

# On Recurrence Axioms

渕野 昌

薄葉 季路

Sakaé Fuchino\* Toshimichi Usuba†

## Abstract

The Recurrence Axiom for a class  $\mathcal{P}$  of posets and a set  $A$  of parameters is an axiom scheme in the language of ZFC asserting that if a statement with parameters from  $A$  is forced by a poset in  $\mathcal{P}$ , then there is a ground containing the parameters and satisfying the statement.

The tightly super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver generic hyperhuge continuum implies the Recurrence Axiom for  $\mathcal{P}$  and  $\mathcal{H}(2^{\aleph_0})$ . The consistency strength of this assumption can be decided thanks to our main theorems asserting that the minimal ground (bedrock) exists under a tightly  $\mathcal{P}$ -generic hyperhuge cardinal  $\kappa$ , and that  $\kappa$  in the bedrock is genuinely hyperhuge, or even super  $C^{(\infty)}$  hyperhuge if  $\kappa$  is a tightly super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver generic hyperhuge definable cardinal.

The Laver Generic Maximum (LGM), one of the strongest combinations of axioms in our context, integrates practically all known set-theoretic principles and axioms in itself, either as its consequences or as theorems holding in (many) grounds of the universe. For instance, double plus version of Martin's Maximum is a consequence of LGM while Cichoń's Maximum is a phenomenon in many grounds of the universe under LGM.

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\* Graduate School of System Informatics, Kobe University  
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan  
fuchino@diamond.kobe-u.ac.jp

† Faculty of Fundamental Science and Engineering, Waseda University  
Okubo 3-4-1, Shinjyuku, Tokyo, 169-8555 Japan  
usuba@waseda.jp

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# 1 Introduction

The Recurrence Axiom for a class  $\mathcal{P}$  of posets and a set  $A$  of parameters is an axiom [theintro](#) scheme in the language of ZFC asserting that if a statement  $\varphi(\bar{a})$  with parameters  $\bar{a}$  in  $A$  is forced by a poset  $\mathbb{P} \in \mathcal{P}$ , then there is a ground (i.e. an inner model from which the universe  $V$  is attainable via set forcing) containing the parameters and satisfying the statement  $\varphi(\bar{a})$ .

Recurrence Axioms can be interpreted as statements about the (eternal?) recurrence in the set generic multiverse in terms of the time-flow along with set forcing extension: everything that can happen in the near future (in form of forcing extension) actually happened already in the past (in a ground). Here, the nearness of the future is measured in terms of extent of the class  $\mathcal{P}$  of posets we may consider. The extent of the set of parameters which we may use in the descriptions of "events" in the future also differentiates the strength of the axiom.

Recurrence Axioms are actually variations of known axioms and principles: they are weakenings of Maximality Principles with corresponding parameters (Proposition 2.2) while they can be characterized as the set-generic versions of Sy Friedman's Inner Ground Hypothesis (Proposition 2.3). See the end of Section 2 for discussions about why we want to keep these axioms in spite of this almost identity with other known principles.

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<https://fuchino.ddd.jp/papers/recurrence-axioms-x.pdf>

In Section 3 we show that the tight Laver-generic ultrahugeness implies  $\Sigma_2$ -fragment of Recurrence Axioms (Theorem 3.1), and  $\Sigma_1$ -fragments of Recurrence Axioms with strong enough combination of  $\mathcal{P}$  and  $A$  decide the size of the continuum: in case of  $\mathcal{P}$  being the class of all posets with  $A = \mathcal{H}(2^{\aleph_0})$  the Continuum Hypothesis (CH) holds while the Recurrence Axiom for stationary preserving  $\mathcal{P}$  with  $A = \mathcal{H}(\kappa_{\text{refl}})$  implies  $2^{\aleph_0} = \aleph_2$  (Theorem 3.3).

In Section 4 we introduce the notion of tightly super  $C^{(\infty)}$ - $\mathcal{P}$ -Laver generic hyperhuge cardinal  $\kappa$  and show that Recurrence Axiom for  $\mathcal{P}$  and  $\mathcal{H}(\kappa)$  follows from the existence of this generic large cardinal (Theorem 4.10). The consistency of the existence of this cardinal is strictly between that of hyperhuge cardinal and a 2-huge cardinal (actually the lower bound can be still raised, see Corollary 5.11). This follows from the main theorems (Theorem 5.2, Theorem 5.3) in Section 5, asserting that the minimal ground (bedrock) under a tightly  $\mathcal{P}$ -generic hyperhuge cardinal  $\kappa$  exists and that  $\kappa$  in the bedrock is genuinely hyperhuge, or even super  $C^{(\infty)}$  hyperhuge if  $\kappa$  is a tightly super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver generic hyperhuge definable cardinal.

This result strengthen the theorem on the existence of bedrock by Usuba under a hyperhuge cardinal in [42].

After examining some of the consequences of the main theorems in Section 6, we discuss in Section 7 the Laver Generic Maximum (LGM), one of the strongest axiom available in our context, which integrates practically all known set-theoretic principles and axioms in itself, either as its consequences or as theorems in (many of) the grounds of the universe. So for example, double plus version of Martin's Maximum ( $\text{MM}^{++}$ ) is a consequence of LGM while Cichoń's Maximum is a phenomenon in many grounds of the universe under LGM.

We tried hard to make the present paper as accessible as possible for a wide audience. The terminology and notations used here are either standard or explained fully in the text. For some basic notions nevertheless left unexplained the reader may consult [33], [36] and/or [34].

## 2 Recurrence Axioms

In the following,  $\mathcal{L}_\in$  denotes the language of ZFC consisting of single binary relation symbol  $\in$ . [intro](#)

In the language of ZFC, we always identify a class  $\mathcal{P}$  with the  $\mathcal{L}_\in$ -formula which defines the class. Thus, if a class  $\mathcal{P}$  is defined by an  $\mathcal{L}_\in$ -formula  $\psi(x)$ , with  $\Vdash_{\mathbb{P}} \text{“}\dot{x} \in \mathcal{P} \dots\text{”}$  for a poset  $\mathbb{P}$ , we simply mean  $\Vdash_{\mathbb{P}} \text{“}\psi(\dot{x}) \dots\text{”}$ . We adopt model

theoretic convention that (in connection with lower case letters) a letter with bar denotes a tuple of objects. Thus,  $\bar{a}$  means  $a_0, \dots, a_{k-1}$  for some natural number  $k$  and write  $\bar{a} \in X$  for  $a_0, \dots, a_{k-1} \in X$ .

We call a class  $\mathcal{P}$  of posets *normal* if it satisfies (2.1):  $\{1\} \in \mathcal{P}$ , and (2.2):  $\mathcal{P}$  is closed with respect to forcing equivalence (i.e. if  $\mathbb{P} \in \mathcal{P}$  and  $\mathbb{P} \sim \mathbb{P}'$  then  $\mathbb{P}' \in \mathcal{P}$ ).

x-intro-0-0

x-intro-1

In the following we assume that all classes  $\mathcal{P}$  of posets we consider are normal. In particular when we say that  $\mathcal{P}$  is a class of posets we assume that  $\mathcal{P} \neq \emptyset$  and it contains the trivial poset.

Some natural classes of posets are not closed under forcing equivalence — notably the class of  $\sigma$ -closed posets. For such classes we simply take the closure of the class with respect to forcing equivalence and replace the class with the closure without mentioning it.

A (normal) class of posets is *iterable* if it also satisfies (2.3): closed with respect to restriction (i.e. if  $\mathbb{P} \in \mathcal{P}$  then  $\mathbb{P} \restriction \mathbb{P} \in \mathcal{P}$  for any  $\mathbb{P} \in \mathbb{P}$ ), and (2.4): for any  $\mathbb{P} \in \mathcal{P}$  and  $\mathbb{P}$ -name  $\dot{Q}$ ,  $\Vdash_{\mathbb{P}} \dot{Q} \in \mathcal{P}$  implies  $\mathbb{P} * \dot{Q} \in \mathcal{P}$ .

x-intro-2

x-intro-3

For a class  $\mathcal{P}$  of posets and a set  $A$  (of parameters),  *$\mathcal{P}$ -Recurrence Axiom with parameters from  $A$*  ( $(\mathcal{P}, A)$ -RcA, for short) is the following assertion formally expressed as an axiom scheme in  $\mathcal{L}_{\in}$ :

(2.5) For any  $\mathcal{L}_{\in}$ -formula  $\varphi(\bar{x})$  and  $\bar{a} \in A$ , if  $\Vdash_{\mathbb{P}} \varphi(\bar{a}^{\vee})$ , then there is a ground  $M$  of  $\mathbb{V}$  such that  $\bar{a} \in M$  and  $M \models \varphi(\bar{a})$ .

x-intro-4

Here, an inner model  $W_0$  of a universe  $W$  is said to be a *ground* of  $W$  if there are a poset  $\mathbb{P} \in W_0$  and  $(W_0, \mathbb{P})$ -generic  $\mathbb{G} \in W$  such that  $W_0[\mathbb{G}] = W$ .  $W_0$  is a  *$\mathcal{P}$ -ground* of  $W$  if  $\mathbb{P}$  as above can be taken such that  $W_0 \models \mathcal{P}$ .

All such grounds are definable. More precisely, the following theorem holds:

**Theorem 2.1** (Reitz [38], Fuchs-Hamkins-Reitz [28]) *There is an  $\mathcal{L}_{\in}$ -formula  $\Phi(x, r)$  such that the following is provable in ZFC:*

p-intro-1

(2.6) for all  $r$ ,  $\Phi(\cdot, r) := \{x : \Phi(x, r)\}$  is a ground in  $\mathbb{V}$ , and

x-intro-10

(2.7) for any ground  $M$  (in  $\mathbb{V}$ ), there is  $r$  such that  $M = \Phi(\cdot, r)$ .  $\square$

x-intro-11

In the following we use this fact often without explicitly mentioning it. As a corollary to Theorem 2.1, we immediately see that being a  $\mathcal{P}$ -ground for a class  $\mathcal{P}$  of posets is a definable property.

The *Strong  $\mathcal{P}$ -Recurrence Axiom with parameters from  $A$*  ( $(\mathcal{P}, A)$ -RcA<sup>+</sup>, for short) holds, if (2.5) holds with  $M$  which is a  $\mathcal{P}$ -ground of  $\mathbb{V}$ .

Actually the Strong Recurrence Axiom is equivalent to an already known axiom: In the following Proposition 2.2, we show that, for any (normal)  $\mathcal{P}$ ,  $(\mathcal{P}, A)\text{-RcA}^+$  is equivalent to the Maximality Principle  $\text{MP}(\mathcal{P}, A)$  in the notation of [17] (see below for definition — the characterization of  $\text{MP}(\mathcal{P}, A)$  corresponding to this proposition as well as the statements corresponding to Proposition 2.3 and Proposition 2.4 were also observed by Barton, Caicedo, Fuchs, Hamkins, Reitz, and Schindler [2]).

For an iterable class  $\mathcal{P}$  of posets, an  $\mathcal{L}_\in$ -formula  $\varphi(\bar{a})$  with parameters  $\bar{a} \in \mathbf{V}$  is said to be a  **$\mathcal{P}$ -button** if there is  $\mathbb{P} \in \mathcal{P}$  such that for any  $\mathbb{P}$ -name  $\dot{Q}$  of poset with  $\Vdash_{\mathbb{P}} \dot{Q} \in \mathcal{P}$ , we have  $\Vdash_{\mathbb{P} * \dot{Q}} \varphi(\bar{a}^\vee)$ .

If  $\varphi(\bar{a})$  is a  $\mathcal{P}$ -button then we call  $\mathbb{P}$  as above a **push of the  $\mathcal{P}$ -button  $\varphi(\bar{a})$** .

For a class  $\mathcal{P}$  of posets and a set  $A$  (of parameters), the **Maximality Principle of  $\mathcal{P}$  and  $A$**  ( $\text{MP}(\mathcal{P}, A)$ , for short) is the following assertion which is formulated as an axiom scheme in  $\mathcal{L}_\in$ :

**$\text{MP}(\mathcal{P}, A)$ :** For any  $\mathcal{L}_\in$ -formula  $\varphi(\bar{x})$  and  $\bar{a} \in A$ , if  $\varphi(\bar{a})$  is a  $\mathcal{P}$ -button then  $\varphi(\bar{a})$  holds.

**Proposition 2.2** *Suppose that  $\mathcal{P}$  is a class of posets and  $A$  a set (of parameters).*

(1)  $(\mathcal{P}, A)\text{-RcA}^+$  is equivalent to  $\text{MP}(\mathcal{P}, A)$ .

(2)  $(\mathcal{P}, A)\text{-RcA}$  is equivalent to the following assertion:

(2.8) For any  $\mathcal{L}_\in$ -formula  $\varphi(\bar{x})$  and  $\bar{a} \in A$ , if  $\varphi(\bar{a})$  is a  $\mathcal{P}$ -button then  $\varphi(\bar{a})$  holds in a ground of  $\mathbf{V}$ .

**Proof.** (1): Suppose first that  $(\mathcal{P}, A)\text{-RcA}^+$  holds. We show that  $\text{MP}(\mathcal{P}, A)$  holds. Assume that  $\mathbb{P} \in \mathcal{P}$  is a push of the  $\mathcal{P}$ -button  $\varphi(\bar{a})$ . Let  $\varphi'(\bar{x})$  be the formula expressing

(2.9) for any  $\dot{Q} \in \mathcal{P}$ ,  $\Vdash_{\dot{Q}} \varphi(\bar{x}^\vee)$  holds.

Then we have  $\Vdash_{\mathbb{P}} \varphi'(\bar{a}^\vee)$ . By  $(\mathcal{P}, A)\text{-RcA}^+$ , there is a  $\mathcal{P}$ -ground  $M$  of  $\mathbf{V}$  such that  $\bar{a} \in M$  and  $M \models \varphi'(\bar{a})$  holds. By the definition (2.9) of  $\varphi'$ , it follows that  $\mathbf{V} \models \varphi(\bar{a})$  holds.

Now suppose that  $\text{MP}(\mathcal{P}, A)$  holds and  $\mathbb{P} \in \mathcal{P}$  is such that  $\Vdash_{\mathbb{P}} \varphi(\bar{a}^\vee)$  for  $\bar{a} \in A$ .

Let  $\varphi''$  be a formula claiming that

(2.10) there is a  $\mathcal{P}$ -ground  $N$  such that  $\bar{x} \in N$  and  $N \models \varphi(\bar{x})$ .

Then  $\varphi''(\bar{a})$  is a  $\mathcal{P}$ -button and  $\mathbb{P}$  is its push.

By  $\text{MP}(\mathcal{P}, A)$ ,  $\varphi''(\bar{a})$  holds in  $\mathbf{V}$  and hence there is a  $\mathcal{P}$ -ground  $M$  of  $\mathbf{V}$  such that  $\bar{a} \in M$  and  $M \models \varphi(\bar{a})$ . This shows that  $(\mathcal{P}, A)\text{-RcA}^+$  holds.

(2): can be proved similarly to (1). Suppose first that  $(\mathcal{P}, A)\text{-RcA}$  holds. We show that (2.8) holds. Assume that  $\mathbb{P} \in \mathcal{P}$  is a push of the  $\mathcal{P}$ -button  $\varphi(\bar{a})$ . Let  $\varphi'(\bar{x})$  be the formula expressing

$$(2.11) \quad \text{for any } \mathbb{Q} \in \mathcal{P}, \Vdash_{\mathbb{Q}} \text{“}\varphi(\bar{x}^\vee)\text{” holds.}$$

x-intro-8

Then we have  $\Vdash_{\mathbb{P}} \text{“}\varphi'(\bar{a}^\vee)\text{”}$ . By  $(\mathcal{P}, A)\text{-RcA}$ , there is a ground  $M$  of  $\mathbf{V}$  such that  $\bar{a} \in M$  and  $M \models \varphi'(\bar{a})$  holds. Since  $\mathcal{P} \ni \{1\}$ , it follows that  $M \models \varphi(\bar{a})$ .

Now suppose that (2.8) holds and  $\mathbb{P} \in \mathcal{P}$  is such that  $\Vdash_{\mathbb{P}} \text{“}\varphi(\bar{a}^\vee)\text{”}$  for  $\bar{a} \in A$ . Let  $\varphi''$  be a formula asserting that

$$(2.12) \quad \text{there is a } \mathcal{P}\text{-ground } N \text{ such that } \bar{x} \in N \text{ and } N \models \varphi(\bar{x}).$$

x-intro-9

Then  $\varphi''(\bar{a})$  is a  $\mathcal{P}$ -button and  $\mathbb{P}$  is its push. Thus, By (2.8),  $\varphi''(\bar{a})$  holds in a ground  $M$  of  $\mathbf{V}$  with  $\bar{a} \in M$ . By the definition (2.12) of  $\varphi''$ , there is a  $\mathcal{P}$ -ground  $N$  of  $M$  such that  $\bar{a} \in N$  and  $N \models \varphi(\bar{a})$ . Since  $N$  is also a ground of  $\mathbf{V}$ , this shows that  $(\mathcal{P}, A)\text{-RcA}$  holds.  $\square$  (Proposition 2.2)

Recurrence Axioms are also related to the Inner Model Hypothesis introduced by Sy Friedman in [6]. *The Inner Model Hypothesis (IMH)* is the following assertion formulated in the language of second-order set theory (e.g. in the context of von Neumann-Bernays-Gödel set theory):

**IMH** : For any statement  $\varphi$  without parameters, if  $\varphi$  holds in an inner model of an inner extension of  $\mathbf{V}$  then  $\varphi$  holds in an inner model of  $\mathbf{V}$ .

Here we say a (not necessarily first-order definable) transitive class  $M$  an *inner model* of  $\mathbf{V}$  if  $M$  is a model of  $\mathbf{ZF}$  and  $\text{On}^M = \text{On}^{\mathbf{V}}$ . In the perspective from such  $M$ , we call  $\mathbf{V}$  an *inner extension* of  $M$ .

We shall call a set-forcing version of this principle *Inner Ground Hypothesis (IGH)*:

For a (definable normal) class  $\mathcal{P}$  of posets and a set  $A$  (of parameters),

**IGH**( $\mathcal{P}, A$ ) : For any  $\mathcal{L}_\in$ -formula  $\varphi = \varphi(\bar{x})$  and  $\bar{a} \in A$ , if  $\mathbb{P} \in \mathcal{P}$  forces “there is a ground  $M$  with  $\bar{a} \in M$  satisfying  $\varphi(\bar{a})$ ”, then there is a ground  $W$  of  $\mathbf{V}$  such that  $\bar{a} \in W$  and  $W \models \varphi(\bar{a})$ .

**Proposition 2.3** *For a class  $\mathcal{P}$  of posets and a set  $A$  (of parameters),  $(\mathcal{P}, A)\text{-RcA}$  holds if and only if  $\text{IGH}(\mathcal{P}, A)$  holds.*

p-intro-2

**Proof.** Suppose that  $(\mathcal{P}, A)\text{-RcA}$  holds. Let  $\varphi = \varphi(\bar{x})$  be an  $\mathcal{L}_\in$ -formula,  $\bar{a} \in A$ , and  $\mathbb{P} \in \mathcal{P}$  be such that  $\Vdash_{\mathbb{P}} \text{“}\varphi(\bar{a}^\vee)\text{”}$  holds in a ground.”

Let  $\varphi'(\bar{x})$  be the  $\mathcal{L}_\in$ -formula asserting that  $\varphi(\bar{x})$  holds in a ground. Then  $\Vdash_{\mathbb{P}} \varphi'(\bar{a}^\vee)$ . By  $(\mathcal{P}, A)$ -RcA, it follows that there is a ground  $W$  of  $V$  such that  $W \models \varphi'(\bar{a}^\vee)$ . Since a ground of a ground is a ground, we conclude that there is a ground  $W_0$  of  $V$  such that  $\bar{a} \in M_0$  and  $W_0 \models \varphi(\bar{a})$ . This shows that  $\text{IGH}(\mathcal{P}, A)$  holds.

Suppose now that  $\text{IGH}(\mathcal{P}, A)$  holds. Assume that  $\Vdash_{\mathbb{P}} \varphi(\bar{a}^\vee)$  for an  $\mathcal{L}_\in$ -formula  $\varphi = \varphi(\bar{x})$ ,  $\bar{a} \in A$ , and  $\mathbb{P} \in \mathcal{P}$ . Then  $\Vdash_{\mathbb{P}} \varphi(\bar{a}^\vee)$  holds in a  $\mathcal{P}$ -ground (of the universe) since  $\Vdash_{\mathbb{P}} \{\mathbb{1}\} \in \mathcal{P}$ . Thus, by  $\text{IGH}(\mathcal{P}, A)$ , there is a ground  $W$  of  $V$  such that  $W \models \varphi(\bar{a})$ . □ (Proposition 2.3)

$(\mathbb{P}, A)$ -RcA<sup>+</sup> ( $\Leftrightarrow$  MP( $\mathcal{P}, A$ ) by Proposition 2.2, (1)) can be also characterized in terms of a strengthening of Inner Ground Hypothesis: For a (definable) class  $\mathcal{P}$  of posets and a set  $A$  (of parameters),

**IGH<sup>+</sup>( $\mathcal{P}, A$ )** : For any  $\mathcal{L}_\in$ -formula  $\varphi = \varphi(\bar{x})$  and  $\bar{a} \in A$  if  $\mathbb{P} \in \mathcal{P}$  forces “there is a  $\mathcal{P}$ -ground  $M$  with  $\bar{a} \in M$  satisfying  $\varphi(\bar{a})$ ”, then there is a  $\mathcal{P}$ -ground  $W$  of  $V$  such that  $\bar{a} \in W$  and  $W \models \varphi(\bar{a})$ .

The following proposition can be proved similarly to Proposition 2.3.

**Proposition 2.4** *For a class  $\mathcal{P}$  of posets and a set  $A$  (of parameters),  $(\mathcal{P}, A)$ -RcA<sup>+</sup> holds if and only if  $\text{IGH}^+(\mathcal{P}, A)$  holds.* p-intro-3

**Proof.** Suppose that  $(\mathcal{P}, A)$ -RcA<sup>+</sup> holds and assume that  $\varphi = \varphi(\bar{x})$  is an  $\mathcal{L}_\in$ -formula,  $\bar{a} \in A$ , and  $\mathbb{P} \in \mathcal{P}$  is such that

$$\Vdash_{\mathbb{P}} \varphi(\bar{a}^\vee) \text{ holds in a } \mathcal{P}\text{-ground } M \text{ with } \bar{a} \in M$$

Let  $\varphi'(\bar{a})$  be the formula expressing “ $\varphi(\bar{x})$  holds in a  $\mathcal{P}$ -ground  $M$  with  $\bar{a} \in M$ ”. Then  $\mathbb{P}$  is a push of the  $\mathbb{P}$ -button  $\varphi'(\bar{a})$ . Thus, by Proposition 2.2, (1),  $\varphi'(\bar{a})$  holds in  $V$ . By definition of  $\varphi'$ , there is a  $\mathcal{P}$ -ground  $W$  of  $V$  such that  $\bar{a} \in W$  and  $W \models \varphi(\bar{a})$ . This shows that  $\text{IGH}^+(\mathcal{P}, A)$  holds.

