More about the Fodor-type Reflection Principle

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Abstract

Continuing the research began in Fuchino, Juhász, Soukup, Szentmiklóssy and Usuba [9], we study further the Fodor-type Reflection Principle (FRP) and its consequences.

We show that FRP is equivalent to the non-existence of almost essentially disjoint ladder system on any stationary subset of a regular uncountable cardinal consisting of ordinals of countable cofinality (Theorem 2.7). Using this characterization, we show that FRP is actually equivalent to many known "mathematical" reflection theorems over ZFC.

For example, it is shown that FRP is equivalent to the statement: *"For any locally countably compact topological space X, if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable, then X itself is also metrizable"* (Theorem 2.8). Another example of statements equivalent to FRP is: *"For any graph G, if all subgraphs of G of cardinality $\leq \aleph_1$ have countable coloring number then G itself also has countable coloring number"* (Theorem 3.1).

We construct models of ZFC separating FRP from Reflection Principle (RP) and Ordinal Reflection Principle (ORP) (Theorems 5.2, 5.4 and 6.1).

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1 Introduction

Let λ be a cardinal of cofinality $\geq \omega_1$. *The Fodor-type Reflection Principle for* λ (FRP(λ)) introduced in [9] is the following statement:

FRP(λ): For any stationary $E \subseteq E^\lambda_\omega$ and mapping $g : E \to [\lambda]^{\leq\aleph_0}$ there is $I \in [\lambda]^{\aleph_1}$ such that

(1.1) $\mathrm{cf}(I) = \omega_1$;

(1.2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap E$;

(1.3) for any regressive $f : E \cap I \to \lambda$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in E \cap I$, there is $\xi^* < \lambda$ such that $f^{-1}{}''\{\xi^*\}$ is stationary in $\sup(I)$.

Here, for cardinals $\kappa < \lambda$, E^λ_κ denotes the set $\{\alpha < \lambda : \mathrm{cf}(\alpha) = \kappa\}$. Similarly, we shall also write $E^\lambda_{<\kappa}$, $E^\lambda_{>\kappa}$, etc. to denote $\{\alpha < \lambda : \mathrm{cf}(\alpha) < \kappa\}$, $\{\alpha < \lambda : \mathrm{cf}(\alpha) > \kappa\}$, etc., respectively.

In [9] it is shown that FRP(λ) is inconsistent for any singular λ. Thus we have to define the global version of the Fodor-type Reflection Principle as:

FRP: FRP(λ) holds for all regular $\lambda > \aleph_1$.

Note that FRP($\aleph_1$) is just a consequence of the Fodor's Lemma in ZFC.

The following local but cumulative version of FRP is proved to be useful (see e.g. Theorem 2.7 below). For any cardinal λ, let FRP($< \lambda$) be the following principle:

FRP($< \lambda$): FRP(κ) holds for all regular uncountable $\kappa < \lambda$.

A mapping $g : S \to [\lambda]^{\aleph_0}$ for some cardinal λ and $S \subseteq \lambda$ is said to be *essentially disjoint* if there is a mapping $f : S \to [\lambda]^{<\aleph_0}$ such that $g(\alpha) \setminus f(\alpha)$, $\alpha \in S$ are pairwise disjoint. g is said to be *almost essentially disjoint* if $g \restriction D$ is essentially disjoint for all $D \in [\mathrm{dom}(g)]^{<|\,\mathrm{dom}(g)\,|}$.

Clearly, if $g \restriction D$ is essentially disjoint and $D' \subseteq D$ then $g \restriction D'$ is also essentially disjoint. Hence if $\mathrm{dom}(g)$ is a regular cardinal λ then g is almost essentially disjoint if and only if $g \restriction \beta$ is essentially disjoint for all $\beta < \lambda$.

For a regular cardinal λ, the following principle was first considered by S. Shelah. The notation with "ADS$^-$" was used in [9] in analogy to the principle ADS in [2]:

ADS$^-$(λ): There are a stationary set $S \subseteq \lambda$ and $g : S \to [\lambda]^{\aleph_0}$ such that

(1.4) $g(\alpha) \subseteq \alpha$ and $\mathrm{otp}(g(\alpha)) = \omega$ for all $\alpha \in S$;

(1.5) g is almost essentially disjoint.

In [9], it is proved that, in the definition of $\mathrm{ADS}^-(\lambda)$, we may assume that $S \subseteq E^\lambda_\omega$.

A mapping $g : S \to [\lambda]^{\aleph_0}$ for some cardinal λ and $S \subseteq E^\lambda_\omega$ is said to be a *ladder system* if $g(\alpha)$ is a cofinal subset of α of order-type ω for all $\alpha \in S$. Note that a ladder system $g : S \to [\lambda]^{\aleph_0}$ is essentially disjoint if and only if there is a regressive function $f : S \to \lambda$ such that $g(\alpha) \setminus f(\alpha)$, $\alpha \in S$ are pairwise disjoint.

(1.4) of the definition of $\mathrm{ADS}^-(\lambda)$, we may drop the condition $\mathrm{otp}(g(\alpha)) = \omega$ or further demand that g be a ladder system (see Lemma 2.5).

In Section 2, we show that $\mathrm{FRP}(< \lambda)$ for any uncountable cardinal λ is equivalent to the statement that $\mathrm{ADS}^-(\mu)$ does not hold for all regular $\mu < \lambda$ (Theorem 2.7). Using this characterization, we show that many known reflection theorems in terms of topology and infinite graph theory are actually equivalent to FRP: see (A) $\sim$ (E') in Theorems 2.8, 3.1 and 4.1. Further equivalent assertions in topology and boolean algebras are given in [7], [8] and [10].

In [9], it is proved that $\mathrm{FRP}(\kappa)$ follows from $\mathrm{RP}([\kappa]^{\aleph_0})$ for all regular uncountable κ where $\mathrm{RP}([\kappa]^{\aleph_0})$ is the following principle:

$\mathrm{RP}([\kappa]^{\aleph_0})$: For any stationary $S \subseteq [\kappa]^{\aleph_0}$, there is an $I \in [\kappa]^{\aleph_1}$ such that

(1.6) $\omega_1 \subseteq I$;

(1.7) $\mathrm{cf}(I) = \omega_1$;

(1.8) $S \cap [I]^{\aleph_0}$ is stationary in $[I]^{\aleph_0}$.

The global version RP of this principle is defined as the assertion that $\mathrm{RP}([\kappa]^{\aleph_0})$ holds for all uncountable cardinal κ of uncountable cofinality.

The principle obtained from $\mathrm{RP}([\kappa]^{\aleph_0})$ by excluding the condition (1.7) is called sometimes simply 'reflection principle'. Adopting the terminology of Foreman and Todorcevic [6] or König, Larson and Yoshinobu [15], we shall call this principle the *Weak Reflection Principle for* $[\kappa]^{\aleph_0}$ and denote it with $\mathrm{WRP}([\kappa]^{\aleph_0})$. Its global version WRP is also defined similarly, that is, WRP holds if and only if $\mathrm{WRP}([\kappa]^{\aleph_0})$ holds for all uncountable κ. Since WRP enjoys the downward transfer property: $\kappa < \kappa' \wedge \mathrm{WRP}([\kappa']^{\aleph_0}) \Rightarrow \mathrm{WRP}([\kappa]^{\aleph_0})$, RP implies WRP.

FRP is preserved by c.c.c. generic extension (see Theorem 3.4 in [9]). Since $\mathrm{WRP}([\aleph_2]^{\aleph_0})$ implies $2^{\aleph_0} \leq \aleph_2$ (Todorcevic, see [13]), we obtain a model of FRP $+\neg\mathrm{WRP}([\aleph_2]^{\aleph_0})$ by starting from a model of FRP and then forcing $2^{\aleph_0} \geq \aleph_3$

by any c.c.c. partial ordering. Unfortunately, it seems that we cannot modify this method to prove the consistency of FRP with ¬RP (or ¬WRP) under $2^{\aleph_0} \le \aleph_2$. In, Section 5, we give several models which separate FRP and WRP under $2^{\aleph_0} \le \aleph_2$.

Let us call the following reflection principle on stationarity of subsets of E^κ_ω the *Ordinal Reflection Principle for* κ and abbreviate it as ORP(κ):

ORP(κ): For any stationary $S \subseteq E^\kappa_\omega$, there is $I \in [\kappa]^{\aleph_1}$ such that (1.6), (1.7) and

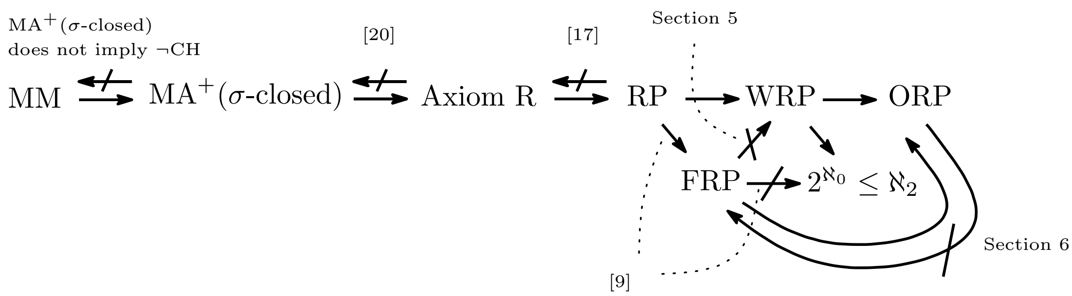
(1.8)' $S \cap I$ is stationary in $\sup(I)$.

The global version ORP of this principle is the assertion that ORP(κ) holds for every cardinal $\kappa > \omega_1$ of cofinality $\ge \omega_1$. FRP(λ) implies ORP(λ). By the transfer property ORP(cf(κ)) $\Rightarrow$ ORP(κ), we also have the implication FRP $\Rightarrow$ ORP. Shelah [23] proved that WRP($[\kappa]^{\aleph_0}$) implies ORP(κ).

In Section 6, we construct a model of ¬FRP(κ) + ORP for arbitrary regular κ assuming MA$^+$(σ-closed).

The reflection principles we discussed above can be put in a yet larger picture: It is obvious that RP follows from Axiom R of Fleissner ([4]) and Axiom R follows from MA$^+$(σ-closed) (Beaudoin [1]).

The known relations between these principles with some results from [9] as well as from Sections 5 and 6 of the present paper are summarized in the following diagram:



Recently we obtained a result which further extends the diagram above: In [11] it is namely proved that FRP follows from Rado Conjecture (RC). The separation of RC and FRP can be easily obtained: Since RC implies $2^{\aleph_0} \le \aleph_2$ ([24]) any model of FRP and $2^{\aleph_0} > \aleph_2$ would separate these principles. Under $2^{\aleph_0} = \aleph_2$ a model of MM would provide such a separation. The model in Theorem 5.4 gives the separation under CH since it is known that RC implies RP($[\aleph_2]^{\aleph_0}$) (see [24]).

2 Further characterizations of FRP

For a set X, we call a sequence $\langle X_\alpha : \alpha < \kappa \rangle$ a *filtration* of X if

(2.1) $\kappa = \mathrm{cf}(|X|)$;

(2.2) $\langle X_\alpha : \alpha < \kappa \rangle$ is a continuously increasing sequence of subsets of X of cardinality $< |X|$; and

(2.3) $X = \bigcup_{\alpha<\kappa} X_\alpha$.

For $I \in [\lambda]^{\aleph_1}$ with $\mathrm{cf}(I) = \omega_1$, we can find a sup-increasing filtration $\langle I_\xi : \xi < \omega_1 \rangle$. Here, for a set S of ordinals, a sequence $\langle S_\xi : \xi < \kappa \rangle$ of subsets of S for $\kappa = \mathrm{cf}(|S|)$ is said to be a *sup-increasing filtration* if $\langle S_\xi : \xi < \kappa \rangle$ is a filtration of S and

(2.4) $\langle \sup(S_\xi) : \xi < \kappa \rangle$ is strictly increasing.

Lemma 2.1. *Suppose that λ is a regular cardinal $> \aleph_1$, $E \subseteq E^\lambda_\omega$ is stationary in λ and $g : E \to [\lambda]^{\aleph_0}$ is such that*

(2.5) *$g(\alpha) \cap \alpha$ is cofinal in α for all $\alpha \in E$.*

If

(2.6) *$I \in [\lambda]^{\aleph_1}$ is closed with respect to g and*

(2.7) $\mathrm{cf}(I) = \omega_1$,

then the following are equivalent:

(a) *For any $f : E \cap I \to I$ such that $f(\alpha) \in g(\alpha) \cap \alpha$ for all $\alpha \in E \cap I$, there is $\xi^* \in I$ such that $f^{-1}{}''\{\xi^*\}$ is stationary in* $\sup(I)$*;*

(b) *For any $f : E \cap I \to I$ such that $f(\alpha) \in g(\alpha) \cap \alpha$ for all $\alpha \in E \cap I$, there is $\xi^* \in I$ such that $f^{-1}{}''\{\xi^*\}$ is cofinal in* $\sup(I)$*;*

(c) *The set $Z = \{x \in [I]^{\aleph_0} :$ $\sup(x) \in E \cap I$ *and* $g(\sup(x)) \cap \sup(x) \subseteq x\}$ *is stationary in $[I]^{\aleph_0}$.*

Note that (a) above is identical with the condition (1.3) in the definition of FRP(λ).

Proof. "(a) $\Rightarrow$ (b)" is clear. So we shall prove "(c) $\Rightarrow$ (a)" and "(b) $\Rightarrow$ (c)".

"(c) $\Rightarrow$ (a)": Assume that I satisfies (c) and suppose that $f : E \cap I \to \lambda$ is such that $f(\alpha) \in g(\alpha) \cap \alpha$ for all $\alpha \in E \cap I$. Since

$$Z' = \{x \in [I]^{\aleph_0} : \sup(x) \in E \cap I \text{ and } f(\sup(x)) \in x\} \supseteq Z,$$

Z' is stationary in $[I]^{\aleph_0}$ by the assumption. By Fodor's Lemma, it follows that there is a $\xi^* \in I$ such that

$$Z'' = \{x \in Z' : \sup(x) \in E \cap I \text{ and } f(\sup(x)) = \xi^*\}$$

is stationary in $[I]^{\aleph_0}$. Then $f^{-1}{}''\{\xi^*\} = \{\alpha \in \sup(I) : \sup(x) = \alpha$ for some $x \in Z''\}$ is stationary in $\sup(I)$. Note that we need here the condition (2.7).

"(b) $\Rightarrow$ (c)": We prove the contraposition. Assume that (c) does not hold. We show that (b) does not hold then. Since (c) does not hold, there is a sup-increasing filtration $\langle I_\eta : \eta < \omega_1 \rangle$ of I such that, for all $\eta < \omega_1$,

(2.8) if $\sup(I_\eta) \in E \cap I$ then $g(\sup(I_\eta)) \cap \sup(I_\eta) \not\subseteq I_\eta$.

Let $f : E \cap I \to \lambda$ be defined by

(2.9) $f(\alpha) = \min\left((g(\alpha) \cap \alpha) \setminus I_{\eta_\alpha}\right)$,

where $\eta_\alpha = \sup\{\eta < \omega_1 : \sup(I_\eta) < \alpha\}$ for $\alpha \in E \cap I$. Note that f is well-defined since, for $\alpha \in E \cap I$, if $\sup(I_{\eta_\alpha}) < \alpha$ then $(g(\alpha) \cap \alpha) \setminus I_{\eta_\alpha} \neq \emptyset$ by (2.5); otherwise, we have again $(g(\alpha) \cap \alpha) \setminus I_{\eta_\alpha} \neq \emptyset$ by (2.8). By the definition (2.9), $f(\alpha) \in g(\alpha) \cap \alpha$ for all $\alpha \in E \cap I$. So the following claim shows that this f is a counterexample to (b):

Claim 2.1.1. *$f^{-1}{}''\{\xi\}$ is bounded in $\sup(I)$ for all $\xi \in I$.*

$\vdash$ For $\xi \in I$, let $\eta^* = \min\{\eta < \omega_1 : \xi \in I_\eta\}$. Then we have $\xi \in I_{\eta^*} \subseteq I_{\eta_\alpha}$ for all $\alpha \in (E \cap I) \setminus \sup(I_{\eta^*})$. By (2.9), it follows that $f(\alpha) \neq \xi$ for all $\alpha \in (E \cap I) \setminus \sup(I_{\eta^*})$. Thus we have $f^{-1}{}''\{\xi\} \subseteq \sup(I_{\eta^*})$. $\dashv$ (Claim 2.1.1)

❑ (Lemma 2.1)

The following proposition gives one of the seemingly weakest assertions among the diverse reformulations of FRP(λ).

Proposition 2.2. *For a regular $\lambda > \aleph_1$,* FRP(λ) *is equivalent to the following assertion:*

$(2.10)_\lambda$ *For any stationary $E \subseteq E^\lambda_\omega$ and a ladder system $g : E \to [\lambda]^{\aleph_0}$, there is an $I \in [\lambda]^{\aleph_1}$ such that*

(2.10a) $\mathrm{cf}(I) = \omega_1$;

(2.10b) *I is closed with respect to g;*

(2.10c) *for any* $f : E \cap I \to I$ *such that* $f(\alpha) \in g(\alpha)$ *for all* $\alpha \in E \cap I$, *there is a* $\xi^* \in I$ *such that* $f^{-1}{}''\{\xi^*\}$ *is cofinal in* $\sup(I)$.

Proof. It is clear that FRP(λ) implies $(2.10)_\lambda$. To show the other implication, assume $(2.10)_\lambda$ and suppose that $E \subseteq E^\lambda_\omega$ is stationary and $g : E \to [\lambda]^{\aleph_0}$. We have to show that there is an I as in the definition of FRP(λ) for these E and g.

Without loss of generality, we may assume that $g(\alpha) \cap \alpha$ is cofinal in α for every $\alpha \in E$.

Let $h : \lambda \to \lambda$ be a λ to 1 surjection and let

$$(2.11) \quad C = \{\alpha < \lambda : \text{(a)} \quad \alpha \text{ is closed with respect to } g \text{ and } h; \quad \text{(b)} \quad \{\gamma < \alpha : h(\gamma) = \beta\} \text{ is cofinal in } \alpha \text{ for all } \beta < \alpha \}.$$

C is a club subset of λ: To see that C is unbounded in λ, note that, for a sufficiently large regular θ and $M \prec \mathcal{H}(\theta)$ with g, $h \in M$ and $\lambda \cap M \in \lambda$, we have $\lambda \cap M \in C$.

It follows that $E_0 = E \cap C$ is stationary. For $\alpha \in E_0$, let $\{\xi^\alpha_i : i \in \omega\}$ be an enumeration of $g(\alpha) \cap \alpha$ and let $\langle \eta^\alpha_i : i \in \omega \rangle$ be a strictly increasing sequence of ordinals cofinal in α such that $h(\eta^\alpha_i) = \xi^\alpha_i$. This is possible since $\alpha \in C$ and thus α satisfies (2.11), (b). Let $g_0 : E_0 \to [\lambda]^{\aleph_0}$ be the ladder system defined by

$$(2.12) \quad g_0(\alpha) = \{\eta^\alpha_i : i \in \omega\} \text{ for } \alpha \in E_0.$$

By the assumption $(2.10)_\lambda$ there is an $I_0 \in [\lambda]^{\aleph_1}$ satisfying $(2.10)_\lambda$ (for $E = E_0$ and $g = g_0$). Let I be the closure of I_0 with respect to g. Since I_0 satisfies (2.10c) for $E = E_0$ and $g = g_0$, $E_0 \cap I_0$ is cofinal in I_0. It follows that $\sup(I_0) \in C$ and, by (2.11), (a), we have $\sup(I) = \sup(I_0)$.

We claim that this I satisfies (1.3). Suppose that $f : E \cap I \to \lambda$ is such that $f(\alpha) \in g(\alpha) \cap \alpha$ for all $\alpha \in E \cap I$. Let $f_0 : I_0 \cap E_0 \to \lambda$ be defined by

$$(2.13) \quad f_0(\alpha) \in g_0(\alpha) \text{ and } h(f_0(\alpha)) = f(\alpha)$$

for all $\alpha \in E_0 \cap I_0$. In particular, we have then

$$(2.14) \quad h \circ f_0 = f \restriction E_0 \cap I_0.$$

Since I_0 was chosen according to $(2.10)_\lambda$, there is $\xi^* \in I_0$ such that $f_0^{-1}{}''\{\xi^*\}$ is cofinal in $\sup(I_0) = \sup(I)$. By (2.14), it follows that $f^{-1}{}''\{h(\xi^*)\}$ is cofinal in $\sup(I)$. Thus we have shown that I satisfies (b) of Lemma 2.1. By Lemma 2.1, it follows that I satisfies (1.3). ❑ (Proposition 2.2)

Lemma 2.3. *For a regular cardinal $\lambda > \aleph_1$, a stationary $E \subseteq E^{\lambda}_{\omega}$, a mapping $g : E \to [\lambda]^{\aleph_0}$ such that $g(\alpha)$ is a cofinal subset of α for all $\alpha \in E$, and $\alpha^* \in E^{\kappa}_{\omega_1}$, the following are equivalent:*

(a) *There is $I \in [\alpha^*]^{\aleph_1}$ such that $\sup(I) = \alpha^*$, I is closed with respect to g and*

$$Z_I = \{x \in [I]^{\aleph_0} : \sup(x) \in E \text{ and } g(\sup(x)) \subseteq x\}$$

is stationary;

(a') *For any $I \in [\alpha^*]^{\aleph_1}$ such that $\sup(I) = \alpha^*$ and I is closed with respect to g as well as with respect to the order topology of α^*, we have that*

$$Z_I = \{x \in [I]^{\aleph_0} : \sup(x) \in E \text{ and } g(\sup(x)) \subseteq x\}$$

is stationary;

(b) *The set*

$$Z_{\alpha^*} = \{x \in [\alpha^*]^{\aleph_0} : \sup(x) \in E \text{ and } g(\sup(x)) \subseteq x\}$$

is stationary.

