\cap M)$*, then* $B\cap M$ *is* $L_{\infty\kappa}$*-free.*

2) Suppose that κ *is regular uncountable and* B *is* $L_{\infty\kappa}$*-free. Let* χ *be sufficiently large. Then for any internally approachable* $M\prec\mathcal{H}(\chi)$ *of cardinality* κ *such that* $B,\kappa\in M$*,* $B\cap M$ *is* $L_{\infty\kappa}$*-free.*

Proof. *1*): Let $\mathcal{D}\in M$ be a subset of $\mathrm{Sub}^{<\kappa}(B)$ with i) and ii') in Proposition 6.15, d). Then $\mathcal{D}\cap M$ is a subset of $\mathrm{Sub}^{<\kappa}(B\cap M)$ satisfying i) and ii') in Proposition 6.15, d). It follows, by Proposition 6.15, that $B\cap M$ is $L_{\infty\kappa}$-free.

2): Let $(M_\alpha)_{\alpha<\kappa}$ be a witness of internally approachability of M. Without loss of generality we may assume that $B,\kappa\in M_0$. Then there is $\mathcal{D}\in M_0$ as in Proposition 6.15, b). Now, we can construct a sequence $(B_\alpha)_{\alpha<\kappa}$ inductively such that

(1) B_α *is minimal*[10] *such that* $B_\alpha\in M_{\alpha+1}$*,* $B_\alpha\in\mathcal{D}$*,* $B_\beta\le_{\mathrm{free}}B_\alpha$ *for every* $\beta<\alpha$ *and* $B\cap M_\alpha\le B_\alpha$*.*

We can carry out this since, if B_β, $\beta<\alpha$ have been chosen according to (1), then $(B_\beta)_{\beta<\alpha}$ is definable in $M_{\alpha+1}$ as we have $(M_\beta)_{\beta<\alpha}\in M_{\alpha+1}$. Hence we have

(2) $(B_\beta)_{\beta<\alpha}\in M_{\alpha+1}$.

For $\alpha<\kappa$, let $C_\alpha=B_\alpha\cap M_\alpha$. Then C_α is free and $C_\alpha\le_{\mathrm{free}}C_\beta$ for all $\alpha<\beta<\kappa$ by Theorem 4.6, *1*). Also $\bigcup_{\alpha<\kappa}C_\alpha=\bigcup_{\alpha<\kappa}B_\alpha\cap M_\alpha\supseteq\bigcup_{\alpha<\kappa}B\cap M_\alpha=B\cap M$. Hence, by Corollary 6.16, $B\cap M$ is $L_{\infty\kappa}$-free. ❑ (Corollary 6.17)

Proposition 6.18 *For an algebra* B*, we have the implication* $a)\Rightarrow b)\Rightarrow c)\Rightarrow d)\Rightarrow e)$ *for the following assertions.*

a) B *is* $L_{\infty\kappa}$*-projective.*

b) There is $\mathcal{D}\subseteq\mathrm{Sub}^{<\kappa}(B)$ *such that*

i) every $D\in\mathcal{D}$ *is projective,*

ii) $\mathcal{D}$ *is cofinal in* $([B]^{<\kappa},\subseteq)$*,*

iii) for any $\mathcal{X}\in[\mathcal{D}]^{<\mathrm{cf}(\kappa)}$ *there is* $D\in\mathcal{D}$ *such that* $A\le_{\mathrm{proj}}D$ *for every* $A\in\mathcal{X}$*.*

c) There is $\mathcal{D}\subseteq\mathrm{Sub}^{<\kappa}(B)$ *such that i) and ii) as in b) and*

iii') for any $\mathcal{X}\in[\mathcal{D}]^{<\aleph_0}$ *there is* $F\in\mathcal{D}$ *such that* $A\le_{\mathrm{proj}}F$ *for every* $A\in\mathcal{X}$*.*

d) There is $\mathcal{D}\subseteq\mathrm{Sub}^{<\kappa}(B)$ *such that i) as in b) and*

ii') for any $A\in\mathcal{D}$*,* $\{\,D\in\mathcal{D}\,:\,A\le_{\mathrm{proj}}D\,\}$ *is cofinal in* $([B]^{<\kappa},\subseteq)$*.*

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[9] note that, by $\kappa\subseteq M$, we have $\mathrm{Sub}^{<\kappa}(B)\cap M\subseteq\mathrm{Sub}^{<\kappa}(B\cap M)$.

[10] with respect to $\le^*$. Remember that we regard $\mathcal{H}(\chi)$ as the structure $(\mathcal{H}(\chi),\in,\le^*)$ for some fixed well-ordering on $\mathcal{H}(\chi)$.

e) $B \oplus \mathrm{Fr}\,\kappa$ *is* $L_{\infty\kappa}$*-free.*

Proof. *a)* ⇒ *b)*: Suppose that $B \equiv_{L_{\infty\kappa}} P$ for some projective algebra P. Let χ be sufficiently large and

$$\mathcal{C} = \{ C \in \mathrm{Sub}^{<\kappa}(P) : C = P \cap M \text{ for some } M \prec \mathcal{H}(\chi) \text{ such that } |M| < \kappa,\ B \in M \}.$$

Then $\mathcal{C}$ is club in $[P]^{<\kappa}$ (so, in particular, cofinal in $[P]^{<\kappa}$ and closed with respect to unions of increasing sequences of countable length). By Theorem 4.6, *2)*, $C \leq_{\mathrm{proj}} C'$ holds for every C, $C' \in \mathcal{C}$ such that $C \leq C'$. Hence, as in the corresponding proof of Proposition 6.15, applying Theorem 6.12, we can show that

$$\mathcal{D} = \{ D \in [B]^{<\kappa} : (A, C) \equiv_{L_{\infty\kappa}} (B, D) \text{ for some } C \in \mathcal{C} \}$$

satisfies the conditions *i)*,*ii)*,*iii)* in *b)*.

The implications *b)* ⇒ *c)* ⇒ *d)* are trivial.

d) ⇒ *e)*: Let $\mathcal{D} \subseteq \mathrm{Sub}^{\kappa}(B)$ be as in *d)* and let

$$\mathcal{D}' = \{ D' \in \mathrm{Sub}^{<\kappa}(B \oplus \mathrm{Fr}\,\kappa) : D' = D \oplus \mathrm{Fr}\,X \text{ for some } D \in \mathcal{D},\ X \in [\kappa]^{<\kappa} \text{ such that } |X| \geq |D| \}.$$

Then $\mathcal{D}'$ satisfies *i)*, *ii')* in Proposition 6.15. *i)* is clear as every $D \in \mathcal{D}$ is projective. To see that $\mathcal{D}'$ satisfies *ii')*, let $D' \in \mathcal{D}'$ be such that $D' = D \oplus \mathrm{Fr}\,X$ for some $D \in \mathcal{D}$ and $X \in [\kappa]^{<\kappa}$. For arbitrary $Y \in [B \oplus \mathrm{Fr}\,\kappa]^{<\kappa}$, there is $D^* \in \mathcal{D}$ and $X^* \in [\kappa]^{<\kappa}$ such that $D \leq_{\mathrm{proj}} D^*$, $X \cap X^* = \emptyset$, $Y \subseteq D^* \oplus \mathrm{Fr}\,(X \cup X^*)$ and $|X^*| \geq |D^* \oplus \mathrm{Fr}\,X|$. Let $D'' = D^* \oplus \mathrm{Fr}\,(X \cup X^*)$. Then $D'' \in \mathcal{D}'$. By Lemma 3.3, *4)* (which clearly also holds in an arbitrary variety $\mathcal{V}$), we have $D' \leq_{\mathrm{proj}} \oplus \mathrm{Fr}\,X$. Hence

$$D' \leq_{\mathrm{free}} (D^* \oplus \mathrm{Fr}\,X) \oplus \mathrm{Fr}\,X^* = D^* \oplus \mathrm{Fr}\,(X \cup X^*) = D''.$$

By Proposition 6.15, it follows that $B \oplus \mathrm{Fr}\,\kappa$ is $L_{\infty\kappa}$-free. ❑ (Proposition 6.18)

6.3 κ^+-free algebras are $L_{\infty\kappa}$-free

We shall prove the assertion above (Theorem 6.20). First, let us introduce another characterization of $L_{\infty\kappa}$-free algebras using an infinitary game. For an algebra B, a *κ-Shelah game* $\mathcal{G}^{\kappa}_{\mathrm{Shelah}}(B)$ is a game of length ω played by Player I and Player II. In a play of $\mathcal{G}^{\kappa}_{\mathrm{Shelah}}(B)$, Player I and II choose $B_n \in \mathrm{Sub}^{<\kappa}(B)$ alternately where B_n for even n (odd n resp.) are moves by Player I (Player II resp.), such that $B_n \leq B_{n+1}$ for every $n \in \omega$. Player II wins if her moves B_{2k+1}, $k \in \omega$ are all free and $B_{2k+1} \leq_{\mathrm{free}} B_{2k+3}$ for every $k \in \omega$.

Lemma 6.19

1) *Player II has a winning strategy in* $\mathcal{G}^{\kappa}_{\mathrm{Shelah}}(B)$ *if and only if* B *is* $L_{\infty\kappa}$*-free.*

2) *For any algebra* B*,* $\mathcal{G}^{\kappa}_{\mathrm{Shelah}}(B)$ *is determined, i.e. one of Players I, II has a winning strategy.*

Proof. *1*): Suppose that σ is a winning strategy for Player II in $\mathcal{G}^{\kappa}_{\mathrm{Shelah}}(B)$. Let

$$\mathcal{D} = \{\, C \in \mathrm{Sub}^{<\kappa}(B) : C \text{ appears as a move of Player II in a play } P \text{ in } \mathcal{G}^{\kappa}_{\mathrm{Shelah}}(B) \text{ such that Player II has applied always } \sigma \text{ for her moves in } P\,\}.$$

Then it is easy to see that $\mathcal{D}$ satisfies the condition *i*), *ii′*) in Proposition 6.15. Conversely, suppose that $\mathcal{D} \subseteq \mathrm{Sub}^{<\kappa}(B)$ satisfies *i*) and *ii′*) in Proposition 6.15. Then Player II wins if she chooses her move $B_{2k+1} \in \mathcal{D}$ such that $B_{2k+1} \leq_{\mathrm{free}} B_{2k+3}$. By *ii′*), this is possible and, by *i*), each B_{2k+1} is free.

2): Suppose that Player I does not have any winning strategy. We show that then the Player II can win. Let us call a partial play $(B_n)_{n<k}$ non-determined for Player I, if he does not have any winning strategy in the play continuing $(B_n)_{n<k}$. Otherwise we say that $(B_n)_{n<k}$ is determined for Player I. Player II can play so that at every $k \in \omega$ the partial play $(B_n)_{n<k}$ is non-determined for Player I: At the first move B_0 by Player I, (B_0) is non-determined since otherwise Player I would have a winning strategy in $\mathcal{G}^{\kappa}_{\mathrm{Shelah}}(B)$. Hence there is B_1 such that $(B_n)_{n<2}$ remains non-determined for Player I. In particular such B_1 is free. Then, by any move B_2 by Player I, $(B_n)_{n<3}$ remains non-determined for Player I since otherwise $(B_n)_{n<2}$ would have been determined for Player I. Now, since $(B_n)_{n<3}$ is non-determined for Player I, there is B_3 such that $(B_n)_{n<4}$ remains non-determined for Player I, in particular such B_3 is free and $B_1 \leq_{\mathrm{free}} B_3$, etc., etc. Clearly Player II wins in this play. ❑ (Lemma 6.19)

The following theorem was stated implicitly in Shelah [40]. The idea of our proof of Theorem 6.20, *1*) is taken from the proof of Theorem 1.3 in Hodges [26].

Theorem 6.20

1) *If* B *is* κ^+*-free then* B *is* $L_{\infty\kappa}$*-free.*

2) *If* B *is* κ^+*-projective then* $B \oplus \mathrm{Fr}\,\kappa$ *is* $L_{\infty\kappa}$*-free.*

Proof. *1*): By Lemma 6.19, *1*), it is enough to show that Player II has a winning strategy in $\mathcal{G}^{\kappa}_{\mathrm{Shelah}}(B)$. By Lemma 6.19, *2*) in turn, it is enough for this to show that Player I does not have any winning strategy in $\mathcal{G}^{\kappa}_{\mathrm{Shelah}}(B)$. Let σ be any strategy for Player I. We show that σ is not a winning strategy for Player I. Since B is κ^+-free, there is a club $\mathcal{C} \subseteq \mathrm{Sub}^{\kappa}(B)$ such that every $C \in \mathcal{C}$ is a free subalgebra of B. Let χ be sufficiently large and M a V_κ-like elementary submodel of $\mathcal{H}(\chi)$ with a continuously increasing witness $(B_\alpha)_{\alpha<\kappa}$ of V_κ-likeness of M such that B, σ, $\mathcal{C}$, $\kappa \in M_0$.There is such M by the proof of Lemma 5.10, *2*). As $\kappa+1 \subseteq M$ by Lemma

5.10, *1*), we have $C \subseteq M$ for all $C \in \mathcal{C} \cap M$. Since $\mathcal{C} \cap M$ is upward directed and of cardinality κ, it follows that $M \cap B = \bigcup(\mathcal{C} \cap M) \in \mathcal{C}$. Let $B^* = B \cap M$. Since $B^* \in \mathcal{C}$, B^* is free. Let X be a free generator of B^*. Now, let $B^\alpha = B \cap M_\alpha$ for $\alpha < \kappa$. Then $B^\alpha \in M$ for every $\alpha < \kappa$ and $(B^\alpha)_{\alpha<\kappa}$ is continuously increasing and $\bigcup_{\alpha<\kappa} B^\alpha = B^*$. Hence

$$\mathcal{D} = \{\, \alpha < \kappa \,:\, B^\alpha = \langle B^\alpha \cap X \rangle_{B^*} \,\}$$

is club in κ. For all $\alpha \in \mathcal{D}$, B^α is free and, for $\alpha, \beta \in \mathcal{D}$ with $\alpha < \beta$, we have $B^\alpha \leq_{\text{free}} B^\beta$. If $(B_0, \ldots, B_{n-1})$ is a partial play in $\mathcal{G}^\kappa_{\text{Shelah}}(B)$ such that $(B_0, \ldots, B_{n-1}) \in M_\alpha$ for some $\alpha < \kappa$ then, since $\sigma \in M_\alpha$, we have $\sigma((B_0, \ldots, B_{n-1})) \in M_\alpha$ and $\sigma((B_0, \ldots, B_{n-1})) \subseteq B^\alpha$. Thus in a play of $\mathcal{G}^\kappa_{\text{Shelah}}(B)$ in which Player I chooses his moves according to σ, Player II can always take her move as one of B^α, $\alpha \in \mathcal{D}$ and win. This shows that σ is not a winning strategy for Player I.

2): If B is κ^+-projective, then $B \oplus \kappa^+$ is κ^+-free by Lemma 8.3. Hence, by *1*), $B \oplus \mathrm{Fr}\,\kappa^+$ is $L_{\infty\kappa}$-free. by Lemma 6.4 and Lemma 6.5, we have $B \oplus \mathrm{Fr}\,\kappa^+ \equiv_{L_{\infty\kappa}} B \oplus \mathrm{Fr}\,\kappa$. Hence $B \oplus \mathrm{Fr}\,\kappa$ is $L_{\infty\kappa}$-free. ❑ (Theorem 6.20)

Recall that a cardinal κ is said to be *weakly Mahlo* if $\{\, \lambda < \kappa \,:\, \lambda$ is weakly inaccessible $\}$ is stationary in κ.

Theorem 6.21 *Suppose that κ is weakly Mahlo. Then*

1) *an algebra B is $L_{\infty\kappa}$-free if and only if B is $L_{\infty\lambda}$-free for every $\lambda < \kappa$;*

2) *if $\{\, \lambda < \kappa \,:\, B$ is λ^+-free$\,\}$ is unbounded in κ then B is $L_{\infty\kappa}$-free.*

Proof. *1*): If B is $L_{\infty\kappa}$-free, then B is clearly $L_{\infty\lambda}$-free for every $\lambda < \kappa$. Conversely, suppose that B is $L_{\infty\lambda}$-free for every $\lambda < \kappa$. Toward a contradiction, let us assume that B is not $L_{\infty\kappa}$-free. Then, by Lemma 6.19, Player I has a winning strategy σ in $\mathcal{G}^\kappa_{\text{Shelah}}(B)$. Let χ be sufficiently large and $(M_\alpha)_{\alpha<\kappa}$ be a continuously increasing sequence of elementary submodels of $\mathcal{H}(\chi)$ such that

(0) $B, \sigma \in M_0$;

(1) $|M_\alpha| < \kappa$ and $|M_\alpha| < |M_{\alpha+1}|$ for every $\alpha < \kappa$;

(2) $(M_\beta)_{\beta\leq\alpha} \in M_{\alpha+1}$ for every $\alpha < \kappa$;

(3) $\bigcup\{\, C \,:\, C \in \mathrm{Sub}^{<\kappa}(B) \cap M_\alpha \,\} \subseteq M_{\alpha+1}$ for every $\alpha < \kappa$;

Since κ is weakly Mahlo, there is a weakly inaccessible $\lambda < \kappa$ such that $|M_\lambda| = \lambda$. By (1) and (2), M_λ is internally approachable. Hence, by Corollary 6.17, *2*), $B \cap M_\lambda$ is $L_{\infty\lambda}$-free. More over by the proof of Corollary 6.17, there is an increasing sequence $(C_\alpha)_{\alpha<\lambda}$ of elements of $\mathrm{Sub}^{<\lambda}(B \cap M_\lambda)$ such that C_α is free and $C_\alpha \leq_{\text{free}} C_\beta$ for $\alpha < \beta < \lambda$, $B \cap M_\lambda = \bigcup_{\alpha<\lambda} C_\lambda$ and, for each $\alpha < \kappa$, $(C_\beta)_{\beta<\alpha} \in M_\lambda$. Now, by (0), there is a play P in $\mathcal{G}^\lambda_{\text{Shelah}}(B \cap M_\lambda)$ such that Player I plays always according to σ and Player II takes C_α for least possible α at each of her moves. Clearly Player II wins in P. Since P is definable we have $P \in M$. Hence

$$M_\lambda \models \text{“}\sigma \text{ is not a winning strategy for Player I”}.$$

This is a contradiction since $M \prec \mathcal{H}(\chi)$.