Suppose now that  $\text{IGH}^+(\mathcal{P}, A)$  holds, and assume that  $\varphi = \varphi(\bar{x})$  is an  $\mathcal{L}_\in$ -formula,  $\bar{a} \in A$  and  $\Vdash_{\mathbb{P}} \varphi(\bar{a}^\vee)$  then (since  $\{\mathbb{1}\} \in \mathcal{P}$ )  $\Vdash_{\mathbb{P}} \varphi(\bar{a}^\vee)$  holds in a  $\mathcal{P}$ -ground  $M$  with  $\bar{a} \in M$ . By  $\text{IGH}^+(\mathcal{P}, A)$ , it follows that there is  $\mathcal{P}$ -ground  $W$  of  $V$  such that  $\bar{a} \in W$  and  $W \models \varphi(\bar{a})$ . This shows that  $(\mathcal{P}, A)$ -RcA<sup>+</sup> holds. □ (Proposition 2.4)

In spite of these almost identity with other known principles, we want to keep the Recurrence Axioms as independent axioms. One of the reasons is that we have the following monotonicity for these axioms which is not the case with the Maximality Principles.

**Lemma 2.5** (Monotonicity of Recurrence Axioms) *For classes of posets  $\mathcal{P}$ ,  $\mathcal{P}'$  and sets  $A$ ,  $A'$  of parameters, if  $\mathcal{P} \subseteq \mathcal{P}'$  and  $A \subseteq A'$ , then we have* p-5

$$(\mathcal{P}', A')\text{-RcA} \Rightarrow (\mathcal{P}, A)\text{-RcA}. \quad \square$$

If we decide that the Recurrence Axioms provide desirable extensions of the axioms of ZFC, then we should try to take the maximal instance of these axioms. (i.e. the one with maximal strength among the instances consistent with ZFC) By Lemma 2.5, this means we should try to take the instance of Recurrence Axioms with the maximal  $\mathcal{P}$  and  $A$  (with respect to inclusion) among the consistent ones.

Lemma 3.3 in the next section suggests that the following (2.13) and (2.14) are candidates of such maximal instances.

Let  $\kappa_{\text{refl}}$  denote the cardinal number  $\max\{2^{\aleph_0}, \aleph_2\}$ .  $\kappa_{\text{refl}}$  appears often as the reflection point of a strong structural reflection principle (see [17]).

(2.13) ZFC +  $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))\text{-RcA}$  for the class  $\mathcal{P}$  of all stationary preserving posets. x-intro-12

(2.14) ZFC +  $(\mathcal{Q}, \mathcal{H}(2^{\aleph_0}))\text{-RcA}$  for the class  $\mathcal{Q}$  of all posets. x-intro-13

The consistency of (2.14) follows from the consistency of ZFC + “there are stationarily many inaccessible cardinals” ([29]). The consistency of (2.13) follows from Lemma 4.5, Theorem 4.7,  $(B')$ , and Theorem 4.10.

The maximality of (2.13) and (2.14) follows from Lemma 3.3, (2') and (5') respectively.

By Lemma 3.3, (4) and (5), (2.13) implies  $2^{\aleph_0} = \aleph_2$ , and (2.14) implies CH. In particular, these two extensions of ZFC are not compatible. However, as we are going to discuss in Section 7, we can combine the plus version of (2.13) with the following weakening of (2.14):

(2.15) ZFC +  $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))\text{-RcA}^+ + (\mathcal{Q}, \mathcal{H}(\omega_1)^{\overline{W}})\text{-RcA}^+$  where  $\mathcal{P}$  is the class of all semi-proper posets,  $\mathcal{Q}$  the class of all posets, and  $\overline{W}$  the bedrock<sup>1)</sup> which is also assumed here to exist. x-intro-14

Note that  $2^{\aleph_0} = \aleph_2$  follows from (2.15). In Section 7, we give an axiom in terms of existence of strong variants of Laver generic large cardinals from which (2.15) follows.

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<sup>1)</sup> For the definition of the bedrock see Section 5.



### 3 Recurrence Axioms in restricted forms and the Continuum Problem

We consider the following restricted forms of Recurrence Axiom: For an iterable Lg-RcA class  $\mathcal{P}$  of posets, a set  $A$  (of parameters), and a set  $\Gamma$  of  $\mathcal{L}_\in$ -formulas,  *$\mathcal{P}$ -Recurrence Axiom for formulas in  $\Gamma$  with parameters from  $A$*  ( $(\mathcal{P}, A)_\Gamma$ -RcA, for short) is the following assertion expressed as an axiom scheme in  $\mathcal{L}_\in$ :

$$(3.1) \quad \text{For any } \varphi(\bar{x}) \in \Gamma \text{ and } \bar{a} \in A, \text{ if } \Vdash_{\mathbb{P}} \varphi(\bar{a}^\vee), \text{ then there is a ground } W \text{ of } \text{span style='color: #0070C0; font-weight: normal; font-size: small; vertical-align: top; margin-left: 10px; display: block; } x\text{-Lg-RcA-a-0}$$

$$V \text{ such that } \bar{a} \in W \text{ and } W \models \varphi(\bar{a}).$$

$(\mathcal{P}, A)_\Gamma$ -RcA<sup>+</sup> corresponding to  $(\mathcal{P}, A)$ -RcA<sup>+</sup> is defined similarly.

Recall that a cardinal  $\kappa$  is ultrahuge if for any  $\lambda > \kappa$ , there are  $j, M \subseteq V$  such ultrahuge that  $j : V \xrightarrow{\sim}_\kappa M$ ,  $j(\kappa) > \lambda$ ,  $j^{(\kappa)}M \subseteq M$  and  $V_{j(\lambda)} \subseteq M$ .

For an iterable class  $\mathcal{P}$  of posets, a cardinal  $\kappa$  is said to be (tightly)  $\mathcal{P}$ -Laver-generically ultrahuge ((tightly)  $\mathcal{P}$ -Laver-gen. ultrahuge, for short), if, for any  $\lambda > \kappa$  and  $\mathbb{P} \in \mathcal{P}$  there is a  $\mathbb{P}$ -name  $\mathbb{Q}$  with  $\Vdash_{\mathbb{P}} \mathbb{Q} \in \mathcal{P}$ , such that for  $(V, \mathbb{P} * \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $j, M \subseteq V[\mathbb{H}]$  such that  $j : V \xrightarrow{\sim}_\kappa M$ ,  $j(\kappa) > \lambda$ ,  $\mathbb{P}, \mathbb{H}, (V_{j(\lambda)})^{V[\mathbb{H}]} \in M$  (and  $\mathbb{P} * \mathbb{Q}$  is forcing equivalent to a poset of size  $\leq j(\kappa)$ ).<sup>2)</sup>

The following theorem stands in contrast to Theorem 6.5 in [17].

**Theorem 3.1** *Suppose that  $\kappa$  is tightly  $\mathcal{P}$ -Laver-gen. ultrahuge for an iterable class p-Lg-RcA-0  $\mathcal{P}$  of posets. Then  $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_2}$ -RcA<sup>+</sup> holds.*

For the proof of Theorem 3.1, we use the following lemma which should be a well-known fact. Nevertheless, we prove the lemma since we could not find a suitable reference.

**Lemma 3.2** *If  $\alpha$  is a limit ordinal and  $V_\alpha$  satisfies a sufficiently large finite fragment of ZFC, then for any  $\mathbb{P} \in V_\alpha$  and  $(V, \mathbb{P})$ -generic  $\mathbb{G}$ , we have  $V_\alpha[\mathbb{G}] = V_\alpha^{V[\mathbb{G}]}$ .* p-Lg-RcA-0-0

**Proof.** “ $\subseteq$ ”: This inclusion holds without the condition on the fragment of ZFC. Also the condition “ $\mathbb{P} \in V_\alpha$ ” is irrelevant for this inclusion.

We show by induction on  $\alpha \in \text{On}$  that  $V_\alpha[\mathbb{G}] \subseteq V_\alpha^{V[\mathbb{G}]}$  holds for all  $\alpha \in \text{On}$ .

The induction steps for  $\alpha = 0$  and limit ordinals  $\alpha$  are trivial. So we assume that  $V_\alpha[\mathbb{G}] \subseteq V_\alpha^{V[\mathbb{G}]}$  holds and show that the same inclusion holds for  $\alpha + 1$ . Suppose  $a \in V_{\alpha+1}[\mathbb{G}]$ . Then  $a = \dot{a}^\mathbb{G}$  for a  $\mathbb{P}$ -name  $\dot{a} \in V_{\alpha+1}$ . Since  $\dot{a} \subseteq V_\alpha$ , each  $\langle \dot{b}, \mathbb{p} \rangle \in \dot{a}$  is

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<sup>2)</sup> In the following, we shall denote this condition simply by “ $|\mathbb{P} * \mathbb{Q}| \leq \lambda$ ”. More generally, fn-0 we shall always write “ $|\mathbb{P}| \leq \lambda$ ” for a poset  $\mathbb{P}$  to mean that “ $\mathbb{P}$  is forcing equivalent to a poset of size  $\leq \lambda$ ”.

an element of  $V_\alpha$ . By induction hypothesis, it follows that  $\tilde{b}^\mathbb{G} \in V_\alpha^{V[\mathbb{G}]}$ . It follows that  $\tilde{a}^\mathbb{G} \subseteq V_\alpha^{V[\mathbb{G}]}$ . Thus  $a = \tilde{a}^\mathbb{G} \in V_{\alpha+1}^{V[\mathbb{G}]}$ .

“ $\supseteq$ ”: Suppose that  $a \in V_\alpha^{V[\mathbb{G}]}$ . Note that we can choose the “sufficiently large finite fragment of ZFC” which should be satisfied in  $V_\alpha$  such that  $(*)$   $V_\alpha^{V[\mathbb{G}]}$  still satisfies a large enough fragment of ZFC, although the fragment may be different from the one  $V_\alpha$  satisfies. In particular we find a cardinal  $\lambda > |\mathbb{P}|$  in  $V_\alpha^{V[\mathbb{G}]}$  (and hence also in  $V[\mathbb{G}]$ ) such that  $a \in \mathcal{H}(\lambda)^{V_\alpha^{V[\mathbb{G}]}} \subseteq \mathcal{H}(\lambda)^{V[\mathbb{G}]} \subseteq V_\alpha^{V[\mathbb{G}]}$ . [Note that  $\mathcal{H}(\lambda)^{V_\alpha^{V[\mathbb{G}]}} = \{a : |\text{trcl}(a)| < \lambda\}^{V_\alpha^{V[\mathbb{G}]}} \subseteq \{a : |\text{trcl}(a)| < \lambda\}^{V[\mathbb{G}]} = \mathcal{H}(\lambda)^{V[\mathbb{G}]}$ .]

Let  $a^* \in \mathcal{H}(\lambda)^{V[\mathbb{G}]}$  be a transitive set such that  $a \in a^*$ . Then  $a^*$  can be coded by a subset of  $\lambda$ . We can find the subset of  $\lambda$  in  $V[\mathbb{G}]$  and this subset has a nice  $\mathbb{P}$ -name which is an element of  $V_\alpha^V$  since  $\mathbb{P} \in V_\alpha$ . This shows that  $a^* \in V_\alpha[\mathbb{G}]$  and hence also  $a \in V_\alpha[\mathbb{G}]$ . □ (Lemma 3.2)

**Proof of Theorem 3.1:** Assume that  $\kappa$  is tightly  $\mathcal{P}$ -Laver gen. ultrahuge for an iterable class  $\mathcal{P}$  of posets.

Suppose that  $\varphi = \varphi(\bar{x})$  is  $\Sigma_2$  formula (in  $\mathcal{L}_\in$ ),  $\bar{a} \in \mathcal{H}(\kappa)$ , and  $\mathbb{P} \in \mathcal{P}$  is such that

$$(3.2) \quad V \models \Vdash_{\mathbb{P}} \varphi(\bar{a}^\vee). \quad \text{x-Lg-RcA-a}$$

Let  $\lambda > \kappa$  be such that  $\mathbb{P} \in V_\lambda$  and

$$(3.3) \quad V_\lambda \prec_{\Sigma_n} V \text{ for a sufficiently large } n. \quad \text{x-Lg-RcA-0}$$

In particular, we may assume that we have chosen the  $n$  above so that a sufficiently large fragment of ZFC holds in  $V_\lambda$  (“sufficiently large” means here in terms of Lemma 3.2).

Let  $\mathbb{Q}$  be a  $\mathbb{P}$ -name such that  $\Vdash_{\mathbb{P}} \mathbb{Q} \in \mathcal{P}$ , and for  $(V, \mathbb{P} * \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $j$ ,  $M \subseteq V[\mathbb{H}]$  with

$$(3.4) \quad j : V \xrightarrow{\prec_\kappa} M, \quad \text{x-Lg-RcA-1}$$

$$(3.5) \quad j(\kappa) > \lambda, \quad \text{x-Lg-RcA-1-0}$$

$$(3.6) \quad \mathbb{P} * \mathbb{Q}, \mathbb{P}, \mathbb{H}, V_{j(\lambda)}^{V[\mathbb{H}]} \in M, \text{ and } \quad \text{x-Lg-RcA-1-1}$$

$$(3.7) \quad |\mathbb{P} * \mathbb{Q}| \leq j(\kappa). \quad \text{x-Lg-RcA-1-2}$$

By (3.7), we may assume that the underlying set of  $\mathbb{P} * \mathbb{Q}$  is  $j(\kappa)$  and  $\mathbb{P} * \mathbb{Q} \in V_{j(\lambda)}^V$ .

Let  $\mathbb{G} := \mathbb{H} \cap \mathbb{P}$ . Note that  $\mathbb{G} \in M$  by (3.6) and we have

Since  $V_{j(\lambda)}^M (= V_{j(\lambda)}^{V[\mathbb{H}]})$  satisfies a sufficiently large fragment of ZFC  
by elementarity of  $j$ , and hence the equality follows by Lemma 3.2

$$(3.8) \quad V_{j(\lambda)}^M \underset{\text{by (3.6)}}{=} V_{j(\lambda)}^{V[\mathbb{H}]} = V_{j(\lambda)}^V[\mathbb{H}]. \quad \text{x-Lg-RcA-2}$$

Thus, by (3.6) and by the definability of grounds, we have  $V_{j(\lambda)}^\vee \in M$  and  $V_{j(\lambda)}^\vee[\mathbb{G}] \in M$ .

**Claim 3.1.1**  $V_{j(\lambda)}^\vee[\mathbb{G}] \models \varphi(\bar{a})$ .

cl-Lg-RcA-0

$\vdash$  By Lemma 3.2,  $V_\lambda^\vee[\mathbb{G}] = V_\lambda^{\vee[\mathbb{G}]}$ , and  $V_{j(\lambda)}^\vee[\mathbb{G}] = V_{j(\lambda)}^{\vee[\mathbb{G}]}$  by (3.8). By (3.3), both  $V_\lambda^\vee[\mathbb{G}]$  and  $V_{j(\lambda)}^\vee[\mathbb{G}]$  satisfy large enough fragment of ZFC. Thus

$$(3.9) \quad V_\lambda^\vee[\mathbb{G}] \prec_{\Sigma_1} V_{j(\lambda)}^\vee[\mathbb{G}].$$

x-Lg-RcA-2-0

By (3.2) and (3.3), we have  $V_\lambda^\vee[\mathbb{G}] \models \varphi(\bar{a})$ . By (3.9) and since  $\varphi$  is  $\Sigma_2$ , it follows that  $V_{j(\lambda)}^\vee[\mathbb{G}] \models \varphi(\bar{a})$ .  $\dashv$  (Claim 3.1.1)

Thus we have

$$(3.10) \quad M \models \text{“there is a } \mathcal{P}\text{-ground } N \text{ of } V_{j(\lambda)} \text{ with } N \models \varphi(\bar{a})\text{”}.$$

x-Lg-RcA-3

By the elementarity (3.4), it follows that

$$(3.11) \quad V \models \text{“there is a } \mathcal{P}\text{-ground } N \text{ of } V_\lambda \text{ with } N \models \varphi(\bar{a})\text{”}.$$

x-Lg-RcA-4

Now by (3.3), it follows that there is a  $\mathcal{P}$ -ground  $W$  of  $V$  such that  $W \models \varphi(\bar{a})$ .

$\square$  (Theorem 3.1)

Some instances of weak forms of Recurrence Axioms decide the size of the continuum.

**Theorem 3.3** Assume that  $\mathcal{P}$  is an iterable class of posets. (1) If  $\mathcal{P}$  contains a poset which adds a real (over the universe), then  $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_1}$ -RcA implies  $\neg\text{CH}$ .

p-Lg-RcA-1

(2) Suppose that  $\mathcal{P}$  contains a poset which forces  $\aleph_2^\vee$  to be equinumerous with  $\aleph_1^\vee$ . Then  $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$ -RcA implies  $2^{\aleph_0} \leq \aleph_2$ .

(2') If  $\mathcal{P}$  contains a posets which forces  $\aleph_2^\vee$  to be equinumerous with  $\aleph_1^\vee$ , then  $(\mathcal{P}, \mathcal{H}((\aleph_2)^+))_{\Sigma_1}$ -RcA does not hold.

(3) If  $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_1}$ -RcA holds then all  $\mathbb{P} \in \mathcal{P}$  preserve  $\aleph_1$  and they are also stationary preserving.

(4) If  $\mathcal{P}$  contains a poset which adds a real as well as a poset which collapses  $\aleph_2^\vee$ , then  $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_1}$ -RcA implies  $2^{\aleph_0} = \aleph_2$ .

(5) If  $\mathcal{P}$  contains a poset which collapses  $\aleph_1^\vee$ , then  $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$ -RcA implies CH.

(5') If  $\mathcal{P}$  contains a poset which collapses  $\aleph_1^\vee$  then  $(\mathcal{P}, \mathcal{H}((2^{\aleph_0})^+))_{\Sigma_1}$ -RcA does not hold.

**Proof.** (1): Assume that  $\mathcal{P}$  is an iterable class of posets containing a poset  $\mathbb{P}$  adding a real and  $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_1}$ -RcA holds. If CH holds, then  $\mathcal{P}(\omega)^\vee \in \mathcal{H}(\kappa_{\text{refl}})$ . Hence “ $\exists x (x \subseteq \omega \wedge x \notin \mathcal{P}(\omega)^\vee)$ ” is a  $\Sigma_1$ -formula with parameters from  $\mathcal{H}(\kappa_{\text{refl}})$  and  $\mathbb{P}$  forces (the formula in forcing language corresponding to) this formula.

By  $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_1}$ -RcA, the formula must hold in a ground. This is a contradiction.

(2): Assume that  $(\mathcal{P}, \mathcal{H}(2^{\aleph_0})_{\Sigma_1})$ -RcA holds and  $\mathbb{P} \in \mathcal{P}$  forces  $\aleph_2^\vee$  to be equinumerous with  $\aleph_1^\vee$ . If  $2^{\aleph_0} > \aleph_2$  then  $\aleph_1^\vee, \aleph_2^\vee \in \mathcal{H}(2^{\aleph_0})$ . Letting  $\psi(x, y)$  a  $\Sigma_1$ -formula saying “ $\exists f (f \text{ is a surjection from } x \text{ to } y)$ ”, we have  $\Vdash_{\mathbb{P}} “\psi((\aleph_1^\vee)^\vee, (\aleph_2^\vee)^\vee)”$ .

By  $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$ -RcA, the formula  $\psi(\aleph_1^\vee, \aleph_2^\vee)$  must hold in a ground. This is a contradiction.

(2’): Assume that  $\mathbb{P} \in \mathcal{P}$  is such that  $\Vdash_{\mathbb{P}} “|\aleph_2^\vee| = |\aleph_1^\vee|”$ , and  $(\mathcal{P}, \mathcal{H}(\aleph_2^+))_{\Sigma_1}$ -RcA holds. Then, since  $\aleph_1, \aleph_2 \in \mathcal{H}(\aleph_2^+)$  and “ $|x| = |y|$ ” is  $\Sigma_1$ , there is a ground  $W$  of  $V$  such that  $W \models |\aleph_2^\vee| = |\aleph_1^\vee|$ . This is a contradiction.

(3): Suppose that  $\mathbb{P} \in \mathcal{P}$  is such that  $\Vdash_{\mathbb{P}} “\aleph_1^\vee \text{ is countable}”$ . Note that  $\omega, \aleph_1 \in \mathcal{H}(\kappa_{\text{refl}})$ . By  $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_1}$ -RcA, it follows that there is a ground  $W$  of  $V$  such that  $W \models “\aleph_1^\vee \text{ is countable}”$ . This is a contradiction.

Suppose now that  $S \subseteq \omega_1$  is stationary and  $\mathbb{P} \in \mathcal{P}$  destroys the stationarity of  $S$ . Note that  $\omega_1, S \in \mathcal{H}(\aleph_2)$ . Let  $\varphi = \varphi(y, z)$  be the  $\Sigma_1$ -formula

$$\exists x (x \text{ is a club subset of the ordinal } y \text{ and } z \cap x = \emptyset).$$

Then we have  $\Vdash_{\mathbb{P}} “\varphi(\omega_1, S)”$ . By  $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_1}$ -RcA, it follows that there is a ground  $W \subseteq V$  such that  $S \in W$  and  $W \models \varphi(\omega_1, S)$ . This is a contradiction to the stationarity of  $S$ .

(4): follows from (1), (2) and (3).

(5): If  $\aleph_1 < 2^{\aleph_0}$ , then  $\aleph_1^\vee \in \mathcal{H}(2^{\aleph_0})$ .

Let  $\mathbb{P} \in \mathcal{P}$  be a poset collapsing  $\aleph_1^\vee$ . That is,  $\Vdash_{\mathbb{P}} “\aleph_1^\vee \text{ is countable}”$ . Since “ $\dots$  is countable” is  $\Sigma_1$ , there is a ground  $M$  such that  $M \models “\aleph_1^\vee \text{ is countable}”$  by  $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$ -RcA. This is a contradiction.