Proof. "(a') $\Rightarrow$ (a)" is clear.

"(a) $\Rightarrow$ (b)": Suppose that Z_{α^*} is not stationary and let $C \subseteq [\alpha^*]^{\aleph_0}$ be a club disjoint from Z_{α^*}. Let $I \in [\alpha^*]^{\aleph_1}$ be such that I is cofinal in α^* and closed with respect to g. Let

(2.15) $C' = \{x \cap I : x \in C \text{ and } \sup(x) = sup(x \cap I)\}$.

Then we can find a $C'' \subseteq C'$ which is a club in $[I]^{\aleph_0}$. C'' is still disjoint from Z_{α^*} and hence also from Z_I. Thus Z_I is not stationary.

"(b) $\Rightarrow$ (a')": Assume that Z_{α^*} is stationary. Let $I \in [\alpha^*]^{\aleph_1}$ be such that $\sup(I) = \alpha^*$ and I is closed with respect to g as well as with respect to the order topology of α^*. We have to show that Z_I is stationary in $[I]^{\aleph_0}$. Suppose that $C \subseteq [I]^{\aleph_0}$ is a club. Let

(2.16) $\tilde{C} = \{x \cup y : x \in C,\ y \in [\alpha^* \setminus I]^{\aleph_0},\ \sup(x) \geq \sup(y)\}$.

Then $\tilde{C}$ is a club in $[\alpha^*]^{\aleph_0}$. Hence, by the assumption, there is $z \in Z_{\alpha^*} \cap \tilde{C}$. Let $x = z \cap I$. By (2.16) and since I is closed with respect to the order topology of α^*, we have $\sup(z) = \sup(x) \in E \cap I$. By closedness of I with respect to g, it follows that $g(\sup(x)) \subseteq I$. Hence $g(\sup(x)) \subseteq z \cap I = x$. Thus we have $x \in Z_I \cap C$. This shows that Z_I is stationary. ❑ (Lemma 2.3)

Proposition 2.4. *For a regular* $\lambda > \aleph_1$, $\mathrm{FRP}(\lambda)$ *is equivalent to the assertion:*

$(2.17)_\lambda$ *For any stationary* $E \subseteq E^\lambda_\omega$ *and a ladder system* $g : E \to [\lambda]^{\aleph_0}$, *there is* $\alpha^* \in E^\lambda_{\omega_1}$ *such that*

$$\{x \in [\alpha^*]^{\aleph_0} : \sup(x) \in E,\ g(\sup(x)) \subseteq x\}$$

is stationary in $[\alpha^*]^{\aleph_0}$.

Proof. By Proposition 2.2, $\mathrm{FRP}(\lambda)$ is equivalent to $(2.10)_\lambda$. The equivalence of $(2.17)_\lambda$ to $(2.10)_\lambda$ follows from Lemma 2.3 and Lemma 2.1. ❑ (Proposition 2.4)

Lemma 2.5. *Suppose that* λ *is a regular cardinal. Then* $\mathrm{ADS}^-(\lambda)$ *is equivalent to the following assertion:*

$(2.18)_\lambda$ *There are a stationary* $E^* \subseteq E^\lambda_\omega$ *and a ladder system* $g^* : E^* \to [\lambda]^{\aleph_0}$ *such that*

(2.18a) g^* *is almost essentially disjoint, i.e.,* $g^* \restriction \alpha$ *is essentially disjoint for all* $\alpha < \lambda$.

Proof. "$(2.18)_\lambda \Rightarrow \mathrm{ADS}^-(\lambda)$" is trivial. The proof of "$\mathrm{ADS}^-(\lambda) \Rightarrow (2.18)_\lambda$" can be done similarly to the proof of Proposition 2.2 using the λ to 1 surjection argument. ❑ (Lemma 2.5)

Proposition 2.6. *Suppose that* FRP *does not hold and let*

$$\lambda^* = \min\{\mu : \mu \text{ is regular and } \mathrm{FRP}(\mu) \text{ does not hold}\}.$$

Then we have $\mathrm{ADS}^-(\lambda^*)$.

In [9], it is proved that, for all regular μ, $\mathrm{FRP}(\mu)$ implies $\neg\mathrm{ADS}^-(\mu)$. Thus Proposition 2.6 implies the following characterization of FRP:

Theorem 2.7. $\mathrm{FRP}(< \lambda)$ *is equivalent to the assertion:*

(2.19) $\mathrm{ADS}^-(\mu)$ *does not hold for all regular* $\mu < \lambda$. ❑

Proof of Proposition 2.6: First note that λ^* is regular and $\geq \aleph_2$. By Proposition 2.2 and Lemma 2.1, there are a stationary $E \subseteq E^{\lambda^*}_\omega$ and a ladder system $g : E \to [\lambda^*]^{\aleph_0}$ such that, for any $I \in [\lambda^*]^{\aleph_1}$ closed with respect to g with $\mathrm{cf}(I) = \omega_1$, we have

(2.20) $Z_I = \{x \in [I]^{\aleph_0} : \sup(x) \in E \cap I \text{ and } g(\sup(x)) \subseteq x\}$ is non-stationary in $[I]^{\aleph_0}$.

Let $\sigma : \lambda^* \to {}^{\aleph_0 >}\lambda^*$ be a λ^* to 1 surjection and

(2.21) $C^* = \{\alpha < \lambda^* \,:\, \text{for all } a \in {}^{\aleph_0 >}\alpha,\ \{\gamma < \alpha \,:\, \sigma(\gamma) = a\} \text{ is cofinal in } \alpha\}$.

Similarly to the argument after (2.11), we can show that C^* is a club subset of λ^*. Thus $E^* = E \cap C^*$ is a stationary subset of λ^*.

For $\alpha \in E^*$, let $\langle \eta_i^\alpha \,:\, i < \omega \rangle$ be an increasing sequence of ordinals cofinal in α such that

(2.22) $\sigma(\eta_i^\alpha) = \langle \xi_k^\alpha \,:\, k \leq i \rangle$

where $\langle \xi_k^\alpha \,:\, k < \omega \rangle$ is a fixed enumeration of $g(\alpha)$. Let $g^* : E^* \to [\lambda^*]^{\aleph_0}$ be the ladder system defined by

(2.23) $g^*(\alpha) = \{\eta_i^\alpha \,:\, i < \omega\}$.

We show that g^* is almost essentially disjoint. To this end, we prove a series of claims.

Claim 2.6.1. *Suppose that $I \in [\lambda^*]^{\aleph_1}$, $\mathrm{cf}(I) = \omega_1$ and I is closed with respect to g and g^*. For a sufficiently large regular θ, let $N \prec \mathcal{H}(\theta)$ be such that $|\, N \,| = \aleph_0$ and I, σ, λ^*, g, $g^* \in N$. Then we have $|\, g^*(\alpha) \cap N \,| < \aleph_0$ for all $\alpha \in (E^* \cap I) \setminus N$.*

$\vdash$ We prove the claim by induction on $\sup(I)$.

Since $g^*(\alpha)$ is a cofinal subset of $\alpha \cap I$ of order-type ω, the claim is trivial if $\alpha > \sup(\alpha \cap N)$. So let us assume

(2.24) $\alpha = \sup(\alpha \cap N)$.

Case I: $\alpha = \sup(I \cap N)$.

By (2.20) and by elementarity of N, there is a sup-increasing filtration $\langle I_\xi \,:\, \xi < \omega_1 \rangle \in N$ of I such that

(2.25) for all $\xi < \omega_1$, if $\sup(I_\xi) \in E$ then $g(\sup(I_\xi)) \not\subseteq I_\xi$.

For $\zeta = N \cap \omega_1$ we have $I \cap N = I_\zeta$ and $\alpha = \sup(I_\zeta)$. By (2.25), there is $k_0 \in \omega$ such that $\xi_{k_0}^\alpha \notin I_\zeta$. Then we have $\eta_i^\alpha \notin N$ for all $i \in \omega \setminus k_0$ by (2.22) and hence by (2.23) $|\, g^*(\alpha) \cap N \,| < \aleph_0$.

Case II: $\alpha < \sup(I \cap N)$.

Let $\delta = \min((I \cap N) \setminus \alpha)$. Then $I \cap \delta \in N$ and $\alpha \in (I \cap \delta) \setminus \sup((I \cap \delta) \cap N)$. Hence $\mathrm{cf}(I \cap \delta) = \omega_1$. Since $\sup(I \cap \delta) < \sup(I)$, we may apply the induction hypothesis to $I \cap \delta$ to conclude $|\, g^*(\alpha) \cap N \,| < \aleph_0$. $\dashv$ (Claim 2.6.1)

Claim 2.6.2. *If $I \in [\lambda^*]^{\aleph_1}$ is such that $\mathrm{cf}(I) = \omega_1$ and I is closed with respect to g and g^*, then there is a sup-increasing filtration $\langle I_\xi : \xi < \omega_1 \rangle$ of I such that, for all $\xi < \omega_1$,*

(2.26) *I_ξ is closed with respect to g and g^*;*

(2.27) *$|\, g^*(\alpha) \cap I_\xi \,| < \aleph_0$ for every $\alpha \in (E^* \cap I) \setminus I_\xi$.*

$\vdash$ Let θ be a sufficiently large regular cardinal and let $\langle N_\xi : \xi < \omega_1 \rangle$ be a continuously increasing chain of countable elementary submodels of $\mathcal{H}(\theta)$ such that I, σ, λ^*, g, $g^* \in N_0$ and $I \subseteq \bigcup_{\xi<\omega_1} N_\xi$.

Then, by Claim 2.6.1, $I_\xi = I \cap N_\xi$ for all $\xi < \omega_1$ are as desired.

$\dashv$ (Claim 2.6.2)

Claim 2.6.3. *For any $I \in [\lambda^*]^{\aleph_1}$, $g^* \restriction (E^* \cap I)$ is essentially disjoint.*

$\vdash$ By blowing up I if necessary, we may assume that $\mathrm{cf}(I) = \omega_1$ and I is closed with respect to g and g^*. Let $\langle I_\xi : \xi < \omega_1 \rangle$ be a filtration of I as in Claim 2.6.2. For $\xi < \omega_1$, let $\langle \rho^\xi_n : n < \omega \rangle$ be an enumeration of $E^* \cap (I_{\xi+1} \setminus I_\xi)$ where we assume without loss of generality that $|\, E^* \cap (I_{\xi+1} \setminus I_\xi) \,| = \aleph_0$ for all $\xi < \omega_1$.

By (2.27), for each $n < \omega$ there is $\zeta^\xi_n < \rho^\xi_n$ such that $(g^*(\rho^\xi_n) \setminus \zeta^\xi_n) \cap I_\xi = \emptyset$. We can find ν^ξ_n with $\zeta^\xi_n \leq \nu^\xi_n < \rho^\xi_n$ such that $g^*(\rho^\xi_n) \setminus \nu^\xi_n$ is disjoint from $g^*(\rho^\xi_i)$ for all $i < n$.

Now, for $\alpha \in E^* \cap I$, let $\xi < \omega_1$ and $n < \omega$ be such that $\alpha = \rho^\xi_n$ and let $f(\alpha) = \nu^\xi_n$. Then $g^*(\alpha) \setminus f(\alpha)$, $\alpha \in E^* \cap I$ are pairwise disjoint. $\dashv$ (Claim 2.6.3)

Claim 2.6.4. *Suppose $\delta < \lambda^*$, $\kappa = \mathrm{cf}(\delta) \geq \aleph_1$ and $W \in [\delta]^\kappa$ is such that W is club in δ and closed with respect to g and g^*. If $\langle W_\xi : \xi < \kappa \rangle$ is a sup-increasing filtration of W then*

$$C = \{\xi < \kappa : \textit{for all } \alpha \in E^* \cap (W \setminus W_\xi), |\, g^*(\alpha) \cap W_\xi \,| < \aleph_0\}$$

contains a club subset of κ.

$\vdash$ Assume, toward a contradiction, that $\kappa \setminus C$ is stationary.

Let $S = (\kappa \setminus C) \cap E^\kappa_\omega$.

Subclaim 2.6.4.1. *S is stationary.*

$\vdash$ If C is bounded then this is clear. Otherwise, let $D = \{\xi < \kappa : C \cap \xi$ is unbounded in $\xi\}$. Then D is a club subset of κ and $D \cap E^\kappa_{>\omega} \subseteq C$. Since $\kappa \setminus C$ is assumed to be stationary, it follows that S must be stationary.

$\dashv$ (Subclaim 2.6.4.1)

Note that, for all $\xi \in S$, there is $\alpha_\xi \in W \setminus W_\xi$ such that

(2.28) $|\, g^*(\alpha_\xi) \cap W_\xi \,| = \aleph_0$.

Without loss of generality, we may assume that $\alpha_\xi \in W_{\xi+1} \setminus W_\xi$.

Let $\pi : \kappa \to W$ be a bijection such that $\pi^{-1}{}''W_\xi \in \kappa$ for all $\xi < \kappa$. Let $\pi^{-1}{}''W_\xi = \beta_\xi$ for $\xi < \kappa$ and $B = \{\beta_\xi : \xi \in S\}$. Then B is a stationary subset of E^κ_ω.

For $\beta \in B$ with $\beta = \beta_\xi$ for some $\xi < \kappa$, let $g'(\beta) = \pi^{-1}{}''(g^*(\alpha_\xi) \cap W_\xi)$. Then, by Claim 2.6.3, $g' \restriction (B \cap I)$ is essentially disjoint for any $I \in [\kappa]^{\aleph_1}$. In particular B and g' make up a counterexample to FRP(κ). But this is a contradiction since we have FRP(κ) by $\mathrm{cf}(\kappa) = \kappa < \lambda^*$. $\dashv$ (Claim 2.6.4)

Now, we can show that $g^* \restriction (E^* \cap \delta)$ is essentially disjoint for all $\delta < \lambda^*$ by induction on δ.

For $\delta = 0$, this is trivial. If the essential disjointness holds for an ordinal $\delta \in \lambda^* \setminus E^*$, then we have $E^* \cap \delta = E^* \cap (\delta + 1)$ and hence the essential disjointness also holds for $\delta + 1$.

If $\delta \in E^*$ then let $f : E^* \cap \delta \to \lambda^*$ be a regressive function witnessing the essential disjointness for δ. For each $\alpha \in E^* \cap \delta$, let $\nu_\alpha < \alpha$ be such that $g^*(\alpha) \setminus \nu_\alpha$ and $g^*(\delta)$ are disjoint and let $f^* : E^* \cap (\delta + 1) \to \lambda^*$ be defined by

$$f^*(\alpha) = \begin{cases} \max\{f(\alpha), \nu_\alpha\}; & \text{if } \alpha \in \delta, \\ 0; & \text{otherwise (i.e. if } \alpha = \delta). \end{cases}$$

Then f^* witnesses the essential disjointness for $\delta + 1$.

Suppose now that δ is a limit ordinal and we have shown the essential disjointness for all $\delta' < \delta$.

Case I. $\mathrm{cf}(\delta) = \omega$.

Let $\langle \delta_n : n < \omega \rangle$ be a strictly increasing sequence of ordinals cofinal in δ. For each $n \in \omega$, let $f_n : E^* \cap (\delta_n + 1) \to \delta_n$ be a regressive function witnessing the essential disjointness of $g^* \restriction (E^* \cap (\delta_n + 1))$. Let $f^*_0 = f_0$ and let $f^*_{n+1} : E^* \cap ((\delta_{n+1} + 1) \setminus (\delta_n + 1)) \to \lambda^*$ be defined by $f^*_{n+1}(\alpha) = \max\{f_{n+1}(\alpha), \delta_n\}$ for $\alpha \in E^* \cap ((\delta_{n+1} + 1) \setminus (\delta_n + 1))$. Then it is easy to see that $f = \bigcup_{n \in \omega} f^*_n$ witnesses the essential disjointness for δ.

Case II. $\mathrm{cf}(\delta) > \omega$.

Let $\kappa = \mathrm{cf}(\delta)$ and let W be a club subset of δ of cardinality κ which is closed with respect to g and g^*. By Claim 2.6.4, there is a sup-increasing filtration $\langle W_\xi : \xi < \kappa \rangle$ of W such that, for every $\xi < \kappa$ and $\alpha \in E^* \cap (W \setminus W_\xi)$, we have $|\, g^*(\alpha) \cap W_\xi \,| < \aleph_0$. For $\xi < \kappa$, let $\eta_\xi = \sup(W_\xi)$.

For $\alpha \in E^* \cap W$, let $\xi_\alpha < \kappa$ be such that $\alpha \in W_{\xi_\alpha+1} \setminus W_{\xi_\alpha}$ and let $\nu_\alpha < \alpha$ be such that $(g^*(\alpha) \setminus \nu_\alpha) \cap W_{\xi_\alpha} = \emptyset$. For $\alpha \in (E^* \cap \delta) \setminus W$, let $\xi^*_\alpha < \kappa$ be such that $\eta_{\xi^*_\alpha} \leq \alpha < \eta_{\xi^*_\alpha+1}$. Note that actually we have $\eta_{\xi^*_\alpha} < \alpha$ in this case: Since $\alpha \notin W$ and W is closed, α is not a limit of W. Thus there is a $\mu_\alpha < \alpha$ such that $(g^*(\alpha) \setminus \mu_\alpha) \cap W = \emptyset$.

By induction hypothesis, there is a regressive $f_\xi : E^* \cap \eta_\xi \to \eta_\xi$ witnessing the essential disjointness of $g^* \restriction (E^* \cap \eta_\xi)$ for every $\xi < \kappa$. Now let $f : E^* \cap \delta \to \delta$ be defined by

$$(2.29) \quad f(\alpha) = \begin{cases} \max\{f_{\xi_\alpha+1}(\alpha), \nu_\alpha\}; & \text{if } \alpha \in W, \\ \max\{f_{\xi^*_\alpha+1}(\alpha), \eta_{\xi^*_\alpha}, \mu_\alpha\}; & \text{otherwise} \end{cases}$$

for $\alpha \in E^* \cap \delta$. Then f witnesses the essential disjointness of $g^* \restriction E^* \cap \delta$.

❏ (Proposition 2.6)

In virtue of FRP in Theorem 2.7 we can also obtain many "mathematical" characterizations of FRP.

For the notions and notations used in (A) and (B) see [9]; for (C) and (C'), see Fleissner [4] or Fuchino [7].

Theorem 2.8. *For a cardinal λ,* FRP$(< \lambda)$ *is equivalent to each of the following assertions over* ZFC*:*

(A) *For every locally separable countably tight topological space X with $L(X) < \lambda$, if all subspaces of X of cardinality $\leq \aleph_1$ are meta-Lindelöf, then X itself is also meta-Lindelöf.*

(B) *For every locally countably compact topological space X with $L(X) < \lambda$, if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable, then X itself is also metrizable.*

(C) *For every T_1-space X of cardinality $< \lambda$ with point countable base, if all subspaces of X of cardinality $\leq \aleph_1$ are left-separated then X itself is also left-separated.*

(C') *For every metrizable space X of cardinality $< \lambda$, if all subspaces of X of cardinality $\leq \aleph_1$ are left-separated then X itself is also left-separated.*

Proof. The implications "FRP$(< \lambda) \Rightarrow$ (A), (B)" are proved in Fuchino et al. [9]. It is also proved in [9] that ADS$^-(\lambda^*)$ implies the existence of a topological space X of cardinality λ^* which is a counter-example to both of (A) and (B). Note that we have $L(X) \leq |X| = \lambda^*$. Hence, it follows from Theorem 2.7 that both of the assertions (A) and (B) are equivalent to FRP$(< \lambda)$.

The implication "FRP($< \lambda$) $\Rightarrow$ (C)" is proved in Fuchino [7]. "(C) $\Rightarrow$ (C')" is trivial noting that every metric spaces are T_1 and have a point countable base by Stone's theorem. For the implication "(C') $\Rightarrow$ FRP($< \lambda$)", assume that $\mathrm{ADS}^-(\lambda^*)$ holds for a regular cardinal λ^* and we show that there is a topological space X of cardinality λ^* which is a counter-example to the assertion of (C').