2): Suppose that B is λ^+-free for unbounded many $\lambda < \kappa$. Then, by Theorem 6.20, B is $L_{\infty\lambda}$-free for unbounded many and hence for every $\lambda < \kappa$. By *1*), it follows that B is $L_{\infty\kappa}$-free. ❑ (Theorem 6.21)

6.4 Shelah's Singular Compactness Theorem

In the rest of this chapter, we prove Shelah's Singular Compactness Theorem in its special case for algebras in a variety $\mathcal{V}$ (Theorem 6.23). Our proof is essentially the same as that presented in Hodges [26]. We shall separate the set-theoretic core of the proof as the following lemma.

Lemma 6.22 *Let B a structure of singular cardinality, say $|B| = \lambda$ with $\kappa = \mathrm{cf}\,\lambda < \lambda$. Suppose that $(\kappa_\alpha)_{\alpha<\kappa}$ is a continuously increasing sequence of cardinals below λ such that $\kappa_0 \geq \kappa$ and $\lambda = \bigcup_{\alpha<\kappa} \kappa_\alpha$. For a sequence $(X_\alpha)_{\alpha<\kappa}$ of subsets of B such that $|X_\alpha| \leq \kappa_\alpha$ for every $\alpha < \kappa$, there is a continuously increasing sequence $(B_\alpha)_{\alpha<\kappa}$ of substructures of B such that $X_\alpha \subseteq B_\alpha$ and $|B_\alpha| = \kappa_\alpha$ for all $\alpha < \kappa$.*

Proof. By induction on $n \in \omega$, we define increasing sequences $(B^n_\alpha)_{\alpha<\kappa}$, $n \in \omega$ of substructures of B such that

(1) $|B^n_\alpha| = \kappa_\alpha$ for all $\alpha < \kappa$.

For $n = 0$, let $(B^0_\alpha)_{\alpha<\kappa}$ be any increasing sequence of substructures such that (1) and

(2) $X_\alpha \subseteq B^0_\alpha$ for all $\alpha < \kappa$.

If $(B^n_\alpha)_{\alpha<\kappa}$ has been chosen, let us fix an enumeration $\{\, b^n_{\alpha,\beta} : \beta < \kappa_\alpha \,\}$ of B^n_α for each $\alpha < \kappa$. For $\alpha < \kappa$, let

$$X^n_\alpha = \{\, b^n_{\alpha',\beta} : \alpha' < \kappa,\ \beta < \kappa_\alpha \,\}.$$

By $\kappa \leq \kappa_\alpha$, we have $|X^n_\alpha| = \kappa_\alpha$. We take an increasing sequence $(B^{n+1}_\alpha)_{\alpha<\kappa}$ of substructures of B so that (1) and

(3) $B^{n+1}_\alpha \supseteq X^n_\alpha$ for all $\alpha < \kappa$.

For $\alpha < \kappa$, let $B_\alpha = \bigcup_{n\in\omega} B^n_\alpha$.

Claim 6.22.1 *$(B_\alpha)_{\alpha<\kappa}$ is as desired.*

$\vdash$ By (2), we have $X_\alpha \subseteq B_\alpha$ and, by (1), $|B_\alpha| = \kappa_\alpha$ for every $\alpha < \kappa$. Hence it is enough to show that $(B_\alpha)_{\alpha<\kappa}$ is continuously increasing. Since each $(B^n_\alpha)_{\alpha<\kappa}$ is increasing, $(B_\alpha)_{\alpha<\kappa}$ is also increasing. Let $\gamma < \kappa$ be a limit. We show that $\bigcup_{\alpha<\gamma} B_\alpha = B_\gamma$. "$\subseteq$" is clear. For "$\supseteq$", let $b \in B_\gamma$. Then there is $n \in \omega$ such that $b \in B^n_\gamma$. Hence $b = b^n_{\gamma,\beta}$ for some $\beta < \kappa_\gamma$. Since $(\kappa_\alpha)_{\alpha<\kappa}$ was continuously increasing, there is some $\alpha^* < \gamma$ such that $\beta < \kappa_{\alpha^*}$. Hence, by (3), we have

$$b = b^n_{\gamma,\beta} \in \{\, b^n_{\alpha',\beta} : \alpha' < \kappa,\, \beta < \kappa_{\alpha^*} \,\} \subseteq B^{n+1}_{\alpha^*}.$$

$\dashv$ (Claim 6.22.1)
❑ (Lemma 6.22)

The assertion for a regular cardinal corresponding to Lemma 6.22 does not hold. For a regular κ, let $X_\alpha = \alpha + 1$, $\alpha < \kappa$. Then for any continuously increasing sequence $(B_\alpha)_{\alpha<\kappa}$ of subsets of κ of size $< \kappa$, if $\bigcup_\alpha B_\alpha = \kappa$, then there are club many $\alpha < \kappa$ such that $B_\alpha = \alpha$. In particular $X_\alpha \not\subseteq B_\alpha$ for such an α.

Theorem 6.23 (Singular Compactness Theorem, Shelah [40], see also Hodges [26]) *Suppose that λ is singular. Then for an algebra B of cardinality λ, the following are equivalent:*

a) *B is free;*

b) *B is $L_{\infty\lambda}$-free;*

c) *B is $L_{\infty\mu}$-free for all $\mu < \lambda$;*

d) *B is μ^+-free for unboundedly many $\mu < \lambda$.*

Proof. *a)* $\Rightarrow$ *b)* $\Rightarrow$ *c)* is trivial.

c) $\Rightarrow$ *a)*: Let B be an algebra of cardinality λ such that B is $L_{\infty\mu}$-free for all $\mu < \lambda$. Let $\kappa = \operatorname{cf}\lambda$ and let $(\kappa_\alpha)_{\alpha<\kappa}$ be a continuously and strictly increasing sequence of cardinals below λ such that $\kappa_0 \geq \kappa$ and $\sup_{\alpha<\kappa} \kappa_\alpha = \lambda$. For each $\alpha < \kappa$, Player II has a winning strategy σ^α in $\mathcal{G}^{\kappa_\alpha}_{\text{Shelah}}(B)$ by Lemma 6.19, *1*).

By induction on $n \in \omega$, we construct sequences $(C^n_\alpha)_{\alpha<\kappa}$, $(B^n_\alpha)_{\alpha<\kappa}$, $(A^n_\alpha)_{\alpha<\kappa}$ of elements of $\mathrm{Sub}^{<\lambda}(B)$ and sequences $(X^n_\alpha)_{\alpha<\kappa}$ of subsets of B for $n \in \omega$ such that

(0) $|C^n_\alpha| = |B^n_\alpha| = |A^n_\alpha| = \kappa_\alpha$ for $\alpha < \kappa$ and $n \in \omega$;

(1) $(C^n_\alpha)_{\alpha<\kappa}$ is continuously increasing and $\bigcup_{\alpha<\kappa} C^n_\alpha = B$;

(2) $C^n_\alpha \leq B^n_\alpha \leq A^n_\alpha \leq C^{n+1}_\alpha$ for all $n \in \omega$;

(3) for every $\alpha < \kappa$, $(B^0_\alpha, A^0_\alpha, B^1_\alpha, A^1_\alpha, \ldots)$ is a play of $\mathcal{G}^{\kappa_\alpha}_{\text{Shelah}}(B)$ in which Player II applied σ^α to choose her moves A^n_α, $n \in \omega$;

Note that the choice of $(A^n_\alpha)_{\alpha<\kappa}$ for $n \in \omega$ is already fixed by (3). Also, by (3), each A^n_α is free and we have

$$A^0_\alpha \leq_{\text{free}} A^1_\alpha \leq_{\text{free}} A^2_\alpha \leq_{\text{free}} \cdots$$

for each $\alpha < \kappa$. Hence we can choose X^n_α, $n \in \omega$, $\alpha < \kappa$ such that

(4) for $n \in \omega$ and $\alpha < \kappa$, X^n_α is a free generator of A^n_α and $(X^n_\alpha)_{n\in\omega}$ is an increasing sequence for every $\alpha < \kappa$.

Let $(C^0_\alpha)_{\alpha<\kappa}$ be any sequence of elements of $\mathrm{Sub}^{<\lambda}(B)$ with (1) and $B^0_\alpha = C^0_\alpha$ for all $\alpha < \kappa$.

Suppose that $(C^m_\alpha)_{\alpha<\kappa}$, $(B^m_\alpha)_{\alpha<\kappa}$, $(A^m_\alpha)_{\alpha<\kappa}$ and $(X^m_\alpha)_{\alpha<\kappa}$ for $m \leq n$ have been chosen. By Lemma 6.22, there is $(C^{n+1}_\alpha)_{\alpha<\kappa}$ such that (1) and $A^n_\alpha \leq C^{n+1}_\alpha$ for every $\alpha < \kappa$.

Suppose now that $(C^m_\alpha)_{\alpha<\kappa}$, $(B^m_\alpha)_{\alpha<\kappa}$, $(A^m_\alpha)_{\alpha<\kappa}$, $(X^m_\alpha)_{\alpha<\kappa}$ for $m \leq n$ and $(C^{n+1}_\alpha)_{\alpha<\kappa}$ have been chosen. Then we choose $(B^{n+1}_\alpha)_{\alpha<\kappa}$ so that

(5) $A^n_{\alpha+1} \cap B^{n+1}_\alpha = \langle X^n_{\alpha+1} \cap B^{n+1}_\alpha \rangle_B$ and

(6) $B^{n+1}_{\alpha+1} \supseteq A^n_\alpha$

for every $\alpha < \kappa$.

For $\alpha < \kappa$, let $B_\alpha = \bigcup_{n\in\omega} A^n_\alpha$. By (2), we have $B_\alpha = \bigcup_{n\in\omega} B^n_\alpha = \bigcup_{n\in\omega} C^n_\alpha$. By (1), $(B_\alpha)_{\alpha<\kappa}$ is continuously increasing and $\bigcup_{\alpha<\kappa} B_\alpha = B$. By (3) and (4), B_α is a free algebra with free generator $\bigcup_{n\in\omega} X^n_\alpha$ for each $\alpha < \kappa$. By (6) and (2), we have $A^{n+1}_{\alpha+1} \supseteq A^n_\alpha$ for every $\alpha < \kappa$. By this and (5), we have

$$\begin{aligned} B_\alpha &= \textstyle\bigcup_{n\in\omega} A^n_\alpha = \bigcup_{n\in\omega}(A^{n+1}_{\alpha+1} \cap A^n_\alpha) \\ &= \textstyle\bigcup_{n\in\omega}(A^{n+1}_{\alpha+1} \cap B^{n+2}_\alpha) = \bigcup_{n\in\omega}(A^n_{\alpha+1} \cap B^{n+1}_\alpha) \\ &= \textstyle\bigcup_{n\in\omega} \langle X^n_{\alpha+1} \cap B^{n+1}_\alpha \rangle_B \\ &= \langle X_{\alpha+1} \cap B_\alpha \rangle_B. \end{aligned}$$

Thus $B_\alpha \leq_{\text{free}} B_{\alpha+1}$ for every $\alpha < \kappa$. By Lemma 3.6 and Lemma 3.7, *1*), it follows that B is free.

a) ⇒ *d*) is trivial.

d) ⇒ *c*): For $\mu < \lambda$ such that B is μ^+-free, B is $L_{\infty\mu}$-free by Theorem 6.20, *1*). Hence *c*) follows, if $\{\, \mu < \lambda : B \text{ is } \mu^+\text{-free} \,\}$ is unbounded below λ.

❑ (Theorem 6.23)

Corollary 6.24 *Suppose that λ is a singular cardinal. If B is an $L_{\infty\lambda}$-projective algebra of cardinality λ, then B is projective.*

Proof. By Proposition 6.18, $B \oplus \mathrm{Fr}\,\lambda$ is $L_{\infty\lambda}$-free. Clearly $|B \oplus \mathrm{Fr}\,\lambda| = \lambda$. Hence, by Theorem 6.23, $B \oplus \mathrm{Fr}\,\lambda$ is free and B is projective. ❑ (Corollary 6.24)

We do not know if the following generalization of Theorem 6.23 holds for an arbitrary variety, in particular for the variety of Boolean algebras.

Problem 6.25 *Suppose that λ is singular. Is it true in an arbitrary variety that B is $L_{\infty\lambda}$-free if and only if B is λ^+-free?* ❑

7 Some constructions of non-free $L_{\infty\kappa}$-free Boolean algebras

7.1 $L_{\infty\aleph_1}$-free Boolean algebras

For a given ordered set P, a subset O of P and $x \in P$, let us write

$$
\begin{aligned}
O\uparrow x &= \{\, y \in O \,:\, y \geq x \,\},\\
O\upuparrows x &= \{\, y \in O \,:\, y > x \,\},\\
O\restriction x &= \{\, y \in O \,:\, y \leq x \,\},\\
O\restriction\restriction x &= \{\, y \in O \,:\, y < x \,\}.
\end{aligned}
$$

Similarly, for a subset X of P, we define
$O\uparrow X = \{\, y \in O \,:\, y \geq x$ for all $x \in X \,\}$, $O\restriction X = \{\, y \in O \,:\, y \leq x$ for all $x \in X \,\}$ etc.

For P, O, x as above,

$$tp^P_O(x) = (O\restriction x, O\uparrow x)$$

is called the type of x over O. We shall write simply $tp_O(x)$ if P is clear from the context. $tp_O(x)$ corresponds to the (model-theoretic) quantifier free type of x over C. In analogy to the terminology in model-theory (see e.g. Shelah [39]) let us call a partial ordering P *κ-stable* Boolean algebra!κ-stable – if, for every subordering O of P of cardinality κ, $\{\, tp_O(x) \,:\, x \in P \,\}$ has cardinality $\leq \kappa$.[11] We shall look at κ-stability of Boolean algebras again in Chapter 8. It is easy to see that every $L_{\infty\aleph_1}$-free Boolean algebra is $\aleph_0$-stable. Since a subordering of an $\aleph_1$-stable partial ordering is also $\aleph_0$-stable, every partial ordering embeddable into an $L_{\infty\aleph_1}$-free Boolean algebra should be $\aleph_0$-stable. As a consequence of this we see that $\mathbb{R}$ is not embeddable into any $L_{\infty\aleph_1}$-free Boolean algebra since $\mathbb{R}$ is not $\aleph_0$-stable. This consideration suggests the following problem:

Problem 7.1 *Is it true that a partial ordering P is embeddable into an $L_{\infty\aleph_1}$-free Boolean algebra if and only if P is $\aleph_0$-stable.* ❑

In the following, we shall present a partial answer to this problem from Fuchino–Koppelberg–Takahashi [19]:

Theorem 7.2 *Let T be a pseudo-tree and B the pseudo-tree algebra over T. Then B is embeddable in an $L_{\infty\aleph_1}$-free Boolean algebra iff B is $\aleph_0$-stable iff T is $\aleph_0$-stable.*

Here, a partially ordered set T is a *pseudo-tree* if the set $T\restriction t$ is linearly ordered for every $t \in T$. The crucial fact used in the proof of Theorem 7.2 is the following.

[11] in this case, we have actually $|\{\, tp_O(x) \,:\, x \in P \,\}| = \kappa$, since $tp_O(o)$ for $o \in O$ are distinct to each other.

Lemma 7.3 *For a Boolean algebra B if there is a set $\mathcal{S}$ of countable subalgebras such that $\mathcal{S}$ is cofinal in $([B]^{\aleph_0}, \subseteq)$ and each $C \in \mathcal{S}$ is relatively complete in B then $B \oplus \mathrm{Fr}\,\omega_1$ is $L_{\infty\aleph_1}$-free.*
In particular B is then embeddable in an $L_{\infty\aleph_1}$-free Boolean algebra.

Proof. Let

$$\tilde{\mathcal{S}} = \{\, S \oplus \mathrm{Fr}\, X \,:\, S \in \mathcal{S}, X \in [\omega_1]^{\aleph_0} \,\}.$$

For any X, $Y \in [\omega_1]^{\aleph_0}$ with $X \subseteq Y$, $|Y \setminus X| = \aleph_0$ and S, $T \in \mathcal{S}$ such that $S \leq T$, since $T \oplus \mathrm{Fr}\, Y$ is isomorphic to $(T \oplus \mathrm{Fr}\, X) \oplus \mathrm{Fr}\,(Y \setminus X)$ over $T \oplus \mathrm{Fr}\, X$, we have $S \oplus \mathrm{Fr}\, X \leq_{\mathrm{free}} T \oplus \mathrm{Fr}\, Y$ by Lemma 3.11. Hence, by Theorem 6.15, $B \oplus \mathrm{Fr}\,\omega_1$ is $L_{\infty\aleph_1}$-free. ❑ (Lemma 7.3)

Let us first introduce some terminology on pseudo-trees we need below. A *branch* in a pseudo-tree T is a maximal linearly ordered subset of T. For a pseudo-tree T the *pseudo-tree algebra* Treealg(T) *over* T is the subalgebra of $\mathcal{P}(T)$ generated by $\{\, T \uparrow t \,:\, t \in T \,\}$. We say that a Boolean algebra B is a pseudo-tree algebra if B is of the form Treealg(T) for some pseudo-tree T. The notion of pseudo-tree algebra has been studied in Koppelberg–Monk [33]. If T is linearly orders, Treealg(T) is also called an *interval algebra over* T and denoted also by Intalg(T).

As for a tree algebra, each element a of a pseudo-tree algebra Treealg(T) can be represented in a normal form (see [31]). We shall call the elements in T which appear in such a normal form representation of a the *end-points of a*.

An element t in a pseudo-tree T is said to be *branched* if t is not maximal in T and there are branches b and b' of T such that $b \neq b'$ and $b \cap b' = T \restriction t$. An initial segment u of a pseudo-tree T is said to be *branched* if u is not a branch of T and there are branches b and b' of T such that $b \neq b'$ and $b \cap b' = u$. Note that if u is not branched then $T \uparrow u$ is downward directed.

The *cofinality* of an upward directed partially ordered set O is the minimal cardinal κ such that there is a cofinal subset of O of order type κ. The *coinitiality* of a downward directed partially ordered set O is the minimal cardinal κ such that there is a coinitial subset of O of order type κ^*.