(5’): Assume that  $\mathbb{P} \in \mathcal{P}$  is such that  $\Vdash_{\mathbb{P}} “\aleph_1^\vee \text{ is countable}”$ , and  $(\mathcal{P}, \mathcal{H}((2^{\aleph_0})^+))$ -RcA holds. Since  $\aleph_1 \in \mathcal{H}((2^{\aleph_0})^+)$ , it follows that there is a ground  $W$  of  $V$  such that  $W \models \aleph_1^\vee \text{ is countable}$ . This is a contradiction.  $\square$  (Theorem 3.3)

In contrast to Theorem 3.3,  $(ccc, \mathcal{H}(\kappa_{\text{refl}}))$ -RcA (or even  $(ccc, \mathcal{H}(\kappa_{\text{refl}}))$ -RcA $^+ \Leftrightarrow \text{MP}(ccc, \mathcal{H}(\kappa_{\text{refl}}))$ ) does not decide the size of the continuum (see Theorem 4.7 and Theorem 4.10).

Recurrence Axioms can be considered as natural requirements claiming that a reflection holds from the set-generic multiverse down to the set-generic archaeology. If we want the maximal amount of the Recurrence, we arrive at either (all posets,  $\mathcal{H}(2^{\aleph_0})$ )-RcA or (semi-proper posets,  $\mathcal{H}(\kappa_{\text{refl}})$ )-RcA according to Theorem 3.3, and these axioms imply CH or  $2^{\aleph_0} = \aleph_2$ , respectively. So in particular they are not compatible to each other.

In Section 7, we shall examine an axiom(scheme) which implies the full (semi-proper posets,  $\mathcal{H}(\kappa_{\text{refl}})$ )-RcA<sup>+</sup> and also a large fragment of (all posets,  $\mathcal{H}(\aleph_1)$ )-RcA<sup>+</sup> as well as MM<sup>++</sup>.

## 4 Tightly super- $C^{(\infty)}$ -Laver generic ultrahuge cardinal

In [17], it is shown that the existence of a (tightly)  $\mathcal{P}$ -Laver-gen. large cardinal c-infnty does not imply Maximality Principle even without parameters. The proof in [17] can be modified to prove the non-implication of  $(\mathcal{P}, \emptyset)_{\Pi_3}$ -RcA from generic large cardinals of various sorts, and this also shows that “ $(\mathcal{P}, \emptyset)_{\Sigma_2}$ -RcA<sup>+</sup>” in Theorem 3.1 is optimal.

In this section we show that the existence of a strong variant of  $\mathcal{P}$ -Laver generic large cardinal  $\kappa$  we are going to introduce below does imply  $\text{MP}(\mathcal{P}, \mathcal{H}(\kappa))$  (see Theorem 4.10). It is essential that the variant of Laver genericity (called “the tightly super  $C^{(\infty)}$ -Laver gen. large cardinal” below) is formulated not in a single formula but rather in an axiom scheme.

For a natural number  $n$ , we call a cardinal  $\kappa$  *super- $C^{(n)}$ -hyperhuge* if for any  $\lambda_0 > \kappa$  there are  $\lambda \geq \lambda_0$  with  $V_\lambda \prec_{\Sigma_n} V$ , and  $j, M \subseteq V$  such that  $j : V \xrightarrow{\prec_\kappa} M$ ,  $j(\kappa) > \lambda$ ,  $j(\lambda)M \subseteq M$  and  $V_{j(\lambda)} \prec_{\Sigma_n} V$ .

$\kappa$  is *super- $C^{(n)}$ -ultrahuge* if the condition above holds with “ $j(\lambda)M \subseteq M$ ” replaced by “ $j(\kappa)M \subseteq M$  and  $V_{j(\lambda)} \subseteq M$ ”.

If  $\kappa$  is super- $C^{(n)}$ -hyperhuge then it is super- $C^{(n)}$ -ultrahuge. This can be shown similarly to Lemma 5.1 in Section 5.

We shall also say that  $\kappa$  is *super- $C^{(\infty)}$ -hyperhuge* (*super  $C^{(\infty)}$ -ultrahuge*, resp.) if it is super- $C^{(n)}$ -hyperhuge (super  $C^{(n)}$ -ultrahuge, resp.) for all natural number  $n$ .

A similar kind of strengthening of the notions of large cardinals which we call here “super- $C^{(n)}$ ” appears also in Boney [3]. It is called in [3] “ $C^{(n)+}$ ” and the notion is considered there in connection with extendibility.

For a natural number  $n$  and an iterable class  $\mathcal{P}$  of posets, a cardinal  $\kappa$  is *super- $C^{(n)}$ - $\mathcal{P}$ -Laver-generically ultrahuge* (super  $C^{(n)}$ - $\mathcal{P}$ -Laver-gen. ultrahuge, for short) if, for any  $\lambda_0 > \kappa$  and for any  $\mathbb{P} \in \mathcal{P}$ , there are a  $\lambda \geq \lambda_0$  with  $V_\lambda \prec_{\Sigma_n} \mathbb{V}$ , a  $\mathcal{P}$ -name  $\mathbb{Q}$  with  $\Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{P}\text{”}$  such that for  $(\mathbb{V}, \mathbb{P} * \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $j, M \subseteq \mathbb{V}[\mathbb{H}]$  with  $j : \mathbb{V} \xrightarrow{\prec}_\kappa M$ ,  $j(\kappa) > \lambda$ ,  $\mathbb{P}, \mathbb{H}, V_{j(\lambda)}^{\mathbb{V}[\mathbb{H}]} \in M$  and  $V_{j(\lambda)}^{\mathbb{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathbb{V}[\mathbb{H}]$ .

A super- $C^{(n)}$ - $\mathcal{P}$ -Laver-generically ultrahuge cardinal  $\kappa$  is *tightly super- $C^{(n)}$ - $\mathcal{P}$ -Laver-generically ultrahuge* (tightly super- $C^{(n)}$ - $\mathcal{P}$ -Laver-gen. ultrahuge, for short), if  $|\mathbb{P} * \mathbb{Q}| \leq j(\kappa)$  (see Footnote 2).

*super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver-gen. ultrahugeness* and *tightly super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver gen. ultrahugeness* are defined similarly to super- $C^{(\infty)}$ -ultrahugeness.

Note that, in general, super- $C^{(\infty)}$ -hyperhugeness and super- $C^{(\infty)}$ -ultrahugeness are notions unformalizable in the language of ZFC without introducing a new constant symbol for  $\kappa$  since we need infinitely many  $\mathcal{L}_\in$ -formulas to formulate them. Exceptions are when we are talking about a cardinal in a set model being with one of these properties like in Lemma 4.5 below (and in such a case “natural number  $n$ ” actually refers to “ $n \in \omega$ ”), or when we are talking about a cardinal definable in  $\mathbb{V}$  having these properties in an inner model like in Theorem 5.8 or Corollary 5.9. In the latter case, the situation is formalizable with infinitely many  $\mathcal{L}_\in$ -sentences.

In contrast, the super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver gen. ultrahugeness of  $\kappa$  is expressible in infinitely many  $\mathcal{L}_\in$ -sentences. This is because a  $\mathcal{P}$ -Laver gen. large cardinal  $\kappa$  for relevant classes  $\mathcal{P}$  of posets is uniquely determined as  $\kappa_{\text{rcfl}}$  or  $2^{\aleph_0}$  (see e.g. [21] or [17]).

**Lemma 4.1** *Suppose that  $\kappa$  is super- $C^{(n)}$ -ultrahuge cardinal. Then we have  $V_\kappa \prec_{\Sigma_{n+1}} \mathbb{V}$ . In particular, in a context in which we can express that  $\kappa$  is super- $C^{(\infty)}$ -ultrahuge cardinal in a (set or class) model  $\mathbb{W}$ , we have  $V_\kappa^{\mathbb{W}} \prec \mathbb{W}$ .*

*p-Lg-RcA-1-a*

**Proof.** We check that  $V_\kappa$  passes Vaught’s test. Let  $\varphi(x, \bar{y})$  be a  $\Pi_n$ -formula. Suppose that  $\bar{b} \in V_\kappa$  and  $a \in \mathbb{V}$  are such that  $\mathbb{V} \models \varphi(a, \bar{b})$ . We want to show that there is  $a' \in V_\kappa$  such that  $\mathbb{V} \models \varphi(a', \bar{b})$ .

Let  $\lambda$  be such that

$$(4.1) \quad a \in V_\lambda,$$

*x-Lg-RcA-4-a-0*

$$(4.2) \quad V_\lambda \prec_{\Sigma_n} \mathbb{V},$$

*x-Lg-RcA-4-a-1*

$$(4.3) \quad \text{there are } j, M \subseteq \mathbb{V} \text{ such that}$$

*x-Lg-RcA-4-a-2*

$$(4.3a) \quad j : \mathbb{V} \xrightarrow{\prec}_\kappa M, \quad (4.3b) \quad j(\kappa) > \lambda, \quad (4.3c) \quad j^{(\kappa)} M \subseteq M, \quad \{x\text{-Lg-RcA-4-a-2}\}\{a\}$$

$$(4.3d) \quad V_{j(\lambda)} \subseteq M, \text{ and } (4.3e) \quad V_{j(\lambda)} \prec_{\Sigma_n} \mathbb{V}. \quad \{x\text{-Lg-RcA-4-a-2}\}\{d\}$$

By (4.3e), we have  $V_{j(\lambda)} \models \varphi(a, \bar{b})$ . By (4.3d), it follows that  $M \models \text{“}V_{j(\lambda)} \models \varphi(a, \bar{b})\text{”}$ .

Noting that  $\bar{b} = j(\bar{b})$ , it follows that

$$M \models \text{“there is } x \in V_{j(\kappa)} \text{ such that } V_{j(\lambda)} \models \varphi(x, j(\bar{b}))\text{”}.$$

By the elementarity (4.3a), it follows

$$\mathbb{V} \models \text{“there is } x \in V_\kappa \text{ such that } V_\lambda \models \varphi(x, \bar{b})\text{”}.$$

Let  $a' \in V_\kappa$  be a witness of this. Then by (4.2) we have  $\mathbb{V} \models \varphi(a', \bar{b})$ .  $\square$  (Lemma 4.1)

**Corollary 4.2** *Suppose that  $\kappa$  is a super- $C^{(\infty)}$ -ultrahuge (-hyperhuge, resp.) cardinal in a (set or class) model  $\mathbb{W}$ . Then for each  $n \in \mathbb{N}$ ,  $\mathbb{W} \models \text{“there are stationarily many super- $C^{(n)}$ -ultrahuge (-hyperhuge, resp.) cardinals”}.$*

*p-Lg-RcA-1-a-0*

**Proof.** Suppose that  $\kappa$  is a super- $C^{(\infty)}$ -ultrahuge cardinal in  $\mathbb{W}$ . By Lemma 4.1, we have  $V_\kappa^{\mathbb{W}} \prec \mathbb{W}$ . Suppose that  $\varphi = \varphi(x, y)$  is an  $\mathcal{L}_\in$ -formula and  $a \in V_\kappa^{\mathbb{W}}$  such that  $V_\kappa^{\mathbb{W}} \models \text{“}\varphi(\cdot, b) \text{ is a club } \subseteq \text{On”}$ . Then, by elementarity,  $\mathbb{W} \models \text{“}\varphi(\cdot, b) \text{ is a club } \subseteq \text{On”}$  and  $\mathbb{W} \models \varphi(\kappa, b)$ . Thus  $\mathbb{W} \models \text{“there is a super- $C^{(n)}$ -ultrahuge cardinal } \mu \text{ such that } \varphi(\mu, b)\text{”}$ . By elementarity, it follows that  $V_\kappa^{\mathbb{W}} \models \text{“there is a super- $C^{(n)}$ -ultrahuge cardinal } \mu \text{ such that } \varphi(\mu, b)\text{”}$ .

Since  $b$  was arbitrary, it follows that  $V_\kappa^{\mathbb{W}} \models \text{“}\forall y \text{ (if } \varphi(\cdot, y) \text{ is a club in On, then there is a super- $C^{(n)}$ -ultrahuge cardinal } \mu \text{ such that } \varphi(\mu, y))\text{”}$ . By elementarity, the same statement also holds in  $\mathbb{W}$ .  $\square$  (Corollary 4.2)

An ultrafilter  $U \subseteq \mathcal{P}(\mathcal{P}(\lambda^*))$  is said to be normal if

$$(4.4) \quad \{x \in \mathcal{P}(\lambda^*) : \alpha \in x\} \in U \text{ for all } \alpha \in \lambda^*, \text{ and}$$

*x-Lg-RcA-4-0*

$$(4.5) \quad \text{for any } \langle X_\alpha : \alpha < \lambda^* \rangle \in {}^{\lambda^*}U, \bigtriangleup_{\alpha < \lambda^*} X_\alpha \in U.$$

*x-Lg-RcA-4-1*

Under (4.4), the condition (4.5) is equivalent to

$$(4.6) \quad \text{For any } X \in U \text{ and any regressive}^3 f : X \rightarrow \lambda^*, \text{ there is } X' \in U \text{ such that } X' \subseteq X \text{ and } f \text{ is constant on } X'.$$

*x-Lg-RcA-4-2*

**Lemma 4.3** *For  $\kappa < \lambda < \kappa^* < \lambda^*$ , the following (A) and (B) are equivalent.*

*p-Lg-RcA-1-0*

(A) *there are  $j, M \subseteq \mathbb{V}$  such that  $j : \mathbb{V} \xrightarrow{\prec}_\kappa M$ ,  $j(\kappa) = \kappa^*$ ,  $j(\lambda) = \lambda^*$ ,  $j^{(\lambda)}M \subseteq M$ .*

(B) *there is a  $\kappa$ -complete normal ultrafilter  $U \subseteq \mathcal{P}(\mathcal{P}(\lambda^*))$  such that*

$$(4.7) \quad X^* := \{x \in \mathcal{P}(\lambda^*) : x \cap \kappa \in \kappa, \text{otp}(x \cap \kappa^*) = \kappa, \text{otp}(x) = \lambda\} \in U.$$

*x-Lg-RcA-4-3*

**A Sketch of the Proof:** (A)  $\Rightarrow$  (B): For  $j$  as in (A), the ultrafilter  $U$  defined by

$$(4.8) \quad U := \{X \in \mathcal{P}(\mathcal{P}(\lambda^*)) : j''\lambda^* \in j(X)\}$$

*x-Lg-RcA-4-3-0*

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<sup>3)</sup>  $f : X \rightarrow \lambda^*$  is *regressive* if  $f(x) \in x$  for all  $x \in X$ .

satisfies (B).

(B)  $\Rightarrow$  (A): For  $U$  as in (B), the elementary embedding  $j_U$  defined by

$$(4.9) \quad j_U : \mathbf{V} \xrightarrow{\sim}_{\kappa} N := mcol(\mathcal{P}(\lambda^*)\mathbf{V}/U); x \mapsto [c_x] \quad \text{x-Lg-RcA-4-4}$$

satisfies the conditions in (A).

Here,  $mcol$  denotes the Mostowski collapse,  $[f]$  the element of  $N$  which corresponds to the equivalence class of  $f \in \mathcal{P}(\lambda^*)\mathbf{V}$  and  $c_x$  the function on  $\mathcal{P}(\lambda^*)$  whose value is constantly  $x$ .  $\square$  (Lemma 4.3)

Similarly to the case of measurable cardinals or supercompact cardinals we can define another function  $k_j$  associated with  $j$  by:

$$(4.10) \quad k_j : N \rightarrow M; [f] \mapsto j(f)(j''\lambda^*). \quad \text{x-Lg-RcA-4-5}$$

**Lemma A 4.1** *Suppose that  $\kappa < \lambda < \kappa^* < \lambda^*$  and  $j : \mathbf{V} \xrightarrow{\sim}_{\kappa} M \subseteq \mathbf{V}$  are as in Lemma 4.3, (A). Let  $U$  be defined by (4.8). Then  $U$  satisfies the condition (4.8) in Lemma 4.3, (B).* p-Lg-RcA-1-0-0

Let  $j_U$  and  $k_j$  be defined as above. Then

- (1)  $k_j$  is well-defined.
- (2)  $k_j$  is an elementary embedding.
- (3)  $k_j \circ j_U = j$ .
- (4)  $k_j \upharpoonright \mathcal{H}((\lambda^*)^+) = id_{\mathcal{H}((\lambda^*)^+)}$ .
- (5)  $j \upharpoonright \mathcal{H}(\lambda^+) = j_U \upharpoonright \mathcal{H}(\lambda^+)$ .

In the following we shall use a (local) notation “ $(\kappa, \lambda)^{(\kappa^*, \lambda^*)}$ ” to denote the condition (B) in Lemma 4.3. Clearly we have

**Lemma 4.4** *For a cardinal  $\kappa$  and a natural number  $n$ ,  $\kappa$  is a super- $C^{(n)}$ -hyperhuge cardinal  $\Leftrightarrow$  for any  $\lambda_0 > \kappa$  there are  $\lambda^* > \kappa^* > \lambda \geq \lambda_0$  such that* p-Lg-RcA-1-1

(C)  $V_\lambda \prec_{\Sigma_n} \mathbf{V}$ ,  $V_{\lambda^*} \prec_{\Sigma_n} \mathbf{V}$ , and  $(\kappa, \lambda)^{(\kappa^*, \lambda^*)}$ .  $\square$

We shall denote the condition (C) by  $[\kappa, \lambda]^{[\kappa^*, \lambda^*, n]}$ . This is also merely a local notation.

**Lemma 4.5** *If  $\kappa$  is 2-huge with the 2-huge elementary embedding  $j$ , that is, there is  $M \subseteq \mathbf{V}$  such that  $j : \mathbf{V} \xrightarrow{\sim}_{\kappa} M \subseteq \mathbf{V}$ , and* p-Lg-RcA-2

$$(4.11) \quad j^2(\kappa)M \subseteq M, \quad \text{x-Lg-RcA-5}$$



then  $V_{j(\kappa)} \models \text{"}\kappa \text{ is super-}C^{(\infty)}\text{-hyperhuge cardinal"}$ , and for each  $n \in \omega$ ,  $V_{j(\kappa)} \models \text{"there are stationarily many super-}C^{(n)}\text{-hyperhuge cardinals"}$ .

**Proof.** Suppose that  $j$  is as above and  $n \in \omega$ .

By elementarity of  $j$  and since  $\kappa = \text{crit}(j)$ , we have  $V_\kappa \prec V_{j(\kappa)}^M$ . By (4.11), it follows that

$$(4.12) \quad V_\kappa \prec V_{j(\kappa)}. \quad \text{x-Lg-RcA-6}$$

By elementarity of  $j$ , this implies  $M \models \text{"}V_{j(\kappa)} \prec V_{j(j(\kappa))}\text{"}$ . By the closedness (4.11) of  $M$  (and since  $j(j(\kappa))$  is inaccessible), it follows that

$$(4.13) \quad V_{j(\kappa)} \prec V_{j(j(\kappa))}. \quad \text{x-Lg-RcA-6-a}$$

For a  $\lambda_0$  with  $\kappa < \lambda_0 < j(\kappa)$ , let  $\lambda_0 \leq \lambda < j(\kappa)$  be such that

$$(4.14) \quad V_\lambda \prec_{\Sigma_n} V_{j(\kappa)}. \quad \text{x-Lg-RcA-6-0}$$

By (4.13), it follows that

$$(4.15) \quad V_\lambda \prec_{\Sigma_n} V_{j(j(\kappa))}. \quad \text{x-Lg-RcA-6-1}$$

We also have  $M \models \text{"}V_{j(\lambda)} \prec_{\Sigma_n} V_{j(j(\kappa))}\text{"}$  by (4.14) and by elementarity of  $j$ . By the closedness (4.11) (and since  $j(j(\kappa))$  is inaccessible), it follows that

$$(4.16) \quad V_{j(\lambda)} \prec_{\Sigma_n} V_{j(j(\kappa))}. \quad \text{x-Lg-RcA-6-0-0}$$

Let

$$(4.17) \quad U := \{X \subseteq \mathcal{P}(j(\lambda)) : j''j(\lambda) \in j(X)\}. \quad \text{x-Lg-RcA-7}$$

The following is easy to check:

**Claim 4.5.1** *In  $V_{j(j(\kappa))}$ ,  $U$  witnesses  $(\kappa, \lambda)^{(j(\kappa), j(\lambda))}$ . Hence we have  $V_{j(j(\kappa))} \models \text{"}[\kappa, \lambda]^{[j(\kappa), j(\lambda), n]}\text{"}$  for all  $n \in \omega$ .*

$\vdash$  Clearly  $U$  is a ultrafilter.

$U$  is  $\kappa$ -complete: Suppose that  $\langle X_\xi : \xi < \mu \rangle \in {}^\mu U$  for some  $\mu < \kappa$ . Then  $j(\langle X_\xi : \xi < \mu \rangle) = \langle j(X_\xi) : \xi < \mu \rangle$  by the elementarity of  $j$  and  $\mu < \text{crit}(j)$ . Since  $X_\xi \in U$ , we have  $j''\lambda \in j(X_\xi)$  for all  $\xi$  by the definition (4.17) of  $U$ . It follows that  $j''j(\lambda) \in \bigcap_{\xi < \mu} j(X_\xi) = j(\bigcap_{\xi < \mu} X_\xi)$ . Thus  $\bigcup_{\xi < \mu} X_\xi \in U$ .

$U \models (4.4)$ : For  $\alpha < j(\lambda)$ ,  $j(\{x \in \mathcal{P}(j(\lambda)) : \alpha \in x\}) = \{x \in \mathcal{P}(j(j(\lambda))) : j(\alpha) \in x\} \ni j''j(\lambda)$ . Hence  $\{x \in \mathcal{P}(j(\lambda)) : \alpha \in x\} \in U$ .

$U \models (4.5)$ : Suppose that  $\vec{X} := \langle X_\alpha : \alpha < j(\lambda) \rangle \in {}^{j(\lambda)}U$ . Then, by elementarity of  $j$ ,  $j(\Delta_{\alpha < j(\lambda)} X_\alpha) = \{x \in \mathcal{P}(j(j(\lambda))) : \forall \alpha \in x (x \in (j(\vec{X}))_\alpha)\}$ . For  $\alpha \in j''j(\lambda)$ ,  $j''j(\lambda) \in j(X_\alpha) = (j(\vec{X}))_\alpha$  by the definition (4.17) of  $U$ . It follows that  $j''j(\lambda) \in j(\Delta_{\alpha < j(\lambda)} X_\alpha)$ . This means  $\Delta_{\alpha < j(\lambda)} X_\alpha \in U$  by the definition (4.17) of  $U$ .

