On geometrical characterizations of $\mathbb{R}$-linear mappings

Sakaé Fuchino (渕野 昌)

Contents

1 Introduction

In this note, we consider several properties characterizing linear mappings $f : X \to Y$ for $\mathbb{R}$-linear spaces X and Y in terms of points and lines in $\mathbb{R}$-linear spaces X and their images in Y by the mappings.

Theorem 4.1 and its corollaries generalize the Fundamental Theorem of Affine Geometry, and characterize the $\mathbb{R}$-linear mappings $f : X \to Y$ with $\dim(f''X) > 1$ as well as the corresponding affine mappings.

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* Graduate School of System Informatics, Kobe University
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan
fuchino@diamond.kobe-u.ac.jp

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This is an extended version of the paper with the same title. All additional details not to be contained in the submitted version of the paper are either typeset in typewriter font (the font this paragraph is typeset) or put in separate appendices. The numbering of the assertions is kept identical with the submitted version.

The most up-to-date file of this extended version is downloadable as: https://fuchino.ddo.jp/papers/linear-mappings-x.pdf

The quite elementary proof of Theorem 4.1 suggests that the result might be a well-known fact possibly originated from the first half of the last century. The author however could not find any appropriate reference to the result in the literature.

Artstein-Avidan and Slomka [1] lists some known variants of the Fundamental Theorem of affine geometry but our results do not seem to be covered by the assertions cited there.

Even in the case that the results presented here turn out later to be a folklore, the author believes that the present note is written in a sufficiently understandable manner that it can serve as a well-written expository article accessible for a wide audience[1)].

In the following, we shall use standard conventions and notation in set theory which might be slightly different from everyday mathematics. In particular, we distinguish sharply between elements $x \in y$ and singletons $\{x\} \subseteq y$. The image of a subset $U \subseteq X$ of the domain of a mapping $f : X \to Y$ is denoted by $f''U$ with is also often denoted as $f[U]$ in the literature. Thus, for a closed interval $[a, b]$, we write $f''[a, b]$ to denote the image of the interval by the function f instead of writing $f[a, b]$ or even, more correctly, $f[[a, b]]$. A natural number n is the set $\{0, 1, ..., n-1\}$ and thus $i \in 2$ means $i = 0$ or $i = 1$.

We work in Z (the modern version of Zermelo's set theory without Axiom of Choice) if not mentioned otherwise.

2 Mappings which preserve proportions on each line

lin-section-1

Let X be a linear space over the scalar field $\mathbb{R}$ (or $\mathbb{R}$-linear space). A subset $P \subseteq X$ is a *point* if P is a singleton. That is, if it is an affine subspace of X of dimension 0. A subset L is a line if it is an affine subspace of X of dimension 0. Thus, $L \subseteq X$ is a line if and only if there are vectors $\mathbb{a}$, $\mathbb{d} \in X$ such that $L = \mathbb{R}\mathbb{d} + \mathbb{a} = \{r\mathbb{d} + \mathbb{a} : r \in \mathbb{R}\}$. Similarly $E \subseteq X$ is a plane if it is an affine subspace of X of dimension 2. Two lines L_0, $L_1 \subseteq X$ are parallel, if either $L_0 = L_1$ or there is a plane $E \subseteq X$ with L_0, $L_1 \subseteq E$ and $L_0 \cap L_1 = \emptyset$. Note that the Parallel

1) Actually, the author began to write the present text as an additional teaching material to his online lecture of an introductory course in linear algebra when he realized that there are several statements like that of Problem 2.5 and Theorem 4.1 whose status was not immediately clear and whose treatment might slightly exceed the capacity of students in the first semester. Most of the details of the following text is still kept elementary and detailed in such a way that even some motivated undergraduate students should be able to follow.

Postulate holds in this setting.

Recall that a mapping $f : X \to Y$ for $\mathbb{R}$-linear spaces X, Y is said to be a *linear mapping*, if we have

(2.1) $f(c\,\mathbbm{a}) = cf(\mathbbm{a})$ for all $\mathbbm{a} \in X$ and $c \in \mathbb{R}$; and lin-1

(2.2) $f(\mathbbm{a}+\mathbbm{b}) = f(\mathbbm{a}) + f(\mathbbm{b})$ for all $\mathbbm{a}$, $\mathbbm{b} \in X$. lin-2

To emphasize that we are talking about linear mappings between $\mathbb{R}$-linear spaces, we shall also say $\mathbb{R}$-linear mappings in contrast to the $\mathbb{Q}$-linear mappings which are linear mappings between $\mathbb{Q}$-linear spaces. Here, a mapping $f : X \to Y$ is a $\mathbb{Q}$-linear mapping, if (2.1)$'$ and (2.2) hold where

(2.1)$'$ $f(c\,\mathbbm{a}) = cf(\mathbbm{a})$ for all $\mathbbm{a} \in X$ and $c \in \mathbb{Q}$.

Note that, for any additive groups X, Y and mapping $f : X \to Y$, if f satisfies (2.2), then $f(\mathbb{0}_X) = \mathbb{0}_Y$ holds, and this fact is proved in most of the standard text books of linear algebra. `[We have` $f(\mathbb{0}_X) = f(\mathbb{0}_X+\mathbb{0}_X) = f(\mathbb{0}_X)+f(\mathbb{0}_X)$`. Thus, by subtracting` $f(\mathbb{0}_X)$ `we obtain` $\mathbb{0}_Y = f(\mathbb{0}_X)$`.]`

The following Lemma is also elementary and well-known.

P-lin-0

Lemma 2.1 *Suppose that X, Y are $\mathbb{R}$-linear spaces, and $f : X \to Y$ a linear mapping. Then*

(2.3) *the image $f''L$ of any line $L \subseteq X$ is either a point or a line in Y; and* lin-3

(2.4) *If $f''L$ for a line $L \subseteq X$ is a line in Y then $f \restriction L$ is 1-1.* lin-3-0

Proof. Suppose that $L = \mathbb{R}\mathbbm{d} + \mathbbm{a}$ with $\mathbbm{d} \neq \mathbb{0}_X$. Then, by linearity of f, $f''L = \{f(r\mathbbm{d}+\mathbbm{a}) : r \in \mathbb{R}\} = \{rf(\mathbbm{d}) + f(\mathbbm{a}) : r \in \mathbb{R}\} = \mathbb{R}f(\mathbbm{d}) + f(\mathbbm{a})$. Thus, $f''L = \{f(\mathbbm{a})\}$ if $f(\mathbbm{d}) = \mathbb{0}_Y$. Otherwise, $f''L$ is the line $\mathbb{R}f(\mathbbm{d}) + f(\mathbbm{a})$. In the latter case, $f \restriction L : L \to Y$; $r\mathbbm{d} + \mathbbm{a} \mapsto rf(\mathbbm{d}) + f(\mathbbm{a})$, $r \in \mathbb{R}$, is 1-1. ❑ (Lemma 2.1)

For two distinct vectors $\mathbbm{a}$, $\mathbbm{b} \in X$, $\mathbbm{a}\,\overline{\ \ }\,\mathbbm{b}$ denotes the line $\mathbb{R}(\mathbbm{b}-\mathbbm{a})+\mathbbm{a} = \{t(\mathbbm{b}-\mathbbm{a})+\mathbbm{a} : t \in \mathbb{R}\}$. $\mathbbm{a}\,\overline{\ \ }\,\mathbbm{b}$ is the unique line L with $\mathbbm{a}$, $\mathbbm{b} \in L$.