A *gap* in a partially ordered set O is a pair (X, Y) of subsets of O such that X is upward directed, Y is downward directed and $X \leq Y$. A gap (X, Y) has the cofinality type (κ, λ^*) if X has cofinality κ and Y has coinitiality λ. A gap (X, Y) in O is *unfilled* if there is no $x \in O$ such that $X \leq x \leq Y$.

In the following lemmas we show that an $\aleph_0$-stable pseudo-tree can be embedded in another pseudo-tree which looks like a normal binary tree and is still $\aleph_0$-stable.

Lemma 7.4 *Let T_0 be an $\aleph_0$-stable pseudo-tree. Then there exists a pseudo-tree $T \supseteq T_0$ such that T is $\aleph_0$-stable and satisfies the following* (a) – (c)*.*

(a) *For all $t, t' \in T$ if $(T \restriction t) \cap (T \restriction t') \neq \emptyset$ then there is the maximal element in $(T \restriction t) \cap (T \restriction t')$ – we shall denote this element by* $\mathrm{bra}_T(t, t')$.

(b) *For every $t \in T$ if t is branched in T there are $t', t'' \in T$ such that $t < t', t''$ and for every $u \in T \upuparrows t$ we have either $t' \leq u$ or $t'' \leq u$.*

(c) *Every unfilled gap in T has the cofinality type (ω, ω^*).*

Proof. Let $T_0 = (T_0, \leq_{T_0})$. Let I be the set of all linearly ordered initial segments in T_0. For each $s \in I$ we insert some new elements X_s between s and $T_0 \upuparrows s$ and define an ordering $\leq_s$ on $T_0 \cup X_s$. The pseudo-tree algebra we look for is then defined as $T = T_0 \cup \bigcup_{s \in I} X_s$ with the ordering generated from $\leq_s$, $s \in I$. Now let $s \in T$.

Case I *s is branched*: Let $Y \subseteq T_0$ be a maximal subset of T with the property that if $y, y' \in Y$ and $y \neq y'$ then $(T_0 \restriction y) \cap (T_0 \restriction y') = s$. Let $\{ t_\alpha : 0 < \alpha \leq \kappa \}$ be a 1-1 enumeration of Y where $\kappa = |Y|$. Let $X_s = \{ u^s_\alpha : 0 < \alpha < \kappa \} \cup \{ v^s_\alpha : 0 < \alpha \leq \kappa \}$, where the u^s_α's and v^s_α's are distinct new elements. Let $\leq_s$ be the partial ordering on $T_0 \cup X_s$ generated by $\leq_{T_0}$ and the relation defined by the following inequalities:

1. $x \leq u^s_1$, for all $x \in s$,
2. $u^s_\alpha \leq u^s_\beta$, for all $0 < \alpha \leq \beta < \kappa$,
3. $u^s_\alpha \leq v^s_\beta$, for all $0 < \alpha < \kappa$ and $\alpha \leq \beta \leq \kappa$.
4. $v^s_\alpha \leq x$, for all $x \in T_0 \uparrow s$ such that $(T_0 \restriction x) \cap (T_0 \restriction t_\alpha) \supset s$.

Case II *s is not branched*: If $T_0 \uparrow s \neq 0$ and either s is of uncountable cofinality or $T_0 \uparrow s$ is of uncountable coinitiality, let $X_s = \{x^s\}$ and $s \leq x^s \leq T_0 \uparrow s$. Otherwise let $X_s = \emptyset$ (and $\leq_s = \leq_{T_0}$). It is easily seen that T satisfies (a) – (c).

Claim 7.4.1 *T is $\aleph_0$-stable.*

$\vdash$ Let us assume by way of contradiction that there are a countable $X \subseteq T$ and an uncountable $Y \subseteq T$ such that the elements of Y realize distinct types over X. Without loss of generality let $X \cap Y = \emptyset$. If $\{ y \in Y : (T \uparrow y) \cap X = \emptyset \}$ is uncountable we may assume without loss of generality that $T \uparrow y \cap X = \emptyset$ for every $y \in Y$. Now by the usual binary tree argument we may also assume without loss of generality that X is order isomorphic to ${}^{\omega>}2$ and for any $x, x' \in X$ with $x \neq x'$ we have $T_0 \restriction x \neq T_0 \restriction x'$. For each $x, x' \in X$ with $x < x'$ let $t_{x,x'} \in T_0$ be such that $x \leq t_{x,x'} \leq x'$. Let $X_0 = \{ t_{x,x'} : x, x' \in X, x < x' \}$. For each $y \in Y$ let $t_y \in T_0 \uparrow y$. Then t_y, $y \in Y$ realize distinct types over X_0. This is a contradiction to the assumption that T_0 is $\aleph_0$-stable. $\dashv$ (Claim 7.4.1)

If $\{ y \in Y : (T \uparrow y) \cap X = \emptyset \}$ is countable we may assume without loss of generality that there is $x_0 \in X$ such that $y < x_0$ for all $y \in Y$ and $X \subseteq T \restriction x_0$. Without loss of generality we may also assume that for any $x, x' \in X$ with $x < x'$ there exists $t \in T_0$ such that $x \leq t \leq x'$. For each $x, x' \in X$ with $x < x'$ let $t_{x,x'} \in T_0$ be such that $x \leq t_{x,x'} \leq x'$. Let $X_0 = \{ t_{x,x'} : x, x' \in X, x < x' \}$. Without loss of generality elements of Y realize distinct types over X_0. Since T_0 is $\aleph_0$-stable, we may also assume that $Y \subseteq T \setminus T_0$.

Case I *Uncountably many $y \in Y$ are of the form u^s_α or v^s_α in* Case I *of the construction of T*: We may then assume without loss of generality that every $y \in T$ is of this form. For each $y \in Y$ if $y = u^s_\alpha$ or $y = v^s_\alpha$ for some $s \in I$, let $y^* = v^s_{\alpha'}$ for some $\alpha' \neq \alpha$ such that $v^s_{\alpha'} \not\leq x_0$ and let $t_y \in T_0 \uparrow y^*$. Then t_y, $y \in Y$ realize distinct types over X_0. This is a contradiction to the assumption that T_0 is $\aleph_0$-stable.

Case II *Only countably many $y \in Y$ are of the form u^s_α or v^s_α in* Case I *of the construction of T*: In this case we may assume without loss of generality that every $y \in Y$ is of the form x^s, as in Case II of the construction of T. Then for each $y \in Y$ either $T_0 \restriction y$ is of uncountable cofinality or $T_0 \uparrow y$ is of uncountable coinitiality. In both cases we can find $t_y \in T_0$ such that $X_0 \restriction y \leq t_y \leq X_0 \uparrow y$. Clearly t_y, $y \in Y$ realize distinct types over X_0. Again this is a contradiction to the fact that T_0 is $\aleph_0$-stable. ❑ (Lemma 7.4)

Lemma 7.5 *Let T_0 be a pseudo-tree such that T_0 is $\aleph_0$-stable and satisfies (a) – (c). Then there exists a pseudo-tree $T \supseteq T_0$ such that T is $\aleph_0$-stable and satisfies (a) – (c) as well as*

(d) For any $t \in T$, if t is not branched and $T \upuparrows t$ has no minimal element then $T \upuparrows t$ has uncountable coinitiality.

Proof. Let T be the pseudo-tree obtained from T_0 by replacing every non-branched $t \in T_0$ such that $T_0 \upuparrows t$ has uncountable coinitiality by an ordering of order type ω. Note that with this construction we add only unfilled gaps of cofinality type (ω, ω^*). ❑ (Lemma 7.5)

Lemma 7.6 *Let T be a pseudo-tree such that T is $\aleph_0$-stable and satisfies (a) – (c). Then for every countable $X \subseteq T$ there exists a countable $Y \subseteq T$ such that $X \subseteq Y$, Y is closed with respect to* $\operatorname{bra} T(\cdot,\cdot)$ *and for any $t, t' \in T$, $(Y \restriction t) \cap (Y \uparrow t')$ has a minimal and a maximal element provided that $(Y \restriction t) \cap (Y \uparrow t') \neq \emptyset$.*

Proof. Let Y' be the closure of Y in T with respect to $\operatorname{bra} T(\cdot,\cdot)$. Clearly Y' is still countable. Let $\mathcal{D}$ be a maximal set of branches in Y' with the property that for every $b \in \mathcal{D}$, $T \restriction b \neq \emptyset$ and for $b, b' \in \mathcal{D}$ if $b \neq b'$ then $(T \restriction b) \cap (T \restriction b') = \emptyset$. Since T is $\aleph_0$-stable, $\mathcal{D}$ is countable. For each $b \in \mathcal{D}$ let $d_b \in T \restriction b$. Now let $\mathcal{U}$ be the set of all branches b in Y' such that $T \uparrow b \neq \emptyset$. T is $\aleph_0$-stable and satisfies (a), so that $\mathcal{U}$ is countable. For each $b \in \mathcal{U}$ let $u_b \in T \uparrow b$. Let $Y'' = Y' \cup \{ d_b : b \in \mathcal{D} \} \cup \{ u_b : b \in \mathcal{U} \}$. Since T is $\aleph_0$-stable, we can enumerate all the types (D, U) in T over Y'' such that $D \neq \emptyset$ and $U \neq \emptyset$ by (D_n, U_n), $n \in \omega$. Let $X_n = \{\, t \in T \,:\, D_n \leq t \leq U_n \,\}$. By condition (c), for every $n \in \omega$ there exist $k_n, l_n \in \{0, \omega\}$ and a subset V_n of X_n with order type ${k_n}^* + l_n$ such that V_n is cofinal and coinitial in X_n. Let $Y = Y'' \cup \bigcup_{n\in\omega} V_n$. ❑ (Lemma 7.6)**Proof of Theorem 7.2** If B is $\aleph_0$-stable then clearly T also satisfies $\aleph_0$-stable. Suppose that T is $\aleph_0$-stable. By Lemmas 7.4, 7.5 and 7.6, we may assume without loss of generality that T satisfies (a) – (d) as well. For $X \subseteq T$, let

$$B_X = \{\, a \in \mathrm{Treealg}(T) \;:\; \text{end-points of } a \text{ are in } X \,\}$$

and

$$S = \{\, B_X \;:\; \begin{array}{l} X \subseteq T,\ X \text{ is countable and closed with respect to } \mathrm{bra}_T(\cdot,\cdot), \\ \text{for any } t, t' \in T,\ (X \restriction t) \cap (X \uparrow t') \text{ has a minimal and a} \\ \text{maximal element provided that } (X \restriction t) \cap (X \uparrow t') \neq \emptyset \end{array} \,\}.$$

By Lemma 7.6, S is cofinal in $[B]^{\aleph_0}$. So by Lemma 7.3 we are done by the following

Claim 7.6.1 *The elements of S are relatively complete subalgebras of* Treealg(T).

$\vdash$ Let $B \in S$, say $B = B_X$ for some $X \subseteq T$ as in the definition of S. It is enough to show that every $a \in \mathrm{Treealg}(T)$ has a projection in B_X. For the simplicity let us consider the case that $a = (T \uparrow t_0) \setminus (T \uparrow t_1)$ for some $t_0, t_1 \in T$, $t_0 < t_1$. The general case can be proved similarly.

If there is no $x \in X$ such that $t_0 \leq x \leq t_1$ then $\emptyset$ is the projection of a in B_X. Otherwise let x_0 be the minimal element of X such that $t_0 \leq x_0 \leq t_1$. If there is no $x \in X$ such that $x_0 < x \leq t_1$ then $\emptyset$ is again the projection of a in B_X. Otherwise let x_1 be the maximal element of X such that $x_0 < x_1 \leq t_1$.

Case I *x_1 is branched*: we shall define $a', a'' \in B_X$ so that $(T \uparrow x_0) \setminus (T \uparrow x_1) \cup a' \cup a''$ will be the projection of a in B_X. Let t' and t'' be as in (b) for $t = x_1$. Assume without loss of generality $t'' \leq t_1$. If $X \uparrow t' \neq \emptyset$ let x' be the minimal element in $X \uparrow t'$. The uniqueness of x' follows from the fact that X is closed with respect to $\mathrm{bra}_T(\cdot,\cdot)$. Let $a' = T \uparrow x'$. If $X \uparrow t' = \emptyset$ let $a' = \emptyset$. Now if $t_1 = t''$ let $a'' = \emptyset$. Otherwise we have $t'' < t_1$. If $(X \uparrow t'') \setminus (X \uparrow t_1) = \emptyset$ let $a'' = \emptyset$. Otherwise, as above, there is the unique minimal element $x'' \in (X \uparrow t'') \setminus (X \uparrow t_1)$. Let $a'' = T \uparrow x''$.

Case II *x_1 is not branched and there is a minimal element $t \in T \upuparrows x_1$*: If $t = t_1$ or $(X \uparrow t) \setminus (X \uparrow t_1) = \emptyset$ then $(T \uparrow x_0) \setminus (T \uparrow x_1)$ is the projection of a in B_X. Otherwise there is a unique minimal element x in $(X \uparrow t) \setminus (X \uparrow t_1)$. $((T \uparrow x_0) \setminus (T \uparrow x_1)) \cup (T \uparrow x)$ is then the projection of a in B_X.

Case III *x_1 is not branched and there is no minimal element in $T \upuparrows x_1$*: By (d) $T \upuparrows x_1$ has cofinality $\geq \omega_1$. Since X is countable there is $t \in T \upuparrows x_1$ such that $X \upuparrows x_1 \subseteq T \uparrow t$. If $(X \uparrow t) \setminus (X \uparrow t_1) = \emptyset$ then $(T \uparrow x_0) \setminus (T \uparrow x_1)$ is the projection of a in B_X. Otherwise let x be the minimal element in $(X \uparrow t) \setminus (X \uparrow t_1)$. Then

$((T \uparrow x_0) \setminus (T \uparrow x_1)) \cup (T \uparrow x)$ is the projection of a in B_X. ⊣ (Claim 7.6.1)
❏ (Theorem 7.2)

Since every ordered set, in particular every cardinal κ is $\aleph_0$-stable and linearly ordered sets are pseudo-trees, we obtain the following

Corollary 7.7 *For any cardinal κ there is an $L_{\infty\aleph_1}$-free Boolean algebra B which does not satisfy the κ-cc.* ❏

In contrast to this result, every $L_{\infty\omega_2}$-free Boolean algebra has precalibre $\aleph_1$ (hence satisfies the ccc).

Since every ω_1-tree is $\aleph_0$-stable and trees are pseudo-trees, we obtain the following corollary of Theorem 7.2:

Corollary 7.8 *Every ω_1-tree T can be embedded into an $L_{\infty\aleph_1}$-free Boolean algebra.* ❏

We can prove the corollary above also directly from the fact that $\text{Treealg}(T) \oplus \text{Fr}\,\omega_1$ is already $L_{\infty\aleph_1}$-free. In particular if T is a Suslin tree we obtain by this construction an $L_{\infty\aleph_1}$-free Boolean algebra which satisfies the ccc but is not absolutely ccc.

7.2 A construction under $V = L$

In this section, we introduce a construction of non-free $L_{\infty\kappa}$-free Boolean algebras (Theorem 7.9). This construction, under $V = L$, yields Boolean algebras which are counter-examples to the theorems under Axiom R given in Chapter 8.2. Thus, we obtain the independence of these theorems from ZFC.

Theorem 7.9 for the case $\kappa = \aleph_1$ and Lemmas 7.11, 7.12 are due to S. Koppelberg. Recall that a stationary subset S of a cardinal κ is said to be *non-reflecting* if $S \cap \alpha$ is not stationary for every $\alpha < \kappa$.

Theorem 7.9 *Suppose that κ is a regular cardinal and S a non-reflecting stationary subset of κ consisting of ordinals of countable cofinality. Then there exists a Boolean algebra A of cardinality κ with a continuously increasing sequence $(A_\alpha)_{\alpha<\kappa}$ in $\text{Sub}^{<\kappa}(A)$ such that*

0) $\bigcup_{\alpha<\kappa} A_\alpha = A$;

i) A_α *is free for every* $\alpha < \kappa$;

ii) $A_\alpha \leq_{\text{free}} A_\beta$ *for every* α, $\beta < \kappa$ *such that* $\alpha \leq \beta$ *and* $\alpha \notin S$;

iii) $\{\alpha < \kappa : A_\alpha \leq_{\neg\text{rc}} A\} = S$.

For the proof of Theorem 7.9, we need the following lemmas:

Lemma 7.10 *Suppose that ξ is a limit ordinal and $(B_\alpha)_{\alpha<\xi}$ a continuously increasing sequence of Boolean algebras such that $B_\alpha \leq_{\rm proj} B_{\alpha+1}$ for all $\alpha < \xi$. Let $B = \bigcup_{\alpha<\xi} B_\alpha$. By Lemma 3.7, 2), we have $B_\alpha \leq_{\rm proj} B$ for every $\alpha < \xi$. Let $B' \geq B$ be such that $B' = B(x)$ for some $x \in B$. If $B_\alpha \leq_{\rm rc} B'$ for every $\alpha < \xi$, then we have $B_\alpha \leq_{\rm proj} B'$ for every $\alpha < \xi$.*

Proof. For an arbitrary $\alpha^* < \xi$, we show that $B_{\alpha^*} \leq_{\rm proj} B'$. By Theorem 4.1, we can find a continuously increasing sequence $(C_\beta)_{\beta<\eta}$ of subalgebras of B for some limit ordinal η such that

(1) $C_0 = B_{\alpha^*}$;

(2) $C_\beta \leq_{\rm rc} B$ and $C_{\beta+1}$ is countably generated over C_β for every $\beta < \eta$;

(3) for all $\alpha < \xi$ such that $\alpha^* \leq \alpha$, there is $\beta_\alpha < \eta$ such that $C_{\beta_\alpha} = B_\alpha$.

Let $(D_\beta)_{\beta<\eta}$ be defined by

(4) $D_0 = C_0$;

(5) $D_\beta = C_\beta(x)$ for $0 < \beta < \eta$.