$X^* \in U$ : Note that

$$j(X^*) = \{x \in \mathcal{P}(j(j(\lambda))) : x \cap j(\kappa) \in j(\kappa), \text{otp}(x \cap j(j(\kappa))) = j(\kappa), \\ \text{otp}(x) = j(\lambda)\}$$

by elementarity of  $j$ . Hence  $j''j(\lambda) \in j(X^*)$  and thus  $X^* \in U$ .

This shows  $V_{j(j(\kappa))} \models “(\kappa, \lambda)^{(j(\kappa), j(\lambda))}”$ . By (4.15) and (4.16), it follows that  $V_{j(j(\kappa))} \models “[\kappa, \lambda]^{[j(\kappa), j(\lambda), n]}”$  ⊢ (Claim 4.5.1)

By Claim 4.5.1,  $V_{j(j(\kappa))} \models \exists x \exists y ([\kappa, \lambda]^{[x, y, n]})$  for all  $n$ . By (4.13), it follows that  $V_{j(j(\kappa))} \models \exists x \exists y ([\kappa, \lambda]^{[x, y, n]})$ . Since  $n \in \omega$  and  $\kappa < \lambda_0 < j(\kappa)$  were arbitrary, it follows that  $V_{j(j(\kappa))} \models “\kappa$  is super- $C^{(\infty)}$ -hyperhuge”.

For a fixed  $n$  and club  $C \subseteq j(\kappa)$ , we have  $V_{j(j(\kappa))} \models “\kappa \in j(C)$  and  $\kappa$  is super- $C^{(n)}$ -hyperhuge”. Thus  $V_{j(j(\kappa))} \models “\exists x (x \in j(C)$  and  $x$  is super- $C^{(n)}$ -hyperhuge)”. It follows that  $V_{j(j(\kappa))} \models “\exists x (x \in C$  and  $x$  is super- $C^{(n)}$ -hyperhuge)”. This shows that  $V_{j(j(\kappa))} \models “\text{there are stationarily many super-}C^{(n)}\text{-hyperhuge cardinals}”$ .  $\square$  (Lemma 4.5)

**Theorem 4.6** (Laver function for a super- $C^{(n)}$ -hyperhuge cardinal) *Suppose that  $\mu$  is an inaccessible cardinal and  $\kappa$  is super- $C^{(\infty)}$ -hyperhuge in  $V_\mu$ . Then there is  $f : \kappa \rightarrow V_\kappa$  such that for any  $n \in \omega$ ,  $a \in V_\mu$ , and  $\lambda_0 \geq \kappa$ , there are  $\lambda_0 < \lambda < \kappa^* < \lambda^* < \mu$  such that  $\kappa < \lambda$ ,  $|trcl(a)| \leq \lambda^*$ ,*

*p-Lg-RcA-3*

$$(4.18) \quad V_\mu \models [\kappa, \lambda]^{[\kappa^*, \lambda^*, n]},$$

*x-Lg-RcA-7-a*

and there is a ultrafilter  $U \subseteq \mathcal{P}(\mathcal{P}(\lambda^*))$  witnessing (4.18) such that  $j_U(f)(\kappa) = a$  where  $j_U$  is given by (4.9).<sup>4)</sup>

**Proof.** The proof is just an adaptation of the proof of Theorem 20.21 in [33].

Suppose toward a contradiction, that the Theorem does not hold. Then for each  $f : \kappa \rightarrow V_\kappa$ , there are  $a_f$ ,  $n_f \in \omega$  and  $\kappa < \lambda_f < \kappa_f^* < \lambda_f^* < \mu$  such that

$$(4.19) \quad |trcl(a_f)| \leq \lambda_f^* (\Leftrightarrow a_f \in \mathcal{H}((\lambda_f^*)^+));$$

*x-Lg-RcA-7-a-0*

$$(4.20) \quad V_\mu \models [\kappa, \lambda_f]^{[\kappa_f^*, \lambda_f^*, n_f]}; \text{ but}$$

*x-Lg-RcA-7-a-1*

$$(4.21) \quad \text{there is no witness } U \text{ of (4.20) with } j_U(f)(\kappa) = a_f.$$

*x-Lg-RcA-7-a-2*

We assume that, for each  $f : \kappa \rightarrow V_\kappa$ ,  $\langle \lambda_f, \kappa_f^*, \lambda_f^*, n_f \rangle$  is chosen to be the minimal possible (with respect to the lexicographical ordering) among those which satisfy (4.19)  $\sim$  (4.21) together with  $f$  and some  $a_f$ .

For a  $\alpha \in \text{On}$ , and  $g : \alpha \rightarrow V_\alpha$ , let  $(*)_{\alpha, g}$  be the assertion

$$(4.22) \quad \alpha \text{ is a cardinal and there are } \alpha < \delta < \alpha^* < \delta^* < \mu, n \in \omega \text{ and } a \text{ such that}$$

*x-Lg-RcA-7-a-3*

<sup>4)</sup> In particular, we have  $j_U(\kappa) = \kappa^*$ ,  $j_U(\lambda) = \lambda^*$ ,  $V_\lambda \prec_{\Sigma_n} V$  and  $V_{\lambda^*} (= V_{j(\lambda)}) \prec_{\Sigma_n} V$ .

$$(4.22a) \quad V_\mu \models [\alpha, \delta]^{[\alpha^*, \delta^*, n]}, \quad a \in \mathcal{H}((\delta^*)^+) \quad \{x\text{-Lg-RcA-7-a-3}\} \{a\}$$

but there is no witness  $U$  of (4.22a) such that  $j_U(g)(\alpha) = a$ .

Let  $f^\dagger : \kappa \rightarrow V_\kappa$  be defined recursively such that for  $\alpha < \kappa$ , if  $(*)_{\alpha, f^\dagger \upharpoonright \alpha}$  holds, then  $f^\dagger(\alpha)$  witnesses  $(*)_{\alpha, f^\dagger \upharpoonright \alpha}$  as  $a$  in (4.22) together with  $\langle \delta, \alpha^*, \delta^*, n \rangle$  which is chosen to be minimal possible (with respect to the lexicographic ordering). Otherwise, we let  $f(\alpha) = \emptyset$ .

Let

$$(4.23) \quad a^\dagger := j_{U_{f^\dagger}}(f^\dagger)(\kappa) \quad x\text{-Lg-RcA-7-a-4}$$

where  $U_{f^\dagger}$  is a witness of  $[\kappa, \lambda_{f^\dagger}]^{[\kappa_{f^\dagger}^*, \lambda_{f^\dagger}^*, n_{f^\dagger}]}$ . By definition of  $f^\dagger$ , elementarity and the assumption of this indirect proof,  $a^\dagger$  together with  $\kappa_{f^\dagger}^*, \lambda_{f^\dagger}^*, n_{f^\dagger}$  is a witness of  $(*)_{\alpha, j_{U_{f^\dagger}}(f^\dagger) \upharpoonright \kappa}$ . But the existence of the ultrafilter  $U_{f^\dagger}$  in (4.23) is a contradiction to  $\underbrace{(\alpha, j_{U_{f^\dagger}}(f^\dagger) \upharpoonright \kappa)}_{= f^\dagger}$  this.

□ (Theorem 4.6)

In [21] and further in [22], [23], [25], we studied in connection with Laver genericity, only classes  $\mathcal{P}$  of posets which are all stationary preserving. This is because the existence of a Laver-gen. large cardinal for one of such classes of posets naturally extends the known reflection properties down to  $< \kappa_{\text{refl}}$ , and strong versions of forcing axioms.

However, we can also consider the class of all posets in connection with Laver genericity. (5) in the following Theorem 4.7 is such an instance of Laver genericity. We shall go more into this scenario in Section 6.

**Theorem 4.7** (1) *Suppose that  $\mu$  is inaccessible and  $\kappa < \mu$  is super- $C^{(\infty)}$ -ultrahuge in  $V_\mu$ . Let  $\mathbb{P} = \text{Col}(\aleph_1, \kappa)$ . Then, in  $V_\mu[\mathbb{G}]$ , for any  $V_\mu, \mathbb{P}$ -generic  $\mathbb{G}$ ,  $\aleph_2^{V_\mu[\mathbb{G}]} (= \kappa)$  is tightly super- $C^{(\infty)}$ - $\sigma$ -closed-Laver-gen. ultrahuge and CH holds.* p-Lg-RcA-4

(2) *Suppose that  $\mu$  is inaccessible and  $\kappa < \mu$  is super- $C^{(\infty)}$ -ultrahuge with a Laver function  $f : \kappa \rightarrow V_\kappa$  for super- $C^{(\infty)}$ -ultrahugeness in  $V_\mu$ . If  $\mathbb{P}$  is the CS-iteration of length  $\kappa$  for forcing PFA along with  $f$ , then, in  $V_\mu[\mathbb{G}]$  for any  $(V_\mu, \mathbb{P})$ -generic  $\mathbb{G}$ ,  $\aleph_2^{V_\mu[\mathbb{G}]} (= \kappa)$  is tightly super- $C^{(\infty)}$ -proper-Laver-gen. ultrahuge and  $2^{\aleph_0} = \aleph_2$  holds.*

(2') *Suppose that  $\mu$  is inaccessible and  $\kappa < \mu$  is super- $C^{(\infty)}$ -ultrahuge with a Laver function  $f : \kappa \rightarrow V_\kappa$  for super- $C^{(\infty)}$ -ultrahugeness in  $V_\mu$ . If  $\mathbb{P}$  is the RCS-iteration of length  $\kappa$  for forcing MM along with  $f$ , then, in  $V_\mu[\mathbb{G}]$  for any  $(V_\mu, \mathbb{P})$ -generic  $\mathbb{G}$ ,  $\aleph_2^{V_\mu[\mathbb{G}]} (= \kappa)$  is tightly super- $C^{(\infty)}$ -semi-proper-Laver-gen. ultrahuge and  $2^{\aleph_0} = \aleph_2$  holds.*

(3) *Suppose that  $\mu$  is inaccessible and  $\kappa$  is super- $C^{(\infty)}$ -ultrahuge with a Laver*

function  $f : \kappa \rightarrow V_\kappa$  for super- $C^{(\infty)}$ -ultrahugeness in  $V_\mu$ . If  $\mathbb{P}$  is a FS-iteration of length  $\kappa$  for forcing MA along with  $f$ , then, in  $V_\mu[\mathbb{G}]$  for any  $(V_\mu, \mathbb{P})$ -generic  $\mathbb{G}$ ,  $2^{\aleph_0} (= \kappa)$  is tightly super- $C^{(\infty)}$ -c.c.c.-Laver-gen. ultrahuge, and  $2^{\aleph_0}$  is very large in  $V_\mu[\mathbb{G}]$ .

(4) Suppose that  $\mu$  is inaccessible and  $\kappa$  is super- $C^{(\infty)}$ -ultrahuge with a Laver function  $f : \kappa \rightarrow V_\kappa$  for super- $C^{(\infty)}$ -ultrahugeness in  $V_\mu$ . If  $\mathbb{P}$  is a FS-iteration of length  $\kappa$  along with  $f$  enumerating “all” posets, then, in  $V_\mu[\mathbb{G}]$  for any  $(V_\mu, \mathbb{P})$ -generic  $\mathbb{G}$ ,  $2^{\aleph_0} (= \aleph_1)$  is tightly super- $C^{(\infty)}$ -all posets-Laver-gen. ultrahuge, and CH holds.<sup>5)</sup>

**Proof.** The proof can be done similarly to that of Theorem 5.2 in [21] using Lemma 4.8 below. In the following we shall only check the case (4).

Suppose that  $f : \kappa \rightarrow V_\kappa$  is a super- $C^{(\infty)}$ -hyperhuge Laver function as in Theorem 4.6.

Let  $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$  be a FS-iteration defined by

$$(4.24) \quad \mathbb{Q}_\beta = \begin{cases} f(\alpha), & \text{if } f(\alpha) \text{ is a } \mathbb{P}_\beta\text{-name of a poset;} \\ \{\mathbb{1}\}, & \text{otherwise} \end{cases}$$

p-Lg-RcA-4-a-0

for  $\beta < \kappa$ .

Let  $\mathbb{G}$  be a  $(V_\mu, \mathbb{P}_\kappa)$ -generic filter. Clearly  $V_\mu[\mathbb{G}] \models “2^{\aleph_0} = \kappa = \aleph_1”$ . We show that  $\kappa$  is tightly super- $C^{(\infty)}$ -all posets-Laver-gen. ultrahuge in  $V_\mu[\mathbb{G}]$ .

Suppose that  $\mathbb{P}$  is a poset in  $V_\mu[\mathbb{G}]$ ,  $\kappa < \lambda_0$  and  $n \in \omega$ . Let  $n' > n$  be sufficiently large and let  $\mathbb{P}$  be a  $\mathbb{P}_\kappa$ -name of  $\mathbb{P}$ .

Working in  $V_\mu$ , we can find

$$(4.25) \quad |\mathbb{P}| < \lambda < \kappa^* < \lambda^* \text{ and } j, M \subseteq V$$

x-Lg-RcA-6-1-a

such that

$$(4.26) \quad j : V \xrightarrow{\sim}_\kappa M,$$

x-Lg-RcA-6-1-0

$$(4.27) \quad j(\kappa) = \kappa^*, j(\lambda) = \lambda^*,$$

x-Lg-RcA-6-1-1

$$(4.28) \quad \lambda^* M \subseteq M,$$

x-Lg-RcA-6-1-2

$$(4.29) \quad V_\lambda \prec_{\Sigma_{n'}} V, V_{\lambda^*} \prec_{\Sigma_{n'}} V, \text{ and}$$

x-Lg-RcA-6-1-3

$$(4.30) \quad j(f)(\kappa) = \mathbb{P}$$

x-Lg-RcA-6-1-4

by definition of  $f$ .

By elementarity (and by the definition (4.24) of  $\mathbb{P}_\kappa$ ),

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<sup>5)</sup> Actually we can obtain a model with the desired property simply by Levy collapsing  $\kappa$  to  $\omega_1$ . We just chose this narrative to emphasize the parallelism to the cases (2), (2') and (3).

$$(4.31) \quad j(\mathbb{P}_\kappa) \sim_{\mathbb{P}_\kappa} (\mathbb{P}_\kappa * \mathbb{P}) * \mathbb{R}$$

x-Lg-RcA-6-2

for a  $(\mathbb{P}_\kappa * \mathbb{P})$ -name  $\mathbb{R}$  of a poset. Note that  $(\mathbb{P}_\kappa * \mathbb{P} * \mathbb{R})/\mathbb{G}$  corresponds to a poset of the form  $\mathbb{P} * \mathbb{Q}$ .

Let  $\mathbb{H}^*$  be  $(\mathbb{V}, (\mathbb{P}_\kappa * \mathbb{P}) * \mathbb{R})$ -generic filter with  $\mathbb{G} \subseteq \mathbb{H}^*$ .  $\mathbb{H}^*$  corresponds to a  $(\mathbb{V}, j(\mathbb{P}_\kappa))$ -generic filter  $\mathbb{H} \supseteq \mathbb{G}$  via the equivalence (4.31).

Let  $\tilde{j}$  be defined by

$$(4.32) \quad \tilde{j} : \mathbb{V}[\mathbb{G}] \rightarrow M[\mathbb{H}]; \quad \dot{a}^\mathbb{G} \mapsto j(\dot{a})^\mathbb{H}$$

x-Lg-RcA-6-3

for all  $\mathbb{P}_\kappa$ -names  $\dot{a}$ .

A standard proof shows that  $f$  is well-defined, and  $j : \mathbb{V}[\mathbb{G}] \xrightarrow{\sim}_\kappa M[\mathbb{H}]$ . By (4.27) and (4.28), we have  $\tilde{j}''\tilde{j}(\lambda) = j''j(\lambda) \in M[\mathbb{H}]$ . Since  $\mathbb{H} \in M[\mathbb{H}]$ , the  $(\mathbb{V}[\mathbb{G}], \mathbb{P} * \mathbb{Q})$ -generic filter corresponding to  $\mathbb{H}$  is also in  $M[\mathbb{H}]$ .

By (4.25), (4.29), by the choice of  $n'$ , and by Lemma 4.8, (1), we have  $V_\lambda^{\mathbb{V}[\mathbb{G}]} \prec_{\Sigma_n} \mathbb{V}[\mathbb{G}]$  and  $V_{\tilde{j}(\lambda)}^{\mathbb{V}[\mathbb{H}]} = V_{\lambda^*}^{\mathbb{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathbb{V}[\mathbb{H}]$ .

Since  $\mathbb{P}$  and  $n$  were arbitrary, this shows that  $\kappa$  is tightly super- $C^{(\infty)}$ -all posets-Laver-gen. ultrahuge in  $V_\mu[\mathbb{G}]$ .  $\square$  (Theorem 4.7)

**Lemma 4.8** (1) *For a natural number  $n$ , if  $n' > n$  is sufficiently large and*

p-Lg-RcA-4-0

$$(4.33): V_\lambda \prec_{\Sigma_{n'}} \mathbb{V},$$

x-Lg-RcA-7-0

*then we have  $V_\lambda[\mathbb{G}] \prec_{\Sigma_n} \mathbb{V}[\mathbb{G}]$  for any poset  $\mathbb{P} \in V_\lambda$  and  $(\mathbb{V}, \mathbb{P})$ -generic  $\mathbb{G}$ .*

(2) *For a natural number  $n$ , if  $n' > n$  is sufficiently large and*

$$(4.34): V_\lambda[\mathbb{G}] \prec_{\Sigma_{n'}} \mathbb{V}[\mathbb{G}] \text{ for some poset } \mathbb{P} \in V_\lambda \text{ and } (\mathbb{V}, \mathbb{P})\text{-generic } \mathbb{G},$$

x-Lg-RcA-7-1

*then we have  $V_\lambda \prec_{\Sigma_n} \mathbb{V}$ .*

**Proof.** (1): Suppose that  $\bar{a} \in V_\lambda[\mathbb{G}]$  and  $\varphi = \varphi(\bar{x})$  is a  $\Sigma_n$ -formula. There are  $\mathbb{P}$ -names  $\bar{a} \in V_\lambda$  such that  $\bar{a} = \bar{a}[\mathbb{G}]$ .

If  $V_\lambda[\mathbb{G}] \models \varphi(\bar{a})$ , there is  $\mathbb{p} \in \mathbb{G}$  such that  $V_\lambda \models \mathbb{p} \Vdash_{\mathbb{P}} \varphi(\bar{a})$ . By (4.33) it follows that  $\mathbb{V} \models \mathbb{p} \Vdash_{\mathbb{P}} \varphi(\bar{a})$ . Thus  $\mathbb{V}[\mathbb{G}] \models \varphi(\bar{a})$ .

The same argument also applies to  $\neg\varphi$ .

(2): We use the  $\mathcal{L}_\in$ -formula  $\Phi(x, y)$  of Theorem 2.1. By (4.34), there is  $r \in V_\lambda[\mathbb{G}]$  such that

$$V_\lambda = \Phi(\cdot, r)^{V_\lambda[\mathbb{G}]} = \Phi(\cdot, r)^{\mathbb{V}[\mathbb{G}]} \cap V_\lambda[\mathbb{G}] \subseteq \Phi(\cdot, r)^{\mathbb{V}[\mathbb{G}]}.$$

For any  $\Sigma_n$ -formula  $\varphi(\bar{x})$  and  $\bar{a} \in \Phi(\cdot, r)^{V_\lambda[\mathbb{G}]}$ . Since  $\varphi^{\Phi(\cdot, r)}$  is a  $\Sigma_{n'}$ -formula (by the choice of  $n'$ ), we have

$$V_\lambda \models \varphi(\bar{a}) \Leftrightarrow V_\lambda[\mathbb{G}] \models \varphi^{\Phi(\cdot, r)}(\bar{a}) \overset{\text{by (4.34)}}{\Leftrightarrow} \mathbb{V}[\mathbb{G}] \models \varphi^{\Phi(\cdot, r)}(\bar{a}) \Leftrightarrow \mathbb{V} \models \varphi(\bar{a}).$$

This shows that  $V_\lambda \prec_{\Sigma_n} V$ .

□ (Lemma 4.8)

In Theorem 4.7, (4), the model constructed in the proof satisfies CH. The next lemma suggests that this does not depend on the specific construction of the model given there.

In the following, we consider a strengthening of tightness of Laver genericity: we say a cardinal  $\kappa$  is tightly<sup>+</sup>  $\mathcal{P}$ -Laver-gen. hyperhuge/ultrahuge if the definition of the tightly  $\mathcal{P}$ -Laver gen. hyperhugeness/ultrahugeness of  $\kappa$  holds with the clause “ $|\mathbb{P} * \mathbb{Q}| \leq j(\kappa)$ ” replaced by “there is a complete Boolean algebra  $\mathbb{B}$  of size  $j(\kappa)$  such that  $\mathbb{P} * \mathbb{Q} \sim \mathbb{B}$ ”. Note that this condition is equivalent to  $2^{<\kappa} = \kappa$ ,  $|\mathbb{P} * \mathbb{Q}| \leq \kappa$  and  $\mathbb{P} * \mathbb{Q}$  has the  $j(\kappa)$ -cc. Note also that all models constructed in the proof of Theorem 4.7 actually satisfy this stronger notion of tightness of the (super- $C^{(n)}$ )  $\mathcal{P}$ -Laver-gen. hyperhugeness or ultrahugeness of  $\kappa$ . Later with much more work we show in Corollary 6.3 that this stronger version of tightness can be dropped from the following Lemma.

**Lemma 4.9** *If  $\mathcal{P}$  is the class of all posets and  $\kappa$  is tightly<sup>+</sup>  $\mathcal{P}$ -Laver-gen. hyperhuge cardinal then  $\kappa = 2^{\aleph_0} = \aleph_1$ .*

p-Lg-RcA-4-1

**Proof.**  $\kappa \leq 2^{\aleph_0}$  holds even without tightness: see e.g. Lemma 3.7 in [17].

To show  $2^{\aleph_0} \leq \kappa$ , let  $\lambda$  and  $\mathbb{Q}$  be such that

(4.35)  $\lambda > 2^{\aleph_0}$ ,  $\kappa$  and  $\lambda$  is large enough such that SCH holds above some  $\mu < \lambda$  (this is possible by Corollary 5.4, (2) in Section 5, and we need here the Laver gen. hyperhugeness of  $\kappa$ ),

x-Lg-RcA-8-a-0

(4.36)  $\mathbb{Q}$  is positive elements of a complete Boolean algebra, and,

x-Lg-RcA-8-a-1

(4.37) for  $(V, \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $j, M \subseteq V[\mathbb{H}]$  such that (1)  $j : V \xrightarrow{\prec_\kappa} M$ , (2)  $j(\kappa) > \lambda$ , (3)  $|\mathbb{Q}| \leq j(\kappa)$ , and (4)  $V_{j(\lambda)}^{V[\mathbb{H}]} \subseteq M$ .

x-Lg-RcA-8-a-2

By (4.36), each  $\mathbb{Q}$ -name  $\dot{r}$  of a real corresponds to a mapping  $f : \omega \rightarrow \mathbb{Q}$ . By (4.35) and by (4.37), (3), there are at most  $j(\kappa)$  many such mappings. Thus we have  $V[\mathbb{H}] \models “2^{\aleph_0} \leq j(\kappa)”$ . By (4.37), (4), it follows  $M \models “2^{\aleph_0} \leq j(\kappa)”$ . By elementarity, it follows that  $V \models “2^{\aleph_0} \leq \kappa”$ .