$\mathbbm{a}\sqcap\mathbbm{b}$ denotes the closed interval between $\mathbbm{a}$ and $\mathbbm{b}$ ($\mathbbm{a}\sqcap\mathbbm{b} = \{t(\mathbbm{b}-\mathbbm{a})+\mathbbm{a} : 0 \leq t \leq 1\}$). $\mathbbm{a}\frown\mathbbm{b}$ denotes the open interval between $\mathbbm{a}$ and $\mathbbm{b}$: $\mathbbm{a}\frown\mathbbm{b} = \mathbbm{a}\sqcap\mathbbm{b} \setminus \{\mathbbm{a},\mathbbm{b}\}$. $\mathbbm{c} \in X$ *divides* $\mathbbm{a}\sqcap\mathbbm{b}$ *in ratio* $r : s$ for r, $s \in \mathbb{R}$ with $r+s \neq 0$, if $\mathbbm{c} = \frac{r}{r+s}(\mathbbm{b}-\mathbbm{a})+\mathbbm{a}$. Note that, if $\frac{r}{r+s} \notin [0,1]$, the point $\mathbbm{c}$ dividing $\mathbbm{a}\sqcap\mathbbm{b}$ in ratio $r : s$ is on the line $\mathbbm{a}\,\overline{\ \ }\,\mathbbm{b}$ outside the interval $\mathbbm{a}\sqcap\mathbbm{b}$ and that any point on the line $\mathbbm{a}\,\overline{\ \ }\,\mathbbm{b}$ can be represented as a point dividing $\mathbbm{a}\sqcap\mathbbm{b}$ in some ratio.

For convenience, we shall also use the notation $\mathbbm{a}\,\overline{\ \ }\,\mathbbm{b}$, $\mathbbm{a}\sqcap\mathbbm{b}$, and $\mathbbm{a}\frown\mathbbm{b}$ for $\mathbbm{a}$, $\mathbbm{b}$ with $\mathbbm{a} = \mathbbm{b}$ defining $\mathbbm{a}\,\overline{\ \ }\,\mathbbm{a} = \mathbbm{a}\sqcap\mathbbm{a} = \{\mathbbm{a}\}$ and $\mathbbm{a}\frown\mathbbm{b} = \emptyset$ in this case.

The proof of Lemma 2.1 actually shows the following:

P-lin-1

Lemma 2.2 *Suppose that X, Y are $\mathbb{R}$-linear spaces, and $f : X \to Y$ a linear mapping. Then, for any $\mathbb{a}$, $\mathbb{b}$, $\mathbb{c} \in X$ with $\mathbb{a} \neq \mathbb{b}$ and $f(\mathbb{a}) \neq f(\mathbb{b})$, and for any r, $s \in \mathbb{R}$, if $\mathbb{c}$ divides $\mathbb{a}{}^\frown\mathbb{b}$ in ratio $r : s$ then $f(\mathbb{c})$ divides $f(\mathbb{a}){}^\frown f(\mathbb{b})$ in ratio $r : s$.*

❑

The property of the linear mapping mentioned in Lemma 2.1 does not characterize linear mappings even when we add the condition $f(\mathbb{0}_X) = \mathbb{0}_Y$:

P-lin-2

Lemma 2.3 *For any non zero-dimensional $\mathbb{R}$-linear spaces X, Y, there are non-linear mappings $f : X \to Y$ satisfying*

(2.3) *the image $f''L$ of any line $L \subseteq X$ is either a point or a line in Y;*

(2.4) *If $f''L$ for a line $L \subseteq X$ is a line in Y then $f \restriction L$ is 1-1; and*

(2.5) $f(\mathbb{0}_X) = \mathbb{0}_Y$. lin-4

Proof. Suppose that $\mathbb{e}_0 \in X \setminus \{\mathbb{0}_X\}$ and $\mathbb{d}_0 \in Y \setminus \{\mathbb{0}_Y\}$. Let B be a linear basis of X with $\mathbb{e}_0 \in B$. For $\mathbb{x} \in X$, let $\varphi(\mathbb{x}) \in \mathbb{R}$ be the $\mathbb{e}_0$-coordinate of $\mathbb{x}$ with respect to B. That is, let $\varphi(\mathbb{x}) \in \mathbb{R}$ be such that there is a linear combination $\mathbb{c}$ of elements of $B \setminus \{\mathbb{e}_0\}$ such that $\mathbb{x} = \varphi(\mathbb{x})\mathbb{e}_0 + \mathbb{c}$.

Let $\psi : \mathbb{R} \to \mathbb{R}$ be any non-linear bijective function with

(2.6) $\psi(0) = 0$. lin-5

Let $f : X \to Y$ be defined by $f(\mathbb{x}) = \psi(\varphi(\mathbb{x}))\mathbb{d}_0$. Clearly f is not a linear mapping. $f(\mathbb{0}_X) = \mathbb{0}_Y$ by (2.6) and the definition of f.

For a line $L \subseteq X$ with $L = \mathbb{R}\mathbb{b} + \mathbb{a}$ for some $\mathbb{a}$, $\mathbb{b} \in X$, if $\varphi(\mathbb{b}) = 0$ then $f''L = \{\psi(\varphi(\mathbb{a}))\mathbb{d}_0\}$. Otherwise, by bijectivity of ψ, we have $f''L = \mathbb{R}\mathbb{d}_0$ and $f \restriction L$ is 1-1. ❑ (Lemma 2.3)

In contrast, the property mentioned in Lemma 2.2 *does* characterize affine mappings. In particular, if we add the condition (2.5), then we obtain a characterization of linear mappings.

P-lin-3

Lemma 2.4 *For any $\mathbb{R}$-linear spaces X, Y, suppose that $f : X \to Y$ is such that f satisfies*

(2.5) $f(\mathbb{0}_X) = \mathbb{0}_Y$;

(2.7) *for any $\mathbb{a}$, $\mathbb{b} \in X$ with $\mathbb{a} \neq \mathbb{b}$ and $f(\mathbb{a}) \neq f(\mathbb{b})$, $\mathbb{c} \in X$, and r, $s \in \mathbb{R}$, if $\mathbb{c}$ divides $\mathbb{a}{}^\frown\mathbb{b}$ in ratio $r : s$, then $f(\mathbb{c})$ divides $f(\mathbb{a}){}^\frown f(\mathbb{b})$ in $r : s$.* lin-6

Then f is a linear mapping.

Proof. Suppose that $f : X \to Y$ satisfies (2.5) and (2.7). If $f''X = \{\mathbb{0}_Y\}$, then f is a linear mapping. Thus, let us assume that there is $\mathbb{b} \in X$ such that $f(\mathbb{b}) \neq \mathbb{0}_Y$. $\mathbb{b} \neq \mathbb{0}_X$ by (2.5).

For $\mathbb{c} \in \mathbb{R}\mathbb{b}$, if $\mathbb{c} = c\,\mathbb{b}$ for some $c \in \mathbb{R}$, then $\mathbb{c}$ divides $\mathbb{0}_X \urcorner \mathbb{b}$ in ratio $c : 1-c$, since $\mathbb{c} = c\,\mathbb{b} = \frac{c}{c+(1-c)}(\mathbb{b} - \mathbb{0}_X) + \mathbb{0}_X$. By (2.7), it follows that $f(c\,\mathbb{b})$ divides $f(\mathbb{0}_X) \urcorner f(\mathbb{b})$ in ratio $c : 1-c$. That is, $f(c\,\mathbb{b}) = \underbrace{\frac{c}{c+(1-c)}}_{=1}(f(\mathbb{b}) - \underbrace{f(\mathbb{0}_X)}_{=\mathbb{0}_Y}) + \underbrace{f(\mathbb{0}_X)}_{=\mathbb{0}_Y} = c f(\mathbb{b})$. Together with (2.5), this implies that f satisfies (2.1).

If $\mathbb{a} = \mathbb{b}$, then $f(\mathbb{a}+\mathbb{b}) = f(2\mathbb{a}) \underbrace{=}_{\text{by (2.1)}} 2f(\mathbb{a}) = f(\mathbb{a}) + f(\mathbb{a}) = f(\mathbb{a}) + f(\mathbb{b})$.