Let E^* and g^* be such that E^* is a stationary subset of $E^{\lambda^*}_\omega$ and g an almost essentially disjoint ladder system on E^*.

The topological space given below is a modification of an example given in Fleissner [4].

Let $X = E^*$ be the metric space with the metric d defined by

$$d(\alpha, \beta) = \frac{1}{k_{\alpha,\beta} + 1}$$

for α, $\beta \in X$ where

$$k_{\alpha,\beta} = \min\{k < \omega : \text{the set of the first } k \text{ elements of } g^*(\alpha) \text{ and the set of the first } k \text{ elements of } g^*(\beta) \text{ are distinct}\}.$$

Let $\mathcal{O}$ be the topology of X which is induced from d. We show that $(X, \mathcal{O})$ is a counter-example to the assertion (C').

Since λ^* is a regular cardinal and subspaces of a left-separated space are also left-separated, the next claim implies that all subspaces of X of cardinality $< \lambda^*$ are left-separated:

Claim 2.8.1. *$E^* \cap \beta$ as a subspace of X is left-separated for all $\beta < \lambda^*$.*

$\vdash$ Let $\beta < \lambda^*$. Since g^* is almost essentially disjoint, there is a regressive $f : E^* \cap \beta \to \beta$ such that $g^*(\alpha) \setminus f(\alpha)$, $\alpha \in E^* \cap \beta$ are pairwise disjoint.

For $n \in \omega$, let

$$D_n = \{\alpha \in E^* \cap \beta : g^*(\alpha) \cap f(\alpha) \text{ consists of the first } n \text{ elements of } g^*(\alpha)\}.$$

Then $E^* \cap \beta = \bigcup_{n\in\omega} D_n$. Each D_n is discrete since $d(\alpha, \alpha') \geq \frac{1}{n+1}$ for every α, $\alpha' \in D_n$. By Theorem 2.2 in [4], it follows that $E^* \cap \beta$ is left-separated.

$\dashv$ (Claim 2.8.1)

To show that $(X, \mathcal{O})$ itself is not left-separated, it is enough to show the following claim. Note that there are stationarily many $\alpha \in E^*$ which can be represented as α_M as below. Also note that it is stated implicitly in Theorem 2.2. in [4] that a T_1-space X with point countable base is left-separated if and only if it is left-separated in order-type $|\,X\,|$.

Claim 2.8.2. *Let θ be a sufficiently large regular cardinal. If $M \prec \mathcal{H}(\theta)$ is such that $|M| < \lambda^*$, λ^*, E^*, $g \in M$ and $\alpha_M = \lambda^* \cap M \in E^*$. Then we have $\alpha_M \in \overline{\alpha_M \cap E^*}$.*

$\vdash$ Let $\langle \alpha_n : n \in \omega \rangle$ be an increasing enumeration of $g^*(\alpha_M)$. For each $k \in \omega$, let ψ_k be the formula asserting

> "there exists an $\alpha \in E^*$ such that the first k elements of $g^*(\alpha)$ are $\alpha_0, \ldots, \alpha_{k-1}$".

$\mathcal{H}(\theta) \models \psi_k$ holds for each $k \in \omega$ since α^* is a witness of all of ψ_k, $k \in \omega$. It follows that $M \models \psi_k$ for each $k \in \omega$ by elementarity. But, by the definition of d, this means that there is $\alpha \in \alpha_M \cap E^*$ such that $d(\alpha_M, \alpha) \leq \frac{1}{k+1}$ for each $k \in \omega$. $\dashv$ (Claim 2.8.2)

❑ (Theorem 2.8)

3 Coloring number of graphs

We consider here a graph G as a pair $\langle G, \mathcal{E} \rangle$ where the elements of G are points and the elements of $\mathcal{E} \subseteq [G]^2$ represent the edges of the graph. For a graph $G = \langle G, \mathcal{E} \rangle$, the coloring number of G is defined by:

(3.1) $\operatorname{col}(G) = \min\{\mu :$ there is a well-ordering $\prec$ of G such that $|\{y \in G : y \prec x \text{ and } \{x,y\} \in \mathcal{E}\}| < \mu$ for all $x \in G\}$.

For a graph $G = \langle G, \mathcal{E} \rangle$ and $I \subseteq G$, we denote with $G \restriction I$ the subgraph $\langle G \cap I, \mathcal{E} \cap [I]^2 \rangle$ of G. Sometimes we also identify a subset G' of G with the subgraph $\langle G', \mathcal{E} \cap [G']^2 \rangle$. For $I \subseteq G$ and $x \in G$ we write:

$$\mathcal{E}_I^x = \{y \in I : \{x, y\} \in \mathcal{E}\}.$$

The following is the main theorem of this section:

Theorem 3.1. *For any cardinal $\lambda \geq \aleph_2$, $\mathrm{FRP}(< \lambda)$ is equivalent to the assertion:*

(D) *For any graph $G = \langle G, \mathcal{E} \rangle$ of cardinality $< \lambda$, if $\operatorname{col}(G \restriction I) \leq \aleph_0$ for all $I \in [G]^{\leq \aleph_1}$, then $\operatorname{col}(G) \leq \aleph_0$.*

Let us begin with the characterization of coloring number $\leq \mu$ by Erdős and Hajnal (Lemma 3.4).

Lemma 3.2. *Suppose that $G = \langle G, \mathcal{E} \rangle$ is a graph and $f : G \to [G]^{<\mu}$ for a cardinal μ is such that*

(3.2) *at least one of* $x \in f(y)$ *and* $y \in f(x)$ *holds for all* $\{x, y\} \in \mathcal{E}$.

If $G' \subseteq G$ *is closed with respect to* f *then* $|\,\mathcal{E}^x_{G'}\,| < \mu$ *for all* $x \in G \setminus G'$.

Proof. Suppose that $x \in G \setminus G'$. If $x' \in G'$ then $f(x') \subseteq G'$ by the closedness of G' with respect to f. Thus we have $x \notin f(x')$. Hence, if $\{x, x'\} \in \mathcal{E}$, then we must have $x' \in f(x)$. It follows that $\mathcal{E}^x_{G'} \subseteq f(x)$ and $|\,\mathcal{E}^x_{G'}\,| \leq |\,f(x)\,| < \mu$.

❑ (Lemma 3.2)

Lemma 3.3. *Suppose that* $G = \langle G, \mathcal{E}\rangle$ *is a graph of cardinality* λ, $\kappa = \mathrm{cf}(\lambda)$ *and* μ *an infinite cardinal. If* $\langle G_\alpha : \alpha < \kappa\rangle$ *is a filtration of* G *such that* $\mathrm{col}(G_\alpha) \leq \mu$ *for all* $\alpha < \kappa$ *and* $|\,\mathcal{E}^x_{G_\alpha}\,| < \mu$ *for all* $\alpha < \kappa$ *and* $x \in G_{\alpha+1} \setminus G_\alpha$, *then we have* $\mathrm{col}(G) \leq \mu$ *and* G *has a well-ordering* $\prec$ *of order-type* λ *witnessing* $\mathrm{col}(G) \leq \mu$.

Proof. For each $\alpha < \kappa$, let $\prec_\alpha$ be a well-ordering of $G_{\alpha+1}$ witnessing $\mathrm{col}(G_{\alpha+1}) \leq \mu$. That is, $\prec_\alpha$ is such that

$$|\,\{y \in G_{\alpha+1} : y \prec_\alpha x \text{ and } \{x, y\} \in \mathcal{E}\}\,| < \mu \text{ for all } x \in G_{\alpha+1}.$$

Let $\prec$ be the well-ordering of G defined by

$$\begin{aligned} x \prec y \;\Leftrightarrow\; & (x, y \in G_{\alpha+1} \setminus G_\alpha \text{ and } x \prec_\alpha y \text{ for some } \alpha < \kappa) \text{ or} \\ & (x \in G_\alpha \text{ and } y \notin G_\alpha \text{ for some } \alpha < \kappa) \end{aligned}$$

for x, $y \in G$. Then $\prec$ witnesses $\mathrm{col}(G) \leq \mu$:

Claim 3.3.1. $|\,\{y \in G : y \prec x$ *and* $\{x, y\} \in \mathcal{E}\}\,| < \mu$ *for all* $x \in G$.

$\vdash$ Suppose $x \in G$ and let $\alpha < \kappa$ be such that $x \in G_{\alpha+1} \setminus G_\alpha$. Then we have

$$\begin{aligned} & \{y \in G : y \prec x \text{ and } \{x, y\} \in \mathcal{E}\} \\ & = \mathcal{E}^x_{G_\alpha} \cup \{y \in G_{\alpha+1} \setminus G_\alpha : y \prec_\alpha x \text{ and } \{x, y\} \in \mathcal{E}\}. \end{aligned}$$

Since $|\,\mathcal{E}^x_{G_\alpha}\,|$, $|\,\{y \in G_{\alpha+1} \setminus G_\alpha : y \prec_\alpha x\}\,| < \mu$ by the assumption of the lemma and the choice of $\prec_\alpha$, it follows that $|\,\{y \in G : y \prec x \text{ and } \{x, y\} \in \mathcal{E}\}\,| < \mu$.

$\dashv$ (Claim 3.3.1)

By the definition of $\prec$, it is clear that $\prec$ is of order-type λ. ❑ (Lemma 3.3)

Lemma 3.4 (Erdős and Hajnal [3])**.** *For a cardinal* μ *and a graph* $G = \langle G, \mathcal{E}\rangle$, *the following are equivalent:*

(a) $\mathrm{col}(G) \leq \mu$;

(b) *there exists a mapping* $f : G \to [G]^{<\mu}$ *satisfying* (3.2);

(c) *there is a well ordering* $\prec$ *of* G *of order-type* $|\,G\,|$ *such that*

(3.3) $\quad |\{y \in G : y \prec x$ *and* $\{x, y\} \in \mathcal{E}\}| < \mu$ *for all* $x \in G$.

Proof. "(c) $\Rightarrow$ (a)" is trivial. We show "(a) $\Rightarrow$ (b)" and "(b) $\Rightarrow$ (c)".

"(a) $\Rightarrow$ (b)": Suppose that $\mathrm{col}(G) \leq \mu$ and $\prec$ is a well-ordering of G witnessing this. Let $f : G \to [G]^{<\mu}$ be defined by

$$f(x) = \{y \in G : y \prec x \text{ and } \{x, y\} \in \mathcal{E}\}$$

for $x \in G$. Then f clearly satisfies (3.2).

"(b) $\Rightarrow$ (c)": It is enough to prove that the following $(3.4)_\lambda$ holds for all cardinal λ by induction on λ:

$(3.4)_\lambda$ for any graph $G = \langle G, \mathcal{E}\rangle$ of cardinality λ with a mapping $f : G \to [G]^{<\mu}$ satisfying (3.2), there is a well-ordering $\prec$ on G of order-type $|G|$ such that (3.3) holds.

For $\lambda \leq \mu$, this is trivial (for a graph $G = \langle G, \mathcal{E}\rangle$ of cardinality μ, any well-ordering of G of order-type μ will do).

Now, assume that $\lambda > \mu$ and we have proved $(3.4)_{\lambda'}$ for all $\lambda' < \lambda$. Suppose that $G = \langle G, \mathcal{E}\rangle$ is a graph of cardinality λ and $f : G \to [G]^{<\mu}$ satisfies (3.2).

Let $\langle G_\alpha : \alpha < \kappa\rangle$ be a filtration of G for $\kappa = \mathrm{cf}(\lambda)$ such that each G_α, $\alpha < \mathrm{cf}(\lambda)$ is closed with respect to f. By the induction hypothesis $\mathrm{col}(G_\alpha) \leq \mu$ for all $\alpha < \kappa$. $|\mathcal{E}^x_{G_\alpha}| < \mu$ for all $\alpha < \kappa$ and $x \in G_{\alpha+1}$ by Lemma 3.2. By Lemma 3.3, it follows that there is a well-ordering $\prec$ of G of order-type $|G|$ satisfying (3.3). ❑ (Lemma 3.4)

Proposition 3.5. *Suppose that* FRP *does not hold and let*

$$\lambda^* = \min\{\lambda : \lambda \textit{ is regular and } \mathrm{FRP}(\lambda) \textit{ does not hold}\}.$$

Then there is a graph $G = \langle G, \mathcal{E}\rangle$ *of cardinality* λ^* *such that* $\mathrm{col}(G) > \aleph_0$ *but all subgraphs of* G *of cardinality* $< \lambda^*$ *have countable coloring number.*

Proof. By Proposition 2.6, there are a stationary $E^* \subseteq E^{\lambda^*}_\omega$ and an almost essentially disjoint ladder system g^* on E^*. Without loss of generality, we may assume that $g^*(\alpha)$ consists of successor ordinals for all $\alpha \in E^*$ since, e.g., we may replace g^* by g^{**} defined by $g^{**}(\alpha) = \{\xi + 1 : \xi \in g^*(\alpha)\}$ for all $\alpha \in E^*$.

Let $D^* = \lambda^* \setminus \mathit{Lim}(\lambda^*)$ and let $G = E^* \cup D^*$ be the graph with its edges $\mathcal{E}$ defined by

(3.5) $\quad \mathcal{E} = \{\{\alpha, \beta\} : \alpha < \beta < \lambda^*, \alpha \in D^*, \beta \in E^* \text{ and } \alpha \in g^*(\beta)\}.$

We show in the following two claims that this $G = \langle G, \mathcal{E}\rangle$ is as desired.

Claim 3.5.1. $\mathrm{col}(G) > \aleph_0$.

Proof. Otherwise there would be a mapping $f : G \to [G]^{<\aleph_0}$ satisfying (3.2). We may assume that $f(\alpha) \subseteq g^*(\alpha) \subseteq D^* \cap \alpha$ for all $\alpha \in E^*$.

Let $f^* : E^* \to \lambda^*$ be the regressive function defined by $f^*(\alpha) = \max f(\alpha)+1$ for $\alpha \in E^*$. By Fodor's Lemma, there are a $\xi^* < \lambda^*$ and a stationary $E' \subseteq E^*$ such that $f^*(\alpha) = \xi^*$ for all $\alpha \in E'$. Similarly, there are a $\xi^{**} < \lambda$ and a stationary $E'' \subseteq E'$ such that $\min(g^*(\alpha) \setminus \xi^*) = \xi^{**}$ for all $\alpha \in E''$. Let $\alpha^* \in E'' \setminus f(\xi^{**})$. Then $\{\xi^{**}, \alpha^*\} \in \mathcal{E}$ but $\alpha^* \notin f(\xi^{**})$ and $\xi^{**} \notin f(\alpha^*)$. This is a contradiction.

$\dashv$ (Claim 3.5.1)

Claim 3.5.2. $\mathrm{col}(G \restriction \beta) \leq \aleph_0$ *for every* $\beta < \lambda^*$.

$\vdash$ For $\beta < \lambda^*$, let $f^* : E^* \cap \beta \to \lambda^*$ be a regressive function witnessing the essential disjointness of $g^* \restriction (E^* \cap \beta)$. Let $f : G \cap \beta \to [G \cap \beta]^{<\aleph_0}$ be defined by

$$f(\alpha) = \begin{cases} \{\gamma\}, & \text{if } \alpha \in D^* \cap \beta \text{ and } \alpha \in g^*(\gamma) \setminus f^*(\gamma) \\ & \text{for some } \gamma \in E^* \cap \beta; \\ g^*(\alpha) \cap f^*(\alpha), & \text{if } \alpha \in E^* \cap \beta; \\ \emptyset, & \text{otherwise} \end{cases}$$

for $\alpha \in G \cap \beta$. Then f satisfies (3.2). $\dashv$ (Claim 3.5.2)

❑ (Proposition 3.5)

The following theorem is going to be used in the proof of Theorem 3.7. Though it can be obtained as a corollary of Shelah's Singular Compactness Theorem ([22]), we give below a direct proof for completeness. Note however that the following Theorem 3.6 does not cover everything about coloring number obtained from Shelah's Singular Compactness Theorem: It also proves Theorem 3.6 for arbitrary infinite κ in place of $\aleph_0$ while our proof does not cover these cases.

Theorem 3.6 (A consequence of the main result in S. Shelah, [22])**.** *Suppose that* $G = \langle G, \mathcal{E} \rangle$ *is a graph of singular cardinality* λ. *If*

(3.6) $\mathrm{col}(G') \leq \aleph_0$ *for all* $G' \in [G]^{<\lambda}$,

then $\mathrm{col}(G) \leq \aleph_0$.

Proof. We prove the theorem in the following two cases:

Case I: $\mathrm{cf}(\lambda) = \omega$.

Let $\langle G_{\alpha,n} : n < \omega \rangle$ and $\langle f_{\alpha,n} : n < \omega \rangle$ for $\alpha < \omega_1$ be sequences constructed by induction on $\alpha < \omega_1$ such that

(3.7) $\langle G_{\alpha,n} : n < \omega \rangle$ is a filtration of G;

(3.8) $f_{\alpha,n} : G_{\alpha,n+1} \to [G_{\alpha,n+1}]^{<\aleph_0}$ and $f_{\alpha,n}$ is a witness of $\mathrm{col}(G_{\alpha,n+1}) \leq \aleph_0$ (which holds by the assumption on G) in the sense of Lemma 3.4, (b);

(3.9) For each $n < \omega$, $\langle G_{\alpha,n} : \alpha < \omega_1 \rangle$ is a continuously increasing sequence;

(3.10) $G_{\alpha+1,n}$ is the closure of $G_{\alpha,n}$ in $G_{\alpha,n+1}$ with respect to $f_{\alpha,n}$.

Let

(3.11) $G_n = \bigcup_{\alpha<\omega_1} G_{\alpha,n}$ for each $n < \omega$.

$\langle G_n : n < \omega \rangle$ is then a filtration of G. By assumption on G, we have $\mathrm{col}(G_n) \leq \aleph_0$ for all $n < \omega$.

Claim 3.6.1. *For any $n < \omega$ and $x \in G_{n+1} \setminus G_n$, we have $|\,\mathcal{E}^x_{G_n}\,| < \aleph_0$.*

$\vdash$ Let $\alpha^* < \omega_1$ be such that $x \in G_{\alpha^*,n+1} \setminus G_{\alpha^*,n}$. Then $x \in G_{\alpha,n+1}$ for all $\alpha^* \leq \alpha < \omega_1$. Since $\langle G_{\alpha,n} : \alpha^* \leq \alpha < \omega_1 \rangle$ is an increasing sequence, $\langle \mathcal{E}^x_{G_{\alpha,n}} : \alpha^* \leq \alpha < \omega_1 \rangle$ is also increasing. By (3.10) and Lemma 3.2, $\mathcal{E}^x_{G_{\alpha,n}}$ for $\alpha^* \leq \alpha < \omega_1$ are finite sets. Hence there is $\alpha^* \leq \alpha^{**} < \omega_1$ and a finite set e such that $\mathcal{E}^x_{G_{\alpha,n}} = e$ for all $\alpha^{**} \leq \alpha < \omega$. By (3.11), it follows that $\mathcal{E}^x_{G_n} = e$. $\dashv$ (Claim 3.6.1)

By Lemma 3.3, it follows that $\mathrm{col}(G) \leq \aleph_0$.

Case II. $\mathrm{cf}(\lambda) > \omega$.

Let $\kappa = \mathrm{cf}(\lambda)$ and $\langle G_\alpha : \alpha < \kappa \rangle$ be a filtration of G. By the assumption, we have $\mathrm{col}(G_\alpha) \leq \aleph_0$ and thus there is a $f_\alpha : G_\alpha \to [G_\alpha]^{<\aleph_0}$ which is a witness of this in the sense of Lemma 3.4 (b) for each $\alpha < \kappa$.

Let $\langle H_\beta : \beta < \kappa \rangle$ be (possibly) another filtration of G such that each H_β is closed with respect to f_α for all $\alpha < \kappa$. Note that we can construct such a filtration since $\kappa < \lambda$. By the assumption of the theorem, we have $\mathrm{col}(H_\beta) \leq \aleph_0$ for all $\beta < \kappa$.

Claim 3.6.2. *For all $\beta < \kappa$ and $x \in G \setminus H_\beta$, we have $|\,\mathcal{E}^x_{H_\beta}\,| < \aleph_0$.*

$\vdash$ Suppose that $x \in G_{\alpha^*}$ for some $\alpha^* < \kappa$. $H_\beta \cap G_\alpha$ is closed with respect tof_α for all $\alpha < \kappa$ by the construction of $\langle H_\beta : \beta < \kappa \rangle$. By Lemma 3.2, it follows that $\langle \mathcal{E}^x_{H_\beta \cap G_\alpha} : \alpha^* \leq \alpha < \kappa \rangle$ is an increasing sequence of finite subsets of G. Hence there is an ordinal $\alpha^* \leq \alpha^{**} < \kappa$ and an $e \in [G]^{<\aleph_0}$ such that $\mathcal{E}^x_{H_\beta \cap G_\alpha} = e$ for all $\alpha^{**} \leq \alpha < \kappa$. Since $H_\beta = \bigcup_{\alpha<\kappa} H_\beta \cap G_\alpha$, it follows that $\mathcal{E}^x_{H_\beta} = e$.
$\dashv$ (Claim 3.6.2)

By Lemma 3.3, it follows that $\mathrm{col}(G) \leq \aleph_0$. ❑ (Theorem 3.6)

In Fleissner [4], the assertion of the following theorem was proved under Axiom R:

Theorem 3.7. (FRP) *For any graph* $G = \langle G, \mathcal{E} \rangle$, *if*

(3.12) $\mathrm{col}(G \restriction I) \leq \aleph_0$ *holds for all* $I \in [G]^{\leq \aleph_1}$,

then $\mathrm{col}(G) \leq \aleph_0$.