Then $(D_\beta)_{\beta<\eta}$ is continuously increasing, $\bigcup_{\beta<\eta} D_\beta = B'$ and $D_{\beta+1}$ is countably generated over D_β for every $\beta < \eta$. We have also $D_\beta \leq_{\rm rc} B'$ for every $\beta < \eta$: for $\beta = 0$, this is clear. For $\beta > 0$, if $\alpha < \xi$ is such that $\alpha^* \leq \alpha$ and $\beta \leq \beta_\alpha$, then we have $C_\beta \leq_{\rm rc} B_\alpha \leq_{\rm proj} B'$. Hence $C_\beta \leq_{\rm rc} B'$. By Lemma 3.1, *3*), it follows that $D_\beta = C_\beta(x) \leq_{\rm rc} B'$. Hence, by Theorem 4.1, *d*), we have $B_{\alpha^*} = D_0 \leq_{\rm proj} B'$. ❑ (Lemma 7.10)

Lemma 7.11 *Suppose that ξ is a limit ordinal and $(B_\alpha)_{\alpha<\xi}$ is an increasing sequence of Boolean algebras such that $B_\alpha \leq_{\rm rc} B_{\alpha+1}$ for every $\alpha < \xi$. Let $B = \bigcup_{\alpha<\xi} B_\alpha$. For $B' \geq B$ of the form $B' = B(x)$ for some $x \in B'$, if $B_\alpha \restriction x$ and $B_\alpha \restriction -x$ have the largest elements s_α and t_α respectively for every $\alpha < \xi$, then we have $B_\alpha \leq_{\rm rc} B'$ for all $\alpha < \xi$.*

Proof. Let $\alpha < \xi$. For $u \in B'$, we show that the lower projection of u onto B_α exists. Let $u = v \cdot x + w \cdot -x$ for some u, $v \in B$. Let $\alpha^* < \xi$ be such that $\alpha \leq \alpha^*$ and v, $w \in B_{\alpha^*}$. Then, for $c \in B_{\alpha^*}$, we have

$$\begin{aligned} c \leq u &\Leftrightarrow c \cdot x \leq v \text{ and } c \cdot -x \leq w \\ &\Leftrightarrow c \cdot -v \leq -x \text{ and } c \cdot -w \leq x \\ &\Leftrightarrow c \cdot -v \leq t_{\alpha^*} \text{ and } c \cdot -w \leq s_{\alpha^*} \\ &\Leftrightarrow c \leq (t_{\alpha^*} + v) \cdot (s_{\alpha^*} + w). \end{aligned}$$

Hence $p^{B_{\alpha^*}}_{B_\alpha}((t_{\alpha^*}+v)\cdot(s_{\alpha^*}+w))$ is the lower projection of u onto B_α. ❑ (Lemma 7.11)

Lemma 7.12 *Suppose that A, B are Boolean algebras such that $A \leq_{\rm rc} B$ and u an element of B independent over A. Then for $a \in A$, $a \neq 1$, there is $b \in B$ such that $a \neq b$ and $p^B_A(b) = a$.*

Proof. Put $b = a + u \cdot -a$. Since $u \cdot -a \neq 0$ by the independence of u over A and $a \cdot (u \cdot -a) = 0$, we have $a \neq b$. For $a^* \in A$, we have

$$\begin{aligned} a^* \leq b &\Leftrightarrow a^* \cdot -a \leq u \\ &\Leftrightarrow a^* \cdot -a = 0 \\ &\Leftrightarrow a^* \leq a \end{aligned}$$

by the independence of u over A. Hence $p^B_A(b) = a$. ❑ (Lemma 7.12)

Proof of Theorem 7.9 We define a continuously and strictly increasing sequence $(A_\alpha)_{\alpha<\kappa}$ inductively so that $A_\alpha \cong \mathrm{Fr}\,(\alpha + \omega)$ and

$(*)_\alpha$ For every $\delta < \alpha$, if $\delta \notin S$ then $A_\delta \leq_{\rm free} A_\alpha$; if $\delta \in S$ then $A_\delta \leq_{\neg{\rm rc}} A_\alpha$

for all $\alpha < \kappa$. Let $A_0 = \mathrm{Fr}\,\omega$. Suppose that A_β, $\beta < \alpha$ have been constructed such that $A_\beta \cong \mathrm{Fr}\,(\beta + \omega)$ and $(*)_\beta$ holds for every $\beta < \alpha$.

Case I. α is a limit: Let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. Since $\alpha \cap S$ is not stationary, there exists a closed unbounded $X \subseteq \alpha$ such that $X \cap S = \emptyset$. For every β, $\beta' \in X$ with $\beta < \beta'$, we have $A_\beta \leq_{\rm free} A_{\beta'}$ by $(*)_\beta$. Hence $A_\beta \leq_{\rm free} A_\alpha$ for every $\beta \in X$. As $A_\alpha = \bigcup_{\beta\in X} A_\beta$ and $A_\beta \cong \mathrm{Fr}\,(\beta + \omega)$ for all $\beta \in X$, it follows that $A_\alpha \cong \mathrm{Fr}\,(\alpha + \omega)$. Also $(*)_\alpha$ holds: for $\delta < \alpha$, if $\delta \notin S$, then for $\beta \in X$ such that $\delta < \beta$ we have $A_\delta \leq_{\rm free} A_\beta$ by $(*)_\beta$. Since $A_\beta \leq_{\rm free} A_\alpha$, it follows that $A_\delta \leq_{\rm free} A_\alpha$. If $\delta \in S$ then $A_\delta \leq_{\neg{\rm rc}} A_{\delta+1}$ by $(*)_{\delta+1}$. It follows that $A_\delta \leq_{\neg{\rm rc}} A_\alpha$.

Case II. $\alpha = \gamma + 1$ for a $\gamma \notin S$: Let $A_\alpha = A_\gamma \oplus \mathrm{Fr}\,\gamma$. Similarly to Case I, we can show that $A_\alpha \cong \mathrm{Fr}\,(\alpha + \omega)$ and $(*)_\alpha$ holds.

Case III. $\alpha = \gamma + 1$ for a $\gamma \in S$: In this case we have that $\mathrm{cf}(\gamma) = \omega$. Let $(\beta_n)_{n\in\omega}$ be an increasing sequence of ordinals such that $\gamma = \bigcup_{n\in\omega} \beta_n$ and $\beta_n \notin S$ for all $n \in \omega$. Then $A_\gamma = \bigcup_{n\in\omega} A_{\beta_n}$. Since $(*)_\beta$ holds for all $\beta \leq \gamma$, we have $A_{\beta_n} \leq_{\rm free} A_{\beta_{n+1}}$ and $A_{\beta_n} \cong \mathrm{Fr}\,(\beta_n + \omega)$.

By Lemma 7.12 there is a strictly increasing sequence $(a_n)_{n<\omega}$ of elements of B_γ such that $a_n \in A_{\beta_n}$ and $p^{A_{\beta_{n+1}}}_{A_{\beta_n}}(a_{n+1}) = a_n$ for all $n < \omega$. Let

$$I = \{\, b \in A_\gamma \,:\, b \leq a_n \text{ for some } n < \omega \,\}$$

and let x be (an element of a enough (model-theoretic) saturated Boolean algebra containing A_γ) such that $\{\, b \in A_\gamma \,:\, b \leq x \,\} = I$ and $\{\, b \in A_\gamma \,:\, b \leq -x \,\} = \{\, 0 \,\}$. Then a_n and 0 are the greatest elements of $A_{\beta_n} \restriction x$ and $A_{\beta_n} \restriction -x$ respectively. Hence by Lemma 7.11, $A_{\beta_n} \leq_{\rm rc} A_\gamma(x)$ for all $n < \omega$. By Lemma 7.10, it follows that $A_{\beta_n} \leq_{\text{proj}} A_\gamma(x)$ for every $n < \omega$. On the other hand, we have $A_\gamma \leq_{\neg\text{rc}} A_\gamma(x)$ since x cannot have the projection onto A_γ since $(a_n)_{n<\omega}$ is cofinal in $A_\gamma \restriction x$.

Let $A_\alpha = A_\gamma(x) \oplus \operatorname{Fr}\gamma$. By definition of $\leq_{\text{proj}}$, we have $A_{\beta_n} \leq_{\text{free}} A_\alpha$ for every $n < \omega$. In particular A_α is free. For $\beta < \alpha$ such that $\beta \notin S$, let $n < \omega$ be such that $\beta \leq \beta_n$. By $(*)_{\beta_n}$ we have $A_\beta \leq_{\text{free}} A_{\beta_n}$. Hence $A_\beta \leq_{\text{free}} A_\alpha$ by Lemma 3.6, *2*). If $\beta < \alpha$ and $\beta \in S$ then we have either $\beta < \gamma$ or $\beta = \gamma$. In the first case $A_\beta \leq_{\neg\text{rc}} A_\gamma$ by $(*)_\gamma$. Hence $A_\beta \leq_{\neg\text{rc}} A_\alpha$. In the second, we get the same conclusion by $A_\gamma \leq_{\neg\text{rc}} A_\gamma(x) \leq A_\alpha$. ❏ (Theorem 7.9)

Proposition 7.13 *Let κ, S, $(A_\alpha)_{\alpha<\kappa}$ and A be as in Theorem 7.9. Then*

1) *A is $L_{\infty\kappa}$-free;*

2) *A is κ-free;*

3) *if S is co-stationary*[12] *in κ then A has the ccc;*

4) *A has the Bockstein separation property;*

5) *A cannot be embedded into any openly generated Boolean algebra.*

Proof. *1*): By *ii*) in Theorem 7.9 and Proposition 6.15. Note that $\kappa \setminus S$ is unbounded in κ. *2*): $\{ A_\alpha : \alpha < \kappa \}$ is a club subset of $\operatorname{Sub}^{<\kappa}(A)$ consisting of free subalgebras of A. *3*): for $\kappa > \aleph_1$ this follows from *1*) and Lemma 6.8. For $\kappa = \aleph_1$, this follows from a theorem by Solovay, e.g. see Lemma 23.7 in Jech [27]. *4*): there is an $L_{\infty\aleph_1}$-formula φ such that, for any ccc Boolean algebra B, $B \models \varphi$ iff B has the Bockstein separation property[13] . Since $\operatorname{Fr}\kappa \models \varphi$, we have $B \models \varphi$. By *3*), it follows that B has the Bockstein separation property. *5*): Suppose that A could be embedded in to a openly generated Boolean algebra B. Then, by *4*) and Proposition 5.20, A should be also openly generated. But it is not the case by Theorem 5.5, *iii*) in Theorem 7.9 and since S is stationary. This is a contradiction. ❏ (Proposition 7.13)

For cardinals κ, λ, let us denote by E^λ_κ the assertion:

E^λ_κ: *there is a non-reflecting stationary $S \subseteq \kappa$ such that* $S \subseteq \{ \alpha < \kappa : \operatorname{cf}(\alpha) = \lambda \}$.

Thus the assumption in Theorem 7.9 is just E^ω_κ. Since $\{ \alpha < \omega_1 : \alpha \text{ is a limit} \}$ is a non-reflecting stationary subset of ω_1, $E^\omega_{\omega_1}$. Since we can always thin out a stationary set to a stationary co-stationary set, we obtain the following:

[12] i.e. $\kappa \setminus S$ is stationary. For $\kappa > \omega_1$ this holds any way since S is non reflecting.

[13] in general, we can find an $L_{\infty\lambda}$-sentence φ such that, for any λ-cc Boolean algebra B, $B \models \varphi$ if and only if B satisfies the Bockstein separation property. This can be seen as follows. For a λ-cc Boolean algebra B, any regular ideal of B can be represented as $X^\perp$ for some $X \subseteq B$ of cardinality $< \lambda$. Hence B satisfies the Bockstein separation property if and only if the following assertion holds in B: "for any $X \subseteq B$ of cardinality $< \lambda$, there is a countable $Y \subseteq B$ such that for any $z \in X^\perp$ there is $y \in Y$ such that $z \leq y$". It is easy to see that this assertion can be formulated in $L_{\infty\lambda}$.

Theorem 7.14 *There is a ccc $L_{\infty\aleph_1}$-free, $\aleph_1$-free Boolean algebra B such that B is not embeddable into any openly generated Boolean algebra.*

Proof. By Theorem 7.9, Theorem 7.13 and the remark above. ❑ (Theorem 7.14)

By Jensen, E^ω_κ holds in L for every regular non weakly-compact κ. For a successor cardinal κ, if $\kappa = \lambda^+$, then E^ω_κ follows from $\Box_\lambda$ (see Kanamori–Magidor [30]).

Theorem 7.15 ($V = L$) *Suppose that κ is a regular non weakly-compact cardinal. Then there is a ccc, $L_{\infty\kappa}$-free and κ-free Boolean algebra A of cardinality κ which cannot be embedded into any openly generated Boolean algebra.*

Proof. By Theorem 7.9, Proposition 7.13 and the remark above.
❑ (Theorem 7.15)

7.3 The construction principle CP+

For a variety $\mathcal{V}$, let $\mathrm{Inc}(\mathcal{V})$ denote the *incompactness spectrum of* $\mathcal{V}$, i.e.

$$\mathrm{Inc}(\mathcal{V}) = \{\, \kappa \in \mathrm{Card} : \text{there is a non-free, } L_{\infty\kappa}\text{-free algebra of cardinality } \kappa \text{ in } \mathcal{V} \,\}.$$

By Theorem 6.23, $\mathrm{Inc}(\mathcal{V})$ consists of regular cardinals. If $V = L$ then, by Theorem 7.15, $\mathrm{Inc}(\mathcal{BA})$ for the variety of Boolean algebras $\mathcal{BA}$ contains all non weakly-compact regular cardinals. Similar construction shows that, under $V = L$, $\mathrm{Inc}(\mathcal{AG})$ for the variety $\mathcal{AG}$ of abelian groups consists of all non weakly-compact regular cardinals.

$\mathrm{Inc}(\mathcal{AG})$ has a nice combinatorial characterization. For a family X (of sets), $f : X \to \bigcup X$ is called *transversal for* X if f is one–to–one choice function, i.e. one–to–one mapping such that $f(x) \in x$ holds for every $x \in X$. For cardinals κ, λ, $\mathrm{NPT}(\kappa, \lambda)$ is the following assertion:

$\mathrm{NPT}(\kappa, \lambda)$: There is $X \subseteq [V]^{<\lambda}$ such that $|X| = \kappa$, there is no transversal for X but every $Y \in [X]^{<\kappa}$ has a transversal.

Theorem 7.16 (Shelah [41]) *For a cardinal κ the following are equivalent*

1) $\kappa \in \mathrm{Inc}(\mathcal{AG})$*;*

2) $\mathrm{NPT}(\kappa, \aleph_1)$. ❑

One of Shelah's recent result says that the class $\mathrm{Inc}(\mathcal{AG})$ is already quite rich already in ZFC without any additional axioms.

Theorem 7.17 (Magidor–Shelah [36]) *For a cardinal κ let $\mathcal{C}_\kappa$ be the closure of $\{\, \kappa \,\}$ under successor of cardinals and the operation $(\lambda, \kappa) \mapsto \lambda^{+(\kappa+1)}$. If $\kappa \in \mathrm{Inc}(\mathcal{AG})$ then $\mathcal{C}_\kappa \subseteq \mathrm{Inc}(\mathcal{AG})$.* ❑

Since it is easy to see that $\aleph_0 \in \mathrm{Inc}(\mathcal{AG})$, it follows (in ZFC) that:

Corollary 7.18 $\{\kappa < \aleph_{\omega^2+1} : \kappa$ *is regular*$\} \subseteq \mathrm{Inc}(\mathcal{AG})$. ❑

For a variety $\mathcal{V}$, (CP+) is the assertion[14] "(CP$_n$) holds for every $n \geq 1$" where (CP$_n$) for $n \geq 1$ is defined by:

(CP$_n$): *There are countably generated free algebras $H \leq K \leq L$ and a partition of ω into n infinite subsets $s^0, \ldots, s^{n-1}$ such that*

1) H is generated freely by $\{h_m : m < \omega\}$ and for every $J \subseteq \omega$, if $J \cap s^k$ is finite for some $k < n$ then the algebra generated by $\{h_m : m \in J\}$ is a free factor of L and

2) $L = K \oplus \mathrm{Fr}\,\omega$ and H is not a free factor of L.

In a variety $\mathcal{V}$ with (CP+), $\mathrm{Inc}(\mathcal{V})$ is at least as rich as $\mathrm{Inc}(\mathcal{AG})$:

Theorem 7.19 (Shelah, see Eklof–Mekler [8]) *For any variety $\mathcal{V}$, if $\mathcal{V}$ satisfies the property* (CP+)*, then* $\mathrm{Inc}(\mathcal{AG}) \subseteq \mathrm{Inc}(\mathcal{V})$*. Moreover, for every $\kappa \in \mathrm{Inc}(\mathcal{AG})$, we can find an $L_{\infty\kappa}$-free, κ-free non-free algebra of cardinality κ in $\mathcal{V}$.* ❑

We shall prove here the following theorem by Shelah:

Theorem 7.20 (Shelah, see Fuchino–Shelah [20]) *The variety $\mathcal{BA}$ of Boolean algebras has the property* (CP+).

By Theorem 7.19, it follows that:

Corollary 7.21 $\mathrm{Inc}(\mathcal{AG}) \subseteq \mathrm{Inc}(\mathcal{BA})$. *Moreover, for every $\kappa \in \mathrm{Inc}(\mathcal{AG})$, we can find an $L_{\infty\kappa}$-free κ-free non-free Boolean algebra of cardinality κ.* ❑

Proof of Theorem 7.20 Let $n \in \omega$ $n \geq 1$. We show that $\mathcal{BA}$ satisfies (CP$_n$). Let $x_{l,h}$, ($l < n$, $h < \omega$), be pairwise distinct sets and let

$$X = \{x_{l,h} : l < n,\ h < \omega\}.$$

Let $A = \mathrm{Fr}\,X$. Let I be the ideal over A generated by $\{\Pi_{l<n} x_{l,h} : h \in \omega\}$ and let $B \geq A$ be such that $B = A(x)$ for some $x \in B$ such that $A \restriction x = I$ and $A \restriction -x = \{0\}$. Let $C = B \oplus \mathrm{Fr}\,\omega$. Clearly B, C are countable and atomless. Hence B, C are free. $A \leq_{\neg\mathrm{free}} C$ since $A \leq_{\neg\mathrm{rc}} B$. Thus the following claim shows that (A, B, C) witnesses the (CP$_n$) of $\mathcal{BA}$.

Claim 7.21.1 *Let $I = \{(l,h) : l < n,\ h < \omega\}$. If $J \subseteq I$ is such that*

$$|J \cap \{(l_0, h) : h < \omega\}| < \aleph_0$$

for some $l_0 < n$ then $A^* \leq_{\mathrm{free}} C$ for $A^* = \langle\{x_{l,h} : (l,h) \in J\}\rangle_A$.