If $\mathbb{a} \neq \mathbb{b}$, then $\mathbb{a}+\mathbb{b}$ divides $2\mathbb{a} \urcorner 2\mathbb{b}$ in ratio $1:1$. Thus $f(\mathbb{a}+\mathbb{b})$ divides $f(2\mathbb{a}) \urcorner f(2\mathbb{b}) \underbrace{=}_{\text{by (2.1)}} 2f(\mathbb{a}) \urcorner 2f(\mathbb{b})$ in ratio 1:1 by (2.7). This means that $f(\mathbb{a}+\mathbb{b}) = \frac{1}{1+1}(2f(\mathbb{b}) - 2f(\mathbb{a})) + 2f(\mathbb{a}) = f(\mathbb{a}) + f(\mathbb{b})$. This shows that f also satisfies (2.2). □ (Lemma 2.4)

Lemma 2.3 and Lemma 2.4 suggest the following question.

open-p

Problem 2.5 *Suppose that X, Y are $\mathbb{R}$-linear spaces and $f : X \to Y$ is such that*

(2.3) *the image $f''L$ of any line $L \subseteq X$ is either a point or a line in Y;*

(2.5) *$f(\mathbb{0}_X) = \mathbb{0}_Y$; and*

(2.8) *there are $\mathbb{a}_0$, $\mathbb{a}_1 \in X$ such that $f(\mathbb{a}_0)$ and $f(\mathbb{a}_1)$ are linearly independent.* lin-8

Does it follow that f is an $\mathbb{R}$-linear mapping?

In the next sections, we shall prove a characterization of linear mappings $f : X \to Y$ with (2.8) by a property which is slightly stronger than the condition (2.8) + (2.8) in Problem 2.5 (see Theorem 4.1).

A mapping $f : X \to Y$ on linear spaces X, Y is said to be an *additive* function if f satisfies (2.2). Note that, for $\mathbb{R}$-linear spaces X, Y, $f : X \to Y$ is additive if and only if it is $\mathbb{Q}$-linear.

Whether all additive functions $f : \mathbb{R} \to \mathbb{R}$ are linear mappings is a question whose answer depends on the axioms of set-theory. In the Zermelo's axiom system ZC of set theory with full Axiom of Choice (AC), we can use a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$, to construct $2^{2^{\aleph_0}}$ many $\mathbb{Q}$-linear (and hence additive) functions from $\mathbb{R}$ to $\mathbb{R}$. Since there are only $2^{\aleph_0}$ many $\mathbb{R}$-linear functions from $\mathbb{R}$ to $\mathbb{R}$, there are $2^{2^{\aleph_0}}$ many additive functions from $\mathbb{R}$ to $\mathbb{R}$ which are not $\mathbb{R}$-linear. Since a continuous function $f : \mathbb{R} \to \mathbb{R}$ is decided by the information on $f \restriction \mathbb{Q}$, the next Lemma follows:

P-lin-4

Lemma 2.6 *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an additive function. Then the following are equivalent:* (a) *f is $\mathbb{R}$-linear.* (b) *f is continuous.* (c) *f is monotonous.* ❑

The equivalence of (a) ⇔ (b) and (c) above follows from the fact that (1) if an additive function is discontinuous at a point, then it is everywhere discontinuous; and (2) a monotone function can be discontinuous at most at countably many points.

More generally, for mappings $f : \mathbb{R}^m \to \mathbb{R}^n$, the equivalence of (a) and (b) in the previous Lemma still holds. Further, by theorems of Steinhaus and Kuratowski, the continuity of f in (b) of Lemma 2.6 in this generalized setting can be replaced either by being a Baire function or by being a measurable function (Kuczma [3]. [2] contains a slightly simplified the proof of these results). Thus we obtain

P-lin-4-a-0

Theorem 2.7 *For any $\mathbb{R}$-linear spaces X, Y such that Y has a linear base* [2]*, and for any aditive $f : X \to Y$, the following are equivalent:* (a) *f is a linear mapping;* (b) *f is continuous;* (c) *f is Lebesgue measurable;* (d) *f is a Baire function.*

Solovay [5] constructed, arguing in the theory ZFC + "there is an inaccessible cardinal", a model of ZF together with the Axiom of Dependent Choice (DC, a weakening of the full AC which covers most of the usages of AC in everyday mathematics) which also satisfies the statement that "all subsets of $\mathbb{R}$ have Baire property and are Lebesgue measurable". Note that, in this model, all functions $f : \mathbb{R}^m \to \mathbb{R}^n$ are Baire functions and also measurable, and thus all additive functions from an $\mathbb{R}$-linear space X to an -linear space Y with a linear base are linear.

Shelah [6] proved that the statement "all subsets of $\mathbb{R}$ are Lebesgue measurable" is equiconsistent with ZF + "there is an inaccessible cardinal". Thus, the inaccessible cardinal in Solovay's model is unavoidable for the statement about Lebesgue measurability. However, Shelah [6] also shows, that ZF + DC + "all subsets of $\mathbb{R}$ have Baire property", hence also the theory ZF + DC + "all additive function on $\mathbb{R}$ is linear", is equiconsistent with ZF. This is one of the most grave signs of asymmetry lying between measure and category in spite of the seeming duality between them e.g. as is seen in Oxtoby [4].

Note that Lebesgue measurability and Baire property of all functions $\mathbb{R} \to \mathbb{R}$ are theorems under ZF + the Axiom of Determinacy (AD). This is a result by Jan Mycielski and Stanisław Swierczkowski for Baire property and by Banach, and Mazur for Lebesgue measurability. Thus, the assertion "all additive function from

[2] Of course, under AC, this extra assumption of the existence of a linear base is superfluous.

an $\mathbb{R}$-linear space to an $\mathbb{R}$-linear space with a linear base is a linear mapping" is a theorem under this axiom system.

The consistency strength of the Axiom of Determinacy, however, is much higher than that of the statement that all subsets of $\mathbb{R}$ are Lebesgue measurable: Woodin's famous theorem states that the Axiom of Determinacy (over ZF) is equiconsistent with infinitely many Woodin cardinals which is much stronger than, e.g. class many measurable cardinals.

Although we shall never rely on these deep results in set-theory in the following, they remind the possibility that a set-theoretic subtlety of this kind can lurk anywhere in a discussion like the following, and warns us that a careful examination *is* necessary.

The next lemma is an easy application of Lemma 2.6:

Lemma 2.8 (Kuczma [3], Theorem 14.4.1) *suppose that $f : \mathbb{R} \to \mathbb{R}$ is an additive function and also multiplicative, i.e. we have that* P-lin-4-a

$$f(rs) = f(r)f(s) \text{ for all } r,\ s \in \mathbb{R}. \tag{2.9}$$ lin-8-0

Then f is either the constant zero function (i.e. $f(r) = 0$ for all $r \in \mathbb{R}$) or the identity function (i.e. $f(r) = r$ for all $r \in \mathbb{R}$).

Proof. For any $x \in \mathbb{R}$,

$$\text{if } x \geq 0, \text{ then } f(x) = f((\sqrt{x})^2) = (f(\sqrt{x}))^2 \geq 0 \tag{2.10}$$ lin-8-1

by the multiplicativity (2.9). It follows that, for any $x,\ y \in \mathbb{R}$ with $x \leq y$,

$$f(y) = f(x + (y - x)) = f(x) + f(y - x) \geq f(x) \tag{2.11}$$ lin-8-2

by additivity of f. Thus f is a monotone function. By Lemma 2.6, and since f is an additive function, there is $c \in \mathbb{R}$ such that $f(x) = cx$ holds for all $x \in \mathbb{R}$.

By (2.9), it follows that

$$c = f(1) = f(1 \cdot 1) = f(1)f(1) = c^2. \tag{2.12}$$ lin-8-3

Since $f(1) \geq 0$ by (2.10), it follows that $c = 0$ or $c = 1$. If $c = 0$, f is the constant function $f(x) = 0$ for all $x \in \mathbb{R}$. If $c = 1$, $f = id_{\mathbb{R}}$. ❑ (Lemma 2.8)

P-lin-4-0

Theorem 2.9 *Suppose that X and Y are $\mathbb{R}$-linear spaces and $f : X \to Y$ is an additive function (i.e. it satisfies (2.2)). If there is a function $\varphi : \mathbb{R} \to \mathbb{R}$ with*

$$f(r\mathbb{a}) = \varphi(r)f(\mathbb{a}) \text{ for all } \mathbb{a} \in X \text{ and } r \in \mathbb{R}, \tag{2.13}$$ lin-9-0

then f is an $\mathbb{R}$-linear mapping.