Proof. We prove by induction on λ that the following $(3.13)_\lambda$ holds for all cardinals λ:

$(3.13)_\lambda$ For any graph $G = \langle G, \mathcal{E} \rangle$ of cardinality λ, if (3.12) holds, then $\mathrm{col}(G) \leq \aleph_0$.

For $\lambda \leq \aleph_1$, $(3.13)_\lambda$ trivially holds.

Suppose that $\lambda > \aleph_1$ and we have proved $(3.13)_{\lambda'}$ for all $\lambda' < \lambda$.

If λ is singular, and G is as in $(3.13)_\lambda$, then we can conclude $\mathrm{col}(G) \leq \aleph_0$ by the induction hypothesis and Theorem 3.6.

Suppose now that λ is regular and assume, toward a contradiction, that there is a graph G of cardinality λ which satisfies (3.12) but $\mathrm{col}(G) > \aleph_0$. Without loss of generality, we may assume that (the underlying set of) G is λ. Note that *col*$(G \restriction \alpha) \leq \aleph_0$ for all $\alpha < \lambda$ by induction hypothesis. Hence

(3.14) $E = \{\alpha \in \lambda \,:\, \text{there is } \beta \in \lambda \setminus \alpha \text{ such that } |\,\mathcal{E}_\alpha^\beta\,| \geq \aleph_0\}$.

is stationary by Lemma 3.3. Let $E^* = E \cap E_\omega^\lambda$.

Claim 3.7.1. E^* *is stationary in* λ.

⊢ Suppose otherwise. Then $E \cap E_{>\omega}^\lambda$ must be stationary. For each $\alpha \in E \cap E_{>\omega}^\lambda$, let $\beta_\alpha \in \lambda \setminus \alpha$ be such that $\mathcal{E}_\alpha^{\beta_\alpha}$ is infinite. Let $c_\alpha \in [\mathcal{E}_\alpha^{\beta_\alpha}]^{\aleph_0}$ and $\xi_\alpha = \sup(c_\alpha)$. Then $\xi_\alpha < \alpha$ since $\mathrm{cf}(\xi_\alpha) \leq \omega$. β_α and c_α witness that $[\xi_\alpha, \alpha) \cap E_\omega^\lambda \subseteq E^*$. By Fodor's Lemma, there are $\xi^* \in E_\omega^\lambda$ and stationary $E^\dagger \subseteq E \cap E_{>\omega}^\lambda$ such that $\xi_\alpha = \xi^*$ for all $\alpha \in E^\dagger$. We have $E_\omega^\lambda \setminus \xi^* = \bigcup_{\alpha \in E^\dagger} [\xi^*, \alpha) \cap E_\omega^\lambda \subseteq E^*$. Thus E^* is stationary. This is a contradiction to the assumption. ⊣ (Claim 3.7.1)

For $\alpha \in E^*$, let $\beta_\alpha \in \lambda \setminus \alpha$ be such that $|\,\mathcal{E}_\alpha^{\beta_\alpha}\,| \geq \aleph_0$ and $c_\alpha \in [\mathcal{E}_\alpha^{\beta_\alpha}]^{\aleph_0}$. Let $g : E^* \to [\lambda]^{\aleph_0}$ be defined by $g(\alpha) = c_\alpha \cup \{\beta_\alpha\}$ for all $\alpha \in E^*$.

By (the original version of) FRP(λ) and Lemma 2.1, there is $I \in [\lambda]^{\aleph_1}$ such that $\mathrm{cf}(I) = \omega_1$, I is closed with respect to g and Z as in Lemma 2.1, (c) (with E there replaced by E^*) is stationary in $[I]^{\aleph_0}$.

Now, we have $\mathrm{col}(G \restriction I) \leq \aleph_0$ by the assumption (3.12). Hence there is a witness $f : I \to [I]^{<\aleph_0}$ of this inequality in the sense of Lemma 3.4, (b). Let $C = \{x \in [I]^{\aleph_0} : x \text{ is closed with respect to } f\}$. Since C is a club in $[I]^{\aleph_0}$, there is an $x \in Z \cap C$. By the definition of Z and g, $|\mathcal{E}_x^\beta| \geq \aleph_0$ for $\beta = \beta_{\sup(x)}$. But since x is closed with respect to f, this is a contradiction to Lemma 3.2.

❑ (Theorem 3.7)

Now we obtain Theorem 3.1 combining the results above:

Proof of Theorem 3.1: "FRP($< \lambda$) $\Rightarrow$ (D)" is just the local but cumulative version of Theorem 3.7. "(D) $\Rightarrow$ FRP($< \lambda$)" follows from Proposition 3.5.

❑ (Theorem 3.1)

4 Collectionwise Hausdorff spaces

In this section, we always assume that topological spaces satisfy T_1.

A topological space X is said to be *collectionwise Hausdorff* (*cwH*, for short) if, for any closed and discrete $D \subseteq X$, there is a family $\mathcal{U}$ of pairwise disjoint open sets such that, for all $d \in D$, there is $U \in \mathcal{U}$ with $D \cap U = \{d\}$. For D and $\mathcal{U}$ as above, we say that D is *simultaneously separated by* $\mathcal{U}$. We also say that D is *simultaneously separated* if it is simultaneously separated by some $\mathcal{U}$.

X is $\leq \lambda$*-cwH* if every subspaces Y of X of size $\leq \lambda$ are cwH.

A topological space X has *local density* $\leq \kappa$, if for every $p \in X$, there is a $Y \in [X]^{\leq\kappa}$ such that $p \in \mathit{int}(\overline{Y})$.

The main theorem of this section is the further characterization of FRP($< \lambda$) now in terms of collectionwise Hausdorff spaces:

Theorem 4.1. *For a cardinal $\lambda \geq \aleph_2$,* FRP($< \lambda$) *is equivalent to the following assertion:*

(E) *For every countably tight topological space X of local density $\leq \aleph_1$, if X is $\leq \aleph_1$-cwH, then every closed discrete subsets of X of size $< \lambda$ are simultaneously separated.*

The next corollary is an immediate consequence of Theorem 4.1:

Corollary 4.2. *The following are equivalent:*

(a) FRP*;*

(b) *For every countably tight topological space X of local density $\leq \aleph_1$, if X is $\leq \aleph_1$-cwH, then X is cwH.* ❑

Fleissner [4] proved the assertion (b) of Corollary 4.2 under Axiom R.

We need the following lemma for the beginning of the induction proof of "FRP($< \lambda$) $\Rightarrow$ (E)" in Theorem 4.1.

Lemma 4.3. *Suppose that X is a space having local density $\leq \kappa$ for an uncountable cardinal κ. If X is $\leq \kappa$-cwH, then every closed discrete subsets of X of cardinality $\leq \kappa$ are simultaneously separated.*

Proof. Suppose that X and κ are as above and $D \in [X]^{\leq\kappa}$ is closed and discrete. Let $\langle p_\beta : \beta < \mu \rangle$ be an enumeration of D with $\mu = |\, D\,| \leq \kappa$ and let $Y_\beta \in [X]^{\leq\kappa}$, $\beta < \mu$ be such that $p_\beta \in U_\beta$ for $U_\beta = int(\overline{Y_\beta})$ for each $\beta < \mu$.

Let $Y = D \cup \bigcup_{\beta<\mu} Y_\beta$. Then $|\, Y\,| \leq \kappa$. Since X is $\leq \kappa$-cwH, the subspace Y of X is cwH. Since D is closed and discrete in Y, there is a sequence $\langle O_\beta : \beta < \mu \rangle$ of open sets in X such that $p_\beta \in O_\beta$ for all $\beta < \mu$ and

(4.1) $\quad O_\beta \cap Y$, $\beta < \mu$ are pairwise disjoint.

Claim 4.3.1. *$O_\beta \cap U_\beta$, $\beta < \mu$ are pairwise disjoint.*

$\vdash$ Suppose that $(O_\beta \cap U_\beta) \cap (O_{\beta'} \cap U_{\beta'}) \neq \emptyset$ for some $\beta < \beta' < \mu$. Since $O_\beta \cap Y_\beta$ is dense in $O_\beta \cap U_\beta$, there is an $x \in (O_\beta \cap Y_\beta) \cap (O_{\beta'} \cap U_{\beta'})$. But then $x \in (O_\beta \cap Y) \cap (O_{\beta'} \cap Y)$. This is a contradiction to (4.1). $\dashv$ (Claim 4.3.1)

Thus $\langle O_\beta \cap U_\beta : \beta < \mu \rangle$ simultaneously separates D in X. ❑ (Lemma 4.3)

The following "Singular Compactness Theorem" is also used as a part of the proof of "FRP(λ) $\Rightarrow$ (E)" of Theorem 4.1. For some other similar singular compactness results on collectionwise Hausdorff spaces, see e.g. Watson [27].

Theorem 4.4. *Let λ be a singular cardinal with $\mu = \mathrm{cf}(\lambda) < \lambda$. Suppose that X is a topological space of local density $\leq \kappa$ for some $\kappa < \lambda$. If*

(4.2) *every closed discrete subsets of X of cardinality $< \lambda$ are simultaneously separated,*

then every closed discrete subsets of X of cardinality $\leq \lambda$ are simultaneously separated.

Proof. Suppose that λ, μ, X and κ are as above and $D \in [X]^\lambda$ is closed and discrete in X. Let $\langle D_i : i < \mu \rangle$ be a filtration of D. Note that, since D is closed and discrete, each D_i is closed and discrete in X.

Since each D_i is of cardinality $\lambda_i < \lambda$, D_i is simultaneously separated by a family $\mathcal{U}_i$ of pairwise disjoint open sets by (4.2).

Since X is of local density $\leq \kappa$, we may assume that

(4.3) each $U \in \mathcal{U}_i$ has a dense subset D_U of size $\leq \kappa$.

Let $\mathcal{U} = \bigcup_{i<\mu} \mathcal{U}_i$ and let $\approx$ be the transitive closure of the relation $U \sim V \Leftrightarrow U \cap V \neq \emptyset$.

By (4.3), we have $|\{V \in \mathcal{U} : U \sim V\}| \leq \kappa + \mu < \lambda$ for all $U \in \mathcal{U}$. It follows that $|\{V \in \mathcal{U} : U \approx V\}| \leq \kappa + \mu < \lambda$ for all $U \in \mathcal{U}$.

Let $\mathbb{E}$ be the set of all equivalence classes of $\approx$. For $e \in \mathbb{E}$, let $O_e = \bigcup e$ and $D'_e = D \cap O_e$. Then O_e, $e \in \mathbb{E}$ are pairwise disjoint open sets, $\bigcup_{e\in\mathbb{E}} D'_e = D$ and $|D'_e| \leq \kappa + \mu < \lambda$ for all $e \in \mathbb{E}$. Since each D'_e for $e \in \mathbb{E}$ is a closed discrete subset of X, D'_e is simultaneously separated by some family $\mathcal{U}'_e$ of pairwise disjoint open sets by (4.2). We may assume that $\bigcup \mathcal{U}'_e \subseteq O_e$ for all $e \in \mathbb{E}$. But then $\mathcal{U}' = \bigcup_{e\in\mathbb{E}} \mathcal{U}'_e$ simultaneously separates D. ❑ (Theorem 4.4)

Proof of Theorem 4.1: "(E) $\Rightarrow$ FRP($<\lambda$)": By Theorem 2.7, it is enough to show the following. Note that $\mathrm{ADS}^-(\mu)$ for a regular uncountable cardinal μ implies that $\mu \geq \aleph_2$.

Claim 4.1.1. *Suppose that* $\mathrm{ADS}^-(\mu)$ *holds for a regular uncountable cardinal* $\mu < \lambda$. *Then there exists a space* X *of cardinality* μ *such that* X *is locally countable, first countable and* $< \mu$*-cwH but not cwH.*

$\vdash$ Let $g : S \to [\mu]^{\aleph_0}$ be an almost essentially disjoint ladder system on some stationary $S \subseteq E^\mu_\omega$. We may assume that $g(\alpha) \cap S = \emptyset$ for all $\alpha \in S$.

Let $T = \bigcup\{g(\alpha) : \alpha \in S\}$ and let $X = S \cup T$ be the topological space with the topology defined by:

(4.4) Every $\beta \in T$ are isolated;

(4.5) For $\alpha \in S$, $\{\{\alpha\} \cup (g(\alpha) \setminus \beta) : \beta < \alpha\}$ is a neighborhood base of α.

Then it is easy to see that this X is as desired. $\dashv$ (Claim 4.1.1)

"FRP($<\lambda$) $\Rightarrow$ (E)": Suppose that X is a countably tight $\leq \aleph_1$-cwH space of local density $\leq \aleph_1$. By induction on $\kappa < \lambda$, we show that

$(*)_\kappa$ if $D \in [X]^{\leq\kappa}$ is closed and discrete then D can be simultaneously separated.

For $\kappa \leq \aleph_1$, $(*)_\kappa$ holds by Lemma 4.3.

Assume now that $\aleph_1 < \kappa < \lambda$ and that we have shown $(*)_\mu$ for all $\mu < \kappa$.

Case I. κ is regular.

Suppose that $D \in [X]^\kappa$ is closed and discrete. We have to show that D is simultaneously separated.

For each $p \in X$, let $E_p \in [X]^{\leq\aleph_1}$ be such that $p \in int(\overline{E_p})$. Let $Y \in [X]^\kappa$ be such that $D \subseteq Y$ and Y is closed with respect to the mapping $p \mapsto E_p$. That is, $E_p \subseteq Y$ holds for all $p \in Y$. Let $\langle Y_\alpha : \alpha < \kappa\rangle$ be a filtration of Y such that, for all $\alpha < \kappa$,

(4.6) Y_α is closed with respect to the mapping $p \mapsto E_p$.

Without loss of generality, we may assume that $Y = \kappa$.

Claim 4.1.2. *$C = \{\alpha < \kappa : Y_\alpha \cap D = \overline{Y_\alpha} \cap D\}$ contains a club.*

$\vdash$ Suppose, toward a contradiction, that $\kappa \setminus C$ is stationary. Since κ is regular, $\{\alpha < \kappa : Y_\alpha = \alpha\}$ is club. By the countable tightness of X, it follows that

$$S = \{\alpha \in E^\kappa_\omega : Y_\alpha = \alpha,\ D \cap Y_\alpha \neq D \cap \overline{Y_\alpha}\}$$

is stationary.

For $\alpha \in S$, let $p_\alpha \in (D \cap \overline{Y_\alpha}) \setminus Y_\alpha$. Again by the countable tightness of X, there is $c_\alpha \in [Y_\alpha]^{\aleph_0}$ such that $p_\alpha \in \overline{c_\alpha}$ for all $\alpha \in S$.

Now let $g : S \to [\kappa]^{\aleph_0}$ be such that $g(\alpha) = c_\alpha \cup \{\beta_\alpha\}$ for $\alpha \in S$ where $\beta_\alpha < \kappa$ is such that $p_\alpha \in Y_{\beta_\alpha}$. By FRP($\kappa$), there is $I \in [\kappa]^{\aleph_1}$ such that $\mathrm{cf}(I) = \omega_1$, I is closed with respect to g and

(4.7) for every $f : S \cap I \to \kappa$ with $f(\alpha) \in g(\alpha) \cap \alpha$ for all $\alpha \in S \cap I$, there exits $\xi < \kappa$ such that $\{\alpha \in S \cap I : f(\alpha) = \xi\}$ is unbounded in $\sup(I)$.

Let $E = \{p_\alpha : \alpha \in S \cap I\}$. Since $E \subseteq D$, E is closed and discrete. Since $|E| \leq \aleph_1$, E is simultaneously separated by a pairwise disjoint family $\mathcal{U}$ of open sets by Lemma 4.3. For $\alpha \in S \cap I$, let $U_\alpha \in \mathcal{U}$ be such that $p_\alpha \in U_\alpha$. Since $p_\alpha \in \overline{c_\alpha}$, $c_\alpha \cap U_\alpha \neq \emptyset$. Let $f(\alpha) \in c_\alpha \cap U_\alpha$ for all $\alpha \in S \cap I$. By (4.7), there are α, $\alpha' \in S \cap I$ such that

(4.8) $\alpha < \beta_\alpha < \alpha'$ and

(4.9) $f(\alpha) = f(\alpha')$.

By the definition of p_α's and (4.8), we have $p_\alpha \neq p_{\alpha'}$ and thus $U_\alpha \neq U_{\alpha'}$ and $U_\alpha \cap U_{\alpha'} = \emptyset$ by the choice of $\mathcal{U}$. On the other hand, we have $f(\alpha) \in U_\alpha \cap U_{\alpha'}$ by (4.9). This is a contradiction. $\dashv$ (Claim 4.1.2)

By moving to a subsequence of $\langle Y_\alpha : \alpha < \kappa\rangle$ with the club subset of κ in Claim 4.1.2 as the new index set, we may assume that

(4.10) $D \cap Y_\alpha = D \cap \overline{Y_\alpha}$ for all $\alpha < \kappa$.

Claim 4.1.3. *$D \cap Y_\alpha = D \cap int(\overline{Y_\alpha})$ for all $\alpha < \kappa$*

$\vdash$ "$\supseteq$" follows from (4.10). For "$\subseteq$", if $p \in D \cap Y_\alpha$, then $E_p \subseteq Y_\alpha$ by (4.6), and $p \in int(\overline{E_p}) \subseteq int(\overline{Y_\alpha})$. Thus $p \in D \cap int(\overline{Y_\alpha})$. $\dashv$ (Claim 4.1.3)

For $\alpha < \kappa$, let $D_\alpha = (D \cap Y_{\alpha+1}) \setminus Y_\alpha$. Since $|\, D_\alpha \,| < \kappa$, D_α is simultaneously separated by a pairwise disjoint family $\mathcal{U}_\alpha$ of open sets. By (4.10) and Claim 4.1.3, we may assume that $U \subseteq int(\overline{Y_{\alpha+1}}) \setminus \overline{Y_\alpha}$ for all $U \in \mathcal{U}_\alpha$. Then $\mathcal{U} = \bigcup_{\alpha<\kappa} \mathcal{U}_\alpha$ separates D simultaneously.

Case II. κ is singular.
In this case, if $D \in [X]^\kappa$ is closed and discrete, then it is simultaneously separated by the induction hypothesis and Theorem 4.4. ❑ (Theorem 4.1)

By Theorem 4.1 and Claim 4.1.1, the following assertion is also equivalent to FRP($< \lambda$):

(E') For every locally countable, first countable topological space X, if X is $\leq \aleph_1$-cwH, then every closed discrete subsets of X of size $< \lambda$ are simultaneously separated.

5 Separation of FRP from RP

As it was already mentioned in the introduction, it is fairly easy to separate FRP from RP: Any model of FRP + $2^{\aleph_0} > \aleph_2$ does not satisfy WRP since WRP($[\omega_2]^{\aleph_0}$) implies $2^{\aleph_0} \leq \aleph_2$.

It is also relatively easy to obtain models of FRP + $\neg$WRP($[\omega_2]^{\aleph_0}$) together with $2^{\aleph_0} \leq \aleph_2$: Suppose that κ is a strongly compact cardinal. Then the collapsing $\mathrm{Col}(\omega_1, <\kappa)$ of κ to ω_2 by countable conditions forces FRP (see Proposition 5.1 below). Forcing further by the partial ordering $\mathbb{C}$ adding a Cohen real over this model still preserve FRP by Theorem 3.4 in [9]. On the other hand $\{x \in [\omega_2]^{\aleph_0} : x \notin V^{\mathrm{Col}(\omega_1,<\kappa)}\}$ is a non-reflecting stationary set in the generic extension by $\mathrm{Col}(\omega_1, <\kappa) * \mathbb{C}$ (see [12]). This construction gives thus a model of FRP + $\neg$WRP($[\omega_2]^{\aleph_0}$) + CH.

Likewise, the forcing by $\mathrm{Col}(\omega_1, <\kappa) * \mathbb{C}_{\aleph_2}$ for a strongly compact κ produces a model of FRP + $\neg$WRP($[\omega_2]^{\aleph_0}$) + $2^{\aleph_0} = \aleph_2$.

In the rest of this section, we give two constructions of models of FRP + $\neg$WRP($[\omega_2]^{\aleph_0}$) + $2^{\aleph_0} \leq \aleph_2$ without adding reals. The first construction (Theorem 5.2) relies on the model of set theory given in Section 5 of Sakai [19].