[14] 'CP' stands for "construction principle".

$\vdash$ By Theorem 4.2, it is enough to show that $A^* \leq_{\mathrm{rc}} B$. By Lemma 7.11, it is enough to prove it for the case that $J \cap \{ (l_0, h) : h < \omega \} = \emptyset$. Let $(J_m)_{m<\omega}$ be the increasing sequence defined by $J_0 = J$ and $J_{m+1} = \{ (l, h) : \text{if } l = l_0 \text{ then } h \leq m \}$. Then $\bigcup_{m<\omega} J_m = I$. Let $A_m = \operatorname{Fr} J_m$ for $m < \omega$. Hence $A_0 = A^*$, $\bigcup_{m<\omega} A_m = A$ and $A_m \leq_{\mathrm{free}} A$ for all $m < \omega$. Further we have $\mathrm{pr}^B_{A_m}(-x) = 0$ for all $m < \omega$; $\mathrm{pr}^b_{A_0}(x) = 0$ and $\mathrm{pr}^B_{A_{m+1}}(x) = \Sigma_{h \leq m}(\Pi_{l<n} x_{l,h})$. Hence $A^* \leq_{\mathrm{rc}} B$ by Lemma 7.11 and this was to show. $\dashv$ (Claim 7.21.1)

$\square$ (Theorem 7.20)

8 κ-projective/openly generated Boolean algebras

8.1 Miscellaneous results on κ-openly generated Boolean algebras

In the following, we shall consider mainly κ-openly generated Boolean algebras, the reason being that κ-openly generated Boolean algebras are much easier to handle with than κ-projective Boolean algebras. This is chiefly because the class of openly generated Boolean algebras is closed under relative complete subalgebras (or even under σ-subalgebras — see Lemma 5.3) while this is not the case for projective Boolean algebras. Let us begin with some basic properties of κ-projective/openly generated Boolean algebras.

Lemma 8.1 *Let κ be a regular cardinal and B a Boolean algebra. Then:*

0) B is $\aleph_1$-free if and only if B is atomless; every Boolean algebra B is $\aleph_1$-projective.

1) If B is κ-free then B is κ-projective.

2) If B is κ-projective then B is κ-openly generated.

3) B is $\aleph_2$-projective if and only if B is $\aleph_2$-openly generated.

4) If B is openly generated then B is $\aleph_2$-projective.

Proof. *0*) The first assertion follows from the fact that a countable Boolean algebra B is free if and only if B is atomless. Form this, if follows that every countable Boolean algebra is projective and hence the second assertion.

1), *2*), *3*) follow immediately from Lemma 2.6, *1*), *2*), *3*).

For *4*) Let χ be sufficiently large and

$$\mathcal{C} = \{\, B \cap M \;:\; M \prec \mathcal{H}(\chi),\, B \in M,\, \omega_1 \subseteq M,\, |M| = \aleph_1 \,\}.$$

By Theorem 2.8, $\mathcal{C}$ contains a club in $[B]^{<\aleph_2}$. By Theorem 5.1, $C \leq_{\rm rc} B$ for all $C \in \mathcal{C}$. Hence, by Lemma 5.3 and *3*), every $C \in \mathcal{C}$ is projective. This shows that B is $\aleph_2$-projective. ❑ (Lemma 8.1)

As we have already seen in Chapter 2, κ-free (κ-projective, κ-openly generated resp.) Boolean algebras can be characterized by existence of a system of subalgebras with certain properties.

Proposition 8.2 *For a regular κ and a Boolean algebra B, the following are equivalent:*

a) B is κ-free (κ-projective, κ-openly generated resp.);

b) there exists an upward directed partial ordering $I = (I, \leq)$ and an indexed family $(B_i)_{i \in I}$ of subalgebras of B such that

0) *$|B_i| < \kappa$ and B_i is free (projective, openly generated resp.) for every $i \in I$;*
1) *$B_i \leq B_j$ for all $i, j \in I$ with $i \leq j$;*
2) *for every increasing chain $(i_\alpha)_{\alpha<\delta}$ in I of length δ for some $\delta < \kappa$, $i^* = \sup\{ i_\alpha : \alpha < \delta \}$ exists and $B_{i^*} = \bigcup_{\alpha<\delta} B_{i_\alpha}$;*
3) $B = \bigcup_{i\in I} B_i$. ❑

Let us call a system $(B_i)_{i\in I}$ as above a *κ-free (κ-projective, κ-openly generated resp.) filtration of B*. The relation between κ-free and κ-projective Boolean algebras parallels to the relation between free and projective Boolean algebras.

Lemma 8.3 *For a regular κ, a Boolean algebra B is κ-projective if and only if $B \oplus \mathrm{Fr}\,\kappa$ is κ-free.*

Proof. Suppose that B is κ-projective. Then there exists a club subset $\mathcal{C}$ of $[B]^{<\kappa}$ consisting of projective subalgebras of B. Let

$$\mathcal{C}' = \{ \langle C \cup \mathrm{Fr}\,\alpha\rangle_{B\oplus \mathrm{Fr}\,\kappa} : C \in \mathcal{C},\ \alpha < \kappa,\ |C| \leq |\alpha| \}.$$

Then $\mathcal{C}'$ is a club subset of $[B \oplus \mathrm{Fr}\,\kappa]^{<\kappa}$. By Theorem 2.5, a), every $C \in \mathcal{C}'$ is free.

Suppose now that $\mathcal{C}'$ is a club subset of $[B \oplus \mathrm{Fr}\,\kappa]^{<\kappa}$ consisting of free subalgebras of $B \oplus \mathrm{Fr}\,\kappa$. Let

$$I = \{ C' \in \mathcal{C}' : C' = C \oplus \mathrm{Fr}\,\alpha \text{ for some } C \leq B \text{ and } \alpha < \kappa \}.$$

Then I is still a club subset of $[B \oplus \mathrm{Fr}\,\kappa]^{<\kappa}$. Let us regard I as a partially ordered set with the subalgebra relation $\leq$. For $C' \in I$, let $C_{C'} = C' \cap B$. Then $(C_{C'})_{C'\in I}$ is κ-projective filtration of B. Hence by Lemma 8.2, B is κ-projective. ❑ (Lemma 8.3)

Lemma 8.4 *Suppose that B is κ-openly generated for some regular $\kappa > \aleph_1$. Then B satisfies the ccc.*

Proof. Otherwise there would be a pairwise disjoint $X \subseteq B^+$ of cardinality $\aleph_1$. Let $\mathcal{C}$ be a club subset of $[B]^{<\kappa}$ consisting of openly generated subalgebras of B. Then there is $C \in \mathcal{C}$ such that $X \subseteq C$. This is a contradiction as C satisfies the ccc by Corollary 5.15. ❑ (Lemma 8.4)

Lemma 8.5 *Suppose that B is κ-openly generated for some regular $\kappa > \aleph_1$. If χ is sufficiently large and M is a V_{ω_1}-like elementary submodel of $\mathcal{H}(\chi)$ such that $B, \kappa \in M$, then $B \cap M \leq_{\mathrm{rc}} B^*$ for some openly generated $B^* \leq B$. In particular, $B \cap M$ is projective.*

Proof. Let $(M_\alpha)_{\alpha<\omega_1}$ be an increasing sequence of elementary submodels of $\mathcal{H}(\chi)$ witnessing the V_{ω_1}-likeness of M. Without loss of generality we may assume B, $\kappa \in M_0$. Let $(B_i)_{i\in I} \in M_0$ be a κ-openly generated filtration of B. For $\alpha < \omega_1$, let

$i_\alpha = \sup I \cap M_\alpha$. i_α exists since $I \cap M_\alpha$ is countable and directed. As $i_\alpha \in M_{\alpha+1}$, we have $B_{i_\alpha} \in M_{\alpha+1} \subseteq M$. Since B_{i_α} is openly generated, we have $B_{i_\alpha} \cap M \leq_{\rm rc} B_{i_\alpha}$ by Theorem 5.1. Let $i^* = \sup I \cap M$. Then $i^* = \sup\{\, i_\alpha : \alpha < \omega_1 \,\}$ and we have $B \cap M = \bigcup_{\alpha<\omega_1}(B_{i_\alpha} \cap M) = B_{i^*} \cap M$. Since B satisfies the ccc by Lemma 8.4, it follows by Lemma 3.9 that $B \cap M \leq_{\rm rc} B_{i^*}$. By Lemma 5.3 and Lemma 2.6, *3*), $B \cap M$ is projective. ❏ (Lemma 8.5)

Lemma 8.6 *If B is κ-openly generated for some regular $\kappa > \aleph_1$ then B satisfies the Bockstein separation property.*

Proof. Suppose that J is a regular ideal on B. Let χ be sufficiently large and M be a V_{ω_1}-like elementary submodel of $\mathcal{H}(\chi)$ such that B, J, $\kappa \in M$. Let $(M_\alpha)_{\alpha<\omega_1}$ be a witness of V_{ω_1}-likeness of M. Since $M \models$ "J is a regular ideal on B", $J \cap M$ is a regular ideal on $B \cap M$. $B \cap M$ is projective by Lemma 8.5. Hence by Proposition 5.18, $J \cap M$ is countably generated. Hence there is $\alpha_0 < \omega_1$ be such that $J \cap M_{\alpha_0}$ generates $J \cap M$. Since $J \cap M_{\alpha_0} \in M_{\alpha_0+1} \subseteq M$, we have

$$M \models \text{“} J \cap M_{\alpha_0} \text{ generates } J \text{”}.$$

Hence by elementarity of M, the countable set $J \cap M_{\alpha_0}$ really generates J. ❏ (Lemma 8.6)

κ-stability of partial orderings was introduced in Section 7.1. In the following we shall look at κ-stability again in the context of almost free Boolean algebras.

Lemma 8.7

1) *Let B be a Boolean algebra. Suppose that $\{\, C \in [B]^\lambda : C \leq_\kappa B \,\}$ is cofinal in $[B]^\lambda$ for some regular cardinal κ such that $2^{<\kappa} \leq \lambda$. Then B is λ-stable.*

2) *If B is $\leq_\kappa$-generated for a regular κ then B is λ-stable for any λ such that $\lambda = \lambda^{<\kappa}$. If B is openly generated then B is λ-stable for any λ.*

3) *If B is κ-openly generated for some $\kappa > \aleph_1$ then B is $\aleph_0$-stable.*

Proof. *1*): Suppose not. Then there is some $C \leq B$ such that $|\{\, tp_C(b) : b \in B \,\}| > |C| = \lambda$. Let $b_\alpha \in B$, $\alpha < \lambda^+$ be such that $tp_C(b_\alpha)$, $\alpha < \lambda^+$ are pairwise distinct. By our assumption, there is $C' \leq \kappa B$ such that $C \leq C'$ and $|C'| = \lambda$. Without loss of generality we may assume that $b_\alpha \in B \setminus M$ for all $\alpha < \lambda^+$. Then $tp_{C'}(b_\alpha)$, $\alpha < \lambda^+$ are still pairwise distinct. Each $tp_{C'}(b_\alpha)$ is decided by cofinal subsets X_α, Y_α of $C' \restriction b_\alpha$ and $C' \restriction -b_\alpha$ respectively of cardinality less than κ. But there are only $2^{<\kappa} \leq \lambda$ possibility of such (X_α, Y_α).

2) follows from *1*) since, for an $\leq_\kappa$-generated Boolean algebra B, we see, e.g. by Theorem 5.1, that $\{\, C \in [B]^\lambda : C \leq_\kappa B \,\}$ is cofinal (actually club) in $[B]^\lambda$.

3): Otherwise there would be a countable $C \leq B$ and $b_\alpha \in B$, $\alpha < \omega_1$ such that $tp_C(b_\alpha)$, $\alpha < \omega_1$ are pairwise distinct. As B is κ-openly generated there is an openly generated $C' \leq B$ such that $C \leq C'$ and $b_\alpha \in C'$ for all $\alpha < \omega_1$. By *2*), this is a contradiction. ❏ (Lemma 8.7)

Theorem 8.8 *Suppose that, for a regular uncountable cardinal κ, a Boolean algebra B satisfies the κ-cc, has the κ^+-Bockstein separation property and μ-stable for every $\mu < \kappa$. If χ is sufficiently large and M a $\mathcal{H}(\kappa)$-like elementary submodel of $\mathcal{H}(\chi)$ such that B, $\kappa \in M$, then $B \cap M \leq_{\rm rc} B$. In particular, if additionally i) and ii) in Thmofcorrection hold (this is the case e.g. under $\neg 0\#$), then B is $\leq_\kappa$-generated.*

Proof. Let B, χ, M be as above. For arbitrary $b_0 \in B$, we show that $q^B_{B\cap M}(b_0)$ exists. Let U be a maximal pairwise disjoint subset of $B\cap M \restriction -b_0$. Since B satisfies the κ-cc, U is of cardinality less than κ. Hence, by $\mathcal{H}(\kappa)$-likeness of M, there is $N \in M$ such that $U \cup \{ B, U \} \subseteq N$, $N \prec M$ and $|N| < \kappa$. Let $T = \{ tp_{B\cap N}(b) : b \in B \}$. By $|B \cap N|$-stability of B, we have $|T| \leq |B \cap N| < \kappa$. Hence there is $N' \in M$ such that $N \cup T \subseteq N'$, $N' \prec M$ and $|N'| < \kappa$. By $T \subseteq N'$, there is some $b_1 \in N'$ such that $tp_{B\cap N}(b_0) = tp_{B\cap N}(b_1)$. Let $K = \{ c \in B\cap N : c\cdot b_1 = 0 \}$. Then $K \in N'$ and $U \subseteq K$. Let $J = K^\perp$. By $J^{\perp\perp} = K^{\perp\perp\perp} = K^\perp = J$, J is a regular ideal on B and $J \in N'$. Since B has the κ^+-Bockstein separation property, there is $X \in N'$ such that $|X| \leq \kappa$ and X is cofinal in J. By Lemma 5.10, *1*), we have $\kappa \subseteq M$ and hence $X \subseteq B \cap M$. Now, since $b_0 \in J$, there is some $d \in X$ such that $b_0 \leq d$. We claim that this d is the upper projection of b_0 onto $B \cap M$. Otherwise there would be $c \in B \cap M$ such that $b_0 \leq c$ and $d \cdot -c \neq 0$. By $d \cdot -c \in B \cap M$, $b_0 \cdot (d \cdot -c) = 0$ and by maximality of U, we could find some $e \in U$ such that $(d \cdot -c) \cdot e \neq 0$. But this is a contradiction to $d \in J$ as $J = K^\perp \subseteq U^\perp$. ❑ (Theorem 8.8)

Corollary 8.9 *Let B be a Boolean algebra and assume that i) and ii) of Theorem 5.12 hold for $\kappa = \aleph_1$. If B satisfies the ccc, has the $\aleph_2$-Bockstein separation property and $\aleph_0$-stable, then B is $\leq_\sigma$-generated.* ❑

Corollary 8.10 *Let B be a Boolean algebra and assume that i) and ii) of Theorem 5.12 hold for $\kappa = \aleph_1$. If B is κ-openly generated for some regular $\kappa > \aleph_1$, then B is $\leq_\sigma$-generated.*

Proof. By Lemma 8.4, Lemma 8.6 and Lemma 8.7, *3*), B is ccc, has the Bockstein separation property and is $\aleph_0$-stable. Hence by Corollary 8.9, B is $\leq_\sigma$-generated. ❑ (Corollary 8.10)

Theorem 8.11 *Suppose that B is κ-openly generated for some $\kappa > \aleph_1$. Then B is λ-stable for every λ.*

Proof. Let B be κ-openly generated for some $\kappa > \aleph_1$. We shall prove by induction on λ that B is λ-stable. For $\lambda \geq |B|$ this is clear anyway. Hence we assume that $\lambda < |B|$ holds.

For $\lambda = \aleph_0$, B is $\aleph_0$-stable by Lemma 8.7, *3*).

Now, suppose that $\lambda > \aleph_0$ and we have shown that B is μ-stable for every $\mu < \lambda$. By Lemma 8.4 and Lemma 8.6, B satisfies the ccc and has the Bockstein separation property. Hence, by Theorem 8.8, $B \cap M \leq_{\rm rc} B$ for any sufficiently large χ and V_λ-like elementary submodel of $\mathcal{H}(\chi)$ such that $B, \lambda \in M$. By Lemma 5.10, *2*),

$$\{ B \cap M : M \text{ is } V_\lambda\text{-like elementary submodel of } \mathcal{H}(\chi) \text{ such that } B, \lambda \in M \}$$

is unbounded in $[B]^\lambda$. Hence by Lemma 8.7, *1*), it follows that B is λ-stable.