Proof. If $f''X = \{\mathbb{0}_Y\}$, then f is a linear mapping. Thus, we may assume that $f''X \neq \{\mathbb{0}_Y\}$.

Then we have $\varphi(1) = 1$. Hence, by Lemma 2.8, it is enough to show that φ is additive and multiplicative.

To show that φ is additive, suppose that r, $s \in \mathbb{R}$. Let $\mathbb{a} \in X$ be such that $f(\mathbb{a}) \neq \mathbb{0}_Y$. By additivity of f, we have $\varphi(r+s)f(\mathbb{a}) = f((r+s)\mathbb{a}) = f(r\mathbb{a} + s\mathbb{a}) = f(r\mathbb{a}) + f(s\mathbb{a}) = \varphi(r)f(\mathbb{a}) + \varphi(s)f(\mathbb{a}) = (\varphi(r) + \varphi(s))f(\mathbb{a})$. It follows that $\varphi(r+s) = \varphi(r) + \varphi(s)$.

Multiplicativity of φ can be shown similarly: Suppose r, $s \in \mathbb{R}$, and let $\mathbb{a} \in X$ be such that $f(\mathbb{a}) \neq \mathbb{0}_Y$. Then we have $\varphi(rs)f(\mathbb{a}) = f(rs\mathbb{a}) = \varphi(r)f(s\mathbb{a}) = \varphi(r)\varphi(s)f(\mathbb{a})$. It follows that $\varphi(rs) = \varphi(r)\varphi(s)$. ❑ (Theorem 2.9)

3 Geometry of line preserving mappings

lin-section-2

Lemma 3.1 *For $\mathbb{R}$-linear spaces X, Y let $f : X \to Y$ be a mapping such that* P-lin-5

(2.3) *the image $f''L$ of any line $L \subseteq X$ is either a point or a line in Y; and*

(3.1) *for any line $L \subseteq X$, if $f''L$ is also a line in Y, then $f \restriction L$ is 1-1.* lin-7

Then, (1) f maps any plane $E \subseteq X$ to either a plane or a line or a point of Y. If $f''E$ is a plane then $f \restriction E$ is 1-1.

(2) Suppose that L_0, $L_1 \subseteq X$ are lines such that $f''L_0$ and $f''L_1$ are also lines in Y. If L_0 and L_1 are parallel to each other then $f''L_0$ and $f''L_1$ are also parallel to each other.

Proof. (1): Suppose that $E \subseteq X$ is a plane. Let $\mathbb{a}$, $\mathbb{b}$, $\mathbb{c} \in E$ be such that $\mathbb{b} - \mathbb{a}$ and $\mathbb{c} - \mathbb{a}$ are independent.

Case 1. $f(\mathbb{a}) = f(\mathbb{b})$ and $f(\mathbb{a}) = f(\mathbb{c})$. By (3.1) and (2.3), we have $f''(\mathbb{a}{}^{-}\mathbb{b}) = f''(\mathbb{a}{}^{-}\mathbb{c}) = \{f(\mathbb{a})\}$. Suppose $\mathbb{p} \in E \setminus (\mathbb{a}{}^{-}\mathbb{b} \cup \mathbb{a}{}^{-}\mathbb{c})$. Then there is a line $L \subseteq X$ going through $\mathbb{p}$ and crossing $\mathbb{a}{}^{-}\mathbb{b}$ and $\mathbb{a}{}^{-}\mathbb{c}$ at two different points. Since the value of f at both of these two points is $f(\mathbb{a})$, $f \restriction L$ is not 1-1. By (3.1) and (2.3) it follows that $f(\mathbb{p}) = f(\mathbb{a})$. Thus we have $f''E = \{f(\mathbb{a})\}$.

Case 2. $f(\mathbb{a}) = f(\mathbb{b})$ and $f(\mathbb{a}) \neq f(\mathbb{c})$. Then $f''(\mathbb{a}{}^{-}\mathbb{b}) = \{f(\mathbb{a})\}$, $f''(\mathbb{a}{}^{-}\mathbb{c}) = f(\mathbb{a}){}^{-}f(\mathbb{c})$, and $f''(\mathbb{a}{}^{-}\mathbb{c})$ is a line in Y. For $\mathbb{p} \in E \setminus (\mathbb{a}{}^{-}\mathbb{b} \cup \mathbb{a}{}^{-}\mathbb{c})$, there is a line $L \subseteq X$ going through $\mathbb{p}$ and crossing $\mathbb{a}{}^{-}\mathbb{b}$ and $\mathbb{a}{}^{-}\mathbb{c}$ at two different points, say at $\{\mathbb{p}_0\}$ and $\{\mathbb{p}_1\}$ respectively. If $f(\mathbb{p}_0) = f(\mathbb{p}_1)$, then we have $f(\mathbb{p}) = f(\mathbb{p}_0) = f(\mathbb{a})$. Otherwise, since $f(\mathbb{p}_0)$ and $f(\mathbb{p}_1)$ belong to $f(\mathbb{a}){}^{-}f(\mathbb{c})$ we have $f(\mathbb{p}) \in f(\mathbb{p}_0){}^{-}f(\mathbb{p}_1) = f(\mathbb{a}){}^{-}f(\mathbb{c})$. This shows that $f''E = f(\mathbb{a}){}^{-}f(\mathbb{c})$.

Case 3. $f(\mathbb{a}) \neq f(\mathbb{b})$ and $f(\mathbb{a}) = f(\mathbb{c})$. This case can be treated similarly to the Case 2. to conclude that $f''E = f(\mathbb{a})^\frown f(\mathbb{b})$.

Case 4. $f(\mathbb{a}) \neq f(\mathbb{b})$ and $f(\mathbb{a}) \neq f(\mathbb{c})$. Then we have $f''(\mathbb{a}^\frown\mathbb{c}) = f(\mathbb{a})^\frown f(\mathbb{b})$, $f''(\mathbb{a}^\frown\mathbb{c}) = f(\mathbb{a})^\frown f(\mathbb{c})$, and both $f''(\mathbb{a}^\frown\mathbb{b})$ and $f''(\mathbb{a}^\frown\mathbb{c})$ are lines in Y. If $f(\mathbb{a})^\frown f(\mathbb{b}) = f(\mathbb{a})^\frown f(\mathbb{c})$ then we can argue similarly to Case 2. and show $f''E = f(\mathbb{a})^\frown f(\mathbb{b})$.

Suppose now $f(\mathbb{a})^\frown f(\mathbb{b}) \neq f(\mathbb{a})^\frown f(\mathbb{c})$. Then we have $f(\mathbb{a})^\frown f(\mathbb{b}) \cap f(\mathbb{a})^\frown f(\mathbb{c})$ $= \{f(\mathbb{a})\}$. Let E^* be the plane with $f(\mathbb{a})^\frown f(\mathbb{b})$, $f(\mathbb{a})^\frown f(\mathbb{c}) \subseteq E^*$.

If $\mathbb{p} \in E \setminus (\mathbb{a}^\frown\mathbb{b} \cup \mathbb{a}^\frown\mathbb{c})$ then we can take a line $L \subseteq X$ going through $\mathbb{p}$ and crossing $\mathbb{a}^\frown\mathbb{b}$ and $\mathbb{a}^\frown\mathbb{c}$ at two different points, say at $\{\mathbb{p}_0\}$ and $\{\mathbb{p}_1\}$ respectively. Since $f(\mathbb{p}_0)$, $f(\mathbb{p}_1) \in f(\mathbb{a})^\frown f(\mathbb{b}) \cup f(\mathbb{a})^\frown f(\mathbb{c})$, we have $f(\mathbb{p}) \in f(\mathbb{p}_0)^\frown f(\mathbb{p}_1) \subseteq E^*$. Conversely, suppose that $\mathbb{q} \in E^*$. Then there is a line L^* in Y going through $\mathbb{q}$ and crossing $f(\mathbb{a})^\frown f(\mathbb{b})$ and $f(\mathbb{a})^\frown f(\mathbb{c})$ at two different points say $\{\mathbb{q}_0\}$ and $\{\mathbb{q}_1\}$. Let $\mathbb{p}_0 \in \mathbb{a}^\frown\mathbb{b}$, $\mathbb{p}_1 \in \mathbb{a}^\frown\mathbb{c}$ be the unique vectors such that $f(\mathbb{p}_0) = \mathbb{q}_0$ and $f(\mathbb{p}_1) = \mathbb{q}_1$. Since $f''(\mathbb{p}_0{}^\frown\mathbb{p}_1) = \mathbb{q}_0{}^\frown\mathbb{q}_1$, there is $\mathbb{p} \in \mathbb{p}_0{}^\frown\mathbb{p}_1 \subseteq E$ such that $f(\mathbb{p}) = \mathbb{q}$.