$\kappa = \aleph_1$ : Otherwise  $\aleph_1 < \kappa$ . In  $V$ , let  $\mathbb{P}$  be the standard poset collapsing  $\aleph_1$  to be countable. Let  $\mathbb{Q}$  be a  $\mathbb{P}$ -name such that for  $(V, \mathbb{P} * \mathbb{Q})$ -generic  $\mathbb{H}$  there are  $j, M \subseteq V[\mathbb{H}]$  such that  $j : V \xrightarrow{\prec_\kappa} M$ , and  $\mathbb{P}, \mathbb{H} \in M$ .

By  $\mathbb{H} \cap \mathbb{P} \in M$ , we have  $M \models “\aleph_1^V = j(\aleph_1^V)$  is countable”. This is a contradiction to the elementarity of  $j$ .

□ (Lemma 4.9)

Recall that, for an iterable  $\mathcal{P}$ ,  $(\mathcal{P}, \mathcal{H}(\kappa))$ -RcA<sup>+</sup> holds if and only if  $\text{MP}(\mathcal{P}, \mathcal{H}(\kappa))$  holds (Proposition 2.2, (1)).

**Theorem 4.10** *Suppose that  $\mathcal{P}$  is an iterable class of posets and  $\kappa$  is super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver-gen. ultrahuge. Then  $(\mathcal{P}, \mathcal{H}(\kappa))$ -RcA<sup>+</sup> holds.*

*p-Lg-RcA-5*

**Proof.** A modification of the proof of Theorem 3.1 works.

Suppose that  $\kappa$  is super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver-gen. ultrahuge,  $\mathbb{P} \in \mathcal{P}$ , and  $\Vdash_{\mathbb{P}} \text{“}\varphi(\bar{a}^\vee)\text{”}$  for an  $\mathcal{L}_\in$ -formula  $\varphi$  and  $\bar{a} \in \mathcal{H}(\kappa)$ . We want to show that  $\varphi(\bar{a})$  holds in some  $\mathcal{P}$ -ground of  $\mathbb{V}$ .

Let  $n$  be a sufficiently large natural number such that the following arguments go through. In particular, we assume that  $V_\alpha^\vee \prec_{\Sigma_n} \mathbb{V}$  implies that “ $\varphi(\bar{x})$ ” and “ $\Vdash \text{“}\varphi(\bar{x}^\vee)\text{”}$ ” are absolute between  $V_\alpha^\vee$  and  $\mathbb{V}$ , and  $V_\alpha^\vee \prec_{\Sigma_n} \mathbb{V}$  also implies that a sufficiently large fragment of ZFC holds in  $V_\alpha$ .

Let  $\mathbb{Q}$  be a  $\mathbb{P}$ -name such that  $\Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{P}\text{”}$  and, for  $(\mathbb{V}, \mathbb{P} * \mathbb{Q})$ -generic  $\mathbb{H}$ , there are a  $\lambda > \kappa$  with

$$(4.38) \quad V_\lambda \prec_{\Sigma_n} \mathbb{V},$$

*x-Lg-RcA-9*

and  $j, M \subseteq \mathbb{V}[\mathbb{H}]$  such that  $j : \mathbb{V} \xrightarrow{\prec}_\kappa M$ ,  $j(\kappa) > \lambda$ ,  $\mathbb{P}, \mathbb{H}, V_{j(\lambda)}^{\mathbb{V}[\mathbb{H}]} \in M$  and  $V_{j(\lambda)}^{\mathbb{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathbb{V}[\mathbb{H}]$ .

By the choice of  $n$ , we have  $V_\lambda \models \Vdash_{\mathbb{P}} \text{“}\varphi(\bar{a}^\vee)\text{”}$ .  $j(V_\lambda^\vee) = V_{j(\lambda)}^M \prec_{\Sigma_n} M$  by elementarity of  $j$ , and  $V_{j(\lambda)}^M = V_{j(\lambda)}^{\mathbb{V}[\mathbb{H}]}$  by the closedness of  $M$ . Since  $V_\lambda \prec_{\Sigma_n} \mathbb{V}$ , we have  $V_\lambda[\mathbb{H}] \prec_{\Sigma_{n_0}} \mathbb{V}[\mathbb{H}]$  for a still large enough  $n_0 \leq n$  by Lemma 4.8, (1). Since  $V_{j(\lambda)}^{\mathbb{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathbb{V}[\mathbb{H}]$ , it follows that  $V_\lambda^{\mathbb{V}[\mathbb{H}]} \prec_{\Sigma_{n_0}} V_{j(\lambda)}^{\mathbb{V}[\mathbb{H}]}$ . Thus

$$(4.39) \quad V_\lambda^\vee \prec_{\Sigma_{n_1}} V_{j(\lambda)}^\vee$$

*x-Lg-RcA-10*

for a still large enough  $n_1 \leq n_0$  by Lemma 4.8, (2).

In particular, we have  $V_{j(\lambda)}^\vee \models \Vdash_{\mathbb{P}} \text{“}\varphi(\bar{a}^\vee)\text{”}$ , and hence  $V_{j(\lambda)}[\mathbb{G}] \models \varphi(\bar{a})$  where  $\mathbb{G}$  is the  $\mathbb{P}$ -part of  $\mathbb{H}$ . Note that by (4.38) and (4.39),  $V_{j(\lambda)}$  satisfies a sufficiently large fragment of ZFC.

Thus we have  $V_{j(\lambda)}[\mathbb{H}] \models \text{“there is a } \mathcal{P}\text{-ground satisfying } \varphi(\bar{a})\text{”}$ , and hence

$$V_{j(\lambda)}^{\mathbb{V}[\mathbb{H}]} \models \text{“there is a } \mathcal{P}\text{-ground satisfying } \varphi(\bar{a})\text{”}$$

by Lemma 3.2. By elementarity, it follows that

$$V_\lambda \models \text{“there is a } \mathcal{P}\text{-ground satisfying } \varphi(\bar{a})\text{”}.$$

Finally, this implies  $\mathbb{V} \models \text{“there is a } \mathcal{P}\text{-ground satisfying } \varphi(\bar{a})\text{”}$  by (4.38).

□ (Theorem 4.10)

## 5 Bedrock of a tightly generic hyperhuge cardinal

Recall that a cardinal  $\kappa$  is *hyperhuge*, if for every  $\lambda > \kappa$ , there is  $j : V \xrightarrow{\lambda} M \subseteq V$  such that  $\lambda < j(\kappa)$  and  $j^{(\lambda)}M \subseteq M$ . By Lemma 4.3, a hyperhuge cardinal  $\kappa$  can be characterized in terms of existence of  $\kappa$ -complete normal ultrafilters with the property (4.7).

For a class  $\mathcal{P}$  of posets, a cardinal  $\kappa$  is *tightly  $\mathcal{P}$ -generic hyperhuge* (tightly  $\mathcal{P}$ -gen. hyperhuge, for short) if for any  $\lambda > \kappa$ , there is  $\mathbb{Q} \in \mathcal{P}$  such that for a  $(V, \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $j, M \subseteq V[\mathbb{H}]$  such that  $j : V \xrightarrow{\lambda} M$ ,  $\lambda < j(\kappa)$ ,  $|\mathbb{Q}| \leq j(\kappa)$ , and  $j''j(\lambda), \mathbb{H} \in M$ .

For a class  $\mathcal{P}$  of posets, a cardinal  $\kappa$  is *tightly  $\mathcal{P}$ -Laver-generically hyperhuge* (tightly  $\mathcal{P}$ -Laver-gen. hyperhuge, for short) if for any  $\lambda > \kappa$ , and  $\mathbb{P} \in \mathcal{P}$  there is a  $\mathbb{P}$ -name  $\mathbb{Q}$  with  $\Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{P}\text{”}$  such that for a  $(V, \mathbb{P} * \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $j, M \subseteq V[\mathbb{H}]$  such that  $j : V \xrightarrow{\lambda} M$ ,  $\lambda < j(\kappa)$ ,  $|\mathbb{P} * \mathbb{Q}| \leq j(\kappa)$ , and  $j''j(\lambda), \mathbb{H} \in M$ .

The following Lemma is easy to prove.

**Lemma 5.1** (1)  $\kappa$  is *hyperhuge* if and only if the following holds:

(5.1) for every  $\lambda > \kappa$ , there is  $j : V \xrightarrow{\lambda} M \subseteq V$  such that  $\lambda < j(\kappa)$ ,  $V_{j(\lambda)} \in M$ , and  $j^{(\lambda)}M \subseteq M$ .

(2) If  $\kappa$  is hyperhuge then it is *ultrahuge*.

(3) Suppose that  $\mathcal{P}$  is a class of posets such that the trivial poset  $\{1\}$  is in  $\mathcal{P}$ . If  $\kappa$  is hyperhuge then  $\kappa$  is tightly  $\mathcal{P}$ -gen. hyperhuge.

(4) For a class  $\mathcal{P}$  of posets,  $\kappa$  is tightly  $\mathcal{P}$ -gen. hyperhuge if and only if the following holds:

(5.2) for any  $\lambda > \kappa$ , there is  $\mathbb{Q} \in \mathcal{P}$  such that for a  $(V, \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $j, M \subseteq V[\mathbb{H}]$  such that  $j : V \xrightarrow{\lambda} M$ ,  $\lambda < j(\kappa)$ ,  $|\mathbb{Q}| \leq j(\kappa)$ , and  $V_{j(\lambda)}^{V[\mathbb{H}]}, j''j(\lambda), \mathbb{H} \in M$ .

(5) For a class  $\mathcal{P}$  of posets,  $\kappa$  is tightly  $\mathcal{P}$ -Laver-gen. hyperhuge if and only if the following holds:

(5.3) for any  $\lambda > \kappa$  and  $\mathbb{P} \in \mathcal{P}$ , there is a  $\mathbb{P}$ -name  $\mathbb{Q}$  with  $\Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{P}\text{”}$  such that for a  $(V, \mathbb{P} * \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $j, M \subseteq V[\mathbb{H}]$  such that  $j : V \xrightarrow{\lambda} M$ ,  $\lambda < j(\kappa)$ ,  $|\mathbb{Q}| \leq j(\kappa)$ , and  $V_{j(\lambda)}^{V[\mathbb{H}]}, j''j(\lambda), \mathbb{H} \in M$ .

(6) For an iterable class  $\mathcal{P}$  of posets, a tightly  $\mathcal{P}$ -Laver generically hyperhuge cardinal is tightly  $\mathcal{P}$ -gen. hyperhuge.



(7) For a class  $\mathcal{P}$  of posets, if  $\kappa$  is (tightly resp.)  $\mathcal{P}$ -Laver-gen. hyperhuge, then it is (tightly resp.)  $\mathcal{P}$ -Laver-gen. ultrahuge.

**Proof.** (1): It is clear that (5.1) implies that  $\kappa$  is hyperhuge.

Suppose that  $\kappa$  is hyperhuge. We show that (5.1) holds. For  $\lambda > \kappa$ , let  $\lambda^* := |V_\lambda|^+$  and  $j : \mathbf{V} \xrightarrow{\prec}_\kappa M$  be such that  $j(\kappa) > \lambda^*$  and

$$(5.4) \quad j(\lambda^*)M \subseteq M. \quad \text{x-bedrock-2}$$

By elementarity, we have

$$(5.5) \quad M \models j(\lambda^*) = |V_{j(\lambda)}|^M. \quad \text{x-bedrock-3}$$

**Claim 5.1.1**  $M \ni V_{j(\lambda)}^M = V_{j(\lambda)}.$  cl-bedrock-0

$\vdash$  We prove  $V_\alpha^M = V_\alpha$  for all  $\omega \leq \alpha \leq j(\lambda)$ .

For  $\alpha = \omega$ , this clearly holds.

Suppose that  $V_\alpha^M = V_\alpha$  for  $\omega \leq \alpha < j(\lambda)$ . Then  $j(\lambda^*) > |V_\alpha^M| = |V_\alpha|$  by (5.5). By (5.4), we have  $V_{\alpha+1} = \mathcal{P}(V_\alpha) \in M$ :  $\mathcal{P}(V_\alpha) \subseteq M$  by (5.4). Since  $M \models \text{ZFC}$  by elementarity of  $j$ , it follows that  $\mathcal{P}(V_\alpha) = \mathcal{P}(V_\alpha)^M \in M$ .

If  $\alpha \leq j(\lambda)$  is a limit and the Claim holds for  $\beta < \alpha$ , then  $\langle V_\beta : \beta < \alpha \rangle \in {}^\alpha M \subseteq M$ . Thus  $V_\alpha = \bigcup_{\beta < \alpha} V_\beta \in M$ .  $\dashv$  (Claim 5.1.1)

$j(\lambda)M \subseteq M$  by  $j(\lambda) < j(\lambda^*)$  and (5.4). Thus  $j$  witnesses (5.1) for  $\lambda$ .

(2): follows from (1). (3): is trivial.

(4): If  $\kappa$  satisfies (5.2) then  $\kappa$  is clearly tightly  $\mathcal{P}$ -gen. hyperhuge.

Suppose that  $\kappa$  is tightly  $\mathcal{P}$ -gen. hyperhuge. For  $\lambda > \kappa$ , let  $\lambda^*$  be such that

$$(5.6) \quad \lambda^* \geq |V_\lambda|^+ \text{ and} \quad \text{x-bedrock-4}$$

$$(5.7) \quad V_{\lambda^*} \prec_{\Sigma_n} \mathbf{V} \text{ for a sufficiently large natural number } n. \quad \text{x-bedrock-5}$$

By (5.2), there is  $\mathbb{Q} \in \mathcal{P}$  such that for  $(\mathbf{V}, \mathbb{Q})$ -generic  $\mathbb{H}$  there are  $j, M \subseteq \mathbf{V}[\mathbb{H}]$  such that

$$(5.8) \quad j : \mathbf{V} \xrightarrow{\prec}_\kappa M, \quad \text{x-bedrock-5-0}$$

$$(5.9) \quad \lambda^* < j(\kappa), \quad \text{x-bedrock-6}$$

$$(5.10) \quad |\mathbb{Q}| \leq j(\kappa), \text{ and} \quad \text{x-bedrock-7}$$

$$(5.11) \quad j''j(\lambda^*), \mathbb{H} \in M. \quad \text{x-bedrock-8}$$

By (5.11), and Lemma 2.5, (5) in [21], we have  $V_{j(\lambda^*)}^\mathbf{V} \in M$ . Hence  $V_{j(\lambda^*)}^\mathbf{V}[\mathbb{H}] \in M$  by (5.11). By (5.7) and (5.11),  $V_{j(\lambda^*)}^\mathbf{V}[\mathbb{H}] = V_{j(\lambda^*)}^{\mathbf{V}[\mathbb{H}]}$ , and thus  $V_{j(\lambda^*)}^{\mathbf{V}[\mathbb{H}]} \in M$ .

This shows that our  $j$  and  $M$  witnesses (5.2) for  $\lambda$ .

(5): Similarly to (4).

(6): Since  $\mathcal{P}$  is iterable, the poset  $\mathbb{P} * \mathbb{Q}$  in the definition of  $\mathcal{P}$ -Laver-gen. hyperhugeness is an element of  $\mathcal{P}$ .

(7): By (5). □ (Lemma 5.1)

For a cardinal  $\kappa$ , a ground  $W$  of the universe  $V$  is called a  $\leq \kappa$ -ground if there is a poset  $\mathbb{P} \in W$  of cardinality  $\leq \kappa$  (in the sense of  $V$ ) and  $(W, \mathbb{P})$ -generic filter  $\mathbb{G}$  such that  $V = W[\mathbb{G}]$ . Note that this definition of  $\leq \kappa$ -ground diverges from the convention of “ $\mathcal{P}$ -ground” in that “of cardinality  $\leq \kappa$ ” is meant here not in the sense of  $W$  but rather of  $V$ . In Proposition 6.1 we will show that in our context, this actually implies “ $\leq \kappa$ -ground” in line with the definition of  $\mathcal{P}$ -ground.

Let

$$(5.12) \quad \overline{W} := \bigcap \{W : W \text{ is a } \leq \kappa\text{-ground}\}.$$

x-bedrock-9

Since there are only set many  $\leq \kappa$ -grounds,  $\overline{W}$  contains a ground by Theorem 1.3 in [42]. We shall call  $\overline{W}$  the  $\leq \kappa$ -mantle of  $V$ .

By Lemma 5.1, (3), the following theorem generalizes Theorem 1.6 in [42].

**Theorem 5.2** *Suppose that  $\mathcal{P}$  is a class of posets. If  $\kappa$  is a tightly  $\mathcal{P}$ -gen. hyperhuge cardinal, then the  $\leq \kappa$ -mantle is the smallest ground of  $V$  (i.e. it is the bedrock of  $V$ ) and it is also a  $\leq \kappa$ -ground.*

p-bedrock-1

**Proof.** Suppose that  $\kappa$  is tightly  $\mathcal{P}$ -gen. hyperhuge and let  $\overline{W}$  be the  $\leq \kappa$ -mantle for this  $\kappa$ .

By Theorem 1.3 in [42] mentioned above, it is enough to show that, for any ground  $W \subseteq \overline{W}$  is actually a  $\leq \kappa$ -ground and hence  $W = \overline{W}$  holds.

Let  $W \subseteq \overline{W}$  be a ground. Let  $\mu$  be the cardinality (in the sense of  $V$ ) of a poset  $\mathbb{S} \in W$  such that there is a  $(W, \mathbb{S})$ -generic  $\mathbb{F}$  such that  $V = W[\mathbb{F}]$ . Without loss of generality,  $\mu \geq \kappa$ .

By Theorem 2.1, there is  $r \in V$  such that  $W = \Phi(\cdot, r)^V$ .

Let  $\theta \geq \mu$  be such that  $r \in V_\theta$ , and for a sufficiently large natural number  $n$ , we have

$$(5.13) \quad V_\theta^V \prec_{\Sigma_n} V.$$

x-bedrock-9-1

By the choice of  $\theta$ , we have

$$(5.14) \quad \Phi(\cdot, r)^{V_\theta^V} = \Phi(\cdot, r)^V \cap V_\theta^V = W \cap V_\theta^V = V_\theta^W.$$

x-bedrock-9-2

There is a  $\mathbb{Q} \in \mathcal{P}$  such that for  $(V, \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $j$ ,  $M \subseteq V[\mathbb{H}]$  with

$$(5.15) \quad j : \mathbf{V} \xrightarrow{\sim}_{\kappa} M, \theta < j(\kappa), \quad \text{x-bedrock-10}$$

$$(5.16) \quad |\mathbb{Q}| \leq j(\kappa), \quad \text{x-bedrock-11}$$

$$(5.17) \quad V_{j(\theta)}^{\mathbf{V}[\mathbb{H}]} \subseteq M, \text{ and} \quad \text{x-bedrock-12}$$

$$(5.18) \quad \mathbb{H}, j''j(\theta) \in M \quad \text{x-bedrock-13}$$

by Lemma 5.1, (4).

Since  $V_{\theta}^{\mathbf{V}} \equiv V_{j(\theta)}^M$  by elementarity, (5.13) implies that  $V_{j(\theta)}^M (= V_{j(\theta)}^{\mathbf{V}[\mathbb{H}]})$  satisfies a sufficiently large fragment of ZFC. Thus it follows that

$$(5.19) \quad V_{j(\theta)}^{\mathbf{V}} \text{ satisfies a sufficiently large fragment of ZFC and also } V_{j(\theta)}^{\mathbf{W}} \text{ satisfies} \quad \text{x-bedrock-16}$$

still a sufficiently large fragment of ZFC.

Thus we have

$$(5.20) \quad \underbrace{V_{j(\theta)}^M}_{\text{by (5.17)}} = \underbrace{V_{j(\theta)}^{\mathbf{V}[\mathbb{H}]}}_{\text{by Lemma 3.2 and the remark above}} = \underbrace{V_{j(\theta)}^{\mathbf{V}[\mathbb{H}]}}_{\text{by Lemma 3.2 and the remark above}} = \underbrace{V_{j(\theta)}^{\mathbf{W}}[\mathbb{F}][\mathbb{H}]}_{\text{by Lemma 3.2 and the remark above}}. \quad \text{x-bedrock-14}$$

Since

$$(5.21) \quad V_{j(\theta)}^M \prec_{\Sigma_n} M \quad \text{x-bedrock-16-0}$$

by elementarity and  $V$  is a ground of  $M$  by Grigorieff's Theorem, it follows from Lemma 4.8, (2) that

$$(5.22) \quad V_{j(\theta)}^{\mathbf{V}} \prec_{\Sigma_{n_0}} \mathbf{V} \text{ for a still large enough } n_0 \leq n. \quad \text{x-bedrock-15}$$

**Claim 5.2.1** (1)  $V_{j(\theta)}^{\mathbf{V}}$  is a generic extension of  $V_{j(\theta)}^{\mathbf{W}}$  by a poset of size  $\leq j(\kappa)$ . cl-bedrock-1  
(2)  $V[\mathbb{H}]$  is a generic extension of  $\mathbf{W}$  by a poset of size  $\leq j(\kappa)$ .  
(3)  $V_{j(\theta)}^M$  is a generic extension of  $V_{j(\theta)}^{\mathbf{W}}$  by a poset of size  $\leq j(\kappa)$ .  
(4)  $M$  is a generic extension of  $j(W) := \Phi(\cdot, j(r))^M$  by a poset of size  $\leq j(\mu)$ .

⊢ (1):  $\mathbf{V}$  is a generic extension of  $W$  by a poset of size  $\mu < \lambda < j(\kappa)$ . Thus the assertion follows from (5.22).