First let us check the following:

Proposition 5.1. *Suppose that $\kappa \le \lambda$, κ is λ-strongly compact and λ is regular. Then $\Vdash_{\mathbb{P}}$ "$\mathrm{FRP}(\lambda)$" holds for $\mathbb{P} = \mathrm{Col}(\aleph_1, <\kappa)$.*

Proof. Let $j : V \preccurlyeq M$ be such that

(5.1) $crit(j) = \kappa$;

(5.2) $M^\kappa \subseteq M$ and

(5.3) $\forall X \in [M]^{\le\lambda}\ \exists Y \in ([M]^{<j(\kappa)})^M\ (X \subseteq Y)$

(see e.g. Theorem 22.17 in [13]). Suppose that $\dot{S}$ is a $\mathbb{P}$-name of a stationary subset of E^λ_ω and $\dot{g}$ a $\mathbb{P}$-name such that

(5.4) $\Vdash_{\mathbb{P}}$ "$\dot{g} : S \to [\lambda]^{\aleph_0}$ and $\dot{g}$ is a ladder system on $\dot{S}$".

Let $\mathbb{P}' = \mathrm{Col}(\aleph_1, <j(\kappa))$. Thus we have $\mathbb{P}' = j(\mathbb{P})$ by (5.2). Let G' be a $(V, \mathbb{P}')$-generic set such that $G \subseteq G'$. j can be then extended to the elementary embedding $j' : V[G] \preccurlyeq M[G']$ defined by $j'(\dot{a}^G) = j(\dot{a})^{G'}$ for each $\mathbb{P}$-name $\dot{a}$.

In $V[G]$, let

$$\tilde{S} = \{x \in [\lambda]^{\aleph_0} : \sup(x) \in \dot{S}^G \text{ and } \dot{g}^G(\sup(x)) \subseteq x\}.$$

Then $\tilde{S}$ is a stationary subset of $[\lambda]^{\aleph_0}$. Note that here we need the regularity of λ. Since $\mathbb{P}'$ is σ-closed, $\tilde{S}$ remains stationary in $V[G']$. It follows that

(5.5) $j'\,''\tilde{S}$ is a stationary subset of $[j''\lambda]^{\aleph_0}$ in $V[G']$.

Let $\alpha^* = \sup(j''\lambda)$. Note that $V[G'] \models \mathrm{cf}(\alpha^*) = \omega_1$. Hence $M[G'] \models \mathrm{cf}(\alpha^*) > \omega$. By $M[G'] \models \alpha^* \le j(\lambda)$ and (5.3), we have $M[G'] \models \mathrm{cf}(\alpha^* < j(\kappa) = \omega_2)$. Hence $M[G'] \models \mathrm{cf}(\alpha^*) = \omega_1$. By (5.3), there is $Y \in M$ such that $Y \in ([\alpha^*]^{<j(\kappa)})^M$ and $j''\lambda \subseteq Y$. In $M[G']$, let $\tilde{Y}$ be the closure of Y with respect to $j'(\dot{g}^G)$. Since $M[G'] \models j(\kappa) = \omega_2$, we have $M[G'] \models \tilde{Y} \in [\alpha^*]^{\aleph_1}$.

In $M[G']$, let $\langle Y_\alpha : \alpha < \omega_1\rangle$ be a filtration of $\tilde{Y}$. Then $\langle Y'_\alpha : \alpha < \omega_1\rangle$ for $Y'_\alpha = Y_\alpha \cap j''\lambda$ is a filtration of $j''\lambda$ (in $V[G']$). By (5.5), the set

$$\{\alpha < \omega_1 : \sup(Y_\alpha) = \sup(Y'_\alpha) \text{ and } j'(\dot{g}^G)(\sup(Y_\alpha)) \subseteq Y'_\alpha\}$$

is stationary in $V[G']$. It follows that $\{\alpha < \omega_1 : j'(\dot{g}^G)(\sup(Y_\alpha)) \subseteq Y_\alpha\}$ is stationary in $V[G']$ and hence also in $M[G']$.

Thus, in $M[G']$, we have

$$M[G'] \models \exists I \in [j(\lambda)]^{\aleph_1}\ (\mathrm{cf}(\sup(I)) = \omega_1,\ I \text{ is closed with respect to } j'(\dot{g}^G) \text{ and } I \text{ has a filtration } \langle I_\alpha : \alpha < \omega_1\rangle \text{ such that } \{\alpha < \omega_1 : j'(\dot{g}^G)(\sup(I_\alpha)) \subseteq I_\alpha\} \text{ is stationary}).$$

By the elementarity of j', it follows that

$$V[G] \models \exists I \in [\lambda]^{\aleph_1} \ \big(\mathrm{cf}(\sup(I)) = \omega_1,\ I \text{ is closed with respect to } \dot{g}^G \text{ and } I \text{ has a filtration } \langle I_\alpha : \alpha < \omega_1 \rangle \text{ such that } \{\alpha < \omega_1 : \dot{g}^G(\sup(I_\alpha)) \subseteq I_\alpha\} \text{ is stationary} \big).$$

Thus, by Theorem 2.2 and Lemma 2.1, we have $V[G] \models \mathrm{FRP}(\lambda)$.

❑ (Proposition 5.1)

Theorem 5.2. *Suppose that there is a supercompact cardinal κ. Then there is a partial ordering $\mathbb{P}$ collapsing κ to ω_2 without adding any new reals such that*

(5.6) $\Vdash_{\mathbb{P}}$ "$\mathrm{FRP} \ + \ \neg\mathrm{WRP}([\omega_3]^{\aleph_0}) \ + \ 2^{\aleph_0} = \aleph_1$".

Note that, since WRP enjoys downward transfer property, (5.6) implies

(5.7) $\Vdash_{\mathbb{P}}$ "$\neg\mathrm{WRP}([\lambda]^{\aleph_0})$ for all $\lambda \geq \omega_3$".

Proof of Theorem 5.2: By Theorem 5.3 in [19], there is a partial ordering $\mathbb{P}_0$ such that $\mathbb{P}_0$ adds no new reals and

$$\Vdash_{\mathbb{P}_0} \text{“}\kappa \text{ is strongly compact”} \quad \text{and} \quad \Vdash_{\mathbb{P}_0 * \mathrm{Col}(\aleph_1, <\kappa)^{\mathbb{P}_0}} \text{“}\neg\mathrm{WRP}([\omega_3]^{\aleph_0})\text{”}.$$

On the other hand, we have $\Vdash_{\mathbb{P}_0 * \mathrm{Col}(\aleph_1, <\kappa)^{\mathbb{P}_0}}$ "FRP" by Proposition 5.1.

❑ (Theorem 5.2)

Note that the model above also satisfies RC (the proof is similar to and much simpler than that of Proposition 5.1). Thus we can conclude that RC does not imply WRP. This solves the Question 7.9 in [25] negatively.

In Theorem 5.2, $\aleph_3$ is the first cardinal κ where the consistence of FRP together with $\neg\mathrm{WRP}([\kappa]^{\aleph_0})$ is shown. T. Miyamoto [18] proved that the consistency strength of $\mathrm{FRP}(\aleph_2)$ is exactly that of the existence of a Mahlo cardinal. His construction also gives a model of $\mathrm{FRP}(\omega_2) + \neg\mathrm{WRP}([\omega_2]^{\aleph_0}) + \mathrm{CH}$. In the following, we construct of a model of FRP and $\neg\mathrm{WRP}([\aleph_2]^{\aleph_0})$ by forcing with a σ-Baire partial ordering over a model with a strongly compact cardinal.

Let us begin with a well-known fact (Fact 5.3) which plays the central role in the following Theorem 5.4 as well as in Theorem 6.1 in the next section. For completeness, we shall give a proof of it. The following proof is a variant of the argument with a game considered in Veličković [26].

Fact 5.3. *Let λ be an ordinal with $\mathrm{cf}(\lambda) > \omega_1$. For any stationary $E \subseteq E^\lambda_\omega$ and any mapping $f : E \to [\lambda]^{\aleph_0}$ such that $f(\alpha)$ is a cofinal subset of α for all $\alpha \in E$, the set*

$$T_f = \{x \in [\lambda]^{\aleph_0} \,:\, f(\sup(x)) \not\subseteq x\}$$

is stationary in $[\lambda]^{\aleph_0}$.

Proof. Suppose that $\mathcal{C} \subseteq [\lambda]^{\aleph_0}$ is a club. We want to show that $T_f \cap \mathcal{C} \neq \emptyset$. Let $g : \lambda^{<\omega} \to \lambda$ be a mapping such that $\mathcal{C}_g \subseteq \mathcal{C}$ where $\mathcal{C}_g = \{x \in [\lambda]^{\aleph_0} :$ x is closed with respect to $g\}$. It is enough to show that $T_f \cap \mathcal{C}_g \neq \emptyset$.

Let $\Game(g)$ be the following game of length ω for the players I and II: A match $\mathfrak{M}$ in $\Game(g)$ looks like this:

$$\mathfrak{M}: \quad \begin{array}{c|cc:cc:c:cc:c} \text{Player I} & \alpha_0 & & \alpha_1 & & \cdots & \alpha_n & & \cdots \\ \hline \text{Player II} & & I_0, \xi_0 & & I_1, \xi_1 & \cdots & & I_n, \xi_n & \cdots \end{array} \qquad (n < \omega)$$

where each I_n is a closed interval $[\beta_n, \gamma_n]$ and

(5.8) $\quad \alpha_n < \beta_n \leq \xi_n < \gamma_n < \alpha_{n+1}$

for all $n \in \omega$.

Player II wins the match if $cl_g(\{\xi_n : n \in \omega\}) \subseteq \gamma_0 \cup \bigcup_{n\in\omega\setminus 1} I_n$. Here $cl_g(s)$ for $s \subseteq \lambda$ denotes the closure of the set s with respect to g. That is, the $\subseteq$-minimal set $s' \subseteq \lambda$ such that $s \subseteq s'$ and $g(u) \in s'$ for all $u \in s'^{<\omega}$.

The proof of the following claim is an adaptation of the corresponding proof in [26]:

Claim 5.3.1 (Veličković). *II has a winning strategy in* $\Game(g)$.

We first show that the claim above implies the theorem. Without loss of generality, we may assume that f is a ladder system on E. Let σ be a winning strategy of II in $\Game(g)$. Let θ be a sufficiently large regular cardinal and let $M \prec \langle \mathcal{H}(\theta), \in \rangle$ be such that

(5.9) $\quad M$ is countable;

(5.10) $\quad g, \sigma \in M$ and

(5.11) $\quad \delta = \sup \lambda \cap M \in E$.

Let

$$\mathfrak{M}^*: \quad \begin{array}{c|cc:cc:c:cc:c} \text{Player I} & \alpha_0 & & \alpha_1 & & \cdots & \alpha_n & & \cdots \\ \hline \text{Player II} & & I_0, \xi_0 & & I_1, \xi_1 & \cdots & & I_n, \xi_n & \cdots \end{array}$$

be a match in $\Game(g)$ such that

(5.12) $\quad$ II played according to σ in $\mathfrak{M}^*$;

(5.13) each initial segment $\langle \alpha_0, \beta_0, \xi_0, \gamma_0, ..., \alpha_n, \beta_n, \xi_n, \gamma_n \rangle$ of $\mathfrak{M}^*$ is in M and

(5.14) the interval (γ_n, α_{n+1}), contains an element δ_n of $f(\delta)$ for $n \in \omega$.

Let $x = cl_g(\{\xi_n : n \in \omega\})$. Then $x \in \mathcal{C}_g$. $\sup(x) = \delta$ by (5.13) and (5.14). By (5.12) and (5.14) we have $x \cap \{\delta_n : n \in \omega\} = \emptyset$. Since $\{\delta_n : n \in \omega\} \subseteq f(\delta)$ by (5.14), it follows that $f(\sup(x)) \not\subseteq x$ and hence $x \in T_f$. This shows that $T_f \cap \mathcal{C}_g \neq \emptyset$ as desired.

$\vdash$ (Claim 5.3.1) Suppose otherwise. Since $\Game(g)$ is an open game for the player I, the player I then has a winning strategy σ by Gale-Stewart Theorem. Let $\langle \delta_n : n \in \omega \rangle$ be a strictly increasing sequence of ordinals below λ such that

(5.15) $\mathrm{cf}(\delta_n) = \omega_1$ and

(5.16) δ_n is closed with respect to g and σ for all $n \in \omega$.

Note that we can take such sequence by $\mathrm{cf}(\lambda) > \omega_1$.

Let $\delta = \sup_{n\in\omega} \delta_n$. Let $x \in [\delta]^{\aleph_0}$ be such that

(5.17) $\sup(x) = \delta$;

(5.18) $[\delta_n, \delta_n + \omega] \subseteq x$ for all $n \in \omega$ and

(5.19) x is closed with respect to σ and g.

Let $\alpha_n, \beta_n, \xi_n \gamma_n \in \delta$ for $n \in \omega$ be such that

(5.20) $\alpha_n < \beta_n = \xi_n < \gamma_n < \alpha_{n+1}$;

(5.21) $\alpha_n = \sigma(\langle \alpha_k, \beta_k, \xi_k, \gamma_k : k < n \rangle)$;

(5.22) $\beta_n = \delta_n$, $\xi_n = \delta_n$ and

(5.23) $\gamma_n = \sup(\delta_{n+1} \cap x)$

for all $n \in \omega$.

By (5.15) and (5.23) we have $\gamma_n < \delta_{n+1}$. If $\langle \alpha_k, \beta_k, \xi_k, \gamma_k : k < n \rangle$ is a sequence below δ_n satisfying (5.20) then, since δ_n is closed with respect to σ, we have

(5.24) $\alpha_n < \delta_n$.

Thus the construction according to (5.21), (5.22), (5.23) provides a sequence satisfying (5.20).

Letting $I_n = [\beta_n, \gamma_n]$ for $n \in \omega$, the sequence

$$\alpha_0,\ I_0, \xi_0,\ \alpha_1,\ I_1, \xi_1, ...,\ \alpha_n,\ I_n, \xi_n, ...$$

is a match in $\Game(g)$ in which the player I played according to the winning strategy σ. However, by (5.19) and (5.23), we have

$$cl_g(\{\xi_n : n \in \omega\}) \subseteq x \subseteq \gamma_0 \cup \bigcup_{n\in\omega\setminus 1} I_n.$$

Thus Player II wins this match. This is a contradiction. $\dashv$ (Claim 5.3.1)

$\square$ (Fact 5.3)

Theorem 5.4. *Suppose that κ is a strongly compact cardinal. Then there exists a κ-c.c. and σ-Baire forcing notion $\mathbb{P}$ which forces*

(5.25) $\kappa = \omega_2$, CH *and* FRP *but* $\neg$WRP$([\omega_2]^{\aleph_0})$.

Proof. The idea of the proof is that we introduce a partial ordering $\mathbb{P}$ such that a $(V, \mathbb{P})$-generic set G collapses κ to ω_2 without adding new countable set and it introduces a mapping f_G and g_G such that f_G maps each $\alpha \in E^\kappa_\omega$ to a cofinal countable subset of α while g_G prevents T_{f_G} defined as in Fact 5.3 from becoming a reflecting stationary set. This destroys WRP$([\omega_2]^{\aleph_0})$ in the generic extension. At the same time, we design $\mathbb{P}$ nice enough to preserve stationarity of certain type of sets so that a proof similar to but slightly more complicated than that of Theorem 5.1 works for this $\mathbb{P}$ to show that $\mathbb{P}$ forces FRP.

Let $\mathbb{P}$ be the set of all pairs $\langle f, g\rangle$ of functions such that

(5.26) $\mathrm{dom}(f) \in [E^\kappa_\omega]^{\leq\aleph_0}$,

(5.27) $f(\alpha) \in [\alpha]^{\aleph_0}$ and $f(\alpha)$ is cofinal in α for all $\alpha \in \mathrm{dom}(f)$,

(5.28) $\mathrm{dom}(g) \in [E^\kappa_{>\omega}]^{\leq\aleph_0}$,

(5.29) $g(\beta) \in [[\beta]^{\aleph_0}]^{\leq\aleph_0}$ and, with respect to $\subseteq$, $g(\beta)$ is well-ordered, closed (that is, the corresponding increasing sequence of countable sets is continuous) and with the maximal element,

(5.30) $\sup(x) \in \mathrm{dom}(f)$ for every $x \in \bigcup\{g(\beta) : \beta \in \mathrm{dom}(g)\}$, and

(5.31) $f(\sup(x)) \subseteq x$ for every $x \in \bigcup\{g(\beta) : \beta \in \mathrm{dom}(g)\}$.

For $\langle f, g\rangle$, $\langle f', g'\rangle \in \mathbb{P}$, we define

(5.32) $\langle f, g\rangle \leq_{\mathbb{P}} \langle f', g'\rangle \;\Leftrightarrow\; f' \subseteq f$, $\mathrm{dom}(g') \subseteq \mathrm{dom}(g)$ and $g(\beta)$ is an end extension of $g'(\beta)$ for all $\beta \in \mathrm{dom}(g')$.

In the following, we show that this $\langle \mathbb{P}, \leq_{\mathbb{P}}\rangle$ is as desired.

Claim 5.4.1. *For any $x^* \in [\kappa]^{\aleph_0}$,*

$$D_{x^*} = \{\langle f, g\rangle \in \mathbb{P} : \begin{array}{ll} \text{(a)} & x^* \cap E^\kappa_\omega \subseteq \mathrm{dom}(f), \\ \text{(b)} & x^* \cap E^\kappa_{>\omega} \subseteq \mathrm{dom}(g), \\ \text{(c)} & \forall \beta \in \mathrm{dom}(g)\, \exists y \in g(\beta)\ (x^* \cap \beta \subseteq y)\, \} \end{array} \tag{5.33}$$

is dense in $\mathbb{P}$.

$\vdash$ Let θ be a sufficiently large regular cardinal. For an arbitrary $\langle f', g'\rangle \in \mathbb{P}$, let $N \prec \mathcal{H}(\theta)$ be countable such that $x^*, \langle f', g'\rangle \in N$.

Let $\langle f, g\rangle$ be the pair of the functions defined by

(5.34) $\mathrm{dom}(f) = (E^\kappa_\omega \cap N) \cup Lim(\kappa \cap N)$ and

$$f(\alpha) = \begin{cases} f'(\alpha), & \text{if } \alpha \in \mathrm{dom}(f'), \\ \alpha \cap N, & \text{if } \alpha \in \mathrm{dom}(f) \setminus \mathrm{dom}(f') \end{cases}$$

for $\alpha \in \mathrm{dom}(f)$;

(5.35) $\mathrm{dom}(g) = E^\kappa_{>\omega} \cap N$ and

$$g(\beta) = \begin{cases} g'(\beta) \cup \{\beta \cap N\}, & \text{if } \beta \in \mathrm{dom}(g'), \\ \{\beta \cap N\}, & \text{if } \beta \in \mathrm{dom}(g) \setminus \mathrm{dom}(g') \end{cases}$$

for $\beta \in \mathrm{dom}(g)$.

$\langle f, g\rangle$ satisfies (a), (b), (c) of (5.33). Since the pair $\langle f, g\rangle$ and $\langle f', g'\rangle$ also satisfies the conditions in the definition (5.32) of $\leq_\mathbb{P}$, it is enough to show that $\langle f, g\rangle \in \mathbb{P}$. That $\langle f, g\rangle$ satisfies (5.26) $\sim$ (5.29) is clear.

For (5.30) and (5.31), suppose that $x \in g(\beta)$ for some $\beta \in \mathrm{dom}(g) \subseteq E^\kappa_{>\omega}$. If $\beta \in \mathrm{dom}(g')$ and $x \in g'(\beta)$ then $\sup(x) \in \mathrm{dom}(f') \subseteq \mathrm{dom}(f)$ and $f(\sup(x)) = f'(\sup(x)) \subseteq x$. Otherwise $x = \beta \cap N$ by (5.35). Then $\sup(x) \in Lim(\beta \cap N) \subseteq \mathrm{dom}(f)$ and, by (5.34), $f(\sup(x)) = \sup(x) \cap N = x$. $\dashv$ (Claim 5.4.1)

Claim 5.4.2. $\mathbb{P}$ *is* σ*-Baire.*

$\vdash$ Suppose that D_i, $i \in \omega$ are dense open sets in $\mathbb{P}$ and $\langle f', g'\rangle \in \mathbb{P}$. We show that there is $\langle f, g\rangle \in \mathbb{P}$ such that $\langle f, g\rangle \leq_\mathbb{P} \langle f', g'\rangle$ and $\langle f, g\rangle \in D_i$ for all $i \in \omega$.

Let θ be a sufficiently large regular cardinal and $N \prec \langle \mathcal{H}(\theta), \in\rangle$ be countable such that $\mathbb{P}, \langle D_i : i \in \omega\rangle, \langle f', g'\rangle \in N$.