To see that $f \restriction E$ is 1-1, suppose that $\mathbb{p}_0$, $\mathbb{p}_1 \in E$ are distinct vectors. If $f(\mathbb{p}_0) = f(\mathbb{p}_1)$ then we can produce the constellation of Case 1. or Case 2. with $\mathbb{p}_0$ and $\mathbb{p}_1$ together with some third point $\in E$. This is a contradiction since we already know that $f''E$ is a plane in Y.

(2): If $f''L_0 = f''L_1$ then $f''L_0$ and $f''L_1$ are parallel.

Assume otherwise. Let E be the plane in X with L_0, $L_1 \subseteq E$. Since $f''L_0$, $f''L_1 \subseteq f''E$, and since $f''L_0$ and $f''L_1$ are two distinct lines in $f''E$, $f''E$ is a plane in Y and $f \restriction E$ is 1-1, by (1). Since $L_0 \cap L_1 = \emptyset$. We also have $f''L_0 \cap f''L_1 = \emptyset$. Thus, also in this case, $f''L_0$ and $f''L_1$ are parallel to each other. ❑ (Lemma 3.1)

P-lin-5-0

Lemma 3.2 *Suppose that X and Y are $\mathbb{R}$-linear spaces, and $f : X \to Y$ a mapping satisfying (2.5), (2.3) and (3.1). If $\mathbb{a}$, $\mathbb{b} \in X$ are such that $\mathbb{a} \neq \mathbb{0}_X$, $\mathbb{b} \neq \mathbb{0}_X$, $f(\mathbb{a}) = \mathbb{0}_Y$, and $f(\mathbb{b}) \neq \mathbb{0}_Y$, then we have $f(\mathbb{a}+\mathbb{b}) = f(\mathbb{b}) = f(\mathbb{a}) + f(\mathbb{b})$.*

Proof. $\mathbb{a}$ and $\mathbb{b}$ are linearly independent: $f \restriction \mathbb{R}\mathbb{b}$ is 1-1 by (2.3) and (3.1). Thus, if $\mathbb{a} \in \mathbb{R}\mathbb{b}$, we would have $f(\mathbb{a}) \neq \mathbb{0}_Y$ by (2.5).

Let $E \subseteq X$ be the plane with $\mathbb{0}_X$, $\mathbb{a}$, $\mathbb{b} \in E$. By Lemma 3.1, (1), we have $f''E = f(\mathbb{b})^\frown\mathbb{0}_Y = \mathbb{R}f(\mathbb{b})$. Note also that $f''\mathbb{R}\mathbb{a} = \{\mathbb{0}_Y\}$.

Suppose, toward a contradiction, that

(3.2) $\quad f(\mathbb{a}+\mathbb{b}) \neq f(\mathbb{b})$. lin-9-1

Then $f \restriction (\mathbb{a}+\mathbb{b})^\frown\mathbb{b}$ is not a constant function. Thus, we have $f''(\mathbb{a}+\mathbb{b})^\frown\mathbb{b} = \mathbb{R}f(\mathbb{b})$. Let $\mathbb{c} \in (\mathbb{a}+\mathbb{b})^\frown\mathbb{b}$ be such that $f(\mathbb{c}) = \mathbb{0}_Y$. Then we have $f''\mathbb{c}^\frown\mathbb{0}_X = \{\mathbb{0}_Y\}$.

For each $\mathbb{d} \in (\mathbb{a}+\mathbb{b})\,\overline{}\,\mathbb{b}$, let $L \subseteq X$ be a line going through $\mathbb{d}$ which crosses $\mathbb{c}\,\overline{}\,\mathbb{0}_X$ and $\mathbb{a}\,\overline{}\,\mathbb{0}_X$ at different points, say at points $\mathbb{p}_0$ and $\mathbb{p}_1$ respectively. Then we have $f(\mathbb{p}_0) = f(\mathbb{p}_1) = \mathbb{0}_Y$. Thus $f''L = \{\mathbb{0}_Y\}$ and $f(\mathbb{d}) = \mathbb{0}_Y$. In particular, we have $f(\mathbb{a}+\mathbb{b}) = f(\mathbb{b}) = \mathbb{0}_Y$. This is a contradiction to the assumption (3.2).

□ (Lemma 3.2)

4 A partial answer to Problem 2.5

lin-section-3

The following theorem gives a positive partial answer to Problem 2.5

P-lin-6

Theorem 4.1 *Suppose that X, Y are $\mathbb{R}$-linear spaces and $f : X \to Y$ satisfies*

(2.5) $f(\mathbb{0}_X) = \mathbb{0}_Y$; *and,*

(2.8) *there are $\mathbb{a}_0$, $\mathbb{a}_1 \in X$ such that $f(\mathbb{a}_0)$ and $f(\mathbb{a}_1)$ are linearly independent.*

Then f is a linear mapping if and only if f satisfies

(2.3) *the image $f''L$ of any line $L \subseteq X$ by f is either a point or a line in Y; and,*

(3.1) *for any line $L \subseteq X$, if $f''L$ is also a line in Y, then $f \restriction L$ is 1-1.*

Proof. Linear mappings satisfy (2.3) and (3.1) (Lemma 2.1).

Suppose that $f : X \to Y$ satisfies (2.5), (2.8), (2.3) and (3.1). By Theorem 2.9, it is enough to show that there is $\varphi : \mathbb{R} \to \mathbb{R}$ with (2.13) for this f, and f is additive.

To prove that there is a mapping $\varphi : \mathbb{R} \to \mathbb{R}$ with (2.13), suppose $\mathbb{a} \in X$. Note that $\mathbb{a}\,\overline{}\,\mathbb{0}_X = \mathbb{R}\mathbb{a}$. If $f(\mathbb{a}) = \mathbb{0}_Y$, then $f''(\mathbb{a}\,\overline{}\,\mathbb{0}_X) = \{\mathbb{0}_Y\}$ by (2.3), (2.5) and (3.1). Thus we have

(4.1) $f(r\mathbb{a}) = \mathbb{0}_Y = r\, f(\mathbb{a})$, for any $r \in \mathbb{R}$ (under $f(\mathbb{a}) = \mathbb{0}_Y$). lin-10

Suppose now that $f(\mathbb{a}) \neq \mathbb{0}_Y$. Then $f''\mathbb{R}\mathbb{a} = f(\mathbb{a})\,\overline{}\,\mathbb{0}_Y$ $(= \mathbb{R}f(\mathbb{a}))$ and $f \restriction \mathbb{R}\mathbb{a}$ is 1-1 by (2.3), (2.5), (3.1). For each $\mathbb{a} \in X$ with $f(\mathbb{a}) \neq \mathbb{0}_Y$ and $r \in \mathbb{R}$, let $\varphi_0(\mathbb{a}, r)$ be $s \in \mathbb{R}$ such that $f(r\mathbb{a}) = \varphi_0(\mathbb{a}, r) f(\mathbb{a})$.