(2): We have  $\mathbf{V}[\mathbb{H}] = \mathbf{W}[\mathbb{F}][\mathbb{H}]$  and  $|\mathbb{S} * \mathbb{Q}| \leq j(\kappa)$  by  $|\mathbb{S}| \leq \mu < \lambda < j(\kappa)$  and by (5.16).

(3): By (5.20) and (1).

(4): By definition of  $\mu$  and elementarity of  $j$ . ⊢ (Claim 5.2.1)

**Claim 5.2.2**  $V_{j(\theta)}^{j(\mathbf{W})} = j(V_{\theta}^{\mathbf{W}}) \subseteq j(V_{\theta}^{\overline{\mathbf{W}}}) \subseteq V_{j(\theta)}^{\mathbf{W}} \subseteq V_{j(\theta)}^{\mathbf{V}}$ . cl-bedrock-1-0

⊢ By (5.17), we have  $V_{j(\theta)}^M = V_{j(\theta)}^{\mathbf{V}[\mathbb{H}]}$ . By Claim 5.2.1, (3), it follows that

$$(5.23) \quad V_{j(\theta)}^W \text{ is a } \leq j(\kappa)\text{-ground of } V_{j(\theta)}^M.$$

x-bedrock-16-1

By elementarity of  $j$  and (5.21),

$$(5.24) \quad j(V_\theta^{\overline{W}}) \text{ is a } \leq j(\kappa)\text{-mantle of } V_{j(\theta)}^M.$$

x-bedrock-18

Thus we have

by (5.23), (5.24) and by the definition of  $\leq j(\kappa)$ -mantle

$$V_{j(\theta)}^{j(W)} = j(V_\theta^W) \subseteq j(V_\theta^{\overline{W}}) \subseteq V_{j(\theta)}^W \subseteq V_{j(\theta)}^V. \quad \dashv \quad (\text{Claim 5.2.2})$$

**Claim 5.2.3**  $j''\lambda \in j(W)$  for every  $\lambda < j(\theta)$ .

cl-bedrock-2

— Since  $\theta$  is a limit cardinal, it is enough to show the claim for all regular  $\lambda < j(\theta)$  with  $\lambda > j(\mu)$ . Let  $\lambda$  be one of such cardinals, and let  $\langle S_\alpha : \alpha < \lambda \rangle$  be a partition of  $(E_\omega^\lambda)^W$  into stationary sets in  $W$ . Since  $|\mathbb{S}| \leq \mu < \lambda$  each  $S_\alpha$  is stationary subset of  $\lambda$  in  $V$ . Since  $V[\mathbb{H}]$  is a generic extension of  $V$  by  $\mathbb{Q}$  of size  $\leq j(\kappa)$ , and  $\lambda > j(\mu) \geq j(\kappa)$ , each  $S_\alpha$ ,  $\alpha < \lambda$  is a stationary subset of  $\lambda$  in  $V[\mathbb{H}]$ . Let

$$(5.25) \quad \langle S'_\alpha : \alpha < j(\lambda) \rangle := j(\langle S_\alpha : \alpha < \lambda \rangle) \in j(W).$$

x-bedrock-17

**Subclaim 5.2.3.1**

sub-bedrock-a

$j''\lambda = \{\beta < \sup(j''\lambda) : S'_\beta \cap \sup(j''\lambda) \text{ is stationary subset of } \sup(j''\lambda) \text{ in } M\}.$

— Suppose  $\alpha \in \lambda$ . To show that  $S'_{j(\alpha)} \cap \sup(j''\lambda) = j(S_\alpha) \cap \sup(j''\lambda)$  is a stationary subset of  $\sup(j''\lambda)$  in  $M$ , suppose  $C \subseteq \sup(j''\lambda)$  is a club subset of  $\sup(j''\lambda)$  in  $M$ . We have to show that  $C$  intersects with  $j(S_\alpha)$ .

Since  $|\mathbb{Q}| \leq j(\kappa) \leq j(\mu) < \lambda$  by the choice of  $\lambda$ , and  $M \subseteq V[\mathbb{H}]$ , there is an unbounded  $D \subseteq \lambda$ ,  $D \in W$  such that for any  $\xi_0, \xi_1 \in D$  with  $\xi_0 < \xi_1$ ,  $[j(\xi_0), j(\xi_1)] \cap C \neq \emptyset$ . Since  $S_\alpha$  is stationary, there is  $\eta \in S_\alpha \cap \lim(D)$ . Since  $cf(\eta) = \omega$ ,  $j(S_\alpha) \ni j(\eta) = \sup(j''\eta)$  the right-most side of this is an element of  $C$  by definition of  $D$ . This shows that  $j(S_\alpha) \cap C \neq \emptyset$ .

Suppose now that  $S'_\beta \cap \sup(j''\lambda)$  is stationary subset of  $\sup(j''\lambda)$  in  $M$  for some  $\beta < \sup(j''\lambda)$ . We want to show that  $\beta = j(\alpha)$  for some  $\alpha < \lambda$ .

Since  $j''\lambda = j''j(\theta) \cap \sup(j''\lambda) \in M$  by (5.18), and thus, also  $\{\langle \alpha, j(\alpha) \rangle : \alpha < \lambda\} \in M$ , we have  $M \models cf(j''\lambda) = \lambda$ , and hence also  $V[\mathbb{H}] \models cf(j''\lambda) = \lambda$ .

By (5.17),  $\mathcal{P}(\lambda)^M = \mathcal{P}(\lambda)^{V[\mathbb{H}]}$ . Hence stationarity of a subset of  $\sup(j''\lambda)$  is absolute between  $M$  and  $V[\mathbb{H}]$ . Thus, our assumption implies that  $S'_\beta \cap \sup(j''\lambda)$  is stationary subset of  $\sup(j''\lambda)$  in  $V[\mathbb{H}]$ .

By (5.13) and elementarity of  $j$ ,  $j(V_\theta^{\overline{W}})$  is a  $\leq j(\kappa)$ -mantle of  $V_{j(\theta)}^M (= V_{j(\theta)}^{V[\mathbb{H}]})$  by (5.17).

**Subsubclaim 5.2.3.1.1** For every  $\eta < \lambda$ , if  $cf^{j(W)}(\eta) = \omega$  then  $cf^V(\eta) = \omega$ .

subsub-bedrock-0

⊢ By Claim 5.2.2.

⊢ (Subsubclaim 5.2.3.1.1)

**Subsubclaim 5.2.3.1.2** *There is an unbounded  $E \subseteq \lambda$  such that  $j \restriction E \in j(\mathbb{W})$ .* subsub-bedrock-1

⊢ By Claim 5.2.1, (4) and  $j(\mu) < \lambda$  by choice of  $\lambda$ ,  $M$  is a generic extension of  $j(\mathbb{W})$  by the  $\lambda$ -c.c. poset  $j(\mathbb{S})$ .<sup>6</sup> Since  $j''\lambda \in M$ , there is a  $j(\mathbb{S})$ -name  $\check{j}$  for  $j \restriction \lambda$ .

Working in  $j(\mathbb{W})$ , suppose that  $\gamma < \lambda$ . By the  $\lambda$ -c.c. of  $j(\mathbb{S})$ , we can find an increasing sequences  $\langle \alpha_n : n < \omega \rangle, \langle \beta_n : n \in \omega \rangle \in j(\mathbb{W})$  such that  $\gamma < \alpha_0$ ,  $\alpha_n < \lambda$  for all  $n \in \omega$  and  $\Vdash_{j(\mathbb{S})} \check{j}(\alpha_n) < \beta_n < \check{j}(\alpha_{n+1})$  for all  $n \in \omega$ .

Let  $\alpha := \sup_{n \in \omega} \alpha_n$  and  $\beta := \sup_{n \in \omega} \beta_n$ . Then  $cf^{j(\mathbb{W})}(\alpha) = \omega$  and hence  $cf^\mathbb{V}(\alpha) = \omega$  by Subsubclaim 5.2.3.1.1. Thus  $\Vdash_{j(\mathbb{S})} \check{j}(\alpha) = \sup_{n \in \omega} \check{j}(\alpha_n) = \sup_{n \in \omega} \beta_n = \beta$ .

This shows that

$$E = \{\alpha < \lambda : \text{there is } \beta \text{ such that } \Vdash_{j(\mathbb{S})} \check{j}(\check{\alpha}) = \check{\beta}\} \in j(\mathbb{W})$$

is a cofinal subset of  $\lambda$  and  $j \restriction E \in j(\mathbb{W})$ .

⊢ (Subsubclaim 5.2.3.1.2)

Now returning to the proof of the second-half of the Subclaim 5.2.3.1, since  $S'_\beta \cap \sup(j''\lambda)$  is stationary and  $\lim(j''E)$  is club in  $\sup(j''\lambda)$ , there is  $\eta \in S'_\beta \cap \lim(j''E)$ . Then  $cf^{j(\mathbb{W})}(\eta) = \omega = cf^\mathbb{V}(\eta)$  (the last equality by Subsubclaim 5.2.3.1.1). Let  $\zeta < \lambda$  be minimal with  $\eta \leq j(\zeta)$ .

We have  $\sup(j''\zeta) = \sup(j''(\zeta \cap E)) = \eta \leq j(\zeta)$ . Since the cofinality of  $\eta$  is  $\omega$  and  $j \restriction (\eta \cap E)$  is increasing and  $\eta \cap E$  is cofinal in  $\eta$ , we have  $j(\zeta) = \sup(j''\zeta) = \eta$ , and hence  $j(\zeta) \in S'_\beta$ . By elementarity, it follows that  $\zeta \in S_\alpha$  for some  $\alpha < \lambda$ . Then  $j(\zeta) \in S'_{j(\alpha)} \cap S'_\beta$ . Since  $S'_\xi$ 's are pairwise disjoint, it follows that  $j(\alpha) = \beta$ . Thus  $\beta \in j''\lambda$  as desired.

⊢ (Subclaim 5.2.3.1)

By Subclaim 5.2.3.1 we have

$$j''\lambda = \{\beta < \sup(j''\lambda) : S'_\beta \cap \sup(j''\lambda) \text{ is stationary subset of } \sup(j''\lambda) \text{ in } M\}.$$

By Claim 5.2.1, (4), and since  $j(\mu) < \lambda$ , it follows that

$$j''\lambda = \{\beta < \sup(j''\lambda) : S'_\beta \cap \sup(j''\lambda) \text{ is stationary subset of } \sup(j''\lambda) \text{ in } j(\mathbb{W})\}.$$

Apparently the right side of the equality is a definition of an element of  $j(\mathbb{W})$ .

⊢ (Claim 5.2.3)

---

<sup>6</sup> Since  $|j(\mathbb{S})| \leq j(\mu) < \lambda$  in  $M$ ,  $j(\mathbb{S})$  has the  $\lambda$ -c.c. in  $M$  and hence it has the  $\lambda$ -c.c. also in  $j(\mathbb{W}) \subseteq M$ .

**Claim 5.2.4**  $V_{j(\theta)}^W = V_{j(\theta)}^{j(W)}$ .

cl-bedrock-3

⊢ “ $\supseteq$ ”: By Claim 5.2.2.

“ $\subseteq$ ”: For  $\lambda < j(\theta)$ , if  $X \in \mathcal{P}(\lambda) \cap W$ , then  $j(X) \in j(W)$ , and  $j''\lambda \in j(W)$  by Claim 5.2.3. Thus  $j''X = j(X) \cap j''\lambda \in j(W)$ . Since we also have  $j \restriction \lambda \in j(W)$ , it follows that  $X \in j(W)$ .

Thus, by induction on  $\alpha \leq j(\theta)$ , we can prove that  $V_\alpha^W \subseteq j(W)$ . ⊢ (Claim 5.2.4)

Now, we have

$$\begin{aligned} & \text{by (5.24) and Claim 5.2.1, (3)} \\ j(V_\theta^W) & \subseteq j(V_\theta^{\overline{W}}) \overset{\text{by Claim 5.2.4}}{\subseteq} V_{j(\theta)}^W = V_{j(\theta)}^{j(W)} = j(V_\theta^W). \end{aligned}$$

by  $V_\theta^W \subseteq V_\theta^{\overline{W}}$  and by elementarity of  $j$       by Claim 5.2.4

It follows  $j(V_\theta^W) = V_{j(\theta)}^W$ . Since  $j(V_\theta^V) = V_{j(\theta)}^M$  is a generic extension of  $V_{j(\theta)} = j(V_\theta^W)$  by a poset of size  $\leq j(\kappa)$  (by Claim 5.2.1, (3)). Thus, by elementarity of  $j$ ,  $V_\theta^V$  is a generic extension of  $V_\theta^W$  by a poset of size  $\leq \kappa$ .

By definition of  $\overline{W}$  and by (5.13), it follows that  $V_\theta^{\overline{W}} \subseteq V_\theta^W$ . Since  $\theta$  was arbitrary, this implies that  $\overline{W} \subseteq W$  and hence  $\overline{W} = W$  as desired. □ (Theorem 5.2)

As already noticed Theorem 5.2 is a generalization of Theorem 1.6 in [42] saying that there is the bedrock (of the universe  $V$ ) if there is a hyperhuge cardinal. In Theorem 1.3 in [43] the assumption of hyperhuge cardinal in this theorem is weakened to the existence of an extendible cardinal.

Though all of three theorems are proved by the same global strategy, the proof of Theorem 5.2 relies on the details, notably that of Claim 5.2.3 in which the tight generic hyperhugeness seems to play an essential role. Thus, at the moment we do not know if the condition of tightly gen. hyperhuge cardinal in Theorem 5.2 can be replaced by some notion of generic large cardinal corresponding to a large cardinal of lower consistency strength.

**Theorem 5.3** *Suppose that  $\mathcal{P}$  is a class of posets. If  $\kappa$  is a tightly  $\mathcal{P}$ -gen. hyperhuge cardinal, then  $\kappa$  is a hyperhuge cardinal in the bedrock  $\overline{W}$  of  $V$ .*

p-bedrock-2

**Proof.** In  $V$ , let  $\lambda > \kappa$  be arbitrary. Let  $\theta > \lambda$  be sufficiently large such that it satisfies  $V_\theta^V \prec_{\Sigma_n} V$  for sufficiently large natural number  $n$  (this is just the condition (5.13) in the proof of Theorem 5.2). Let  $Q \in \mathcal{P}$  be such that for a  $(V, Q)$ -generic  $\mathbb{H}$ , there are  $j$ ,  $M \subseteq V[\mathbb{H}]$  such that  $j : V \xrightarrow{\prec}_\kappa M$ ,  $\theta < j(\kappa)$ ,  $j''j(\theta) \in M$ , and  $V_{j(\theta)}^{V[\mathbb{H}]} = V_{j(\theta)}^M$ . Note that  $W$  in the proof of Theorem 5.2 coincides with the bedrock  $\overline{W}$  of  $V$ .

Thus we have (5.26):  $j''j(\lambda^*) \in j(\overline{W})$  for all  $\lambda^* < j(\theta)$  by Claim 5.2.3, and (5.27):  $j(V_\theta^{\overline{W}}) = V_{j(\theta)}^{\overline{W}}$  by Claim 5.2.4.

x-bedrock-19

x-bedrock-20

Let  $\langle X_\alpha : \alpha < \gamma \rangle$  be an enumeration of  $\mathcal{P}(\mathcal{P}(j(\lambda)))^{\bar{W}}$  in  $\bar{W}$ . Note that  $\gamma < j(\theta)$  by the choice of  $\theta$ .

By (5.27), we have  $\langle X_\alpha : \alpha < \gamma \rangle \in j(\bar{W})$  and  $j''j(\lambda) \in j(\bar{W})$  by (5.26).

Also for any  $\lambda^* < j(\theta)$  and  $A \in \mathcal{H}((\lambda^*)^+)$  we have  $j \restriction A \in j(\bar{W})$  by (5.26) and Lemma 2.5 in [21]. It follows that

$$(5.28) \quad I := \{\alpha < \lambda : j''j(\lambda) \in j(X_\alpha)\} \in j(\bar{W}).$$

x-bedrock-21

Let  $U = \{X_\alpha : \alpha \in I\}$ . Then  $U \in j(\bar{W})$ . Furthermore we have

$$(5.29) \quad U \in V_{j(\theta)}^{j(\bar{W})} = j(V_\theta^{\bar{W}}) \overset{\text{by (5.27)}}{=} V_{j(\theta)}^{\bar{W}} \subseteq \bar{W}.$$

It is a routine to check that  $U$  is a  $\kappa$ -complete normal ultrafilter over  $\mathcal{P}(j(\lambda))^{\bar{W}}$  and  $\{x \in \mathcal{P}(j(\lambda)) : x \cap \kappa \in \kappa, otp(x \cap j(\kappa)) = \kappa, otp(x) = \lambda\} \in U$  holds.

Since  $\lambda$  was arbitrary, this implies that  $\kappa$  is hyperhuge in  $\bar{W}$  by Lemma 4.3.

□ (Theorem 5.3)

**Corollary 5.4** *Suppose that  $\mathcal{P}$  is an arbitrary class of posets and  $\kappa$  is a tightly  $\mathcal{P}$ -gen. hyperhuge cardinal. Then (1) there are cofinally many huge cardinals.  
(2) SCH holds above some cardinal.*

p-bedrock-3

**Proof.** Suppose that  $\kappa$  is a tightly  $\mathcal{P}$ -gen. hyperhuge cardinal. By Theorem 5.2 there is the bedrock  $\bar{W}$  and  $\kappa$  is hyperhuge cardinal in  $\bar{W}$ .

(1): Since the existence of a hyperhuge cardinal implies the existence of cofinally many huge cardinals (it is easy to show that the target  $j(\kappa)$  of hyperhuge embedding for a sufficiently large inaccessible  $\lambda$  is a huge cardinal), there are cofinally many huge cardinals in  $\bar{W}$ . Since  $V$  is attained by a set forcing starting from  $\bar{W}$ , a final segment of these huge cardinals survive in  $V$ .

(2): By Theorem 20.8 in [33], SCH holds above  $\kappa$  in  $\bar{W}$ . Since  $V$  is a set generic extension of  $\bar{W}$ , SCH should hold above some cardinal  $\mu \geq \kappa$ . □ (Corollary 5.4)

Compare Corollary 5.4, (1) with Corollary 4.2 and Corollary 6.7.

For iterable stationary preserving  $\mathcal{P}$  containing all proper posets, Theorem 5.4, (2) holds already under the  $\mathcal{P}$ -Laver-gen. supercompactness of  $\kappa$ . The reason is that in such case PFA holds (see Theorem 5.7 in [21]), and by Viale [44], SCH follows from it.

In the following Corollary, we adopt the notation of Ikegami-Trang in [32] on their version of Maximality Principle.

**Corollary 5.5** *Suppose that  $\mathcal{P}$  is the class of semi-proper posets. If  $\kappa$  is a tightly  $\mathcal{P}$ -Laver gen. hyperhuge cardinal, then  $\text{MP}_{\Pi_2}(\omega_1, \text{all posets})$  holds.*

*p-bedrock-4*

**Proof.** Theorem 1.8, (1) in [32] states that the conclusion of the present corollary holds under  $\text{MM}^{++}$  and proper class many Woodin cardinals.

If  $\kappa$  is a  $\mathcal{P}$ -Laver gen. supercompact cardinal, then  $\kappa = \aleph_2$  and  $\text{MM}^{++}$  holds (see [21]). By Corollary 5.4, tightly  $\mathcal{P}$ -Laver gen. hyperhugeness implies that there are proper class many Woodin cardinals.  $\square$  (Corollary 5.5)

**Corollary 5.6** *Suppose that  $\mathcal{P}$  is the class of all posets. Then the following theories are equiconsistent:*

*p-bedrock-5*

- (a)  $\text{ZFC} + \text{"there is a hyperhuge cardinal"}$ .
- (b)  $\text{ZFC} + \text{"there is a tightly } \mathcal{P}\text{-Laver gen. hyperhuge cardinal"}$ .
- (c)  $\text{ZFC} + \text{"there is a tightly } \mathcal{P}\text{-gen. hyperhuge cardinal"}$ .
- (d)  $\text{ZFC} + \text{"bedrock } \overline{W} \text{ exists and } \omega_1 \text{ is a hyperhuge cardinal in } \overline{W}"$ .  $\square$

**Corollary 5.7** *Suppose that  $\mathcal{P}$  is one of the following classes of posets: all semi-proper posets; all proper posets; all ccc posets; all  $\sigma$ -closed posets. Then the following theories are equiconsistent:*

*p-bedrock-6*

- (a)  $\text{ZFC} + \text{"there is a hyperhuge cardinal"}$ .
- (b)  $\text{ZFC} + \text{"there is a tightly } \mathcal{P}\text{-Laver gen. hyperhuge cardinal"}$ .
- (c)  $\text{ZFC} + \text{"there is a tightly } \mathcal{P}\text{-gen. hyperhuge cardinal"}$ .
- (d)  $\text{ZFC} + \text{"bedrock } \overline{W} \text{ exists and } \kappa_{\text{refl}} \text{ is a hyperhuge cardinal in } \overline{W}"$ .  $\square$

The proof of Theorem 5.3 can be modified to obtain the following:

**Theorem 5.8** *Suppose that  $\mathcal{P}$  is a class of posets. If a definable cardinal  $\kappa$  is a tightly super- $C^{(\infty)}$ - $\mathcal{P}$ -gen. hyperhuge cardinal, then  $\kappa$  is super- $C^{(\infty)}$ -hyperhuge in the bedrock  $\overline{W}$  of  $V$ .*

*p-bedrock-2-0*

The definability of the cardinal  $\kappa$  (e.g. as  $\omega_1$ ,  $2^{\aleph_0}$  etc.) in Theorem 5.8 actually is needed so that the conclusion of the theorem is formalizable (in infinitely many formulas).

**Corollary 5.9** *Suppose that  $\mathcal{P}$  is one of the following classes of posets: all semi-proper posets; all proper posets; all ccc posets; all  $\sigma$ -closed posets. Then the following theories are equiconsistent:*

*p-bedrock-8*

- (a)  $\text{ZFC} + \text{"}c \text{ is a super-}C^{(\infty)} \text{ hyperhuge cardinal" where } c \text{ is a new constant symbol but "... is super-}C^{(\infty)} \text{ hyperhuge ..."} \text{ is formulated in an infinite collection of formulas in } \mathcal{L}_\in$ .



- (b) ZFC + “there is a tightly super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver gen. hyperhuge cardinal”.
- (c) ZFC + “bedrock  $\overline{W}$  exists and  $\kappa_{\text{refl}}^V$  is a super  $C^{(\infty)}$ -hyperhuge cardinal in  $\overline{W}$ ”.