❑ (Theorem 8.11)

Theorem 8.12 *For a regular cardinal $\kappa > \aleph_1$ and a Boolean algebra B the following are equivalent.*

a) *B is κ-openly generated.*

b) *B has the ccc, satisfies the Bockstein separation property and μ-stable for every $\mu < \kappa$. $\{C : C \leq B, |C| < \kappa, C$ is openly generated$\}$ is cofinal (with respect to $\subseteq$) in $[B]^{<\kappa}$.*

c) *For some, or equivalently any sufficiently large χ and for any regular uncountable $\mu < \kappa$, if M is $\mathcal{H}(\mu)$-like elementary submodel of $\mathcal{H}(\chi)$ such that B, $\mu \in M$, then $B \cap M \leq_{\rm rc} B$ and, if additionally $|M| < \kappa$, then $B \cap M$ is openly generated.*

d) *For some, or equivalently any sufficiently large χ and for any regular uncountable $\mu < \kappa$, if M is V_μ-like elementary submodel of $\mathcal{H}(\chi)$ such that B, $\mu \in M$, then $B \cap M \leq_\sigma B$ and $B \cap M$ is openly generated.*

e) *For some, or equivalently any sufficiently large χ, if M is an elementary submodel of $\mathcal{H}(\chi)$ such that $B \in M$, $\omega_1 \subseteq M$ and $|M| < \kappa$, then $B \cap M \leq_\sigma B$ and $B \cap M$ is openly generated.*

Proof. *a*) $\Rightarrow$ *b*): The first assertion of *b*) follows from Lemma 8.4, Lemma 8.6 and Theorem 8.11. The second one is trivial.

b) $\Rightarrow$ *c*): Suppose that B is as in *b*) and χ, M are as in *c*). Then by Theorem 8.8, we have $B \cap M \leq_{\rm rc} B$. If $|B \cap M| < \kappa$, then, there is an openly generated $C \leq B$ such that $B \cap M \leq C$. $B \cap M \leq_{\rm rc} C$ by Lemma 3.1, *2*). Hence by Lemma 5.3, $B \cap M$ is also openly generated.

c) $\Rightarrow$ *d*): Trivial.

d) $\Rightarrow$ *e*): Suppose that B is as in *d*) and χ, M are as in *e*). By Proposition 5.11, B is $\leq_\sigma$-generated. Hence, by Theorem 5.1, *c*), we have $B \cap M \leq_\sigma B$. (Note that ω_1 is definable and hence $\omega_1 \in M$ holds for any elementary submodel of $\mathcal{H}(\chi)$.) Since $|M| = \mu < \kappa$, there is a V_μ-like elementary submodel M' of $\mathcal{H}(\chi)$ such that $M \subseteq M'$ by Lemma 5.10, *2*). Since $B \cap M \leq_\sigma B \cap M'$ and $B \cap M'$ is openly generated by the assumption, it follows by Lemma 5.3 that $B \cap M$ is also openly generated.

e) $\Rightarrow$ *a*): Suppose that B satisfies the condition *d*). Let $\mathcal{C} = \{ M \in [\mathcal{H}(\chi)]^{<\kappa} : M \prec \mathcal{H}(\chi), B \in M, \omega_1 \subseteq M \}$. Then $\mathcal{C}$ is club in $[\mathcal{H}(\chi)]^{<\kappa}$. Hence, by Lemma 2.8, $\mathcal{C}' = \{ B \cap M : M \in \mathcal{C} \}$ contains a club in $[B]^{<\kappa}$. By the assumption each $C \in \mathcal{C}'$ is openly generated. ❑ (Theorem 8.12)

Corollary 8.13 *For any regular κ, λ with $\kappa \leq \lambda$, if B is λ-openly generated, then B is κ-openly generated.*

Proof. By Theorem 8.12, e). ❑ (Corollary 8.13)

At the moment, we do not know if the assertion corresponding to Corollary 8.13 also holds for κ-projective Boolean algebras.

Problem 8.14 *For regular κ, λ with $\kappa \leq \lambda$, is it true that every λ-projective Boolean algebra is κ-projective?*

Proposition 8.15 *Suppose that κ is a regular cardinal greater than $\aleph_1$. If B is a κ-openly generated Boolean algebra and $A \leq_\sigma B$, then A is also κ-openly generated.*

Proof. We shall show that A satisfies the condition of Theorem 8.12, e). Let χ be sufficiently large and M be an elementary submodel of $\mathcal{H}(\chi)$ such that $A \in M$. Without loss of generality we may assume[15] κ, $B \in M$. By Theorem 8.12, we have $B \cap M \leq_\sigma B$. Since $A \leq_\sigma B$, we have $M \models$ "$A \leq_\sigma B$". It follows that $A \cap M \leq_\sigma B \cap M$. Hence $A \cap M \leq_\sigma B$. In particular $A \cap M \leq_\sigma A$. If $|M| < \kappa$, then $|B \cap M| < \kappa$. Hence there is openly generated $C \leq B$ such that $A \cap M \leq C$. By $A \cap M \leq_\sigma C$ and Lemma 5.3, it follows that $A \cap M$ is openly generated.
❑ (Proposition 8.15)

For regular $\kappa > \aleph_1$, the assertion of Theorem 8.12, b) can be formulated[16] in $L_{\infty\kappa}$. Hence we obtain the following.

Proposition 8.16 *For regular $\kappa > \aleph_1$ and a Boolean algebra B, if B is $L_{\infty\kappa}$-projective then B is κ-openly generated.* ❑

The following theorem generalizes Theorem 2.7 in Fuchino–Koppelberg–Takahashi [19].

Theorem 8.17 *1) If B is an $L_{\infty\aleph_2}$-projective Boolean algebra, then B is $\aleph_2$-projective.*

2) If B is an $L_{\infty\aleph_2}$-free Boolean algebra, then B is $\aleph_2$-free.

Proof. *1*): By Proposition 8.16, B is $\aleph_2$-openly generated. Hence, by Lemma 8.1, *3*), B is $\aleph_2$-projective.

2): Let χ be sufficiently large and M be an elementary submodel of $\mathcal{H}(\chi)$ such

[15] in M, the statement "*there is some regular cardinal $\underline{\kappa}$ and a Boolean algebra $\underline{B}$ such that $A \leq_\sigma \underline{B}$ and $\underline{B}$ is $\underline{\kappa}$-openly generated*" holds. Hence there are κ', $B' \in M$ witnessing the statement above. We may rename κ', and B' to κ, B respectively and treat them as if they were the original κ and B. Similar arguments will be used in the following without any further comments.

[16] see the footnote on p.59.

that $B \in M$, $\omega_1 \subseteq M$ and $|M| = \aleph_1$. Then, by *1*) and Theorem 8.12, we have $B \cap M \leq_\sigma B$. By $L_{\infty\aleph_2}$-freeness of B, $B \cap M$ is contained in some free subalgebra of B. By Lemma 5.3, it follows that $B \cap M$ is openly generated and hence projective by $|B \cap M| \leq \aleph_1$. Since $B \cap M$ is $L_{\infty\aleph_1}$-free by Corollary 6.17, $B \cap M$ is free by Corollary 4.3. Since there are club many subalgebras of B of cardinality $\aleph_1$ of the form $B \cap M$, it follows that B is $\aleph_1$-free. ❑ (Theorem 8.17)

Concerning unions of chains of κ-openly generated Boolean algebras, we have the following theorem.

Theorem 8.18

1) Suppose that $(B_n)_{n\in\omega}$ is an increasing sequence of $\aleph_2$-openly generated Boolean algebras such that $B_n \leq_\sigma B_{n+1}$ for every $n \in \omega$. Then $B = \bigcup_{n\in\omega} B_n$ is $\aleph_2$-openly generated.

2) Let κ, λ be regular cardinals such that $\aleph_1 < \kappa \leq \lambda$. Suppose that $(B_\alpha)_{\alpha<\lambda}$ is an continuously increasing sequence of κ-openly generated Boolean algebras such that $B_\alpha \leq_\sigma B_{\alpha+1}$ for every $\alpha < \lambda$. Then $B = \bigcup_{\alpha<\lambda} B_\alpha$ is κ-openly generated.

Proof. *1*): Let χ be sufficiently large and M a V_{ω_1}-like elementary submodel of $\mathcal{H}(\chi)$ such that $B \in M$. By Theorem 8.12, it is enough to show that $B \cap M \leq_{\mathrm{rc}} B$ and $B \cap M$ is openly generated. Without loss of generality we may assume that $(B_n)_{n\in\omega} \in M$. Hence $B_n \in M$ for each $n \in \omega$. Since B_n is $\aleph_2$-openly generated, we have $B_n \cap M \leq_{\mathrm{rc}} B_n$ for $n \in \omega$. Now, take $b \in B$. Then there is $n^* \in \omega$ such that $b \in B_{n^*}$. Let $b^* = q^{B_{n^*}}_{B_{n^*}\cap M}(b)$. We show that b^* is the upper projection of b onto $B \cap M$. Let $b' \in B \cap M$ be such that $b \leq b'$. Since $B_{n^*} \restriction b'$ is countably generated, there is a countable $X \in M$ such that $X \subseteq B_{n^*} \restriction b'$ and X is cofinal in $B_{n^*} \restriction b'$. As X is countable we have $X \subseteq M$. Since $b \in B_{n^*} \restriction b'$, there is $x \in X$ such that $b \leq x$. By definition of b^* it follows that $b \leq b^* \leq x$. As $x \leq b'$ on the other hand, we obtain $b^* \leq b'$ as desired. This shows $B \cap M \leq B$. Now, since $B_n \leq_\sigma B_{n+1}$, we have $B \models$ "$B_n \leq_\sigma B_{n+1}$". Hence $B_n \cap M \leq_\sigma B_{n+1} \cap M$ for all $n \in \omega$. Since $B_n \cap M$ is openly generated for every $n \in \omega$, it follows by Theorem 5.7 that $B \cap M = \bigcup_{n\in\omega} B_n \cap M$ is openly generated as well.

2): Let χ be sufficiently large and $\mu < \kappa$ an arbitrary uncountable regular cardinal and M a V_μ-like elementary submodel of $\mathcal{H}(\chi)$ such that B, $\mu \in M$. By Theorem 8.12, it is enough to show that $B \cap M \leq_\sigma B$ and $B \cap M$ is openly generated. Let $(M_\beta)_{\beta<\mu}$ be a witness of V_μ-likeness of M. Without loss of generality we may assume that $(B_\alpha)_{\alpha<\lambda} \in M_0$. For $\beta < \mu$, let $\alpha_\beta = \sup \lambda \cap M_\beta$. As $M_\beta \in M_{\beta+1}$, we have $\alpha_\beta \in M_{\beta+1}$ and hence $B_{\alpha_\beta} \in M_{\beta+1} \subseteq M$. Since each B_{α_β} is κ=openly generated, it follows by Theorem 8.12 that $B_{\alpha_\beta} \cap M \leq_{\mathrm{rc}} B_{\alpha_\beta}$. Let $\alpha^* = \sup_{\beta<\mu} \alpha_\beta$. Then $B \cap M = B_{\alpha^*} \cap M$. By an argument similar to the one in the proof of *1*), we can show that $B \cap M \leq B_{\alpha^*}$ holds. Since $B_{\alpha^*} \leq_\sigma B$, it follows that $B \cap M \leq_\sigma B$. Since $|B \cap M| \leq \mu < \kappa$, there is an openly generated $C \leq B_{\alpha^*}$ such that $B \cap M \leq C$. Since $B \cap M \leq_\sigma C$, it follows by Theorem 5.7 that $B \cap M$ is openly generated.

❑ (Theorem 8.18)

8.2 Axiom R

This chapter is devoted to some independence results. The independence of the assertions considered here will be established by Boolean algebras constructed in Chapter 7 under E^{ω}_{κ} on one hand and by consequences of Axiom R (see the definition below) on the other.

For regular κ and a set X, $\mathcal{T} \subseteq [X]^{\kappa}$ is said to be *tight* if, for every increasing sequence $(u_\alpha)_{\alpha<\kappa}$ of elements of $\mathcal{T}$, the union $\bigcup_{\alpha<\kappa} u_\alpha$ is also an element of $\mathcal{T}$.

Fleissner's Axiom R ([10]) is the following statement.

(Axiom R): *Suppose* $\lambda \geq \aleph_2$, $\mathrm{cf}\,\lambda > \omega$ *and* $\mathcal{T} \subseteq [\lambda]^{\aleph_1}$ *is cofinal in* $([\lambda]^{\aleph_1}, \subseteq)$ *and tight. Then for any stationary* $S \subseteq [\lambda]^{\aleph_0}$*, there is* $X \in \mathcal{T}$ *such that* $S \cap [X]^{\aleph_0}$ *is stationary in* $[X]^{\aleph_0}$*.*

MA^+(σ-closed) is the axiom which asserts that, for any σ-closed partial ordering P, P-name $\dot{S}$ of stationary subset of ω_1 and dense subsets D_α, $\alpha < \omega_1$ of P, there exists a filter G on P such that $G \cap D_\alpha \neq \emptyset$ for every $\alpha < \omega_1$ and

$$\dot{S}^G = \{\, \alpha < \omega_1 \,:\, p \Vdash_P \text{“}\alpha \in \dot{S}\text{” for some } p \in G \,\}$$

is stationary in ω. It is known (and we shall also see this later) that we need the strength of very large cardinals for the proof of the consistency of Axiom R. On the other hand, it is known that the implication: MM $\Rightarrow$ MA^+(σ-closed) $\Rightarrow$ Axiom R holds (see [4], [42]). In contrast to MM from which $2^{\aleph_0} = \aleph_2$ follows, a model of MA^+(σ-closed) + CH can be constructed starting from a super-compact cardinal.

The following theorem was originally proved under MA^+(σ-closed) (see [14]). Qi Feng then pointed out that the theorem already holds under Axiom R.

Theorem 8.19 (Axiom R) (Qi Feng, see Fuchino [16]) *For a Boolean algebra B the following are equivalent:*

1) *B is openly generated;*

2) *B is $\aleph_2$-projective.*

Proof. *1)* $\Rightarrow$ *2)*: By Lemma 8.1, *4)*.

For *2)* $\Rightarrow$ *1)*, suppose that B is $\aleph_2$-projective. If $|B| \leq \aleph_1$, then B is openly generated. Hence we may assume $|B| \geq \aleph_2$. We consider first:

Case I: $\mathrm{cf}\,|B| > \omega$.
Let

$$\mathcal{T} = \{\, A \,:\, A \leq_{\mathrm{rc}} B,\ A \text{ is projective and } |A| = \aleph_1 \,\}.$$

By Lemma 2.6, *3)* and Theorem 8.12, $\mathcal{T}$ is cofinal in $([B]^{\aleph_1}, \subseteq)$. $\mathcal{T}$ is tight: suppose that $(A_\alpha)_{\alpha<\omega_1}$ is an increasing sequence in $\mathcal{T}$. Let $A = \bigcup_{\alpha<\omega_1} A_\alpha$. If A were not relatively complete in B then there would be $b \in B$ such that b has no lower projection onto A. Hence $(p^B_{A_\alpha}(b))_{\alpha<\omega_1}$ is non eventually constant increasing sequence in A. But this is a contradiction since B satisfies the ccc by Lemma 8.4. Hence we have $A \leq_{\rm rc} B$. Since $\mathcal{T}$ is cofinal in $[B]^{\aleph_1}$, there is some $A' \leq B$ such that $A \leq A'$ and A' is projective. Since $A \leq_{\rm rc} A'$ and $|A| = \aleph_1$ it follows that A is also projective. Hence $A \in \mathcal{T}$.

Now, if B were not openly generated, then

$$S = \{\, C \leq B \,:\, |C| = \aleph_0 \text{ and } C \text{ is not relatively complete in } B\,\}$$

would be stationary in $[B]^{\aleph_0}$. Hence by Axiom R, there would be $A \in \mathcal{T}$ such that $S \cap [A]^{\aleph_0}$ is stationary in $[A]^{\aleph_0}$. But, since $A \leq_{\rm rc} B$, each $C \in S \cap [A]^{\aleph_0}$ is not relatively complete in A. This is a contradiction to the fact that A is projective.

Case II: $\mathrm{cf}\,|B| = \omega$.
Let $\kappa = |B|$ and $\kappa = \sup_{n\in\omega} \kappa_n$ for an increasing sequence $(\kappa_n)_{n\in\omega}$ of uncountable regular cardinals. Since B is σ-filtered by Corollary 8.10, there is a $\aleph_1$-Freese-Nation mapping $f : B \to [B]^{\leq\aleph_0}$ by Theorem 5.5. Let $(B_n)_{n\in\omega}$ be an increasing sequence of subalgebras of B such that

1) $|B_n| = \kappa_n$ for every $n \in \omega$;

2) B_n is closed with respect to f for every $n \in \omega$ and

3) $B = \bigcup_{n\in\omega} B_n$.

By *2*), and by Lemma 5.4, *3*), $B_n \leq_\sigma B$ holds for every $n \in \omega$. Hence by Proposition 8.15, B_n is $\aleph_2$-projective for every $n \in \omega$. Now, by *1*) and Case I, B_n, $n \in \omega$ are openly generated. Hence by Theorem 5.7, it follows that B is also openly generated.

❑ (Theorem 8.19)

From Theorem 8.19 it follows that there is no Boolean algebra with the property *2*) and *5*) in Proposition 7.13[17] for any κ. In particular, under $V = L$, Theorem 8.19 does not hold. This shows that the assertion of Theorem 8.19 is independent from ZFC[18]. Theorem 7.15 gives also counterexamples to the following Theorems 8.21, 8.27. Hence the assertions of these theorems are also independent form ZFC. By the remark before Theorem 7.15, we obtain the following.

Corollary 8.20 (Axiom R) $\square_\kappa$ *does not hold for any* $\kappa \geq \aleph_1$. ❑

It is known that the consistency strength of a very large large cardinal is needed to establish the conclusion of Corollary 8.20 (see e.g. [30]).

Theorem 8.21 (Axiom R) *Every* $L_{\infty\aleph_2}$*-projective Boolean algebra is openly generated.*

[17] note that, by Lemma 8.1 and Corollary 8.13, it follows from *2*) that A is $\aleph_2$-projective. By Proposition 5.8, it follows from *5*), that A is openly generated.

[18] of course under the assumption that ZFC + Axiom R is consistent. By Corollary 8.20 we need the consistency strength of some very large cardinal for the consistency.

Proof. By Proposition 8.16 and Theorem 8.19. ❑ (Theorem 8.21)

Theorem 8.21 restricted to subalgebras of openly generated Boolean algebras holds already in ZFC.

Proposition 8.22 *Every $L_{\infty\aleph_1}$-projective Boolean algebra embeddable into an openly generated Boolean algebra is openly generated.*

Proof. Suppose that B is an $L_{\infty\aleph_1}$-projective Boolean algebra and C be an openly generated Boolean algebra such that $B \leq C$. Then as a subalgebra of a ccc Boolean algebra C, B satisfies the ccc. Hence, by the footnote on p.59, B satisfies the Bockstein separation property. By Corollary 5.22, it follows that B is openly generated. ❑ (Proposition 8.22)

Corollary 8.23 *Suppose that B is an $L_{\infty\aleph_1}$-projective ($L_{\infty\aleph_1}$-free resp.) Boolean algebra of cardinality $\aleph_1$. If B is embeddable into an openly generated Boolean algebra (or even into a dyadic one) then B is projective (free resp.).*

Proof. Suppose that B is $L_{\infty\aleph_1}$-projective Boolean algebra of cardinality $\aleph_1$ and is embeddable into an openly generated Boolean algebra. Then, by Proposition 8.22, B is openly generated. Hence, by Lemma 2.6, *3*), B is projective. If B is $L_{\infty\aleph_1}$-free, then for any countable $C \leq B$, there is $b \in B$ independent over C. By Corollary 4.3, it follows that B is free. ❑ (Corollary 8.23)

Problem 8.24 *Is Corollary 8.23 also true for $L_{\infty\kappa}$-projective/free Boolean algebras for $\kappa > \aleph_1$?* ❑

Theorem 8.21 for $L_{\infty\aleph_1}$-projective Boolean algebras is not true. An $L_{\infty\aleph_1}$-projective Boolean algebra can be even non ccc. Even if an $L_{\infty\aleph_1}$-projective Boolean algebra B of cardinality $\aleph_1$ satisfies the ccc, by Lemma 2.6,*3*), it is not openly generated unless it is projective. However, ccc $L_{\infty\aleph_1}$-projective Boolean algebras are $\leq_\sigma$-generated. To see this, let us first consider the next lemma. !!The next lemma and hence also the consequences of the lemma uses Proposition 5.11 which actually needs some additional assumptions like □-principles!!