Cl-lin-0

Claim 4.1.1 *$\varphi_0(\mathbb{a}, r)$ does not depend on $\mathbb{a} \in X$ with $f(\mathbb{a}) \neq \mathbb{0}_Y$.*

$\vdash$ we first prove the following:

Subclaim 4.1.1.1 *For any $\mathbb{a}$, $\mathbb{b} \in X$ such that $f(\mathbb{a})$ and $f(\mathbb{b})$ are independent in Y, we have $\varphi_0(\mathbb{a}, r) = \varphi_0(\mathbb{b}, r)$ for all $r \in \mathbb{R}$.*

$\vdash$ For $\mathbb{a} \in X$ with $f(\mathbb{a}) \neq \mathbb{0}_Y$, $f(0\mathbb{a}) = \mathbb{0}_Y$ by (2.5). Thus $\varphi_0(\mathbb{a}, 0) = 0$ for all such $\mathbb{a} \in X$.

For $r \neq 0$, let us consider the four lines $L_0 = \mathbb{a}^\frown \mathbb{0}_X$, $L_1 = \mathbb{b}^\frown \mathbb{0}_X$, $L_2 = \mathbb{a}^\frown \mathbb{b}$, and $L_3 = r\mathbb{a}^\frown r\mathbb{b}$. We have $L_0 \cap L_1 = \{\mathbb{0}_X\}$, $L_0 \cap L_2 = \{\mathbb{a}\}$, $L_1 \cap L_2 = \{\mathbb{b}\}$, $L_0 \cap L_3 = \{r\mathbb{a}\}$, $L_1 \cap L_3 = \{r\mathbb{b}\}$, and, L_2 and L_3 are parallel to each other. By Lemma 3.1, it follows that $f''L_0 = f(\mathbb{a})^\frown \mathbb{0}_Y$, $f''L_1 = f(\mathbb{b})^\frown \mathbb{0}_Y$, $f''L_2 = f(\mathbb{a})^\frown f(\mathbb{b})$, and $f''L_3 = f(r\mathbb{a})^\frown f(r\mathbb{b}) = \varphi_0(\mathbb{a}, r)f(\mathbb{a})^\frown \varphi_0(\mathbb{b}, r)f(\mathbb{b})$. $f''L_0 \cap f''L_1 = \{\mathbb{0}_Y\}$, $f''L_0 \cap f''L_2 = \{f(\mathbb{a})\}$, $f''L_1 \cap f''L_2 = \{f(\mathbb{b})\}$, $f''L_0 \cap f''L_3 = \{f(r\mathbb{a})\} = \{\varphi_0(\mathbb{a}, r)f(\mathbb{a})\}$, $f''L_1 \cap f''L_3 = \{f(r\mathbb{b})\} = \{\varphi_0(\mathbb{b}, r)f(\mathbb{b})\}$, and, $f''L_2$ and $f''L_3$ are parallel to each other. The last condition is only possible when $\varphi_0(\mathbb{a}, r) = \varphi_0(\mathbb{b}, r)$. $\dashv$ (Subclaim 4.1.1.1)

Let $\mathbb{a}_0$, $\mathbb{a}_1 \in X$ be as in (2.8). By Subclaim 4.1.1.1, we have $\varphi_0(\mathbb{a}_0, r) = \varphi_0(\mathbb{a}_1, r)$ for any $r \in \mathbb{R}$. Suppose that $\mathbb{a} \in X$ is such that $f(\mathbb{a}) \neq \mathbb{0}_Y$. Then there is $i \in 2$, such that $f(\mathbb{a})$ and $f(\mathbb{a}_i)$ are linearly independent. By Subclaim 4.1.1.1, we have $\varphi_0(\mathbb{a}, r) = \varphi_0(\mathbb{a}_i, r)$. Thus, for any $\mathbb{a} \in X$ with $f(\mathbb{a}) \neq \mathbb{0}_Y$ and $r \in \mathbb{R}$, $\varphi_0(\mathbb{a}, r) = \varphi_0(\mathbb{a}_0, r)$. $\dashv$ (Claim 4.1.1)

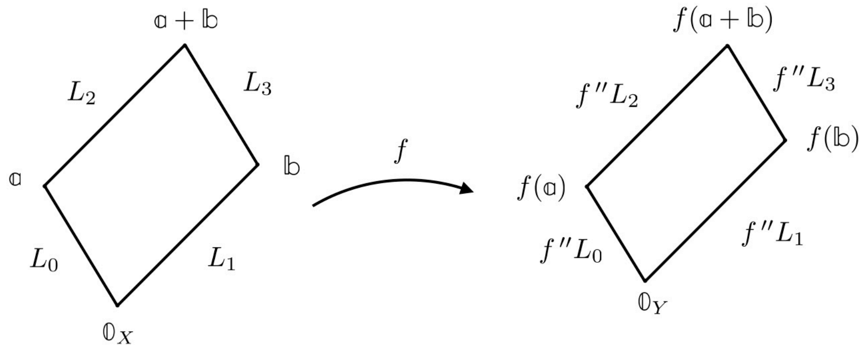
Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by $\varphi(r) = \varphi_0(\mathbb{a}_0, r)$. By Claim 4.1.1 and the argument above it, this φ satisfies (2.13).

To prove the additivity of f, suppose $\mathbb{a}$, $\mathbb{b} \in X$.

Case 1. $f(\mathbb{a})$ and $f(\mathbb{b})$ are independent in Y. In this case, $\mathbb{a}$ and $\mathbb{b}$ are independent and, letting E be the plane in X with $\mathbb{a}$, $\mathbb{b}$, $\mathbb{0}_X \in E$, we have that $f \restriction E$ is a 1-1 mapping and $f''E$ is the plain in Y with $f(\mathbb{a})$, $f(\mathbb{b})$, $\mathbb{0}_Y$ $(= f(\mathbb{0}_X))$ $\in f''E$ by Lemma 3.1, (1).

Let $L_0 = \mathbb{a}^\frown \mathbb{0}_X$, $L_1 = \mathbb{b}^\frown \mathbb{0}_X$, $L_2 = \mathbb{a}^\frown(\mathbb{a}+\mathbb{b})$, $L_3 = \mathbb{b}^\frown(\mathbb{a}+\mathbb{b})$. Then we have $L_0 \cap L_1 = \{\mathbb{0}_X\}$, $L_0 \cap L_2 = \{\mathbb{a}\}$, $L_1 \cap L_3 = \{\mathbb{b}\}$, and $L_2 \cap L_3 = \{\mathbb{a}+\mathbb{b}\}$. We also have that L_0 and L_3 are parallel to each other, and, L_1 and L_2 are parallel to each other.

By (2.3) and (2.5), and since $f \restriction E$ is 1-1, it follows that $f''L_0 = f(\mathbb{a})^\frown \mathbb{0}_Y$, $f''L_1 = f(\mathbb{b})^\frown \mathbb{0}_Y$, $f''L_2 = f(\mathbb{a})^\frown f(\mathbb{a}+\mathbb{b})$, $f''L_3 = f(\mathbb{b})^\frown f(\mathbb{a}+\mathbb{b})$. We also have $f''L_0 \cap f''L_1 = \{\mathbb{0}_Y\}$, $f''L_0 \cap {}''L_2 = \{f(\mathbb{a})\}$, $f''L_1 \cap f''L_3 = \{f(\mathbb{b})\}$, and $f''L_2 \cap f''L_3 = \{f(\mathbb{a}+\mathbb{b})\}$. $f''L_0$ and $f''L_3$ are parallel to each other, and, $f''L_1$ and $f''L_2$ are parallel to each other by Lemma 3.1, (2). This implies that $f(\mathbb{a}+\mathbb{b}) = f(\mathbb{a}) + f(\mathbb{b})$.