**Theorem 5.10** *Suppose that  $\kappa$  is tightly super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver gen. hyperhuge for an iterable  $\mathcal{P}$ . Then, for each  $n \in \mathbb{N}$ , there are stationarily many super- $C^{(n)}$ -hyperhuge cardinals.*

*p-bedrock-8-0*

**Proof.** Let  $n \in \mathbb{N}$ . By a modification of Theorem 5.3, we can show that  $\kappa$  is super- $C^{(\infty)}$ -hyperhuge in the bedrock  $\overline{W}$ . Thus there are stationarily many super- $C^{(n)}$ -hyperhuge cardinals in  $\overline{W}$  by Corollary 4.2. Since  $V$  is a set generic extension of  $\overline{W}$ , the stationarity of these classes is preserved by the generic extension.  $\square$  (Theorem 5.10)

**Corollary 5.11** *The consistency strength of the existence of a tightly super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver gen. hyperhuge cardinal for one of the iterable  $\mathcal{P}$ 's in Theorem 4.7 is strictly between that of the existence of a super  $C^{(n)}$ -hyperhuge cardinal, and that of the existence of a 2-huge cardinal.*

*p-bedrock-8-1*

**Proof.** Suppose that  $\kappa$  is tightly super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver gen. hyperhuge for an iterable  $\mathcal{P}$ . Then by Theorem 5.10, there is a super- $C^{(n)}$ -hyperhuge  $\lambda > \kappa$  with  $V_\lambda \prec_{\Sigma_{n'}}$  for sufficiently large  $n' > n$ . any super- $C^{(n)}$ -hyperhuge  $\lambda_0 < \lambda$  is super- $C^{(n)}$ -hyperhuge in  $V_\lambda$  by elementarity. Thus  $V_\lambda$  is a model of ZFC + “there is a super-  $C^{(n)}$ -hyperhuge cardinal”.

If  $\kappa$  is 2-huge then by Lemma 4.5 and Theorem 4.7, there is a set model with a tightly super- $C^{(\infty)}$ - $\mathcal{P}$ -Laver gen. hyperhuge cardinal.  $\square$  (Corollary 5.11)

## 6 Bedrock and Laver genericity

**Proposition 6.1** *Suppose that  $\mathcal{P}$  is a class of posets and  $\kappa$  is tightly  $\mathcal{P}$ -gen. ultra-huge cardinal. By Theorem 5.2 there is the bedrock  $\overline{W}$  of  $V$ .*

*edrock-Lg*

*p-bedrock-9*

*We have  $(\kappa^+)^{\overline{W}} = (\kappa^+)^V$  and  $V$  is a set generic extension  $\overline{W}$  by some poset  $\mathbb{P} \in \overline{W}$  such that  $\overline{W} \models |\mathbb{P}| \leq \kappa$ .*

**Proof.** Let  $\mathbb{P}_0 \in \overline{W}$  be such that there is a  $(\overline{W}, \mathbb{P}_0)$ -generic  $\mathbb{G}_0$  such that  $V = \overline{W}[\mathbb{G}_0]$ . By the proof of Theorem 5.2,  $\mathbb{P}_0$  can be chosen such that (6.1):  $V \models |\mathbb{P}_0| \leq \kappa$ .

*x-bedrock-21-0*

Let (6.2):  $\theta > \kappa + |\mathbb{P}_0|$  (in  $V$ ) be large enough, and such that it satisfies (5.13) in the proof of Theorem 5.2 for sufficiently large natural number  $n$ . Let  $\mathbb{Q} \in \mathcal{P}$  be such that for  $(V, \mathbb{Q})$ -generic  $\mathbb{H}$  there are  $j, N \subseteq V[\mathbb{H}]$  such that  $j : V \xrightarrow{\prec_\kappa} M$ ,  $j(\kappa) > \theta$ , (6.3):  $|\mathbb{Q}| \leq j(\kappa)$ ,  $\mathbb{H}, j''j(\theta) \in M$ , and  $V_{j(\theta)}^{V[\mathbb{H}]} \subseteq M$ .

*x-bedrock-22*

*x-bedrock-22-0*

Since  $|\mathbb{P}_0| < \theta < j(\kappa)$  we have

$$(6.4) \quad \overline{W} \models |\mathbb{P}_0| < j(\kappa).$$

x-bedrock-22-1

Thus

$$(6.5) \quad \begin{aligned} (j(\kappa)^+)^{j(\overline{W})} &= (j(\kappa)^+)^{V_{j(\theta)}^{j(\overline{W})}} \stackrel{\text{by (6.4)}}{=} (j(\kappa)^+)^{V_{j(\theta)}^{\overline{W}}} \stackrel{\text{by Claim 5.2.4}}{=} (j(\kappa)^+)^{V_{j(\theta)}^V} \\ &\stackrel{\text{by (6.3)}}{=} (j(\kappa)^+)^{V_{j(\theta)}^{V[\mathbb{H}]}}. \end{aligned}$$

x-bedrock-23

By elementarity it follows that  $(\kappa^+)^W = (\kappa^+)^V$ . By (6.1) this implies that

$$\overline{W} \models |\mathbb{P}_0| \leq \kappa.$$

□ (Proposition 6.1)

**Proposition 6.2** *Suppose that  $\kappa$  is tightly  $\mathcal{P}$ -gen. hyperhuge for a class  $\mathcal{P}$  of posets.*

p-bedrock-10

*By Proposition 6.1, there is a poset  $\mathbb{P}$  in the bedrock  $\overline{W}$  with  $\overline{W} \models |\mathbb{P}| \leq \kappa$  such that  $V = \overline{W}[\mathbb{G}]$  for a  $(\overline{W}, \mathbb{P})$ -generic  $\mathbb{G}$ .*

*For any bounded  $b \subseteq \kappa$  in  $V$ , there is  $\mathbb{P}_b \in \overline{W}$  with  $\overline{W} \models |\mathbb{P}_b| < \kappa$ ,  $\mathbb{P}_b \subseteq \mathbb{P}$ , such that  $\mathbb{G} \cap \mathbb{P}_b$  is  $(\overline{W}, \mathbb{P}_b)$ -generic and  $\overline{W}[\mathbb{G} \cap \mathbb{P}_b] \ni b$ .*

**Proof.** Since  $\overline{W} \models “|\mathbb{P}| \leq \kappa”$  we may assume, without loss of generality, that the underlying set of  $\mathbb{P}$  is  $\kappa$ . Let  $\theta > |\mathbb{P}|$  be large enough and such that  $V_\theta \prec_{\Sigma_n} V$  for a large enough  $n$  in accordance with Lemma 3.2.

Let  $\mathbb{Q} \in \mathcal{P}$  be such that for  $(V, \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $j, M \subseteq V[\mathbb{H}]$  such that  $j : V \xrightarrow{\prec_\kappa} M$ ,  $\theta < j(\kappa)$ ,  $|\mathbb{Q}| \leq j(\kappa)$ ,  $V_{j(\theta)}^{V[\mathbb{H}]} \subseteq M$  and  $j''j(\theta) \in M$  (see Lemma 5.1, (5)).

By Claim 5.2.4, we have (6.6):  $j(V_\theta^{\overline{W}}) (= V_{j(\theta)}^{j(\overline{W})}) = V_{j(\theta)}^{\overline{W}}$ . By the choice of  $\theta$ , we have (6.7):  $V_{j(\theta)}^V = V_{j(\theta)}^{\overline{W}}[\mathbb{G}]$ .

x-bedrock-24

x-bedrock-25

Since the underlying set of  $\mathbb{P}$  is  $\kappa$  we have  $j(\mathbb{P}) \cap \kappa = \mathbb{P}$  and  $j(\mathbb{G}) \cap \kappa = \mathbb{G}$ . By (6.6) and (6.7)  $b = j(b) \in V_{j(\theta)}^V = \overset{\text{by (6.6) and (6.7)}}{=} j(V_\theta^{\overline{W}})[\mathbb{G}] = j(V_\theta^{\overline{W}})[j(\mathbb{G} \cap \kappa)]$ , and since  $|\mathbb{P}| < \theta < j(\kappa)$ , we have

$$\begin{aligned} M \models “\text{there is } P \in j(\overline{W}) \text{ with } P \subseteq j(\mathbb{P}) \text{ and } j(\overline{W}) \models “|P| < j(\kappa)” \\ \text{such that } j(\overline{W})[j(\mathbb{G}) \cap P] \ni j(b)” \end{aligned}$$

By elementarity, it follows that

$$\begin{aligned} V \models “\text{there is } P \in \overline{W} \text{ and } G \subseteq P \text{ with } P \subseteq \mathbb{P} \text{ and } \overline{W} \models “|P| < \kappa” \\ \text{such that } G \text{ is } (\overline{W}, P)\text{-generic filter and } \overline{W}[\mathbb{G} \cap P] \ni j(b)” \end{aligned}$$

as desired.

□ (Proposition 6.2)

**Corollary 6.3** *Suppose  $\kappa$  is a tightly  $\mathcal{P}$ -gen. hyperhuge cardinal for a class  $\mathcal{P}$  of posets. Then we have  $2^{<\kappa} = \kappa$ .*

p-bedrock-11

**Proof.**  $\kappa$  is hyperhuge in the bedrock  $\overline{W}$  of  $V$  by Theorem 5.3. Thus,  $\overline{W} \models 2^{<\kappa} = \kappa$ . It follows that essentially there are at most  $\kappa$  many (nice)  $\mathbb{P}$ -names of elements of  $[\kappa]^{<\kappa}$  for posets  $\mathbb{P}$  of size  $<\kappa$ . Since all elements of  $([\kappa]^{<\kappa})^V$  are realizations of names of this kind by Proposition 6.2, it follows that  $V \models 2^{<\kappa} = \kappa$ .  $\square$  (Corollary 6.3)

Hamkins [29] proved that if  $\mathcal{P}$  contains  $\text{Col}(\omega, \lambda)$  for every  $\lambda$ , and  $(\mathcal{P}, \mathcal{H}(\aleph_1))$ -RcA holds, then we have  $L_\kappa \prec L$  for  $\kappa = \omega_1$ . Practically the same proof concludes that the mantle  $W$  (the intersection of all grounds which is shown to be a model of ZFC in [42]) also satisfies  $V_\kappa^W \prec W$  for  $\kappa = \omega_1^V$ . Note that, since grounds are downward directed, the mantle of the universe  $V$  is also the mantle of any ground  $W$  of  $V$ .

Actually, we can say a little bit more.

**Lemma 6.4** *Let  $W^*$  be the mantle of  $V$ . (1) Suppose that  $(\mathcal{P}, \mathcal{H}(2^{\aleph_0})^V)$ -RcA holds for a class  $\mathcal{P}$  of posets such that either  $\mathcal{P}$  contains posets collapsing  $\lambda$  to be countable for cofinally many cardinals  $\lambda$ , or it contains posets forcing  $2^{\aleph_0}$  arbitrary large adding reals without collapsing cardinals below the number of the reals added.*

*p-bedrock-12*

*Then  $V_\alpha^{W^*}$  for all  $\alpha < 2^{\aleph_0}$  is of cardinality  $< 2^{\aleph_0}$  in  $V$ .*

*(2) Suppose that  $(\mathcal{P}, \mathcal{H}(\omega_1)^V)$ -RcA holds for a class  $\mathcal{P}$  of posets such that  $\mathcal{P}$  contains posets collapsing  $\lambda$  to be countable for cofinally many cardinals  $\lambda$ .*

*Then  $V_\alpha^{W^*}$  for all  $\alpha < (\omega_1)^V$  is countable in  $V$ .*

**Proof.** (1): For  $\alpha < 2^{\aleph_0}$ , “ $V_\alpha^{W^*}$  is of cardinality  $< 2^{\aleph_0}$ ” can be formulated as an  $\mathcal{L}_\in$ -formula with the parameter  $\alpha \in \mathcal{H}(2^{\aleph_0})$ , and it is forcable by a poset in  $\mathcal{P}$ . By  $(\mathcal{P}, \mathcal{H}(2^{\aleph_0})^V)$ -RcA, there is a ground  $W$  of  $V$  ( $W^* \subseteq W \subseteq V$ ) such that the statement above holds in  $W$ . Then  $W \models$  “ $V_\alpha^{W^*}$  is of cardinality  $< 2^{\aleph_0}$ ”, and hence  $V \models$  “ $V_\alpha^{W^*}$  is of cardinality  $< 2^{\aleph_0}$ ”.

(2): A proof similar to that of (1) will do.

For  $\alpha < \omega_1$ , “ $V_\alpha^{W^*}$  is of cardinality  $< \aleph_1$ ” is a statement represented as an  $\mathcal{L}_\in$ -formula with the parameter  $\alpha \in \mathcal{H}(\aleph_1)$ , and it is forcable by a poset in  $\mathcal{P}$ . By  $(\mathcal{P}, \mathcal{H}(\omega_1)^V)$ -RcA, there is a ground  $W$  of  $V$  ( $W^* \subseteq W \subseteq V$ ) such that the statement above holds in  $W$ . Then  $W \models$  “ $V_\alpha^{W^*}$  is countable”. and hence  $V \models$  “ $V_\alpha^{W^*}$  is countable”.  $\square$  (Lemma 6.4)

Compare the following proposition with Lemma 4.1:

**Proposition 6.5** *Let  $W^*$  be the mantle of  $V$ . (1) Suppose that  $(\mathcal{P}, \mathcal{H}(2^{\aleph_0})^V)$ -RcA holds for  $\mathcal{P}$  as in Lemma 6.4, (1). That is, either  $\mathcal{P}$  contains posets collapsing  $\lambda$  to be countable for cofinally many cardinals  $\lambda$ , or it contains posets forcing  $2^{\aleph_0}$  arbitrary large adding reals without collapsing small cardinals.*

*p-bedrock-13*

Then we have  $V_\kappa^{W^*} \prec W^*$  for  $\kappa = (2^{\aleph_0})^\vee$ .

(2) Suppose that  $(\mathcal{P}, \mathcal{H}((\aleph_1)^\vee))$ -RcA holds for a class  $\mathcal{P}$  of posets such that  $\mathcal{P}$  contains posets collapsing  $\lambda$  to make it countable for cofinally many cardinals  $\lambda$  in On.

Then we have  $V_\kappa^{W^*} \prec W^*$  for  $\kappa = \omega_1^\vee$ .

**Proof.** (1): We show that  $(V_{(2^{\aleph_0})^\vee})^{W^*}$  in  $W^*$  passes the Vaught's test. Suppose  $W^* \models \varphi(\bar{a}, b)$  where  $\bar{a} \in (V_{(2^{\aleph_0})^\vee})^{W^*}$  and  $b \in W^*$ . We have  $\bar{a} \in \mathcal{H}(2^{\aleph_0})^\vee$  by Lemma 6.4, (1). The statement

$$\psi := \exists y(y \in W^* \wedge |trcl(y)| < 2^{\aleph_0} \wedge W^* \models \varphi(\bar{a}, y))$$

is an  $\mathcal{L}_\in$ -formula the parameters  $\bar{a} \in \mathcal{H}(2^{\aleph_0})^\vee$  and forcable by a poset in  $\mathcal{P}$  (just by collapsing a large enough cardinal to countable). By  $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))$ -RcA, it follows that there is a ground  $W$  of  $V$  (so  $W^* \subseteq W \subseteq V$ ) such that (6.8):  $W \models \psi$ .

x-bedrock-26

Let  $b' \in W$  be a witness for (6.8). Then we have (6.9):  $W \models |trcl(b')| < 2^{\aleph_0}$ , and  $W^* \models \varphi(\bar{a}, b')$ .

x-bedrock-27

Since  $(2^{\aleph_0})^W \leq (2^{\aleph_0})^\vee$ , it follows that  $W^* \models |trcl(b')| < (2^{\aleph_0})^\vee$  by (6.9), and thus  $b' \in (V_{(2^{\aleph_0})^\vee})^{W^*}$ .

(2): can be proved similarly to (1).

We show that  $(V_{\omega_1^\vee})^{W^*}$  in  $W^*$  passes the Vaught's test. Suppose  $W^* \models \varphi(\bar{a}, b)$  where  $\bar{a} \in (V_{\omega_1^\vee})^{W^*}$  and  $b \in W^*$ . We have  $\bar{a} \in \mathcal{H}(\aleph_1)^\vee$  by Lemma 6.4, (2). The statement

$$\psi := \exists y(y \in W^* \wedge |trcl(y)| < \aleph_1 \wedge W^* \models \varphi(\bar{a}, y))$$

is  $\mathcal{L}_\in$ -formula with the parameters  $\bar{a} \in \mathcal{H}(\aleph_1)^\vee$  and forcable by a poset in  $\mathcal{P}$ . By  $(\mathcal{P}, \mathcal{H}(\aleph_1)^\vee)$ -RcA, it follows that there is a ground  $W$  of  $V$  (so  $W^* \subseteq W \subseteq V$ ) such that (N6.1):  $W \models \psi$ .

x-bedrock-26-0

Let  $b' \in W$  be a witness for (N6.1). Then we have (N6.2):  $V \models |trcl(b')| < \aleph_1$ , and  $W^* \models \varphi(\bar{a}, b')$ .

x-bedrock-27-0

Since  $\omega_1^W \leq \omega_1^\vee$ , it follows that  $W^* \models |trcl(b')| < (\aleph_1)^\vee$  by (N6.2), and thus  $b' \in (V_{\omega_1^\vee})^{W^*}$ .

□ (Proposition 6.5)

**Theorem 6.6** Suppose that  $\mathcal{P}$  is a class of posets and  $\kappa := (\omega_1)^\vee$  is tightly  $\mathcal{P}$ -gen. hyperhuge. Then the following are equivalent:

p-bedrock-16

- (a) (all posets,  $\mathcal{H}(\kappa)$ )-RcA holds.
- (b) (all posets,  $\mathcal{H}(\kappa)$ )-RcA<sup>+</sup> holds.
- (c)  $V_\kappa^{\bar{W}} \prec \bar{W}$  where  $\bar{W}$  is the bedrock of  $V$ .

**Proof.** (a)  $\Leftrightarrow$  (b): is trivial. (a)  $\Rightarrow$  (c): By Proposition 6.5, (2).

(c)  $\Rightarrow$  (b): Assume (6.10):  $V_\kappa^{\overline{W}} \prec \overline{W}$ , and  $\Vdash_{\mathbb{P}} \text{“}\varphi(\overline{a}^\vee)\text{”}$ , for a poset  $\mathbb{P}$ , an  $\mathcal{L}_\in$ -formula  $\varphi = \varphi(\overline{x})$ , and  $\overline{a} \in \mathcal{H}(\kappa)^\vee$ . x-bedrock-29

By Proposition 6.2, there are poset  $\mathbb{Q} \in \overline{W}$  with  $\overline{W} \models |\mathbb{Q}| < \kappa$ , and  $(\overline{W}, \mathbb{Q})$ -generic  $\mathbb{G} \in \mathbf{V}$  such that  $\overline{a} \in \overline{W}[\mathbb{G}]$ . Let  $\overline{a}$  be a  $\mathbb{Q}$ -name of  $\overline{a}$ .

By the choice of  $\varphi$  and  $\overline{a}$ , we have

$$\overline{W} \models \Vdash_{\mathbb{Q}} \text{“there is a poset } P \text{ which forces } \varphi(\overline{a}^\vee)\text{”}.$$

By the elementarity (6.10), it follows that  $V_\kappa^{\overline{W}} \models \Vdash_{\mathbb{Q}} \text{“there is a poset } P \text{ which forces } \varphi(\overline{a}^\vee)\text{”}$ . Thus, there is a  $\mathbb{Q}$ -name  $\mathbb{R} \in V_\kappa^{\overline{W}}$  such that  $\Vdash_{\mathbb{Q} * \mathbb{R}} \text{“}\varphi(\overline{a}^\vee)\text{”}$ .

We have  $\mathbb{Q} * \mathbb{R}, \mathcal{P}(\mathbb{Q} * \mathbb{R})^{\overline{W}} \in V_\kappa^{\overline{W}} = \mathcal{H}(\kappa)^{\overline{W}} \subseteq \mathcal{H}(\kappa)^\vee$ , and  $\kappa$  is  $\omega_1$  in  $\mathbf{V}$ . Hence, we can construct a  $(\overline{W}[\mathbb{G}], \mathbb{R}[\mathbb{G}])$ -generic  $\mathbb{G}' \in \mathbf{V}$  in  $\omega$  steps in  $\mathbf{V}$ .  $\overline{W}[\mathbb{G}][\mathbb{G}']$  is a ground in  $\mathbf{V}$  and  $\overline{W}[\mathbb{G}][\mathbb{G}'] \models \varphi(\overline{a})$ . □ (Theorem 6.6)

The following corollary is proved similarly to Corollary 4.2.

**Corollary 6.7** *Suppose that  $\mathcal{P}$  is a class of posets and  $\omega_1$  is tightly  $\mathcal{P}$ -gen. hyperhuge. If (all posets,  $\mathcal{H}(\aleph_1)$ )-RcA holds then there are stationarily many hyperhuge cardinals. More precisely, under this condition, for any club subclass  $\mathcal{C}$  of On defined with a parameter, there is a hyperhuge cardinal in  $\mathcal{C}$ . In particular, there are class many hyperhuge cardinals.* p-bedrock-17

**Proof.** Since  $\mathbf{V}$  is a set generic extension of the bedrock  $\overline{W}$ , it is enough to show that there are stationarily many hyperhuge cardinals in  $\overline{W}$  (in the same sense as in the statement of the corollary).

Suppose this is not the case. Then there is an  $\mathcal{L}_\in$ -formula  $\Phi = \Phi(x, y)$  such that

$$\begin{aligned} \overline{W} \models & \text{“}\exists y (\Phi(\cdot, y) \text{ is a club in On} \\ & \text{but } \Phi(\cdot, y) \text{ does not contain any hyperhuge cardinal)”.} \end{aligned}$$

Since we have  $V_{(\omega_1)^\vee}^{\overline{W}} \prec \overline{W}$  by Theorem 6.6, it follows that

$$\begin{aligned} (\aleph 6.3) \quad V_{(\omega_1)^\vee}^{\overline{W}} \models & \text{“}\exists y (\Phi(\cdot, y) \text{ is a club in On} \\ & \text{but } \Phi(\cdot, y) \text{ does not contain any hyperhuge cardinal)”.} \end{aligned}$$
x-bedrock-30

Let  $b \in V_{(\omega_1)^\vee}^{\overline{W}}$  be a witness of (6.3). Then we have  $(\omega_1)^\vee \in \Phi(\cdot, b)$  by the closedness of  $\Phi(\cdot, b)$ . But this is a contradiction since  $(\omega_1)^\vee$  is a hyperhuge cardinal by Theorem 5.4. □ (Corollary 6.7)

The following proposition is a variation of Theorem 5.7 in [21].