Lemma 8.25 *For a ccc Boolean algebra B, if $\mathcal{C} = \{C \in [B]^{\aleph_0} : C \leq_{\mathrm{rc}} B\}$ is cofinal in $[B]^{\aleph_0}$, then B is $\leq_\sigma$-generated.*

Proof. Let χ be sufficiently large and M a V_{ω_1}-like elementary submodel of $\mathcal{H}(\chi)$ such that $B \in M$. By Proposition 5.11, it is enough to show that $B \cap M \leq_\sigma B$ holds. Actually, we can even show $B \cap M \leq_{\mathrm{rc}} B$. Let $(M_\alpha)_{\alpha<\omega_1}$ be a witness of V_{ω_1}-likeness of M. Without loss of generality, we may assume that B, $\mathcal{C} \in M_0$. For $\alpha < \omega_1$, since $B \cap M_\alpha \in M_{\alpha+1}$ and $M_{\alpha+1} \models$ "$B \cap M_\alpha$ is countable", there is $C_\alpha \in \mathcal{C} \cap M_{\alpha+1}$ such that $B \cap M_\alpha \subseteq C_\alpha$. Then $(C_\alpha)_{\alpha<\omega_1}$ is an increasing sequence of relatively complete subalgebras of B and $B \cap M = \bigcup_{\alpha<\omega_1} C_\alpha$. Hence, by Lemma 3.9, it follows that $B \cap M \leq_{\mathrm{rc}} B$. ❑ (Lemma 8.25)

Proposition 8.26 *If B is a ccc $L_{\infty\aleph_1}$-projective Boolean algebra, then B is $\leq_\sigma$-generated.*

Proof. By Theorem 2.5, if A is projective Boolean algebra, then $\{ C \in [A]^{\aleph_0} : C \leq_{\mathrm{rc}} A \}$ is cofinal (even club) in $[A]^{\aleph_0}$. It follows, by $L_{\infty\aleph_1}$-projectivity of B, that $\{ C \in [B]^{\aleph_0} : C \leq_{\mathrm{rc}} B \}$ is cofinal in $[B]^{\aleph_0}$. Hence, by Lemma 8.25, B is $\leq_\sigma$-generated. ❏ (Proposition 8.26)

For $\kappa = \aleph_1$, Theorem 5.7 can be still improved under Axiom R:

Theorem 8.27 (Axiom R) *Suppose that $(B_\alpha)_{\alpha<\delta}$ is a continuously increasing sequence of openly generated Boolean algebras for some ordinal δ such that $\mathrm{cf}\,\delta \neq \omega_1$ and $B_\alpha \leq_\sigma B_{\alpha+1}$ for all $\alpha < \delta$. Then $B = \bigcup_{\alpha<\delta} B_\alpha$ is also openly generated.*

Proof. By Theorem 8.18 and Theorem 8.19. ❏ (Theorem 8.27)

9 Forcing

In this chapter we collect some results on almost freeness which stand in connection with the method of the forcing.

9.1 Potentially free/projective Boolean algebras

Openly generated Boolean algebras can be also characterized in terms of potentially projectiveness.

Theorem 9.1 *For a Boolean algebra B, the following are equivalent:*

a) *B is openly generated;*

b) *B is proper–potentially projective;*

c) *if P is a proper partial ordering collapsing $|B|$ to $\leq \aleph_1$, then*

$$\Vdash_P \text{“}B \text{ is projective”}$$

.

Proof. *a)* ⇒ *c)*: Let B be openly generated. Then, by Theorem 5.5, there is an $\aleph_0$-Freese-Nation mapping $f : B \to [B]^{<\aleph_0}$. Clearly we have $\Vdash_P$ “f is an $\aleph_0$-Freese-Nation mapping on B”. Hence $\Vdash_P$ “B is openly generated”. Since $\Vdash_P$ “$|B| \leq \aleph_1$”, it follows from Lemma 2.6, *3)* that $\Vdash_P$ “B is projective”.

c) ⇒ *b)* is clear.

b) ⇒ *a)*: Let P be a proper partial ordering such that $\Vdash_P$ “B is projective”. If B were not openly generated, then $\mathcal{S} = \{ C \in \mathrm{Sub}^{\aleph_0}(B) : C \leq_{\neg\mathrm{rc}} B \}$ would be stationary in $[B]^{\aleph_0}$. Hence, by properness of P, it follows that $\Vdash_P$ “$\mathcal{S}$ is stationary in $[B]^{\aleph_0}$”. Since we have $\Vdash_P$ “no $C \in \mathcal{S}$ is relatively complete in B”, we obtain $\Vdash_P$ “B is not openly generated”. In particular, $\Vdash_P$ “B is not projective”. This is a contradiction to our assumption. ❑ (Theorem 9.1)

Theorem 9.2 *For an algebra B, the following are equivalent*

a) *B is $\aleph_0$-distributive-potentially free;*

b) *there is a stationary $\mathcal{D} \subseteq [B]^{\aleph_0}$ such that $\mathcal{D} \subseteq \mathrm{Sub}^{\aleph_0}(B)$, every $D \in \mathcal{D}$ is atomless (hence free) and, for every $C \in \mathcal{D}$, $\{ D \in \mathcal{D} : C \leq_{\mathrm{free}} D \}$ is cofinal in $[B]^{\aleph_0}$. (In particular, B is $L_{\infty\aleph_0}$-free by Proposition 6.15.)*

Proof. *a)* ⇒ *b)*: Let P be an $\aleph_0$-distributive partial ordering such that $\Vdash_P$ “B is free”. Let $\dot{X}$ be a P-name of free generator of B. Let

$$\mathcal{D} = \{ [Y]_B : Y \in [B]^{\aleph_0},\ p \Vdash_P \text{“}Y \subseteq \dot{X}\text{”} \text{ for some } p \in P \}.$$

Then each $C \in \mathcal{D}$ is atomless since we have $p \Vdash_P$ “C is atomless” for some $p \in P$.

Claim 9.2.1 *$\mathcal{D}$ is stationary in $[B]^{\aleph_0}$.*

$\vdash$ Otherwise there would be a club $\mathcal{C} \subseteq [B]^{\aleph_0}$ disjoint from $\mathcal{D}$. Since P is $\aleph_0$-distributive, $\mathcal{C}$ remains club in V^P. Since $\Vdash_P$ "there are club many countable subalgebra of B of the form $[Y]_B$ for some $Y \in [\dot{X}]^{\aleph_0}$" and by $\aleph_0$-distributivity of P, there is $p \in P$ and $Y \in [B]^{\aleph_0}$ such that $p \Vdash_P$ "$Y \subseteq \dot{X}$ and $[Y]_B \in \mathcal{C}$". Hence $[Y]_B \in \mathcal{D}$ and $[Y]_B$ is really an element of $\mathcal{C}$. This is a contradiction to the choice of $\mathcal{C}$. $\dashv$ (Claim 9.2.1)

Claim 9.2.2 *For each $C \in \mathcal{D}$, $\{ D \in \mathcal{D} : C \leq_{\text{free}} D \}$ is cofinal in $[B]^{\aleph_0}$.*

$\vdash$ Let $C = [Y]_B$ with $p \Vdash_P$ "$Y \subseteq \dot{X}$" for some $p \in P$. For $U \in [B]^{\aleph_0}$, since $\Vdash_P$ "$[\dot{X}]_B = B$" and P is $\aleph_0$-distributive, there is some $q \leq p$ and $Z \in [B]^{\aleph_0}$ such that $U \subseteq [Z]_B$ and $q \Vdash_P$ "$Y \subseteq Z \subseteq \dot{X}$". Let $D = [Z]_B$. Then we have $D \in \mathcal{D}$, $C \leq_{\text{free}} D$ and $U \subseteq D$. $\dashv$ (Claim 9.2.2)

Thus $\mathcal{D}$ is as in *b*).

b) $\Rightarrow$ *a*): Suppose that B and $\mathcal{D}$ are as in *b*). Let P be a σ-closed partial ordering collapsing $|B|$ to $\aleph_1$. In V^P, $\mathcal{D}$ is still stationary in $[B]^{\aleph_0}$. In V^P, let $\dot{Q}$ be an $\aleph_0$-distributive partial ordering shooting a club in $\mathcal{D}$. Thus, in $V^{P*\dot{Q}}$, there is a club $\mathcal{C} \subseteq \mathcal{D}$. In $V^{P*\dot{Q}}$, we can construct a continuously increasing sequence $(B_\alpha)_{\alpha<\omega_1}$ of elements of $\mathcal{C}$ such that $B_\alpha \leq_{\text{free}} B_{\alpha+1}$ for every $\alpha < \omega_1$. Returning to V, this means that $\Vdash_{P*\dot{A}}$ "B is free" holds. ❑ (Theorem 9.2)

With ccc-potentially freeness/projectivity we obtain nothing new. For Boolean algebras alone, we could give an easier proof to the following theorem by using Theorem 4.1.

Proposition 9.3 ([12]) *Let A be an algebra in a variety V.*

1) If there exists a ccc partial ordering P such that $\Vdash_P$ "A is free" then A is really free. In particular, a Boolean algebra A is ccc-potentially free if and only if B is really free.

2) If there exists a ccc partial ordering P such that $\Vdash_P$ "A is projective" then A is really projective. In particular, a Boolean algebra A is ccc-potentially projective if and only if B is really projective.

Proof. *1*): Let P be as above and $\dot{X}$ be a P-name such that $\Vdash_P$ "$\dot{X}$ is a free generator of A".

Claim 9.3.1 *Let $a \in A$ and B be a subalgebra of A such that $\Vdash_P$ "$B = [\dot{X} \cap B]_A$". Then there is a subalgebra B' of A such that B' is countably generated over B, $a \in B'$ and $\Vdash_P$ "$B' = [\dot{X} \cap B']_A$". If B is free then B' is also free and any free generator of B can be extended to a free generator of B'.*

$\vdash$ For each $b \in A$ let Y_b be a countable subset of A such that $\Vdash_P$ "$[\dot{X} \cap Y_b]_A \ni b$". This is possible since P satisfies the ccc. Let Y be a countable subset of B such that $a \in [Y]_A$ and $Y_b \subseteq Y$ for every $b \in Y$. Then $B' = [B \cup Y]_A$ is as desired.

Now suppose that U is a free generator of B. Let $Y = \{ y_n : n \in \omega \}$. Let $p_n \in P$, $k_n \in \omega$ and $u_{n,i} \in B' \setminus B$ for $i \leq k_n$ and $n \in \omega$ be such that

$$p_n \geq p_{n+1} \quad \text{for every } n \in \omega;$$
$$p_n \Vdash_P \text{“} u_{n,0}, \ldots, u_{n,k_n} \in \dot{X} \text{”} \quad \text{for every } n \in \omega;$$
$$p_n \Vdash_P \text{“} y_n \in [B \cup \{ u_{n,0}, \ldots, u_{n,k_n} \}]_A \text{”} \quad \text{for all } n \in \omega.$$

It follows that

$$B' = [B \cup \{ u_{n,i} : i \leq k_n, n \in \omega \}]_A.$$

Clearly $U \cup \{ u_{n,i} : i \leq k_n, n \in \omega \}$ is then a free generator of B'. $\dashv$ (Claim 9.3.1)

Let $\kappa = |A|$. By Claim 9.3.1 we can construct sequences $(A_\alpha)_{\alpha<\beta}$ and $(X_\alpha)_{\alpha<\beta}$ for some $\beta \leq \kappa$ inductively so that:

$(A_\alpha)_{\alpha<\beta}$ is a continuously increasing sequence of subalgebras of A;

$(X_\alpha)_{\alpha<\beta}$ is a continuously increasing sequence of subsets of A and X_α is a free generator of A_α for all $\alpha < \beta$;

$\Vdash_P$ "$A_\alpha = [\dot{X} \cap A_\alpha]_A$" for all $\alpha < \beta$;

$\bigcup_{\alpha<\beta} A_\alpha = A$.

Then $\bigcup_{\alpha<\beta} X_\alpha$ is a free generator of A.

2): Suppose that $\Vdash_P$ "A is projective" for a ccc partial ordering P. Then $\Vdash_P$ "$A \oplus \mathrm{Fr}\,\kappa$ is free" for $\kappa = |A|$. By *1*), it follows that $A \oplus \mathrm{Fr}\,\kappa$ is really free. Hence A is projective. ❑ (Proposition 9.3)

An assertion similar to Proposition 9.3 also holds for potentially openly generatedness:

Proposition 9.4 *A Boolean algebra B is proper-potentially openly generated if and only if B is really openly generated.*

Proof. Suppose that $\Vdash_P$ "B is openly generated" for some proper partial ordering P. In V^P, let $\dot{Q}$ be a proper partial ordering collapsing $|B|$ to $\leq \aleph_1$. By Theorem 9.1, we have $\Vdash_{P*\dot{Q}}$ "B is projective". Since $P * \dot{Q}$ is proper it follows again by Theorem 9.1 that B is openly generated. ❑ (Proposition 9.4)

9.2 Large cardinals

In [44], Ščepin asked if $\aleph_3$-projectiveness implies projectiveness. The answer to this question is negative by Theorem 7.20. It seems that we need the consistency strength of some large cardinal to get a cardinal κ such that κ-projectiveness implies projectiveness. In the following we shall give some of known results concerning this kind of compactness. We shall formulate the following Theorems 9.5, 9.7 and 9.8 only for "κ-freeness" and "freeness" but these theorems and their corresponding proofs are valid for "κ-projectivity" and "projectivity" as well. For Boolean algebras they even hold for "κ-openly generatedness" and "openly generatedness".

Theorem 9.5 *Let κ be a weakly compact cardinal. Then every κ-free algebra A of cardinality κ is free.*

Proof. Suppose that A is κ-free and $|A| = \kappa$. Then there is a continuously increasing sequence of elements of $\mathrm{Sub}^{<\kappa}(A)$ such that each A_α is free and $A = \bigcup_{\alpha<\kappa} A_\alpha$. Toward a contradiction, let us assume that A is not free. Let

$$S = \{\,\alpha < \kappa \,:\, \{\,\beta < \kappa \,:\, \alpha < \beta,\ A_\alpha \leq_{\neg\text{free}} A_\beta\,\} \text{ is stationary}\,\}.$$

Claim 9.5.1 *S is stationary.*

$\vdash$ Otherwise there would be a club $C \subseteq \kappa$ such that for every $\alpha \in C$ the set $\{\,\beta < \kappa \,:\, \alpha < \beta,\ A_\alpha \leq_{\text{free}} A_\beta\,\}$ contains a club set $C_\alpha \subseteq \kappa$. Let

$$C' = C \cap \triangle_{\alpha<\kappa} C_\alpha.$$

Then C' is still club and for any α, $\beta \in C$ such that $\alpha < \beta$, we have $A_\alpha \leq_{\text{free}} A_\beta$. By Lemma 3.7, *1*), it follows that A is free which contradicts our assumption.

$\dashv$ (Claim 9.5.1)

Since κ is weakly compact, there is a regular $\lambda < \kappa$ such that $|A_\nu| < \lambda$ and $S_\nu \cap \lambda$ is stationary in λ for any $\nu \in S \cap \lambda$ (see e.g. Proposition 4.6 in Kanamori [29]). It follows that A_λ is not free. This is a contradiction. ❑ (Theorem 9.5)

Theorem 9.6 *1*) *Let κ be weakly compact. For an algebra A of cardinality κ, if $\{\,C \in \mathrm{Sub}^{<\kappa}(A) \,:\, C$ is free$\,\}$ is cofinal in $[A]^{<\kappa}$ then A is embeddable into a free algebra. In particular, if A is an $L_{\infty\kappa}$-free algebra of cardinality κ, then A is embeddable into a free algebra.*

2) *Let κ be strongly compact. For an algebra A, if $\{\,C \in \mathrm{Sub}^{<\kappa}(A) \,:\, C$ is free$\,\}$ is cofinal in $[A]^{<\kappa}$ then A is embeddable into a free algebra. In particular, if A is an $L_{\infty\kappa}$-free algebra, then A is embeddable into a free algebra.*

Proof. *a*): Let L_0 be the language of algebras in the variety and let $L = L_0 \cup \{ U \} \cup \{ c_a : a \in A \}$ where U is a unary relation symbols and c_a a new constant symbol for each $a \in A$. Let

$$\begin{aligned} T = \{ &\text{“}U \text{ is independent subset”} \} \\ &\cup \{ \varphi(c_{a_1}, \ldots, c_{a_n}) : \varphi \text{ is a quantifier free } L_0\text{-formula such that} \\ &\qquad\qquad A \models \varphi[a_1, \ldots, a_n] \text{ for some } a_1, \ldots, a_n \in A \} \\ &\cup \{ \text{“}c_a = t(\overline{u}) \text{ for some } L_0\text{-term } t \text{ and } \overline{u} \in U\text{”} : a \in A \}. \end{aligned}$$

It is easy to see that T can be formulated in $L_{\kappa\kappa}$, $|T| = \kappa$ and T is κ-satisfiable. Since κ is weakly compact, it follows that T is satisfiable. Clearly, any model of T gives an embedding of A into a free algebra.