Case 2. $\mathbb{a}$ and $\mathbb{b}$ are not linearly independent. If one of $\mathbb{a}$, $\mathbb{b}$ equals to $\mathbb{0}_X$ then the additivity

$$f(\mathbb{a}+\mathbb{b}) = f(\mathbb{a}) + f(\mathbb{b}) \tag{4.2}$$ lin-11

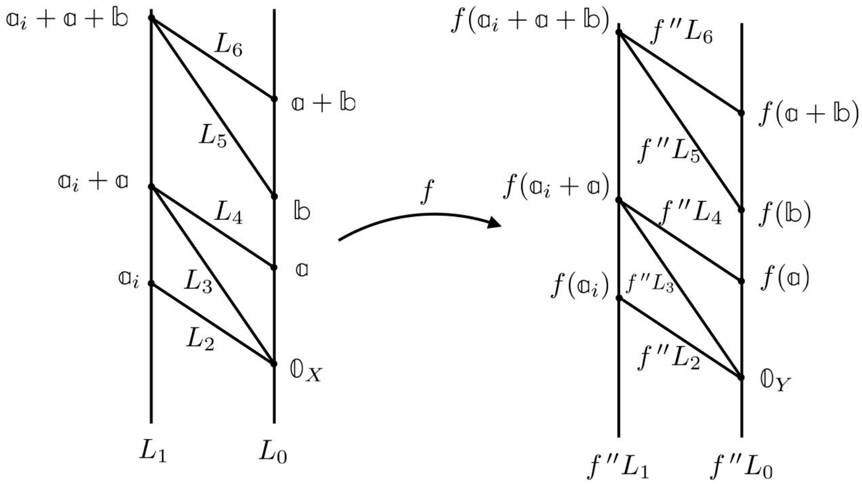
trivially holds by (2.5). Thus we may assume without loss of generality that $\mathbb{a} \neq \mathbb{0}_X$, $\mathbb{b} \neq \mathbb{0}_X$, and $\mathbb{b} \in \mathbb{R}\mathbb{a}$. If $f(\mathbb{a}) = \mathbb{0}_Y$ or $f(\mathbb{b}) = \mathbb{0}_Y$, then $f''\mathbb{R}\mathbb{a} = \{\mathbb{0}_Y\}$ by (2.3), (2.4) and (2.5) and the equation (4.2) again holds. Thus we may assume that $f(\mathbb{a}) \neq \mathbb{0}_Y$, $f(\mathbb{b}) \neq \mathbb{0}_Y$ and $f(\mathbb{b}) \in \mathbb{R}f(\mathbb{a})$.

Let $\mathbb{a}_0$, $\mathbb{a}_1 \in X$ be as in (2.8). Then there is $i \in 2$ such that $f(\mathbb{a}_i)$ and $f(\mathbb{a})$ are independent. Let E be the plane with $\mathbb{a}_i$, $\mathbb{a}$, $\mathbb{0}_X \in E$. By Lemma 3.1, (1), $f \restriction E$ is 1-1 and $f''E$ is a plane with $f(\mathbb{a}_i)$, $f(\mathbb{a})$, $f(\mathbb{b})$, $\mathbb{0}_Y \in f''E$.

Let $L_0 = \mathbb{a}\,{}^{\overline{\ \ }}\,\mathbb{0}_X$, $L_1 = (\mathbb{a}_i + \mathbb{a})\,{}^{\overline{\ \ }}\,\mathbb{a}_i$, $L_2 = \mathbb{a}_i\,{}^{\overline{\ \ }}\,\mathbb{0}_X$, $L_3 = (\mathbb{a}_i + \mathbb{a})\,{}^{\overline{\ \ }}\,\mathbb{0}_X$, $L_4 = (\mathbb{a}_i + \mathbb{a})\,{}^{\overline{\ \ }}\,\mathbb{a}$, $L_5 = (\mathbb{a}_i + \mathbb{a} + \mathbb{b})\,{}^{\overline{\ \ }}\,\mathbb{b}$, and $L_6 = (\mathbb{a}_i + \mathbb{a} + \mathbb{b})\,{}^{\overline{\ \ }}\,(\mathbb{a} + \mathbb{b})$.

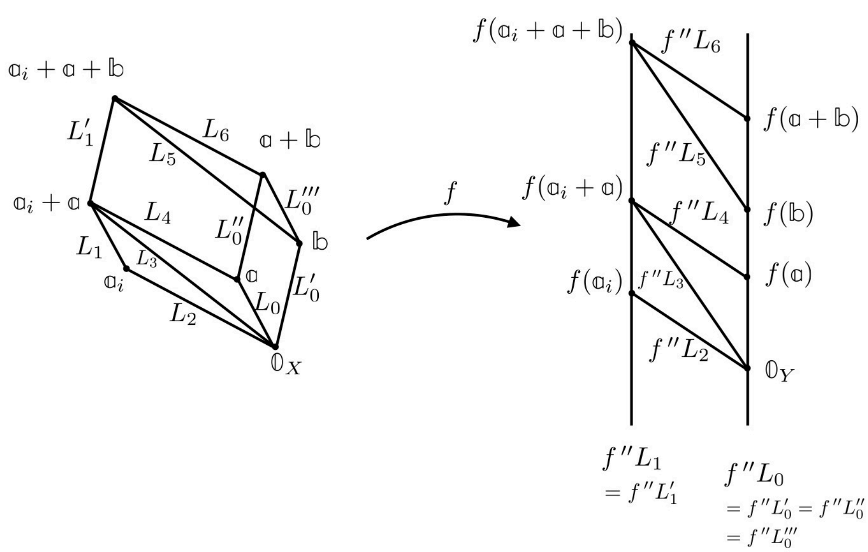
We have $L_0 \cap L_2 \cap L_3 = \{\mathbb{0}_X\}$, $L_0 \cap L_4 = \{\mathbb{a}\}$, $L_0 \cap L_5 = \{\mathbb{b}\}$, $L_0 \cap L_6 = \{\mathbb{a}+\mathbb{b}\}$, $L_1 \cap L_2 = \{\mathbb{a}_i\}$, $L_1 \cap L_3 \cap L_4 = \{\mathbb{a}_i + \mathbb{a}\}$, and $L_1 \cap L_5 \cap L_6 = \{\mathbb{a}_i + \mathbb{a} + \mathbb{b}\}$.

We also have that L_0 and L_1 are parallel to each other, L_2, L_4 and L_6 are parallel to each other, as well as L_3 and L_5 are parallel to each other. Since f transfers this constellation keeping all the parallelisms, it follows, similarly to the Case 1., we can conclude in turn that $f(\mathbb{a}_i + \mathbb{a}) = f(\mathbb{a}_i) + f(\mathbb{a})$, $f(\mathbb{a}_i + \mathbb{a} + \mathbb{b}) = f(\mathbb{a}_i) + f(\mathbb{a}) + f(\mathbb{b})$, and finally $f(\mathbb{a} + \mathbb{b}) = f(\mathbb{a}) + f(\mathbb{b})$.



Case 3. $\mathbb{a}$ and $\mathbb{b}$ are independent in X but $f(\mathbb{a})$ and $f(\mathbb{b})$ are not independent in Y. If $f(\mathbb{a}) = f(\mathbb{b}) = \mathbb{0}_Y$ then $f(\mathbb{a} + \mathbb{b}) = \mathbb{0}_Y = f(\mathbb{a}) + f(\mathbb{b})$ by the the Case 1 in the proof of Lemma 3.1, (1). If one of $f(\mathbb{a})$ and $f(\mathbb{b})$ is $\mathbb{0}_Y$, then we have $f(\mathbb{a} + \mathbb{b}) = f(\mathbb{a}) + f(\mathbb{b})$ by Lemma 3.2. Thus we may assume that $f(\mathbb{a}) \neq \mathbb{0}_Y$ and $f(\mathbb{b}) \neq \mathbb{0}_Y$. Let $E \subseteq X$ be the plane with $\mathbb{0}_X$, $\mathbb{a}$, $\mathbb{b} \in E$. By the assumption of the present case, we have $f''E = \mathbb{R}f(\mathbb{a}) = \mathbb{R}f(\mathbb{b})$. Thus, there is $i \in 2$ such that $f(\mathbb{a}_i)$ is independent from $f(\mathbb{a})$ (and also from $f(\mathbb{b})$).