**Proposition 6.8** *Suppose that  $\kappa$  is tightly  $\mathcal{P}$ -Laver gen. supercompact for an iterable class  $\mathcal{P}$  of posets. Then we have  $\text{MA}(\mathcal{P}, < \kappa)$ .*

*p-bedrock-14*

**Proof.** Suppose that  $\mathbb{P} \in \mathcal{P}$  and  $\mathcal{D}$  is a family of dense subsets of  $\mathbb{P}$  with  $|\mathcal{D}| < \kappa$ . Let  $\mathbb{Q}$  be a  $\mathbb{P}$ -name of a poset such that, for  $(\mathbb{V}, \mathbb{P} * \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $j$ ,  $M \subseteq \mathbb{V}[\mathbb{H}]$  such that  $j : \mathbb{V} \xrightarrow{\sim}_{\kappa} M$ ,  $\mathcal{P}, \mathbb{H} \in M$ ,  $j''\lambda \in M$ , and  $|\mathbb{P} * \mathbb{Q}| \leq j(\kappa)$ .

Note that  $j(\mathcal{D}) = \{j(D) : D \in \mathcal{D}\}$ . Let  $\mathbb{G}$  be the  $\mathbb{P}$  part of  $\mathbb{H}$ . We have  $\mathbb{G} \in M$ . Thus

$$\mathbb{G}^* = \{\mathbb{P} \in j(\mathbb{P}) : j(\mathbb{Q}) \leq_{j(\mathbb{P})} \mathbb{P} \text{ for some } \mathbb{Q} \in \mathbb{G}\}$$

is an element in  $M$ . Since  $\mathbb{G}^*$  is  $j(\mathcal{D})$ -generic filter over  $j(\mathbb{P})$ ,

$$M \models \exists G (G \text{ is a } j(\mathcal{D})\text{-generic filter over } j(\mathbb{P})).$$

By elementarity, it follows that

$$\mathbb{V} \models \exists G (G \text{ is a } \mathcal{D}\text{-generic filter over } \mathbb{P}).$$

□ (Proposition 6.8)

**Lemma 6.9** *Suppose that  $\kappa$  is a tightly  $\mathcal{P}$ -gen. hyperhuge cardinal for a class  $\mathcal{P}$  of posets, and  $a \in \mathcal{H}(\kappa)$  is such that  $\mathbb{V} \models \psi(a)$  for some  $\mathcal{L}_\in$ -formula  $\psi = \psi(x)$ . Let  $\overline{W}$  be the bedrock of  $\mathbb{V}$ .*

*p-bedrock-15*

*Then there is  $\mathbb{P}^* \in V_\kappa^{\overline{W}}$  with  $\overline{W} \models |\mathbb{P}^*| < \kappa$ , and  $(\overline{W}, \mathbb{P}^*)$ -generic  $\mathbb{G}^* \in \mathbb{V}$  such that  $a \in \overline{W}[\mathbb{G}^*]$ ,  $\overline{W}[\mathbb{G}^*] \models \psi(a)$ , and  $\overline{W}[\mathbb{G}^*]$  is a  $\mathcal{P}$ -ground of  $\mathbb{V}$ .*

**Proof.** Assume that  $\mathbb{V} = \overline{W}[\mathbb{G}]$  where  $\mathbb{G}$  is a  $(\overline{W}, \mathbb{P})$ -generic filter over a  $\mathbb{P} \in \overline{W}$  with  $\overline{W} \models |\mathbb{P}| \leq \kappa$  (Proposition 6.1). Without loss of generality, we shall assume that the underlying set of  $\mathbb{P}$  is  $\kappa$ .

Let  $\lambda > \kappa$  be such that (6.11):  $V_\lambda \prec_{\Sigma_n} \mathbb{V}$  for a sufficiently large  $n$ .<sup>7)</sup>

*x-bedrock-27-1*

Let  $\mathbb{Q} \in \mathcal{P}$  be such that, for  $(\mathbb{V}, \mathbb{Q})$ -generic  $\mathbb{H}$ , there are  $j$ ,  $M \subseteq \mathbb{V}[\mathbb{H}]$  such that  $j : \mathbb{V} \xrightarrow{\sim}_{\kappa} M$ ,  $j(\kappa) > \lambda$ , (6.12):  $V_{j(\lambda)}^{\mathbb{V}[\mathbb{H}]} \in M$ , and  $|\mathbb{Q}| = j(\kappa)$  (see Lemma 5.1, (5)).

*x-bedrock-28*

Note that we have  $V_{j(\lambda)}^M \prec_{\Sigma_n} M$  by elementarity. By Lemma 3.2, it follows that  $M \models V_{j(\lambda)} = V_{j(\lambda)}^{\mathbb{V}[\mathbb{H}]}$ . Also, by Lemma 3.2 and (6.12), we have  $V_{j(\lambda)}^{\mathbb{V}[\mathbb{H}]} = V_{j(\lambda)}^{\mathbb{V}[\mathbb{H}]} = V_{j(\lambda)}^M$ .

Thus, for  $\delta < \kappa$ , noting  $j(\delta) = \delta$ , we have

$$M \models \text{“} V_{j(\lambda)} \text{ is a } \mathcal{P}\text{-generic extension of a ground } W \text{ which is a model of } \psi(j(a)), \text{ and } W \text{ is a } \mathcal{P}\text{-generic extension of } V_{j(\lambda)}^{\overline{W}} \text{ for some poset } P \text{ of size } < j(\kappa)\text{.”}$$

<sup>7)</sup> Here “sufficiently large  $n$ ” refers, among other things, largeness in terms of Lemma 3.2, and the absoluteness of “ $\mathbb{P} \in \mathcal{P}$ ”.

By elementarity, it follows that

$$\begin{aligned} V \models & \text{“} V_\lambda \text{ is a } \mathcal{P}\text{-generic extension of a ground } W \text{ which is a model of} \\ & \psi(a), \text{ and } W \text{ is a } P\text{-generic extension of } V_\lambda^{\overline{W}} \text{ for some poset} \\ & P \text{ of size } < \kappa. \text{”} \end{aligned}$$

Now by the choice (6.11) of  $\lambda$ , it follows that

$$\begin{aligned} V & \text{ is a } \mathcal{P}\text{-generic extension of a ground } W \text{ which is a model of } \psi(a), \text{ and} \\ W & \text{ is a } \mathbb{P}^*\text{-generic extension of } \overline{W} \text{ for a poset } \mathbb{P}^* \text{ of size } < \kappa. \quad \square \text{ (Lemma 6.9)} \end{aligned}$$

A poset  $\mathbb{P}$  has pre-caliber  $\kappa$  if for any  $A \in [\mathbb{P}]^{\geq \kappa}$  there is  $B \in [A]^{\geq \kappa}$  such that  $B$  is centered (i.e.  $b$  has a lower bound in  $\mathbb{P}$  for and  $b \in [B]^{< \aleph_0}$ ).

**Lemma A 6.1** (Lemma III, 3.35, in [36])  *$\text{MA}(\aleph_1)$  implies that all ccc poset  $\mathbb{P}$  has pre-caliber  $\aleph_1$ .*

*p-bedrock-18*

**Proof.** Suppose  $\mathbb{p}_\alpha \in \mathbb{P}$  for  $\alpha < \omega_1$ . It is enough to find a filter containing uncountably many of  $\mathbb{p}_\alpha$ 's.

For each  $\alpha < \omega_1$ , let  $D_\alpha = \{\mathbb{q} \in \mathbb{P} : \mathbb{q} \leq_{\mathbb{P}} \mathbb{p}_\beta \text{ for some } \beta \geq \alpha\}$ .

If there is no  $\mathbb{s} \in \mathbb{P}$  such that uncountably many  $D_\alpha$ 's are dense below  $\mathbb{s}$ , we can construct an strictly increasing sequence  $\langle \beta_\alpha : \alpha < \omega_1 \rangle$  in  $\omega_1$  such that  $\mathbb{p}_{\beta_\alpha}$  is incompatible with all  $\mathbb{p}_{\beta_\gamma}$ ,  $\gamma < \alpha$ . But then  $\{\mathbb{p}_{\beta_\alpha} : \alpha < \omega_1\}$  is a pairwise incompatible uncountable set which is a contradiction to the ccc of  $\mathbb{P}$ .

Thus, there is  $\mathbb{s} \in \mathbb{P}$  such that there are uncountably many  $\alpha$ 's such that  $D_\alpha$  is dense below  $\mathbb{s}$ . Then since  $D_\alpha$ ,  $\alpha < \omega_1$  build a decreasing sequence,  $D_\alpha$  is dense below  $\mathbb{s}$  for all  $\alpha < \omega_1$ . Let  $\mathcal{D} := \{D_\alpha : \alpha < \omega_1\}$  and let  $\mathbb{G}$  be a  $\mathcal{D}$ -generic filter over  $\mathbb{P}$ . Then  $\mathbb{p}_\alpha \in \mathbb{G}$  for all  $\alpha \in I$  as desired.  $\square$  (Lemma A 6.1)

**Corollary A 6.2**  *$\text{MA}(\aleph_1)$  implies that, for all ccc posets  $\mathbb{P}$ ,  $\mathbb{Q}$ , the product  $\mathbb{P} \times \mathbb{Q}$  has the ccc.*

*p-bedrock-19*

**Proof.** It is enough to show that  $\mathbb{P} \times \mathbb{Q}$  has pre-caliber  $\aleph_1$  by Lemma A 6.1. Suppose that  $\langle \mathbb{p}_\alpha, \mathbb{q}_\alpha \rangle \in \mathbb{P} \times \mathbb{Q}$  for  $\alpha < \omega_1$ . Since  $\mathbb{P}$  has pre-caliber  $\aleph_1$ , there is  $I_0 \in [\omega_1]^{\aleph_1}$  such that  $\{\mathbb{p}_\alpha : \alpha \in I_0\}$  is centered in  $\mathbb{P}$ . Since  $\mathbb{Q}$  also has pre-caliber  $\aleph_1$  by Lemma A 6.1, there is  $I_1 \subseteq [I_0]^{\aleph_1}$  such that  $\{\mathbb{q}_\alpha : \alpha \in I_1\}$  is centered in  $\mathbb{Q}$ .  $\{\langle \mathbb{p}_\alpha, \mathbb{q}_\alpha \rangle : \alpha \in I_1\}$  is centered in  $\mathbb{P} \times \mathbb{Q}$ .  $\square$  (Corollary A 6.2)

The following Lemma is classical:

**Lemma 6.10** (1) *Suppose that  $V \models \text{“}\mathbb{P} \times \mathbb{Q} \text{ is ccc”}$  and  $\mathbb{G}$  is a  $(V, \mathbb{P})$ -generic filter, then  $V[\mathbb{G}] \models \text{“}\mathbb{Q} \text{ is ccc”}$ .*

*p-bedrock-20*

(2) *Suppose that  $V \models \text{MA}(\aleph_1)$ . If  $V \models \text{“}\mathbb{P} \text{ and } \mathbb{Q} \text{ are ccc”}$  and  $\mathbb{G}$  is a  $(V, \mathbb{P})$ -*

generic filter, then  $V[G] \models \text{"}\mathbb{Q} \text{ is ccc"}$ .

**Proof.** (1): Suppose otherwise and there is a  $f \in V[G]$  such that  $V[G] \models \text{"}f : \omega_1 \rightarrow \mathbb{Q} \text{ is such that, } f(\alpha), f(\beta) \text{ for distinct } \alpha, \beta < \omega_1 \text{ are pairwise incompatible"}$ .

Let  $\tilde{f}$  be a  $\mathbb{P}$ -name of  $f$  and let  $\mathbb{p}_\alpha \in \mathbb{P}$ ,  $\mathbb{q}_\alpha \in \mathbb{Q}$  for  $\alpha < \omega_1$  are such that  $\mathbb{p}_\alpha \Vdash_{\mathbb{P}} \text{"}\tilde{f}(\alpha) = \mathbb{q}_0\text{"}$ . Then  $\{\langle \mathbb{p}_\alpha, \mathbb{q}_\alpha \rangle : \alpha < \omega_1\}$  is pairwise incompatible in  $\mathbb{P} \times \mathbb{Q}$ . This is a contradiction to the assumption.

(2): By (1) and Corollary A 6.2.

□ (Lemma 6.10)

**Theorem 6.11** *Suppose that  $\kappa$  is tightly ccc-Laver-gen. hyperhuge. Then the following are equivalent:*

*p-bedrock-21*

- (a)  $(\text{ccc}, \mathcal{H}(\kappa))\text{-RcA}$  holds.
- (b)  $(\text{ccc}, \mathcal{H}(\kappa))\text{-RcA}^+$  holds.
- (c)  $V_\kappa^{\overline{W}} \prec \overline{W}$  where  $\overline{W}$  is the bedrock of  $V$ .

**Proof.** Note that  $\kappa = 2^{\aleph_0}$  by Theorem 5.8 in [21]. Thus, by Proposition 6.8, MA holds.

(b)  $\Rightarrow$  (a): is trivial. (a)  $\Rightarrow$  (c): By Proposition 6.5, (1).

(c)  $\Rightarrow$  (b): Suppose (in  $V$ ) that (6.13):  $\Vdash_{\mathbb{P}} \text{"}\varphi(\bar{a})\text{"}$  for a ccc poset  $\mathbb{P}$ ,  $\mathcal{L}_{\in-}$  formula  $\varphi = \varphi(\bar{x})$ , and  $\bar{a} \in \mathcal{H}(\kappa)$ .

*x-bedrock-31*

By Lemma 6.9 there is  $\mathbb{P}^* \in \overline{W}$  with  $\overline{W} \models |\mathbb{P}^*| < \kappa$  and  $(\overline{W}, \mathbb{P}^*)$ -generic  $\mathbb{G}^* \in V$  such that (6.14):  $\bar{a} \in \overline{W}[\mathbb{G}^*]$ , (6.15):  $\overline{W}[\mathbb{G}^*] \models \text{MA}$ , and (6.16):  $\overline{W}$  is a ccc-ground of  $V$ .

*x-bedrock-32 x-bedrock-33*

*x-bedrock-34*

By (6.13) and (6.16), we have

$$\overline{W}[\mathbb{G}^*] \models \text{"there is a ccc poset } P \text{ such that } \Vdash_P \text{"}\varphi(\bar{a})\text{"}."$$

By the assumption (c), we have (6.17):  $V_\kappa^{\overline{W}[\mathbb{G}^*]} \prec \overline{W}[\mathbb{G}^*]$ . Hence

*x-bedrock-35*

$$(6.18) \quad V_\kappa^{\overline{W}[\mathbb{G}^*]} \models \text{"there is a ccc poset } P \text{ such that } \Vdash_P \text{"}\varphi(\bar{a})\text{"}."$$

*x-bedrock-36*

Let  $\mathbb{Q}^*$  be a witness of (6.18). Then  $\overline{W}[\mathbb{G}^*] \models |\mathbb{Q}^*| < \kappa$  and  $\overline{W}[\mathbb{G}^*] \models \text{"}\mathbb{Q}^* \text{ is ccc"}$  by (6.17). By (6.16) and Lemma 6.10, we have  $V \models \text{"}\mathbb{Q}^* \text{ is ccc"}$ . Since  $\mathcal{P}(\mathbb{Q}^*)^{\overline{W}[\mathbb{G}^*]} \in V_\kappa^{\overline{W}[\mathbb{G}^*]}$  and  $\kappa = 2^{\aleph_0}$  in  $V$ , MA (in  $V$ ) implies that there is a  $(\overline{W}, \mathbb{Q}^*)$ -generic  $\mathbb{H}^* \in V$ .

$\overline{W}[\mathbb{G}^*][\mathbb{H}^*] \models \varphi(\bar{a})$  since  $\mathbb{Q}^*$  is a witness of (6.18), and  $\overline{W}[\mathbb{G}^*][\mathbb{H}^*]$  is a ccc-ground of  $V$  by Lemma 6.10.

□ (Theorem 6.11)

## 7 The Laver-Generic Maximum

*LGM*



Suppose that  $\kappa_0 < \kappa_1 < \kappa^*$  are regular cardinals such that  $V_{\kappa^*} \models \text{ZFC}$  and  $\kappa_0, \kappa_1$  are super- $C^{(\infty)}$ -hyperhuge cardinals in  $V_{\kappa^*}$ . Note that the existence of one 2-huge cardinal imply the consistency of this constellation (see Lemma 4.5).

Consider the following construction:

first make  $\kappa_0$  a tightly super- $C^{(\infty)}$ -all posets-Laver gen. hyperhuge by a poset of size  $\kappa_0$  as in Theorem 4.7, (4). Then we force  $\kappa_1$  to be a tightly super- $C^{(\infty)}$ -semi-proper-Laver gen. hyperhuge by a poset of size  $\kappa_1$  as in Theorem 4.7, (2').

The resulting model satisfies:

(7.1) ZFC

x-bedrock-37

- + “ $\omega_2 = 2^{\aleph_0}$  is the tightly super- $C^{(\infty)}$ -semi-proper-Laver gen. hyperhuge cardinal”
- + “There is a semi-proper ground  $W$  of the universe  $V$  such that  $(2^{\aleph_0})^W = \omega_1 = \omega_1^V$  is the tightly super- $C^{(\infty)}$ -all-posets-Laver gen. hyperhuge cardinal in  $W$ ”.

We want to call (7.1) Laver Generic Maximum (LGM), of cause not because of the maximality of possible consistency strength among similar assertions (this is not true since we can also switch in some other notion of large cardinal stronger than the hyperhugeness) but rather because this combination of the properties implies that practically all set-theoretic assertions known to be consistent with ZFC are realized either as consequences of (7.1) or as theorems in (many of) grounds of  $V$  or some other inner models of  $V$ .

So we have under the LGM (7.1) that

- The bedrock  $\overline{W}$  exists (Theorem 5.2).
- $\omega_1^V$  and  $\omega_2^V$  are super- $C^{(\infty)}$ -hyperhuge cardinals in  $\overline{W}$  (Theorem 5.8).
- $V_{(\omega_1)^V}^{\overline{W}} \prec \overline{W}$ . (Theorem 4.1).
- $V_{(\omega_2)^V}^{\overline{W}} \prec \overline{W}$ . (Theorem 4.1).
- (semi-proper,  $\mathcal{H}(2^{\aleph_0})$ )- $\text{RcA}^+$  holds (Theorem 4.10).
- (all posets,  $\mathcal{H}(\aleph_1)^{\overline{W}}$ )- $\text{RcA}^+$  holds (Theorem 4.10, and since  $\text{RcA}^+$  for all posets with the same parameters is preserved by generic extensions).
- For each natural number  $n$  there are stationarily many super- $C^{(n)}$ -hyperhuge cardinals (this holds in  $\overline{W}$  by Corollary 4.2, and since the statement is preserved by set forcing, it also holds in  $V$ ).

- For any natural number  $n$  and any  $a \in \mathcal{H}(2^{\aleph_0})$ , there is a semi-proper ground  $W$  with  $a \in W$  such that  $W \models “2^{\aleph_0}$  is the tightly super- $C^{(n)}$ -ccc-Laver gen. hyperhuge cardinal”. (there is a super- $C^{(n)}$ -hyperhuge cardinal  
by the previous item and hence we can force the statement by a ccc forcing (by a variation of Theorem 4.7, (3)). (semi-proper,  $\mathcal{H}(2^{\aleph_0})$ )- $\text{RcA}^+$  now implies the existence of a semi-proper ground  $W$  with  $a \in W$  satisfying the statement).
- Unbounded Resurrection Axiom of Tsaprounis in [40] for semi-proper (see [17]).
- $\text{MM}^{++}$  (by Theorem 5.7 in [21]).
- $2^{\aleph_0} = \omega_2$  (either by  $\text{MM}^{++}$  or by Theorem 3.3, (4)).
- For any known instance  $\text{CM}$  of Cichoń’s Maximum (even one of those in which some mild large cardinals are involved) and any  $a \in \mathcal{H}(\aleph_2)$  there is a semi-proper ground  $W$  with  $a \in W$  such that  $W \models \text{CM}$  (by (semi-proper,  $\mathcal{H}(2^{\aleph_0})$ )- $\text{RcA}^+$ ).
- ...

If we can accept the tightly super- $C^{(\infty)}$ -semi-proper-Laver gen. hyperhuge continuum as a natural and/or even desirable set-theoretic assumption, (7.1) may be considered as the final solution to the continuum problem (and actually much more) in terms of the properties listed above (for a discussion about an argument supporting the naturalness of the tightly super- $C^{(\infty)}$ -semi-proper-Laver gen. hyperhuge continuum, see also section 2 of [18]).

On the other hand, it also seems that Theorem 6.6 and Theorem 6.11 suggest the naturalness of each of the combinations:

$$(7.2) \quad \begin{array}{l} \text{ZFC} \\ + “\omega_1 = 2^{\aleph_0} \text{ is the tightly super-} C^{(\infty)}\text{-all posets-Laver gen. hyperhuge} \\ \text{cardinal}”. \end{array} \quad \text{x-bedrock-37-0}$$

$$(7.3) \quad \begin{array}{l} \text{ZFC} \\ + “2^{\aleph_0} \text{ is the tightly super-} C^{(\infty)}\text{-ccc Laver gen. hyperhuge cardinal}” \\ + “\text{There is a ccc-ground } W \text{ of the universe } V \text{ such that } (2^{\aleph_0})^W = \omega_2^W \\ \text{is the tightly semi-proper-Laver gen. hyperhuge cardinal in } W” \\ + “\text{There is a semi-proper-ground } W \text{ of the universe } V \text{ such that} \\ (2^{\aleph_0})^W = \omega_1 = \omega_1^V \text{ is the tightly super-} C^{(\infty)}\text{-all-posets-Laver gen.} \\ \text{hyperhuge cardinal in } W”. \end{array} \quad \text{x-bedrock-38}$$

The combinations axioms (7.2) and (7.3) will be also examined further in the subsequent papers. Note that (7.3) implies that the continuum is extremely large (weakly Mahlo and much more, see [21], [26]), and that the Fodor-type Reflection Principle (FRP) holds (FRP holds in the ccc-ground  $W$  with the tightly semi-proper-Laver gen. hyperhuge continuum, since this implies  $MM^{++}$  in  $W$  (by Theorem 5.7 in [21])). Since FRP is preserved by ccc generic extension,  $V$  also satisfies it).

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