Let $\langle x_i : i \in \omega\rangle$ be an increasing sequence in $[\kappa]^{\aleph_0} \cap N$ such that

$$\textstyle\bigcup_{i<\omega} x_i = \kappa \cap N. \tag{5.36}$$

By Claim 5.4.1, by closedness of D_i's and by the elementarity of N, we can find a decreasing sequence $\langle f_i, g_i\rangle$, $i \in \omega$ in $\mathbb{P}$ such that

(5.37) $\langle f_0, g_0\rangle \leq_{\mathbb{P}} \langle f', g'\rangle$;

for all $i \in \omega$,

(5.38) $\langle f_i, g_i\rangle \in D_i \cap N$, and

(5.39) $E^{\kappa}_{\omega} \cap x_i \subseteq \mathrm{dom}(f_i)$, $E^{\kappa}_{>\omega} \cap x_i \subseteq \mathrm{dom}(g_i)$ and $x_i \cap \beta \subseteq \bigcup g_i(\beta)$ for all $\beta \in \mathrm{dom}(g_i)$.

By (5.36) and (5.39), we have

(5.40) $\bigcup_{i\in\omega} \mathrm{dom}(f_i) = E^{\kappa}_{\omega} \cap N, \qquad \bigcup_{i\in\omega} \mathrm{dom}(g_i) = E^{\kappa}_{>\omega} \cap N$ and

(5.41) $\bigcup\{\bigcup g_i(\beta) \,:\, i < \omega, \beta \in \mathrm{dom}(g_i)\} = \beta \cap N$

for all $\beta \in E^{\kappa}_{>\omega} \cap N$.

Let $\langle f, g\rangle$ be the pair of functions f, g such that

(5.42) $\mathrm{dom}(f) = (E^{\kappa}_{\omega} \cap N) \cup Lim(\kappa \cap N)$, and

$$f(\alpha) = \begin{cases} f_i(\alpha), & \text{if } \alpha \in \mathrm{dom}(f_i) \text{ for some } i \in \omega, \\ N \cap \alpha, & \text{otherwise,} \end{cases}$$

for $\alpha \in \mathrm{dom}(f)$,

(5.43) $\mathrm{dom}(g) = E^{\kappa}_{>\omega} \cap N$, and

$$g(\beta) = \bigcup\{g_i(\beta) \,:\, i < \omega,\ \beta \in \mathrm{dom}(g_i)\} \cup \{\beta \cap N\} \quad \text{for all } \beta \in \mathrm{dom}(g).$$

Then we have $\langle f, g\rangle \in \mathbb{P}$: It is clear that $\langle f, g\rangle$ satisfies (5.26) ~ (5.29). For (5.30) and (5.31), suppose that $x \in \bigcup\{g(\beta) \,:\, \beta \in \mathrm{dom}(g)\}$. If $x \in g_i(\beta)$ for some $i \in \omega$ and $\beta \in \mathrm{dom}(g_i)$, then $\sup(x) \in \mathrm{dom}(f_i)$ and $f_i(\sup(x)) \subseteq x$, since $\langle f_i, g_i\rangle \in \mathbb{P}$. It follows that $\sup(x) \in \mathrm{dom}(f)$ and $f(sup(x)) \subseteq x$ by (5.42). Otherwise, $x = \beta \cap N$ for some $\beta \in \mathrm{dom}(g)$ and $\sup(x) \in Lim(\kappa \cap N)$. Then $f(\sup(x)) = \sup(x) \cap N$ by (5.42) and we again have $f(\sup(x)) \subseteq x$.

$\langle f, g\rangle \leq_{\mathbb{P}} \langle f', g'\rangle$ by (5.37) and $\langle f, g\rangle \in D_n$ for all $n \in \omega$ by (5.38).

$\dashv$ (Claim 5.4.2)

Claim 5.4.3. *Suppose* $\langle f, g\rangle, \langle f', g'\rangle \in \mathbb{P}$. *Then*

$\langle f, g\rangle$ *and* $\langle f', g'\rangle$ *are compatible* $\Leftrightarrow$

(5.44) $f \restriction (\mathrm{dom}(f) \cap \mathrm{dom}(f')) = f' \restriction (\mathrm{dom}(f) \cap \mathrm{dom}(f'))$, *and*

(5.45) *for every* $\beta \in \mathrm{dom}(g) \cap \mathrm{dom}(g')$, *one of* $g(\beta)$ *and* $g'(\beta)$ *is an initial segment of the other.*

$\vdash$ "$\Rightarrow$ is clear by the definition of $\leq_{\mathbb{P}}$.

For "$\Leftarrow$", suppose that $\langle f, g\rangle, \langle f', g'\rangle \in \mathbb{P}$ satisfy (5.44) and (5.45). Let $\langle f^*, g^*\rangle$ be defined by

(5.46) $f^* = f \cup f'$,

(5.47) $\mathrm{dom}(g^*) = \mathrm{dom}(g) \cup \mathrm{dom}(g')$, and

$$g^*(\beta) = \begin{cases} g(\beta), & \text{if } \beta \in \mathrm{dom}(g) \setminus \mathrm{dom}(g'),\\ g'(\beta), & \text{if } \beta \in \mathrm{dom}(g') \setminus \mathrm{dom}(g),\\ g(\beta) \cup g'(\beta), & \text{if } \beta \in \mathrm{dom}(g) \cap \mathrm{dom}(g') \end{cases}$$

for $\beta \in \mathrm{dom}(g^*)$.

Then $\langle f^*, g^*\rangle \in \mathbb{P}$ and $\langle f^*, g^*\rangle \leq_{\mathbb{P}} \langle f, g\rangle, \langle f', g'\rangle$. $\dashv$ (Claim 5.4.3)

Claim 5.4.4. *$\mathbb{P}$ satisfies the κ-Knaster property. In particular, $\mathbb{P}$ satisfies the κ-c.c.*

$\vdash$ Suppose $A \in [\mathbb{P}]^\kappa$. Then by Δ-System Lemma and Pigeon Hole Principle, we can find $r_0, r_1 \in [\kappa]^{\aleph_0}$ and $B \in [A]^\kappa$ such that

(5.48) for any distinct $\langle f, g\rangle$ and $\langle f', g'\rangle \in B$, we have $\mathrm{dom}(f) \cap \mathrm{dom}(f') = r_0$ and $f \restriction r_0 = f' \restriction r_0$,

(5.49) for any distinct $\langle f, g\rangle$ and $\langle f', g'\rangle \in B$, we have $\mathrm{dom}(g) \cap \mathrm{dom}(g') = r_1$ and $g \restriction r_1 = g' \restriction r_1$.

By Claim 5.4.3, the elements of B are pairwise compatible. $\dashv$ (Claim 5.4.4)

For $(V, \mathbb{P})$-generic G, let

(5.50) $f_G = \bigcup\{f : \langle f, g\rangle \in G \text{ for some } g\}$ and

(5.51) $g_G = \bigcup\{g : \langle f, g\rangle \in G \text{ for some } f\}$.

Claim 5.4.5. *In $V[G]$, we have the following:*

(1) $\omega_1^{V[G]} = \omega_1^V$.

(2) $f_G : E^\kappa_\omega \to [\kappa]^{\aleph_0}$, *$f_G(\alpha) \in [\alpha]^{\aleph_0}$ and $f_G(\alpha)$ is cofinal in α for all $\alpha \in E^\kappa_\omega$.*

(3) $\mathrm{dom}(g_G) = E^\kappa_{>\omega}$, *$g_G(\beta)$ is a continuously increasing cofinal sequence in $[\beta]^{\aleph_0}$ of length ω_1 with $\bigcup g(\beta) = \beta$ for all $\beta \in E^\kappa_\omega$ and*

(5.52) $f_G(\sup(x)) \subseteq x$ *for $x \in g_G(\beta)$.*

(4) $\kappa = \omega_2$ *and* $2^{\aleph_0} = \aleph_1$.

(5) $S = \{x \in [\kappa]^{\aleph_0} : f_G(\sup(x)) \not\subseteq x\}$ *(which is a stationary set in $[\kappa]^{\aleph_0}$ by Fact 5.3) is non-reflecting stationary. That is, for every limit ordinal $\alpha < \kappa$, $S \cap [\alpha]^{\aleph_0}$ is non-stationary in $[\alpha]^{\aleph_0}$. In particular, $V[G]$ is a model of* $\neg\mathrm{WRP}([\omega_2]^{\aleph_0})$.

⊢ (1): By Claim 5.4.2.

(2) and (3) follow from the definition of $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ and Claim 5.4.1. In particular, (5.52) follows from (5.31).

(4) follows from (3) and (1).

(5): For $\alpha \in E^{\kappa}_{\omega}$, it is clear that $S \cap [\alpha]^{\aleph_0}$ is non-stationary. For $\alpha \in E^{\kappa}_{>\omega}$, $g_G(\alpha)$ is a closed unbounded subset of $[\alpha]^{\aleph_0}$ by (3) and $g_G(\alpha)$ is disjoint from $S \cap [\alpha]^{\aleph_0}$ by (5.52). ⊣ (Claim 5.4.5)

In the rest of the proof, we show that FRP holds in $V[G]$.

Let $\lambda \geq \kappa$ be a regular cardinal and $j : V \to M$ be a λ-compact elementary embedding satisfying (5.1), (5.2) and (5.3). By (5.2), we have $\mathbb{P} \in M$ and, by (5.1), $j(p) = p$ for all $p \in \mathbb{P}$. By elementarity, we have

(5.53) $M \models j(\mathbb{P}) = \{\langle f, g\rangle \,:\, \langle f, g\rangle \models \text{(5.26)'} \sim \text{(5.31)'}\}$ and

(5.54) $M \models j(\leq_{\mathbb{P}})$
$$= \{\langle\langle f, g\rangle, \langle f', g'\rangle\rangle \in (j(\mathbb{P}))^2 \,:\, \langle\langle f, g\rangle, \langle f', g'\rangle\rangle \models \text{(5.32)'}\}$$

where (5.26)' ∼ (5.31)', (5.32)' are conditions obtained from (5.26) ∼ (5.31), (5.32) by replacing κ appearing there by $j(\kappa)$. Hence by (5.2), it follows that

(5.55) $j(\mathbb{P}) = \{\langle f, g\rangle \,:\, \langle f, g\rangle \models \text{(5.26)'} \sim \text{(5.31)'}\}$ and
$$j(\leq_{\mathbb{P}}) = \{\langle\langle f, g\rangle, \langle f', g'\rangle\rangle \in (j(\mathbb{P}))^2 \,:\, \langle\langle f, g\rangle, \langle f', g'\rangle\rangle \models \text{(5.32)'}\}.$$

In particular, the proof of Claim 5.4.2 also applies to $\langle j(\mathbb{P}), j(\leq_{\mathbb{P}})\rangle$ and we can conclude that $j(\mathbb{P}) = \langle j(\mathbb{P}), j(\leq_{\mathbb{P}})\rangle$ is also σ-Baire. In the following we shall denote $j(\leq_{\mathbb{P}})$ also by $\leq_{j(\mathbb{P})}$.

Claim 5.4.6. *$j \restriction \mathbb{P} : \mathbb{P} \to j(\mathbb{P})$; $p \mapsto p$ is a complete embedding.*

⊢ Suppose that $A \subseteq \mathbb{P}$ is maximal pairwise incompatible. By Claim 5.4.4, we have $|\,A\,| < \kappa$. Hence

(5.56) $j(A) = j''A = A$.

By the elementarity, we have

(5.57) $M \models$ "$j(A)$ is maximal pairwise incompatible in $j(\mathbb{P})$".

So, by (5.56), it follows that $j''A = A$ is maximal pairwise incompatible in $j(\mathbb{P})$ (in V). ⊣ (Claim 5.4.6)

For $\langle f, g\rangle \in j(\mathbb{P})$, let

(5.58) $\langle f, g\rangle \restriction \mathbb{P} = \langle f \restriction \kappa, g \restriction \kappa\rangle$.

Note that $\langle f, g\rangle \restriction \mathbb{P} \in V$ and hence $\langle f, g\rangle \restriction \mathbb{P} \in \mathbb{P}$ and $\langle f, g\rangle \restriction \mathbb{P} \leq_{j(\mathbb{P})} \langle f, g\rangle$.

Let $\mathbb{Q} = j(\mathbb{P})/G$. By Claim 5.4.3, we have

$$(5.59)\quad \begin{aligned}\mathbb{Q} &= \{p \in j(\mathbb{P}) : p \text{ is compatible with all } j(r) = r \in G\}\\ &= \{\langle f, g\rangle \in j(\mathbb{P}) : \langle f, g\rangle \restriction \mathbb{P} \in G\}.\end{aligned}$$

Working in $V[G]$, let $E \subseteq E^{\lambda}_{\omega}$ be stationary and $g : E \to [\lambda]^{\aleph_0}$ be a ladder system. We are going to show that there is $\alpha^* \in E^{\lambda}_{\omega_1}$ witnessing $(2.17)_{\lambda}$ for these E and g.

Let

(5.60) $S = \{x \in [\lambda]^{\aleph_0} : \sup(x) \in E,\ f_G(\sup(x \cap \kappa)) \cup g(\sup(x)) \subseteq x\}$.

Then S is a stationary subset of $[\lambda]^{\aleph_0}$.

Claim 5.4.7. *$\mathbb{Q}$ preserves the stationarity of S.*

$\vdash$ Suppose that $\langle f^\dagger, g^\dagger\rangle \in \mathbb{Q}$ and $\dot{h}$ is a $\mathbb{Q}$-name such that $\Vdash_{\mathbb{Q}}$ "$\dot{h} : \lambda^{<\omega} \to \lambda$". It is enough to show that there is $\langle f^*, g^*\rangle \leq_{\mathbb{Q}} \langle f^\dagger, g^\dagger\rangle$ and $x^* \in S$ such that $\langle f^*, g^*\rangle \Vdash_{\mathbb{Q}}$ "x^* is closed with respect to $\dot{h}$".

Let θ be a sufficiently large regular cardinal and let $N \prec \mathcal{H}(\theta)$ be countable such that $\kappa, j(\kappa), \mathbb{Q}, \langle f^\dagger, g^\dagger\rangle, \lambda, g, \dot{h} \in N$ and

(5.61) $\lambda \cap N \in S$.

For $\vec{\xi} \in (\lambda \cap N)^{<\omega}$, let

(5.62) $\bar{D}_{\vec{\xi}} = \{q \in \mathbb{Q} : q \text{ decides } \dot{h}(\vec{\xi})\}$.

Note that $\bar{D}_{\vec{\xi}}$ is a dense open subset of $\mathbb{Q}$.

Let D_n, $n \in \omega$ be an enumeration of $\{\bar{D}_{\vec{\xi}} : \vec{\xi} \in (\lambda \cap N)^{<\omega}\}$ and let $x_i \in [j(\kappa)]^{\aleph_0} \cap N$, $i \in \omega$ be such that $\bigcup_{i\in\omega} x_i = j(\kappa) \cap N$.

Let $\langle\langle f_i, g_i\rangle : i \in \omega\rangle$ be a descending sequence in $\mathbb{Q} \cap N$ such that

(5.63) $\langle f_0, g_0\rangle \leq_{\mathbb{Q}} \langle f^\dagger, g^\dagger\rangle$,

and, for all $i \in \omega$,

(5.64) $\langle f_i, g_i\rangle \in D_i$,

(5.65) $E^{j(\kappa)}_{\omega} \cap x_i \subseteq \mathrm{dom}(f_i)$,

(5.66) $E^{j(\kappa)}_{>\omega} \cap x_i \subseteq \mathrm{dom}(g_i)$,

(5.67) $x_i \cap \beta \subseteq \bigcup g_i(\beta)$ for all $\beta \in \mathrm{dom}(g_i)$.

Subclaim 5.4.7.1. *$\langle\langle f_i, g_i\rangle : i \in \omega\rangle$ has a lower bound $\langle f^*, g^*\rangle$ in $\mathbb{Q}$. In particular, $\langle f^*, g^*\rangle \leq_{\mathbb{Q}} \langle f^\dagger, g^\dagger\rangle$ by* (5.63).

$\vdash$ Let $\langle f^*, g^*\rangle$ be defined by the following:

(5.68) $\mathrm{dom}(f^*) = (E^{j(\kappa)}_\omega \cap N) \cup Lim(j(\kappa) \cap N)$ and

$$f^*(\alpha) = \begin{cases} f_i(\alpha), & \text{if } \alpha \in \mathrm{dom}(f_i) \text{ for some } i \in \omega & \text{(a)} \\ \alpha \cap N, & \text{if } \alpha \notin \bigcup_{i\in\omega} \mathrm{dom}(f_i) = E^{j(\kappa)}_\omega \cap N \text{ and } \alpha > \kappa & \text{(b)} \\ f_G(\alpha), & \text{otherwise} & \text{(c)} \end{cases}$$

for $\alpha \in \mathrm{dom}(f^*)$;

(5.69) $\mathrm{dom}(g^*) = E^{j(\kappa)}_{>\omega} \cap N$ and

$$g^*(\beta) = \bigcup\{g_i(\beta) : \beta \in \mathrm{dom}(g^*),\ i \in \omega,\ \beta \in \mathrm{dom}(g_i)\} \cup \{\beta \cap N\}$$

for $\beta \in \mathrm{dom}(g^*)$.

We check first that $\langle f^*, g^*\rangle \in j(\mathbb{P})$. It is clear that $\langle f^*, g^*\rangle$ satisfies (5.26) ~ (5.29) (with κ there replaced by $j(\kappa)$): For (5.27), note that $\bigcup_{i\in\omega} \mathrm{dom}(f_i) = E^{j(\kappa)}_\omega \cap N$ by $f_i \in N$ for $i \in \omega$ and (5.65), and $f_G(\alpha)$ is cofinal in α for all $\alpha \in E^\kappa_\omega$. For (5.29), note that $\bigcup\bigcup\{g_i(\beta) : \beta \in \mathrm{dom}(g^*),\ i \in \omega,\ \beta \in \mathrm{dom}(g_i)\} = \beta \cap N$ by $g_i \in N$ for $i \in \omega$, (5.66) and (5.67).

To see that $\langle f^*, g^*\rangle$ satisfies (5.30) and (5.31) (also with κ replaced by $j(\kappa)$), let $x \in g^*(\beta)$ for some $\beta \in \mathrm{dom}(g^*)$.

If $x \in g_i(\beta)$ for some $i \in \omega$, then it is clear that this x satisfies the conditions in (5.30) and (5.31) for $\langle f^*, g^*\rangle$ since $\langle f_i, g_i\rangle \in \mathbb{Q}$.

So assume that $x \notin \bigcup_{i\in\omega} g_i(\beta)$. Then, by (5.69), $x = \beta \cap N$ and $\sup(x) \in Lim(j(\kappa) \cap N) \setminus (E^{j(\kappa)}_\omega) \cap N)$.

If $\beta > \kappa$, then $\sup(x) > \kappa$ and $f^*(\sup(x)) = \beta \cap N = x$, by (5.68), (b).

If $\beta = \kappa$, then $x = \kappa \cap N$ and since $\lambda \cap N \in S$, $f^*(\sup(x)) = f_G(\sup(x)) = f_G(\sup(\kappa \cap N)) \subseteq x$ by (5.68), (c) and (5.60) (note that $x = \kappa \cap N = (\lambda \cap N) \cap \kappa$).

If $\beta < \kappa$, there is a $\langle f', g'\rangle \in G$ such that

(5.70) $\sup(x) \in \mathrm{dom}(f')$, $\beta \in \mathrm{dom}(g')$ and

(5.71) $x \subseteq \bigcup g'(\beta)$

by Claim 5.4.1. For each $i \in \omega$ such that $\beta \in \mathrm{dom}(g_i)$, we have $\bigcup g_i(\beta) \in g'(\beta)$ since $\langle f_i, g_i\rangle \restriction \mathbb{P} \in G$ and $\langle f_i, g_i\rangle \in N$, and by (5.71).

Thus $x = \beta \cap N = \bigcup\{\bigcup g_i(\beta) : i \in \omega,\ \beta \in \mathrm{dom}(g_i)\} \in g'(\beta)$ by the closedness of $g'(\beta)$ (see (5.29)). It follows that $f^*(\sup(x)) = f_G(\sup(x)) = f'(\sup(x)) \subseteq x$.

By the argument above, it also follows that $\langle f^*, g^*\rangle \restriction \mathbb{P} \in G$ and hence $\langle f^*, g^*\rangle \in \mathbb{Q}$.

By the definition of $\langle f^*, g^*\rangle$ it is clear that $\langle f^*, g^*\rangle \leq_{\mathbb{Q}} \langle f_i, g_i\rangle$ for $i \in \omega$.

$\dashv$ (Subclaim 5.4.7.1)

By (5.64), $\langle f^*, g^*\rangle$ decides $\dot{h} \restriction \lambda^{<\omega} \cap N$ to be a mapping from $\lambda^{<\omega} \cap N$ to $\lambda \cap N$. Thus, letting $x^* = \lambda \cap N$, we have $x^* \in S$ by the choice of N (see (5.61)) and $\langle f^*, g^*\rangle \Vdash_{\mathbb{Q}}$ "x^* is closed with respect to $\dot{h}$". $\dashv$ (Claim 5.4.7)

Now, let H be a $(V[G], \mathbb{Q})$-generic filter. Then $V[G][H] = V[H]$ and the λ-compact elementary embedding $j : V \to H$ can be extended to the elementary embedding $\tilde{j} : V[G] \to M[H]$ by $\tilde{j}(\dot{a}_G) = (j(\dot{a}))_H$ for all $\mathbb{P}$-name $\dot{a}$.