In this case, some of the lines connecting $\mathbb{0}_X$ $\mathbb{a}$, $\mathbb{b}$, $\mathbb{a}_i$, $\mathbb{a}_i + \mathbb{a}$, $\mathbb{a}_i + \mathbb{a} + \mathbb{b}$, $\mathbb{a} + \mathbb{b}$ are sent to the same lines by f. However, parallelism of some lines survive to conclude in turn that $f(\mathbb{a}_i + \mathbb{a}) = f(\mathbb{a}_i) + f(\mathbb{a})$, $f(\mathbb{a}_i + \mathbb{a} + \mathbb{b}) = f(\mathbb{a}_i) + f(\mathbb{a}) + f(\mathbb{b})$, and finally $f(\mathbb{a} + \mathbb{b}) = f(\mathbb{a}) + f(\mathbb{b})$ just as in Case 2.



❑ (Theorem 4.1)

Theorem 4.1 can be easily generalized to a characterization of affine mappings (i.e mappings $g : X \to Y$ for $\mathbb{R}$-linear spaces which can be represented as $\mathbb{a} \mapsto f(\mathbb{a}) + \mathbb{b}$ for a linear mapping $f : X \to Y$ and $\mathbb{b} \in Y$) with the following condition corresponding to (2.8):

P-lin-7

Corollary 4.2 *Suppose that X, Y are $\mathbb{R}$-linear spaces and $g : X \to Y$ satisfies*

(2.8)′ *there are $\mathbb{a}^*$ $\mathbb{a}_0^*$, $\mathbb{a}_1^* \in X$ such that $g(\mathbb{a}_0^*) - g(\mathbb{a}^*)$ and $g(\mathbb{a}_1^*) - g(\mathbb{a}^*)$ are linearly independent.*

Then g is an affine mapping if and only if g satisfies

(2.3) *the image $g''L$ of any line $L \subseteq X$ by g is either a point or a line in Y; and*

(3.1) *for any line $L \subseteq X$, if $g''L$ is also a line in Y, then $g \restriction L$ is 1-1.*

Proof. If $g : X \to Y$ is an affine mapping then g clearly satisfies (2.3) and (3.1).

Conversely, suppose that $g : X \to Y$ satisfies (2.8)′, (2.3) and (3.1). Let $\mathbb{a}^*$, $\mathbb{a}_0^*$, $\mathbb{a}_1^* \in X$ be as in (2.8)′. Let $f : X \to Y$ be defined by

(4.3) $f(\mathbb{a}) = g(\mathbb{a} + \mathbb{a}^*) - g(\mathbb{a}^*)$ for $\mathbb{a} \in X$. lin-11-0

Clearly f also satisfies (2.3) and (3.1). We have $f(\mathbb{0}_X) = g(\mathbb{a}^*) - g(\mathbb{a}^*) = \mathbb{0}_Y$. $f(\mathbb{a}_i^* - \mathbb{a}^*) = g(\mathbb{a}_i^* - \mathbb{a}^* + \mathbb{a}^*) - g(\mathbb{a}^*) = g(\mathbb{a}_i^*) - g(\mathbb{a}^*)$ for $i \in 2$. Thus, letting $\mathbb{a}_i = \mathbb{a}_i^* - \mathbb{a}^*$ for $i \in 2$, we see that f satisfies (2.8). By Theorem 4.1, it follows that f is a linear mapping. Since we have

(4.4) $g(\mathbb{a}) = g((\mathbb{a} - \mathbb{a}^*) + \mathbb{a}^*) = f(\mathbb{a} - \mathbb{a}^*) + g(\mathbb{a}^*) = f(\mathbb{a}) + f(-\mathbb{a}^*) + g(\mathbb{a}^*) = f(\mathbb{a}) + g(\mathbb{0}_X)$

for all $\mathbb{a} \in X$, g is an affine mapping. ❏ (Corollary 4.2)

The following (4.5) apparently holds for any affine mapping $f : X \to Y$. Conversely, it is easy to see that (2.3) and (4.5) imply (2.4). Thus we obtain:

P-lin-8

Corollary 4.3 *Suppose that X, Y are $\mathbb{R}$-linear spaces and $g : X \to Y$ satisfies*

(2.8)′ *there are $\mathbb{a}^*$ $\mathbb{a}_0^*$, $\mathbb{a}_1^* \in X$ such that $g(\mathbb{a}_0^*) - g(\mathbb{a}^*)$ and $g(\mathbb{a}_1^*) - g(\mathbb{a}^*)$ are linearly independent.*

Then g is an affine mapping if and only if g satisfies

(2.3) *the image $g''L$ of any line $L \subseteq X$ by g is either a point or a line in Y; and,* (4.5) *for any* $\mathbb{a}$, $\mathbb{b}$, $\mathbb{c} \in X$ *with* $\mathbb{c} \in \neg\frown \mathbb{b}$, $g(\mathbb{a})$*, either* $g(\mathbb{a}) = g(\mathbb{b}) = g(\mathbb{c})$*, or* $g(\mathbb{c}) \in g(\mathbb{a}) \frown g(\mathbb{b})$ *holds.* ❑ lin-12

The following characterization of linear mappings consisting of a mixture of algebraic and geometric properties is also easy to prove.

P-lin-9

Proposition 4.4 *Suppose that* X *and* Y *are* $\mathbb{R}$*-linear spaces and* $f : X \to Y$*. Then* f *is linear if and only if*

(4.6) f *is an additive function; and*

(4.5)′ *for any* $\mathbb{a}$, $\mathbb{c} \in X$*, if* $\mathbb{c} \in \mathbb{a} \frown \mathbb{0}_X$*, then, either* $f(\mathbb{a}) = f(\mathbb{c}) = \mathbb{0}_Y$ *or* $\mathbb{c} \in f(\mathbb{a}) \frown \mathbb{0}_Y$*.*

Proof. For $\mathbb{a} \in X$, let $\varphi_{\mathbb{a}} : \mathbb{R} \to \mathbb{R}$ be such that

($\aleph$4.1) $f(r\mathbb{a}) = \varphi_{\mathbb{a}}(r) f(\mathbb{a})$ for all $r \in \mathbb{R}$. lin-13

We assume, without loss of generality, that $\varphi_{\mathbb{a}}(r) = r$ for all $r \in \mathbb{R}$, if $f(\mathbb{a}) = \mathbb{0}_Y$.

Claim 4.4.1 $\varphi_{\mathbb{a}}$ *is additive and monotonically increasing.*

$\vdash$ If $f(\mathbb{a}) = \mathbb{0}_Y$, the assertion is trivial. So assume that $f(\mathbb{a}) \neq \mathbb{0}_Y$.

For r, $s \in \mathbb{R}$, we have $\varphi_{\mathbb{a}}(r+s) f(\mathbb{a}) = f((r+s)\mathbb{a}) = f(r\mathbb{a} + s\mathbb{a}) = f(r\mathbb{a}) + f(s\mathbb{a}) = \varphi_{\mathbb{a}}(r) f(\mathbb{a}) + \varphi_{\mathbb{a}}(s) f(\mathbb{a}) = (\varphi_{\mathbb{a}}(r) + \varphi_{\mathbb{a}}(s)) f(\mathbb{a})$. This implies that $\varphi_{\mathbb{a}}(r+s) = \varphi_{\mathbb{a}}(r) + \varphi_{\mathbb{a}}(s)$.

To show that $\varphi_{\mathbb{a}}$ is monotonically increasing, suppose that r, $s \in \mathbb{R}$ with $r < s$. For simplicity, consider the case $1 < r < s$. All other cases can be treated similarly. Since $\mathbb{a} \in r\mathbb{a} \frown \mathbb{0}_X$, we have $f(\mathbb{a}) \in f(r\mathbb{a}) \frown \mathbb{0}_Y$ by (4.5)′. By definition ($\aleph$4.1), it follows that $1 < \varphi_{\mathbb{a}}(r)$. Since $r\mathbb{a} \in s\mathbb{a} \frown \mathbb{0}_X$, it follows that $\varphi_{\mathbb{a}}(r) < \varphi_{\mathbb{a}}(s)$. $\dashv$ (Claim 4.4.1)

By Lemma 2.6, it follows that $\varphi_{\mathbb{a}}$ is $\mathbb{R}$-linear since $\varphi_{\mathbb{a}}(1) = 1$, $\varphi_{\mathbb{a}}$ must be the identity function. ❑ (Proposition 4.4)

References

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