2): Similarly to *1*). ❑ (Theorem 9.6)

Theorem 9.7 *Let κ be supercompact. For an algebra B, if B is κ-free then B is free.*

Proof. Suppose that there exists a non-free κ-free algebra B. Let $(B_i)_{i\in I}$ be a κ-free filtration of B for an index set $I = (I, \leq)$ and let $\lambda \geq |B|, |I|$. Let $j : V \to M$ be an elementary embedding with critical point κ such that $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$. Let $j((B_i)_{i\in I}) = (B^*_{i^*})_{i^*\in I^*}$. By $|j[B]|, |j[I]| \leq \lambda$ we have that $j[B], j[I] \in M$. Since $j[B]$ ($\cong B$) is not free, $M \models$“$j[B]$ is not free”. By $|B_i| < \kappa$ we have $B^*_{j(i)} = j[B_i]$ for all $i \in I$. Hence we have that $M \models$“$j[B] = \bigcup_{i^*\in j[I]} B^*_{i^*}$”. Since $j(\kappa) > \lambda \leq |j[I]|^M$ and by the elementarity of j, it follows that $M \models$“$j[B] = B^*_{i^*}$ for some $i^* \in I^*$”. Hence again by the elementarity of j, $M \models$“$j[B]$ is free”. This is a contradiction. ❑ (Theorem 9.7)

Theorem 9.8 (Shelah, see Ben-David [5]) *If the existence of a supercompact cardinal is consistent with* ZFC *then the following assertion is also consistent with* ZFC*:*

> *For any algebra B, if B is $2^{\aleph_0}$-free then B is free.*

Proof. Let κ be a supercompact cardinal. Let $P = \mathrm{Fn}(\kappa, 2)$ and let G be a V-generic filter over P where V is our ground model.

We shall show that $V[G]$ satisfies the assertion in the theorem: otherwise, in $V[G]$, there would be a non-free κ-free B in $V[G]$. Let $(B_i)_{i\in I}$ be a κ-free filtration of B in $V[G]$. Without loss of generality we may assume that the underlying sets of B and I are some cardinals λ and μ such that $\lambda \geq \mu$. Further we may assume that 0 and 1 of the algebra B are the ordinals 0 and 1 respectively, and these facts for the corresponding P-names are already forced by 1_P. Let $\dot{a}$, $\dot{m}$, $\dot{c}$ be “nice” P-names (in the sense of [35], p.208) of addition, multiplication and complement operation in B respectively and $\dot{r}$ be a P-name of the partial ordering of I. By the ccc of P we have: $|\dot{a}| = |\dot{m}| = |\dot{c}| = \lambda$ and $|\dot{r}| \leq \lambda$. For $\alpha \in \mu$ let $\dot{B}_\alpha$ be a nice

P-name of (the underlying set of) B_α. Again by the ccc of P, $\dot{B}_\alpha$ has cardinality $< \kappa$. Now let $j : V \to M$ be an elementary embedding with critical point κ such that $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$ hold. Let $\dot{a}^* = j(\dot{a})$, $\dot{m}^* = j(\dot{m})$, $\dot{c}^* = j(\dot{c})$, $\dot{r}^* = j(\dot{r})$, $\dot{S}^* = j((\dot{B}_\alpha)_{\alpha<\mu})$ and $P^* = j(P)$. Then we have that $M \models$"$P^* = \mathrm{Fn}(j(\kappa), 2)$" and $M \models\ \Vdash_P$ "$\dot{S}^*$ with the index set $\mu^* = (j(\mu), \dot{r}^*)$ is $j(\kappa)$-free filtration of $(j(\lambda), \dot{a}^*, \dot{m}^*, \dot{c}^*, 0, 1)$".

Let $\dot{a}^\dagger = j[\dot{a}]$, $\dot{m}^\dagger = j[\dot{m}]$, $\dot{c}^\dagger = j[\dot{c}]$, $\dot{r}^\dagger = j[\dot{r}]$ and $P^\dagger = j[P]$. Since $|\dot{a}|,\ldots, |P| \leq \lambda$ we have $\dot{a}^\dagger,\ldots, P^\dagger \in M$. $P^\dagger = \mathrm{Fn}(j[\kappa], 2)$ and $\dot{a}^\dagger,\ldots, \dot{r}^\dagger$ are $P^\dagger$-names. Since $\Vdash_{P^\dagger}$ "$(j[\lambda], \dot{a}^\dagger, \dot{m}^\dagger, \dot{c}^\dagger, 0, 1)$ is a non-free algebra" we have that

$$M \models\ \Vdash_{P^\dagger} \text{“}(j[\lambda], \dot{a}^\dagger, \dot{m}^\dagger, \dot{c}^\dagger, 0, 1) \text{ is a non-free algebra”}.$$

Hence it follows by Proposition 9.3 that

$$M \models\ \Vdash_{P^*} \text{“}(j[\lambda], \dot{a}^\dagger, \dot{m}^\dagger, \dot{c}^\dagger, 0, 1) \text{ is a non-free algebra”}.$$

On the other hand, since

$$M \models\ \Vdash_{P^*} \text{“}(j[\lambda], \dot{a}^\dagger, \dot{m}^\dagger, \dot{c}^\dagger, 0, 1) = \bigcup_{\alpha<\mu} j(\dot{B}_\alpha)\text{”}$$

and

$$M \models \text{“}j[\mu] \text{ is directed and } |j[\mu]| < j(\kappa)\text{”},$$

it follows that

$$M \models\ \Vdash_{P^*} \text{“}(j[\lambda], \dot{a}^\dagger, \dot{m}^\dagger, \dot{c}^\dagger, 0, 1) \text{ is an element of } \dot{S}^*\text{”}.$$

Hence

$$M \models\ \Vdash_{P^*} \text{“}(j[\lambda], \dot{a}^\dagger, \dot{m}^\dagger, \dot{c}^\dagger, 0, 1) \text{ is a free algebra”}.$$

This is a contradiction. ❑ (Theorem 9.8)

9.3 A 0-1 law on subalgebras of openly generated Boolean algebras

By the characterization of openly generated Boolean algebras given in Theorem 9.1, we obtain often very elegant proofs of ZFC results on openly generated Boolean algebras using the forcing. The proof of the following Theorem 9.11 is such an example.

Let $\mathcal{T}$ be the set of all terms in the language of Boolean algebras.

Lemma 9.9 (Shelah, in Fuchino–Shelah [20]) *Let κ be a regular uncountable cardinal and let B be a dyadic Boolean algebra of size $\geq \kappa$. Then either $\mathcal{G} = \{ A \in \mathrm{Sub}^{<\kappa}(B) : A \leq_{\mathrm{rc}} B \}$ contains or is disjoint from a club set in $\mathrm{Sub}^{<\kappa}(B)$.*

Proof. Let $B \leq \mathrm{Fr}\, X$. For $Y \subseteq X$ let B_Y denote the Boolean algebra $\langle Y \rangle \cap B$. Let $\tau = \tau(x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathcal{T}$.

For $\bar{a} = (a_1, \ldots, a_n)$ where $\{ a_1, \ldots, a_n \} \in [X]^n$, let

$$\begin{aligned} \mathcal{C}_{\tau,\bar{a}} = \{ B_Y \ : Y \in [X]^{<\kappa},\ Y \supset \{a_1, \ldots, a_n\},\ &\text{for any } \{ b_1, \ldots, b_m \} \in [X \setminus Y]^m \\ &\text{if } t = \tau(b_1, \ldots, b_m, a_1, \ldots, a_n) \in B, \quad \text{then } p^B_{B_Y}(t) \text{ exists} \}. \end{aligned}$$

Note that, if there are $b_1, \ldots, b_m \in X \setminus Y$ such that $t = \tau(b_1, \ldots, b_m, a_1, \ldots, a_n) \in B$ and t has its projection c in B_Y, then c is the projection of $d \in B$ of the form $\tau(d_1, \ldots, d_m, a_1, \ldots, a_n)$.

Claim 9.9.1 *Either $\mathcal{C}_{\tau,\bar{a}}$ contains or is disjoint from a club set.*

$\vdash$ If $\mathcal{C}_{\tau,\bar{a}}$ is bounded in $\mathrm{Sub}^{<\kappa}(B)$ then $\mathrm{Sub}^{<\kappa}(B) \setminus \mathcal{C}_{\tau,\bar{a}}$ contains a club set. Assume that $\mathcal{C}_{\tau,\bar{a}}$ is unbounded in $\mathrm{Sub}^{<\kappa}(B)$.

For each $\sigma = \sigma(x_1, \ldots, x_k, y_1, \ldots, y_n) \in \mathcal{T}$ let

$$\begin{aligned} \mathcal{C}_{\tau,\bar{a},\sigma} = \{ B_Y \ : Y \in [X]^{<\kappa},\ Y \supseteq \{a_1, \ldots, a_n\},\ &\text{for any } \{ b_1, \ldots, b_m \} \in [X \setminus Y]^m \\ &\text{if } t = \tau(b_1, \ldots, b_m, a_1, \ldots, a_n) \in B \text{ then } p^B_{B_Y}(t) \text{ exists and} \\ &\text{the projection } p^B_{B_Y}(t) \text{ is of the form } \sigma(c_1, \ldots, c_k, a_1, \ldots, a_n) \\ &\text{for some } \{ c_1, \ldots, c_k \} \in [Y \setminus \{ a_1, \ldots, a_n \}]^k \quad \}. \end{aligned}$$

Clearly

$$\mathcal{C}_{\tau,\bar{a}} = \bigcup_{\sigma \in \mathcal{T}} \mathcal{C}_{\tau,\bar{a},\sigma} .$$

So there is a $\sigma_0 \in \mathcal{T}$ such that $\mathcal{C} = \mathcal{C}_{\tau,\bar{a},\sigma_0}$ is unbounded. We claim that $\mathcal{C}$ is closed.

Let $(B_\alpha)_{\alpha<\lambda}$ be an increasing sequence in $\mathcal{C}$ for some $\lambda < \kappa$. For $\alpha < \lambda$ let $Y_\alpha \in [X]^{<\kappa}$, be such that $B_\alpha = B_{Y_\alpha}$ and let $Y = \bigcup_{\alpha<\lambda} Y_\alpha$. Then $\bigcup_{\alpha<\lambda} B_\alpha = B_Y$. Let $b_1, \ldots, b_m \in X \setminus Y$ be such that $t = \tau(b_1, \ldots, b_m, a_1, \ldots, a_n) \in B$. By the definition of $\mathcal{C}$ there are $c^\alpha_1, \ldots, c^\alpha_k \in Y_\alpha \setminus \{ a_1, \ldots, a_n \}$ such that

$$p^B_{B_\alpha}(t) = \sigma_0(c^\alpha_1, \ldots, c^\alpha_k, a_1, \ldots, a_n).$$

Hence we have that $m(p^B_{B_\alpha}(t)) = m(p^B_{B_{\alpha'}}(t))$ for any $\alpha, \alpha' < \lambda$, where m is the canonical finitely additive strictly positive measure on $\mathrm{Fr}\, X$. Since

$$p^B_{B_0}(t) \leq p^B_{B_1}(t) \leq p^B_{B_2}(t) \leq \cdots ,$$

it follows that

$$p^B_{B_0}(t) = p^B_{B_1}(t) = p^B_{B_2}(t) = \cdots .$$

Hence $p^B_{B_Y}(t) = p^B_{B_0}(t)$. It follows that $B_Y \in \mathcal{C}$. ⊣ (Claim 9.9.1)

Now if there are $\tau \in \mathcal{T}$ and $\bar{a} = (a_1, \ldots, a_n)$, $a_1, \ldots, a_n \in X$ such that $\mathcal{C}_{\tau,\bar{a}}$ is disjoint from some club set then, since $\mathcal{G}$ is almost contained in $\mathcal{C}_{\tau,\bar{a}}$, $\mathcal{G}$ is also disjoint from a club set. Otherwise, by Claim 9.9.1, for any term $\tau \in \mathcal{T}$ and $\bar{a} = (a_1, \ldots, a_n)$, $a_1, \ldots, a_n \in X$, the set $\mathcal{C}_{\tau,\bar{a}}$ contains a club subset. Since

$$\mathcal{G} \supseteq \bigcap_{\tau \in \mathcal{T}} \{ B_Y : Y \in [X]^{<\kappa}, B_Y \in \bigcap \{ \mathcal{C}_{\tau,\bar{a}} : \bar{a} \in {}^nY \} \},$$

it follows that $\mathcal{G}$ contains a club set. ❏ (Lemma 9.9)

Corollary 9.10 *Let κ be a regular cardinal $> \aleph_1$ and let B be a dyadic Boolean algebra. Then either $\mathcal{G} = \{ A \in \mathrm{Sub}^{<\kappa}(B) : A \leq_{\mathrm{rc}} B \}$ is bounded or $\mathcal{G}$ contains a club subset in $[B]^{<\kappa}$.*

Proof. We may assume that $|B| \geq \kappa$. Suppose that $\mathcal{G}$ is not bounded in $[B]^{<\kappa}$. By Lemma 9.9 it is enough to show that $\mathcal{G}$ is stationary. Let $\mathcal{C}$ be a club set in $[B]^{<\kappa}$. Let $(B_\alpha)_{\alpha<\omega_1}$ and $(B_\alpha)_{\alpha<\omega_1}$ be increasing sequence in $\mathcal{G}$ and $\mathcal{C}$ respectively such that

$$C_\alpha \subseteq B_\alpha \subseteq C_{\alpha+1}$$

for every $\alpha < \omega_1$ holds. If $C_\alpha \in \mathcal{G}$ for some $\alpha < \omega_1$ we are done. So assume that $C_\alpha \notin \mathcal{G}$ for every $\alpha < \omega_1$. Let $C = \bigcup_{\alpha<\omega_1} B_\alpha = \bigcup_{\alpha<\omega_1} C_\alpha$. Since $\mathcal{C}$ is closed we have $C \in \mathcal{C}$. On the other hand, by Lemma 3.9, we have $C \in \mathcal{G}$. ❏ (Corollary 9.10)

Theorem 9.11 (Fuchino-Shelah [20]) *If B is a subalgebra of an openly generated Boolean algebra then either*

$$\mathcal{C} = \{ C \leq B : |C| = \aleph_0, C \textit{ is relatively complete in } B \}$$

contains a closed unbounded subset of $[B]^{\aleph_0}$ or is disjoint from a closed unbounded subset of $[B]^{\aleph_0}$.

Proof. Suppose that B is a subalgebra of a openly generated Boolean algebra A. Assume that $\mathcal{C} = \{ C \leq B : |C| = \aleph_0, C$ is relatively complete in $B \}$ does not contain any closed unbounded subset of $[B]^{\aleph_0}$. Let P be a σ-closed partial ordering such that $\Vdash_P$ "$|A| \leq \aleph_1$". Then $\Vdash_P$ "A is a projective Boolean algebra" by Theorem 9.1. By the assumption and properness of P, we have $\Vdash_P$ "$[B]^{\aleph_0} \setminus \mathcal{C}$ is stationary". Hence by Theorem 9.9, it follows that

$$\Vdash_P \text{“}\mathcal{C} \text{ is disjoint from a closed unbounded set of } [B]^{\aleph_0}\text{”}.$$

Let χ be sufficiently large such that B, P, $\Vdash_P \in \mathcal{H}(\chi)$.

Claim 9.11.1 *For any countable $M \prec \mathcal{H}(\chi)$ such that B, P, $\Vdash_P \in \mathcal{H}(\chi)$, we have $B \cap M \leq_{\neg\mathrm{rc}} B$.*

$\vdash$ Clearly, we have $B \cap M \leq B$. Let $B \cap M = \{ b_n : n \in \omega \}$. Take a P-name $\dot{\mathcal{C}} \in M$ such that

$$\Vdash_P \text{“}\dot{\mathcal{C}} \text{ is closed unbounded subset of } [B]^{\aleph_0} \text{ and } \forall C \in \dot{\mathcal{C}}(C \leq_{\neg\mathrm{rc}} B)\text{”}.$$

Let $f : \omega \to \omega^2$, $n \mapsto (f_0(n), f_1(n))$ be a surjection such that $f_0(n) \leq n$ holds for every $n \in \omega$. We can construct sequences $(p_n)_{n\in\omega}$, $(\dot{B}_n)_{n\in\omega}$ and $(\dot{b}_{n,m})_{n,m\in\omega}$ inductively such that

(1) $(p_n)_{n\in\omega}$ is a decreasing sequence in P and $p_n \in M$ for every $n \in \omega$;

(2) $(\dot{B}_n)_{n\in\omega}$ and $(\dot{b}_{n,m})_{n,m\in\omega}$ are sequence of P-names and $\dot{B}_n$, $\dot{b}_{n,m} \in M$ for all $n, m \in \omega$;

(3) $p_n \Vdash_P$ “$\dot{B}_n \in \dot{\mathcal{C}}$, $\dot{B}_n = \{ \dot{b}_{n,m} : m \in \omega \}$” for all $n \in \omega$;

(4) $p_{n+1} \Vdash_P$ “$\dot{B}_n \leq \dot{B}_{n+1}$, $b_n \in \dot{B}_{n+1}$” for every $n \in \omega$;

(5) p_{n+1} decides $\dot{b}_{f_0(n),f_1(n)}$
(i.e. there is $b \in B$ such that $p_{n+1} \Vdash_P$ “$\dot{b}_{f_0(n),f_1(n)} = b$”. Note that, we have $b \in M$ since p_{n+1}, $\Vdash_P$, $\dot{b}_{f_0(n),f_1(n)} \in M$).

As P is σ-closed, there is $p \in P$ such that $p \leq p_n$ for every $n \in \omega$. By (5), p decides each of $\dot{b}_{n,m}$, $n, m \in \omega$. Hence, for every $n \in \omega$, there is a countable $C_n \leq B$ such that $p \Vdash_P$ “$\dot{B}_n = C_n$” and $C_n \leq B \cap M$ by the remark after (5). Let $C = \bigcup_{n\in\omega} C_n$. By (4), we have $b_n \in C_{n+1}$ for every $n \in \omega$. Hence $C = B \cap M$. By (3), we have $p \Vdash_P$ “$C_n \in \dot{\mathcal{C}}$” for every $n \in \omega$. Hence $p \Vdash_P$ “$C \in \dot{\mathcal{C}}$”. It follows that $p \Vdash_P$ “$C \leq_{\neg\mathrm{rc}} B$”. Thus we have $B \cap M = C \leq_{\neg\mathrm{rc}} B$. $\dashv$ (Claim 9.11.1)

Since there are club many countable subalgebras of B of the form $B \cap M$ for M as in Claim 9.11.1, it follows that $\{ C \leq B : |C| = \aleph_0,\ C \leq_{\neg\mathrm{rc}} B \}$ contains a club subset of $[B]^{\aleph_0}$ ❑ (Theorem 9.11)

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