Let $\alpha^* = \sup(j''\lambda)$. Note that $\alpha^* < j(\lambda)$ since $M \models \mathrm{cf}(\alpha^*) < j(\kappa) < j(\lambda)$ by (5.3) while $M \models j(\lambda)$ is a regular cardinal. Since $M \models \mathrm{cf}(\alpha^*) \geq \omega_1$, we have $M[H] \models \mathrm{cf}(\alpha^*) = \omega_1$.

By Claim 5.4.7, S is stationary in $[\lambda]^{\aleph_0}$. Since $j \restriction \lambda : \lambda \to \alpha^*$ sends λ cofinal in α and preserves limits of increasing sequences of length ω,

$$\begin{aligned} S^* &= \{x \in [\alpha^*]^{\aleph_0} : \tilde{j}(g)(\sup(x)) \subseteq x\} \\ &\supseteq \{x \in [\alpha^*]^{\aleph_0} : x \supseteq z \text{ and } \sup(x) = \sup(z) \text{ for some } z \in \tilde{j}''S\} \end{aligned}$$

is stationary in $V[H]$. Note that $S^* \in M[H]$. Hence S^* is also stationary in $M[H]$. Thus we have

$$M[H] \models \tilde{j}(g) \text{ satisfies } (2.17)_{j(\lambda)}.$$

By elementarity, it follows that

$$V[G] \models g \text{ satisfies } (2.17)_{\lambda}.$$

By Proposition 2.4, this shows that $V[G] \models \mathrm{FRP}$. ❑ (**Theorem 5.4**)

6 Separation of ORP from FRP

In this section, we prove the following theorem:

Theorem 6.1. *Suppose that* $\mathrm{MA}^+(\sigma\text{-closed})$ *holds. Then, for every regular* $\kappa \geq \omega_2$*, there is a* κ*-strategically closed partial ordering* $\mathbb{P}_*$ *such that*

(1) $\Vdash_{\mathbb{P}_*}$ "$\mathrm{ORP}(\kappa) + \neg\mathrm{FRP}(\kappa)$".

Furthermore, we have

(2) *for all regular cardinals* $\lambda > 2^{<\kappa}$ *as well as for all regular cardinals* $\omega_1 < \lambda < \kappa$*, we have* $\Vdash_{\mathbb{P}_*}$ "FRP(λ)".

By κ*-strategically closedness and since* $\mathbb{P}_*$ *can be chosen such that it collapses all cardinals in the interval* $(\kappa, 2^{<\kappa}]$*, it follows that*

(3) $\mathbb{P}_*$ *forces that* κ *is the unique regular cardinal* $\geq \aleph_2$ *where Fodor-type Reflection Principle does not hold. In particular, we have* $\Vdash_{\mathbb{P}_*}$ "ORP".

Here, a partial ordering $\mathbb{P}$ is said to be α*-strategically closed* for an ordinal α if the Player II has a winning strategy in the following infinite game $\Game_\alpha(\mathbb{P})$: In a match of $\Game_\alpha(\mathbb{P})$, Player I and II play elements of $\mathbb{P}$ forming a decreasing sequence in $\mathbb{P}$:

Player I	p_0	p_1	$\cdots$	p_ξ	$\cdots$
Player II	q_0	q_1	$\cdots$	q_ξ	$\cdots$

$(\xi < \alpha)$

where

$$p_0 \geq_{\mathbb{P}} q_0 \geq_{\mathbb{P}} p_1 \geq_{\mathbb{P}} q_1 \geq_{\mathbb{P}} \cdots \geq_{\mathbb{P}} p_\xi \geq_{\mathbb{P}} q_\xi \geq_{\mathbb{P}} p_{\xi+1} \geq_{\mathbb{P}} \cdots$$

and such that, in each of the η'th innings for limit $\eta < \alpha$, only Player II may play (to simplify the notation we assume $p_\xi = q_\xi$ but this move is chosen by Player II for all limit ordinal $\xi < \alpha$).

Player II wins the match in $\Game_\alpha(\mathbb{P})$ if the game can be played in all of the ξ'th innings for $\xi < \alpha$.

If $\mathbb{P}$ is α-strategically closed and $\lambda < \alpha$ is a cardinal, it is easy to see that $\mathbb{P}$ does not add any new set of ordinals of size $\leq \lambda$.

A partial ordering $\mathbb{P}$ is said to be *strongly* α*-strategically closed* if Player II has a winning strategy in $\Game^+_\alpha(\mathbb{P})$ which is defined just as $\Game_\alpha(\mathbb{P})$ except that Player I may begin each of the η'th innings for limit $\eta < \alpha$.

A typical example of strongly κ-strategically closed partial orderings for a cardinal κ is a partial ordering $\mathbb{P}$ with a κ-closed dense subset D. Such a partial ordering $\mathbb{P}$ is indeed strongly κ-strategically closed since taking moves always from D is a winning strategy for Player II in $\Game^+_\kappa(\mathbb{P})$.

In [16], α-strategical closedness and strong α-strategical closedness are called α-game closedness and strong α-game closedness respectively.

Let κ be a regular cardinal $\geq \omega_2$. In the proof of Theorem 6.1 below, we show that the following partial ordering $\mathbb{P}_*$ has the desired property under $\mathrm{MA}^+(\sigma\text{-closed})$.

Let $\mathbb{P}_*$ consist of all functions p such that

(6.1) $\mathrm{dom}(p) \subseteq E^\kappa_\omega$ and $|\,p\,| < \kappa$;

(6.2) p is a ladder system on $\mathrm{dom}(p)$;

(6.3) for all $\beta \in E^\kappa_{\omega_1} \cap \sup(\mathrm{dom}(p)) + 1$, there is a club $C \subseteq \beta$ and a 1-1 choice function on $\{p(\alpha) \,:\, \alpha \in \mathrm{dom}(p) \cap C\}$.

For p_0, $p_1 \in \mathbb{P}_*$, we define $p_1 \leq_{\mathbb{P}_*} p_0$ if $p_0 \subseteq p_1$, $\mathrm{dom}(p_1)$ is an end extension of $\mathrm{dom}(p_0)$ and

(6.4) $|\,\mathrm{dom}(p_1) \setminus \mathrm{dom}(p_0)\,| > \aleph_0$ if $p_0 \neq p_1$.

The following lemma can be proved still without using MA^+(σ-closed).

Lemma 6.2. (1) *For each $\xi < \kappa$, the set*

$$D_\xi = \{p \in \mathbb{P}_* \,:\, \sup(\mathrm{dom}(p)) > \xi,\ \mathrm{dom}(p) \text{ has the maximal element}\}$$

is dense in $\mathbb{P}_*$.

(2) *$\mathbb{P}_*$ is σ-closed.*

(3) *$\mathbb{P}_*$ is κ-strategically closed.*

(4) *Let $\dot{S}_G$ be a $\mathbb{P}_*$-name of*

(6.5) $S_G = \bigcup\{\mathrm{dom}(p) \,:\, p \in G\}$

for $(V, \mathbb{P}_*)$-generic filter G. Then we have $\Vdash_{\mathbb{P}_*}$ "$\dot{S}_G \subseteq E^\kappa_\omega$", $\Vdash_{\mathbb{P}_*}$ "$\dot{S}_G$ is stationary" and $\Vdash_{\mathbb{P}_*}$ "$E^\kappa_\omega \setminus \dot{S}_G$ is stationary".

(5) $\Vdash_{\mathbb{P}_*}$ "$\neg\mathrm{FRP}(\kappa)$".

(6) *$\mathbb{P}_*$ collapses all cardinals in the interval $(\kappa, 2^{<\kappa}]$.*

Proof. (1): Suppose $\xi < \kappa$ and $p \in \mathbb{P}_*$. Let $\eta = \max\{\xi, \sup(\mathrm{dom}(p))\}$. Let q be any ladder system extending p with

(6.6) $$\mathrm{dom}(q) = \mathrm{dom}(p) \cup \{\alpha \in E^\kappa_\omega \,:\, \eta < \alpha \leq \eta + (\omega_1 + \omega) \quad \alpha = \gamma + \omega \text{ for some } \gamma < \alpha\}.$$

Then $q \in \mathbb{P}_*$ and thus $q \leq_{\mathbb{P}_*} p$, and $q \in D_\xi$: To see that q satisfies the condition (6.3) for $\beta = \eta + \omega_1$, let $C = \{\alpha \in E^\beta_\omega \,:\, \eta < \alpha$ and $\alpha \neq \gamma + \omega$ for any $\gamma \in \alpha\}$. C is then a club below β disjoint from $\mathrm{dom}(q)$.

(2): If q_n, $n \in \omega$ is a descending sequence in $\mathbb{P}_*$ then $\bigcup_{n\in\omega} q_n$ is the lower bound of the sequence.

For (3), Let σ be any strategy of Player II in $\Game_\kappa(\mathbb{P}_*)$ such that σ satisfies the following at the ξ'th inning for $\xi < \kappa$.

Case I. ξ is a successor ordinal. Suppose that p_ξ is the ξ'th move of Player I. Then, let q_ξ, the ξ'th move of Player II to be taken according to σ, be any $q \in \mathbb{P}_*$ such that $q \leq_{\mathbb{P}_*} p_\xi$ and $q \neq p_\xi$. Note that we can always take such q_ξ by (1).

Case II. ξ is a limit ordinal. If the set of the previous moves $\{p_\eta, q_\eta : \eta < \xi\}$ has a lower bound, then

$$q = \bigcup_{\eta<\xi} q_\eta \cup \{\langle \beta, b\rangle\}$$

is also a lower bound of this set where $\beta = \sup(\bigcup_{\eta<\xi} \mathrm{dom}(q_\eta))) + \omega$ and $b \in [\beta]^{\aleph_0}$ is a cofinal subset of β of order type ω. Let ξ'th move q_ξ of the player II to be taken according to σ be such q.

We show that σ as above is a winning strategy for Player II. Since we already saw that Player II can take his move at successor steps according to σ, it is enough to show that, for any limit $\xi < \kappa$ and any (partial) match $\langle p_\eta, q_\eta : \eta < \xi\rangle$ in $\eth_\kappa(\mathbb{P}_*)$ where Player II has played according to σ, there is a lower bound of $\{p_\eta, q_\eta : \eta < \xi\}$. If $\mathrm{cf}(\xi) = \omega$ then this holds by (2). So suppose that $\mathrm{cf}(\xi) \neq \omega$. Then the set C of all limit points of $\{\sup(\mathrm{dom}(q_\eta)) : \eta < \xi\}$ is a club in $\beta = \sup(\bigcup_{\eta<\xi} \mathrm{dom}(q_\eta))$ and $C \cap (\bigcup_{\eta<\xi} \mathrm{dom}(q_\eta)) = \emptyset$ by the definition of q_η's at limit ordinals $\eta < \xi$ taken according to σ. It follows that $q = \bigcup_{\eta<\xi} q_\eta$ satisfies (6.3). Thus $q \in \mathbb{P}_*$ and q is a lower bound of $\{q_\eta : \eta < \xi\}$.

(4): $\Vdash_{\mathbb{P}_*}$ "$\dot{S}_G \subseteq E^\kappa_\omega$" follows from (6.1). To show $\Vdash_{\mathbb{P}_*}$ "$\dot{S}_G$ is stationary", let $\dot{C}$ be a $\mathbb{P}_*$-name of a club subset of κ. We have to show that $\Vdash_{\mathbb{P}_*}$ "$\dot{S}_G \cap \dot{C} \neq \emptyset$". For any $p \in \mathbb{P}_*$, let $\langle p_n : n \in \omega\rangle$ be a decreasing sequence in $\mathbb{P}_*$ and $\langle \alpha_n : n \in \omega\rangle$ an increasing sequence of ordinals below κ such that

(6.7) $p_0 \leq_{\mathbb{P}_*} p$;

(6.8) $p_n \Vdash_{\mathbb{P}_*}$ "$\alpha_n \in \dot{C}$"; and

(6.9) $\sup(\mathrm{dom}(p_{n+1})) > \alpha_n$.

Let $\alpha = \sup_{n\in\omega} \alpha_n$. We have $\alpha = \sup(\bigcup_{n\in\omega} p_n)$ by (6.9). Let

(6.10) $q = \bigcup_{n<\omega} p_n \cup \{\langle \alpha, \{\alpha_n : n \in \omega\}\rangle\}$.

Then $q \in \mathbb{P}_*$, $q \leq_{\mathbb{P}_*} p_n$ for all $n \in \omega$, and hence $q \leq_{\mathbb{P}_*} p$ by (6.7), and $q \Vdash_{\mathbb{P}_*}$ "$\alpha \in \dot{S}_G \cap \dot{C}$" by (6.10) and (6.8). This shows that $\Vdash_{\mathbb{P}_*}$ "$\dot{S}_G \cap \dot{C} \neq \emptyset$". It follows that $\Vdash_{\mathbb{P}_*}$ "$\dot{S}_G$ is stationary".

$\Vdash_{\mathbb{P}_*}$ "$E^\kappa_\omega \setminus \dot{S}_G$ is stationary" can be proved similarly.

(5): Let G be a $(V, \mathbb{P}_*)$-generic filter. In $V[G]$, S_G of (6.5) is stationary by (4). $S_G \subseteq E^\kappa_\omega$ by (6.1). $g_G = \bigcup G$ is a ladder system on S_G by (6.2) and this ladder system is a counter example to $\mathrm{FRP}(\kappa)$ by (6.3).

(6): It is enough to show that $\mathbb{P}_*$ adds a surjection from a subset of κ to ${}^{\kappa>}2$.

We work in $V[G]$. For $\alpha \in S_G$, let $s(\alpha)$ be the element of ${}^{\xi}2$ where ξ is such that the element of S_G next to α is $\alpha+(\xi+\omega)$ and, for $\eta < \xi$, $s(\alpha)(\eta) = 0$ if and only if the first element of $g_G(\beta)$ is even where β is the η'th element from below of $S_G \setminus (\alpha + (\xi + \omega + 1))$. By the genericity of S_G and by a slight modification of the proofs of (1) and (3), it is easy to see that $s : S_G \to {}^{\kappa>}2$ is a surjection.

❑ (Lemma 6.2)

We need the following Fact 6.3 for the proof of Theorem 6.1.

Fact 6.3. (B. König and Y. Yoshinobu [16]) $\mathrm{MA}^+(\sigma$-*closed*) *is preserved by strongly* $\omega_1 + 1$-*strategically closed forcing.* ❑

For the proof of Fact 6.3 the reader may refer to [16].

Let G be a $(V, \mathbb{P}_*)$-generic filter. Working in $V[G]$, let S_G be defined as in (6.5). $g_G = \bigcup G$ is a ladder system on S_G.

Let $\mathbb{Q}_0$ be the following partial ordering: Elements of $\mathbb{Q}_0$ are pairs $\langle q, f\rangle$ such that

(6.11) q is a strictly increasing continuous mapping from a successor ordinal $< \omega_1$ to κ;

(6.12) $\mathrm{rng}(q) \subseteq S_G$;

(6.13) $\mathrm{dom}(f) = \mathrm{rng}(q)$; and

(6.14) f is an injective choice function on $\{g_G(\alpha) : \alpha \in \mathrm{rng}(q)\}$.

For $\langle q_0, f_0\rangle, \langle q_1, f_1\rangle \in \mathbb{Q}_0$,

(6.15) $\langle q_1, f_1\rangle \leq_{\mathbb{Q}_0} \langle q_0, f_0\rangle$ if $q_0 \subseteq q_1$ and $f_0 \subseteq f_1$.

Lemma 6.4. (1) *For $\alpha < \omega_1$ and $\beta < \kappa$, the set*

$$\{\langle q, f\rangle \in \mathbb{Q}_0 : \mathrm{dom}(q) \geq \alpha \text{ and } \max(\mathrm{rng}(q)) \geq \beta\}$$

is dense in $\mathbb{Q}_0$.

(2) *$\mathbb{Q}_0$ adds the canonical strictly and continuously increasing mapping π from ω_1 to S_G whose image is cofinal in κ, as well as the canonical injective choice function φ on $\{g_G(\alpha) : \alpha \in \mathrm{rng}(\pi)\}$.*

(3) *For any stationary $S \subseteq S_G$, $\mathbb{Q}_0$ forces that $S \cap \pi''\omega_1$ for π as in (the proof of) (2) is stationary.*

Proof. (1) is clear and (2) follows immediately from (1) and the definition of $\mathbb{Q}_0$. More specifically,

(6.16) $\pi = \bigcup\{q : \langle q, f\rangle \in H \text{ for some } f\}$ and
$\varphi = \bigcup\{f : \langle q, f\rangle \in H \text{ for some } q\}$

for a $(\mathbb{Q}_0, V[G])$-generic filter H are the canonical objects.

(3): Suppose that $\langle q, f\rangle \in \mathbb{Q}_0$ and $\dot{C}$ is a $\mathbb{Q}_0$-name of a club subset of κ. Let θ be a sufficiently large regular cardinal. By Fact 5.3, there is a countable elementary submodel M of $\mathcal{H}(\theta)$ such that

(6.17) S, g_G, $\langle q, f\rangle$, $\dot{C}$, $\cdots \in M$;

(6.18) $\sup(\kappa \cap M) \in S$; and

(6.19) $g_G(\sup(\kappa \cap M)) \not\subseteq \kappa \cap M$.

Let $\langle \gamma_n : n \in \omega\rangle$ be an increasing sequence of countable ordinals cofinal in $\omega_1 \cap M$ and $\langle \delta_n : n \in \omega\rangle$ be an increasing sequence of ordinals $\in \kappa \cap M$ cofinal in $\kappa \cap M$. Let $\langle q_n, f_n\rangle \in \mathbb{Q}_0 \cap M$, $n \in \omega$ be a decreasing sequence such that

(6.20) $\langle q_0, f_0\rangle \leq_{\mathbb{Q}_0} \langle q, f\rangle$;

(6.21) $\langle q_n, f_n\rangle \Vdash_{\mathbb{Q}_0}$ "$\alpha_n \in \dot{C}$" for some $\alpha_n \in (\kappa \cap M) \setminus \delta_n$ for all $n \in \omega$; and

(6.22) $\sup(\mathrm{dom}(q_n)) \geq \gamma_n$ and $\sup(\mathrm{rng}(q_n)) \geq \delta_n$ for all $n \in \omega$.

By (6.19), there is some

(6.23) $\alpha^* \in g_G(\sup(\kappa \cap M)) \setminus M$.

Let

(6.24) $q^* = \bigcup_{n\in\omega} q_n \cup \{\langle \sup(\omega_1 \cap M), \sup(\kappa \cap M)\rangle\}$ and

(6.25) $f^* = \bigcup_{n\in\omega} f_n \cup \{\langle \sup(\kappa \cap M), \alpha^*\rangle\}$.

Note that $\sup(\kappa \cap M) \in S_G$ by (6.18) and $S \subseteq S_G$. Hence $\mathrm{rng}(q^*) \subseteq S_G$. Since $\langle q_n, f_n\rangle \in M$, we have $q_n \in M$ for each $n \in \omega$ and

(6.26) $f_n(\beta) \in M$ for all $n \in \omega$ and $\beta \in \mathrm{rng}(q_n)$.

By (6.22), it follows that

(6.27) $\sup(\bigcup_{n\in\omega} \mathrm{dom}(q_n)) = \sup(\omega_1 \cap M)$ and
$\sup(\bigcup_{n\in\omega} \mathrm{rng}(q_n)) = \sup(\kappa \cap M)$.

By (6.23) and (6.26), f^* is an injective choice function of $\{g(\beta) : \beta \in \mathrm{rng}(q^*)\}$.

Thus we have $\langle q^*, f^*\rangle \in \mathbb{Q}_0$ and $\langle q^*, f^*\rangle \le_{\mathbb{Q}_0} \langle q_n, f_n\rangle$ for $n \in \omega$. By (6.20), $\langle q^*, f^*\rangle \le_{\mathbb{Q}_0} \langle q, f\rangle$. By (6.27), (6.21), and since $\dot{C}$ is a $\mathbb{Q}_0$-name of a club set $\subseteq \kappa$, we have $\langle q^*, f^*\rangle \Vdash_{\mathbb{Q}_0}$ " $\sup(\kappa \cap M) \in \dot{C}$ ". By (6.18) and (6.24), $\langle q^*, f^*\rangle \Vdash_{\mathbb{Q}_0}$ " $\sup(\kappa \cap M) \in S \cap \dot{\pi}''\omega_1$ ". Thus, $\langle q^*, f^*\rangle \Vdash_{\mathbb{Q}_0}$ " $S \cap \dot{\pi}''\omega_1 \cap \dot{C} \neq \emptyset$ " where $\dot{\pi}$ for a $\mathbb{Q}_0$-name of π in (6.16). ❑ (Lemma 6.4)

Stepping back to V, let $\dot{\mathbb{Q}}_0$ be a $\mathbb{P}_*$-name of $\mathbb{Q}_0$. By Lemma 6.4, we have $\Vdash_{\mathbb{P}_* * \dot{\mathbb{Q}}_0}$ " $\dot{S} \cap \dot{\pi}''\omega_1$ is stationary" for all $\mathbb{P}_*$-name $\dot{S}$ of stationary subset of $\dot{S}_G$.

Let

(6.28) $D_0 = \{\langle p, \langle \dot{q}, \dot{f}\rangle\rangle \in \mathbb{P}_* * \dot{\mathbb{Q}}_0 :$ p decides $\dot{q}$ and $\dot{f}$ to be some q and f in V and $\sup(\mathrm{dom}(p)) = \sup(\mathrm{rng}(q))\}$.

By the definition of D_0, each element $\langle p, \langle \dot{q}, \dot{f}\rangle\rangle$ of D_0 corresponds uniquely to $\langle p, \langle q, f\rangle\rangle$ (with q, $f \in V$) where $p \Vdash_{\mathbb{P}_*}$ " $q = \dot{q}$ and $f = \dot{f}$ ", and hence

(6.29) q is a strictly and continuously increasing mapping in V with $\mathrm{dom}(q) \in \omega_1 \setminus Lim(\omega_1)$;

(6.30) $\mathrm{rng}(q) \subseteq \mathrm{dom}(p)$ and $\sup(\mathrm{rng}(q)) = \sup(\mathrm{dom}(p))$;

(6.31) $\mathrm{dom}(f) = \mathrm{rng}(q)$; and

(6.32) f is an injective choice function on $\{p(\alpha) : \alpha \in \mathrm{rng}(q)\}$.

Let D_0^* be the collection of all such $\langle p, \langle q, f\rangle\rangle$. Then the partial ordering $\le_{D_0^*}$ on D_0^* corresponding to $\le_{\mathbb{P}_* * \dot{\mathbb{Q}}_0} \restriction D_0$ is

(6.33) $\langle p_1, \langle q_1, f_1\rangle\rangle \le_{D_0^*} \langle p_0, \langle q_0, f_0\rangle\rangle \;\Leftrightarrow\; p_1 \le_{\mathbb{P}_*} p_0,\ q_1 \supseteq q_0$ and $f_1 \supseteq f_0$.

Lemma 6.5. (1) *D_0 is dense in $\mathbb{P}_* * \dot{\mathbb{Q}}_0$.*
(2) *D_0 is σ-closed (with respect to $\le_{\mathbb{P}_* * \dot{\mathbb{Q}}_0} \restriction D_0$).*

Proof. (1) follows from the κ-strategical closedness of $\mathbb{P}_*$.

For (2), it is enough to show that $\langle D_0^*, \le_{D_0^*}\rangle$ is σ-closed.

Suppose that $\mathcal{S} = \langle\langle p_n, \langle q_n, f_n\rangle\rangle : n < \omega\rangle$ is a strictly decreasing chain in $\langle D_0^*, \le_{D_0^*}\rangle$. We have to show that $\mathcal{S}$ has a lower bound in D_0^* with respect to $\le_{D_0^*}$. Let $p^* = \bigcup_{n<\omega} p_n$, $q^* = \bigcup_{n<\omega} q_n$ and $f^* = \bigcup_{n<\omega} f_n$. Let $\gamma^* = \sup(\mathrm{dom}(q^*))$ and $\delta^* = \sup(\mathrm{dom}(p^*))$. We have $\delta^* = \sup(\mathrm{rng}(q^*)) = \sup(\mathrm{dom}(f^*))$. By (6.4) and by the definition of $\le_{D_0^*}$, $\delta^* \setminus \mathrm{rng}(f^*)$ is cofinal in δ^* and $\mathrm{cf}(\delta^*) = \omega$. So we can take a subset c^* of $\delta^* \setminus \mathrm{rng}(f^*)$ of order type ω which is cofinal in δ^*. Pick an element δ_0 of c^* and, let $p^\dagger = p^* \cup \{\langle \delta^*, c^*\rangle\}$, $q^\dagger = q^* \cup \{\langle \gamma^*, \delta^*\rangle\}$ and $f^\dagger = f^* \cup \{\langle \delta^*, \delta_0\rangle\}$. Then we have $\langle p^\dagger, \langle q^\dagger, f^\dagger\rangle\rangle \in D_0^*$ and $\langle p^\dagger, \langle q^\dagger, f^\dagger\rangle\rangle$ is a lower bound of $\mathcal{S}$. ❑ (Lemma 6.5)

Lemma 6.6 (MA$^+$(σ-closed)). $\mathbb{P}_*$ *forces that every stationary subsets of* S_G *reflect.*

Proof. Let $\dot{S}$ be a $\mathbb{P}_*$-name of a stationary subset of S_G and $p_0 \in \mathbb{P}_*$. We show that there are $p_1 \leq_{\mathbb{P}_*} p_0$, $\delta^* \in E^\kappa_{\omega_1}$ and $S^* \subseteq \delta^*$ (in V) stationary in δ^* such that $p^* \Vdash_{\mathbb{P}_*}$ "$S^* \subseteq \dot{S}$".

Let us denote by p_0 and $\dot{S}$ also the element of $\mathbb{P}_* * \dot{\mathbb{Q}}_0$ and the $\mathbb{P}_* * \dot{\mathbb{Q}}_0$-name corresponding to them. Let $\dot{\pi}$ be a $\mathbb{P}_* * \dot{\mathbb{Q}}_0$-name of the canonical mapping π from ω_1 to S_G and $\dot{\varphi}$ a $\mathbb{P}_* * \dot{\mathbb{Q}}_0$-name of the canonical injective choice function φ for g_G added generically by $\dot{\mathbb{Q}}_0$ as in Lemma 6.4, (2).

Let $\dot{T}$ be the $\mathbb{P}_* * \dot{\mathbb{Q}}_0$-name of $\dot{\pi}^{-1}{}''\dot{S}$. By Lemma 6.4, (3), and since $\dot{\pi}$ is forced to be continuous with $\dot{\pi}''\omega_1$ cofinal in S_G, we have

(6.34) $\Vdash_{\mathbb{P}_* * \dot{\mathbb{Q}}_0}$ "$\dot{T}$ is a stationary subset of ω_1".

By MA$^+$(σ-closed) and by Lemma 6.5, there is a filter F on $\mathbb{P}_* * \dot{\mathbb{Q}}_0$ such that

(6.35) F is generated from $F \cap D_0$;

(6.36) $\langle p_0, \langle \check{\emptyset}, \check{\emptyset} \rangle \rangle \in F$;

(6.37) $\dot{T}^F$ is a stationary subset of ω_1; and

(6.38) $\dot{\pi}^F$ is a strictly and continuously increasing function from ω_1 to κ.

Let

(6.39) $p^* = \bigcup\{p : \langle p, \langle q, f \rangle \rangle \in F \cap D_0$ for some q and $f\}$,
$q^* = \bigcup\{q : \langle p, \langle q, f \rangle \rangle \in F \cap D_0$ for some p and $f\}$ and
$f^* = \bigcup\{f : \langle p, \langle q, f \rangle \rangle \in F \cap D_0$ for some p and $q\}$.

By (6.37) and (6.38), $S^* = \dot{\pi}^F{}''\dot{T}^F$ is stationary subset of $\delta^* = \sup(\mathrm{rng}(q^*))$. By (6.38), $\mathrm{cf}(\delta^*) = \omega_1$. Since F is a filter, p^* is a ladder system on $\mathrm{dom}(p^*) \subseteq E^\kappa_\omega$ and $\sup(\mathrm{dom}(p^*)) = \delta^*$. f^* witnesses that p^* satisfies (6.3) for $\beta = \sup(\mathrm{dom}(p)) = \delta^*$ and hence we have $p^* \in \mathbb{P}_*$. Clearly $p^* \leq_{\mathbb{P}_*} p_0$ and $p^* \Vdash_{\mathbb{P}_*}$ "$S^* \subseteq \dot{S}$".
❑ (Lemma 6.6)

Now we are going to prove that every stationary subsets of $E^\kappa_\omega \setminus S_G$ reflect in $V[G]$. Working in $V[G]$ again for some $(V, \mathbb{P}_*)$-generic filter G, let S_G and g_G be as before. Let $\mathbb{Q}_1$ be the following partial ordering: The elements of $\mathbb{Q}_1$ are mappings q such that

(6.40) q is a strictly and continuously increasing mapping from a successor ordinal $< \omega_1$ to κ; and

(6.41) $q''Lim(\mathrm{dom}(q)) \subseteq E^\kappa_\omega \setminus S_G$.

Note that $E^\kappa_\omega \setminus S_G$ is stationary by Lemma 6.2, (4).

The following lemma can be proved similarly to Lemma 6.4.

Lemma 6.7. (1) *For $\alpha < \omega_1$ and $\beta < \kappa$, the set*

$$\{q \in \mathbb{Q}_1 \,:\, \mathrm{dom}(q) \geq \alpha \text{ and } \max(\mathrm{rng}(q)) \geq \beta\}$$

is dense in $\mathbb{Q}_1$.

(2) *$\mathbb{Q}_1$ adds the canonical strictly and continuously increasing mapping $\pi_1 = \bigcup H$ for a $(V[G], \mathbb{Q}_1)$-generic filter H. $\pi_1 : \omega_1 \to \kappa$, $\pi_1''\omega_1$ is cofinal in κ and $\pi_1''Lim(\omega_1) \subseteq E^\kappa_\omega \setminus S_G$.*

(3) *For any stationary $S \subseteq E^\kappa_\omega \setminus S_G$, $\mathbb{Q}_1$ forces that $S \cap \pi_1''\omega_1$ for π_1 as in (2) is stationary.* ❑

Stepping back to V, let $\dot{\mathbb{Q}}_1$ be a $\mathbb{P}_*$-name of $\mathbb{Q}_1$. By Lemma 6.7, we have $\Vdash_{\mathbb{P}_* * \dot{\mathbb{Q}}_1}$ "$\dot{S} \cap \dot{\pi}_1''\omega_1$ is stationary" for all $\mathbb{P}_*$-name $\dot{S}$ of stationary subset of $E^\kappa_\omega \setminus \dot{S}_G$ and $\mathbb{P}_1$-name $\dot{\pi}_1$ of π_1.

Let

$$(6.42) \quad D_1 = \{\langle p, \dot{q}\rangle \in \mathbb{P}_* * \dot{\mathbb{Q}}_1 \,:\, p \text{ decides } \dot{q} \text{ to be some } q \text{ in } V \text{ and } \sup(\mathrm{dom}(p)) = \sup(\mathrm{rng}(q))\}.$$

By the definition of D_1, each element $\langle p, \dot{q}\rangle$ of D_1 corresponds uniquely to $\langle p, q\rangle$ (with $q \in V$ and $p \Vdash_{\mathbb{P}_*}$ "$\dot{q} = q$") where

(6.43) q is a continuous strictly increasing mapping in V with $\mathrm{dom}(q)$ being a successor ordinal in ω_1;

(6.44) $q''Lim(\mathrm{dom}(q)) \subseteq \sup(\mathrm{dom}(p)) \setminus \mathrm{dom}(p)$; and

(6.45) $\sup(\mathrm{rng}(q)) = \sup(\mathrm{dom}(p))$.

Let D_1^* be the collection of all such $\langle p, q\rangle$. Then the partial ordering $\leq_{D_1^*}$ on D_1^* corresponding to $\leq_{\mathbb{P}_* * \dot{\mathbb{Q}}_1} \restriction D_1$ is

(6.46) $\langle p_1, q_1\rangle \leq_{D_1^*} \langle p_1, q_1\rangle \;\Leftrightarrow\; p_1 \leq_{\mathbb{P}_*} p_0$ and $q_1 \supseteq q_0$.

Lemma 6.8. (1) *D_1 is dense in $\mathbb{P}_* * \dot{\mathbb{Q}}_1$.*

(2) *D_1 is σ-closed (with respect to $\leq_{\mathbb{P}_* * \dot{\mathbb{Q}}_1} \restriction D_1$).*

Proof. We only prove (2) since (1) follows immediately form the κ-strategical closedness of $\mathbb{P}_*$.

It is enough to show that $\langle D_1^*, \leq_{D_1^*}\rangle$ is σ-closed. Suppose that $\mathcal{S} = \langle\langle p_n, q_n\rangle : n < \omega\rangle$ is a strictly decreasing chain in $\langle D_1^*, \leq_{D_1^*}\rangle$. We have to show that $\mathcal{S}$ has a lower bound in D_1^* with respect to $\leq_{D_1^*}$. Let $p^* = \bigcup_{n<\omega} p_n$ and $q^* = \bigcup_{n<\omega} q_n$. Let $\gamma^* = \sup(\mathrm{dom}(q^*))$ and $\delta^* = \sup(\mathrm{dom}(p^*))$. Let $c^* = \{\delta^* + n : n \in \omega\}$, $p^\dagger = p^* \cup \{\langle\delta^* + \omega, c^*\rangle\}$ and $q^\dagger = q^* \cup \{\langle\gamma^*, \delta^*\rangle, \langle\gamma^* + 1, \delta^* + \omega\rangle\}$. Then we have $\langle p^\dagger, q^\dagger\rangle \in D_1^*$ and $\langle p^\dagger, q^\dagger\rangle$ is a lower bound of $\mathcal{S}$. ❑ (Lemma 6.8)

By an argument parallel to the proof of Lemma 6.6 we can now prove the following:

Lemma 6.9 (MA$^+$(σ-closed))**.** $\mathbb{P}_*$ *forces that every stationary subsets of* $E^\kappa_\omega \backslash S_G$ *reflect.* ❑

Now we show that $\mathbb{P}_*$ forces FRP(λ) for all regular cardinals $> 2^{<\kappa}$. We consider again a two step iteration over $\mathbb{P}_*$.

Working again in $V[G]$ for a $(V, \mathbb{P}_*)$-generic filter G, let $\mathbb{Q}_2$ be the standard forcing for shooting a club through $\kappa \setminus S_G$ by conditions of size $< \kappa$: The elements q of $\mathbb{Q}_2$ are closed bounded subsets of κ such that $Lim(q) \subseteq \kappa \setminus S_G$ (with the maximal element) and, for q_0, $q_1 \in \mathbb{Q}_2$, $q_1 \leq_{\mathbb{Q}_2} q_0$ if q_1 is an end extension of q_0.

In V, let $\dot{\mathbb{Q}}_2$ be a $\mathbb{P}_*$-name of $\mathbb{Q}_2$ and let

$$(6.47) \quad \begin{aligned} D_2 = \{\langle p, \dot{q}\rangle \in \mathbb{P}_* * \dot{\mathbb{Q}}_2 &: p \text{ decides } \dot{q} \text{ to be some } q \text{ in } V \\ &\quad \text{and } \sup(\mathrm{dom}(p)) = \sup(\mathrm{rng}(q))\} \text{ and} \\ D_2^* = \{\langle p, q\rangle &: \text{there is a } \langle p, \dot{q}\rangle \in D_2 \text{ such that} \\ &\quad p \text{ decides } \dot{q} \text{ to be } q\} \end{aligned}$$

D_2^* with the ordering defined by

$$(6.48) \quad \langle p_1, q_1\rangle \leq_{D_2^*} \langle p_0, q_0\rangle \text{ if } p_1 \leq_{\mathbb{P}_*} p_0 \text{ and } q_1 \text{ is an end extension of } q_0$$

for $\langle p_0, q_0\rangle$, $\langle p_1, q_1\rangle \in D_2^*$ is isomorphic to $\langle D_2, \leq_{\mathbb{P}_* * \dot{\mathbb{Q}}_2}\restriction D_2\rangle$ modulo possible multitude of $\mathbb{P}_*$-names for q in $\langle p, q\rangle \in D_2^*$.

We can prove the following similarly to the corresponding lemmas for $\dot{\mathbb{Q}}_0$ and $\dot{\mathbb{Q}}_1$:

Lemma 6.10. (1) D_2 *is dense in* $\mathbb{P}_* * \dot{\mathbb{Q}}_2$.
(2) D_2 *is* κ*-closed.* ❑

Lemma 6.11 (MA$^+$(σ-closed))**.** $\mathbb{P}_*$ *forces* FRP(λ) *for all regular* $\lambda > 2^{<\kappa}$.

Proof. By $|\, D_2^* \,| = 2^{<\kappa}$ and by Lemma 6.10, (1), $\mathbb{P}_* * \dot{\mathbb{Q}}_2$ has the $(2^{<\kappa})^+$-c.c. Hence all (regular) cardinals $> 2^{<\kappa}$ are preserved.

By Lemma 6.10, (1) and (2), $\mathbb{P}_* * \dot{\mathbb{Q}}_2$ is strongly $< \kappa$-strategically closed. Hence, by Fact 6.3, we have $\Vdash_{\mathbb{P}_* * \dot{\mathbb{Q}}_2}$ "$\mathrm{MA}^+(\sigma\text{-closed})$". In particular, we have $\Vdash_{\mathbb{P}_* * \dot{\mathbb{Q}}_2}$ "$\mathrm{FRP}(\lambda)$".

Suppose that $p_0 \in \mathbb{P}_*$, $\dot{S}$ is a $\mathbb{P}_*$-name of a stationary subset of E^λ_ω and $\dot{g}$ a $\mathbb{P}_*$-name for a ladder system on $\dot{S}$. Since $\Vdash_{\mathbb{P}_*}$ "$|\, \dot{\mathbb{Q}}_2 \,| = 2^{<\kappa}$", $\dot{S}$ seen as a $\mathbb{P}_* * \dot{\mathbb{Q}}_2$-name is forced to be a stationary subset of E^λ_ω. By $\mathrm{FRP}(\lambda)$ in $V^{\mathbb{P}_* * \dot{\mathbb{Q}}_2}$, there is a $\mathbb{P}_* * \dot{\mathbb{Q}}_2$-name $\langle \dot{I}_\alpha \,:\, \alpha < \omega_1 \rangle$ of a filtration of a subset $\dot{I}$ of λ of cardinality $\aleph_1$ such that $\mathbb{P}_* * \dot{\mathbb{Q}}_2$ forces

(6.49) $\mathrm{cf}(\dot{I}) = \omega_1$;

(6.50) $\dot{I}$ is closed with respect to $\dot{g}$; and

(6.51) $\{\alpha \in \omega_1 \,:\, \sup(\dot{I}_\alpha) \in \dot{S}$ and $\dot{g}(\sup(\dot{I}_\alpha)) \subseteq I_\alpha\}$ is stationary.

By Lemma 6.10, we have $\Vdash_{\mathbb{P}_*}$ "$\dot{\mathbb{Q}}_2$ is $< \kappa$-Baire". Hence $\langle I_\alpha \,:\, \alpha < \omega_1 \rangle \in V^{\mathbb{P}_*}$ and this sequence witnesses $\mathrm{FRP}(\lambda)$ for S and g in $V^{\mathbb{P}_*}$. ❑ (Lemma 6.11)

We can now prove Theorem 6.1 just by recapitulating all the results we obtained so far.

Proof of Theorem 6.1: We show that the partial ordering $\mathbb{P}_*$ defined in (6.1) $\sim$ (6.4) has the desired property (1) $\sim$ (3).

(1): We have $\Vdash_{\mathbb{P}_*}$ "$\neg\mathrm{FRP}(\kappa)$" by Lemma 6.2, (5). To show $\Vdash_{\mathbb{P}_*}$ "$\mathrm{ORP}(\kappa)$", suppose that $p \in \mathbb{P}_*$ and $\dot{S}$ is a $\mathbb{P}_*$-name of a stationary subset of E^κ_ω. Then there is $p' \leq_{\mathbb{P}_*} p$, such that either $p' \Vdash_{\mathbb{P}_*}$ "$\dot{S} \cap \dot{S}_G$ is stationary" or $p' \Vdash_{\mathbb{P}_*}$ "$\dot{S} \setminus \dot{S}_G$ is stationary" holds in either case we can conclude that p' forces that $\dot{S}$ reflects by Lemma 6.6 or Lemma 6.9.

(2): Since $\mathbb{P}_*$ is κ-strategically closed by Lemma 6.2, (3), $\mathrm{FRP}(\lambda)$ is preserved for all regular λ with $\aleph_1 < \lambda < \kappa$. For regular $\lambda > 2^{<\kappa}$, we have $\Vdash_{\mathbb{P}_*}$ "$\mathrm{FRP}(\lambda)$" by Lemma 6.11.

(3): For regular $\aleph_2 \leq \lambda < \kappa$, we have $\Vdash_{\mathbb{P}_*}$ "$\mathrm{FRP}(\lambda)$" by $\mathrm{MA}^+(\sigma\text{-closed})$ and κ strategically closedness of $\mathbb{P}_*$. For $\lambda > \kappa$ forced to be a regular cardinal by $\mathbb{P}_*$, Lemma 6.2, (6) and (2) above imply $\Vdash_{\mathbb{P}_*}$ "$\mathrm{FRP}(\lambda)$" and this together with (1) implies $\Vdash_{\mathbb{P}_*}$ "ORP". ❑ (Theorem 6.1)

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