On Recurrence Axioms

渕野 昌　　薄葉 季路

Sakaé Fuchino* Toshimichi Usuba†

Abstract

The Recurrence Axiom for a class $\mathcal{P}$ of posets and a set A of parameters is an axiom scheme in the language of ZFC asserting that if a statement with parameters from A is forced by a poset in $\mathcal{P}$, then there is a ground containing the parameters and satisfying the statement.

The tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver generic hyperhuge continuum implies the Recurrence Axiom for $\mathcal{P}$ and $\mathcal{H}(2^{\aleph_0})$. The consistency strength of this assumption can be decided thanks to our main theorems asserting that the minimal ground (bedrock) exists under a tightly $\mathcal{P}$-generic hyperhuge cardinal κ, and that κ in the bedrock is genuinely hyperhuge, or even super $C^{(\infty)}$ hyperhuge if κ is a tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver generic hyperhuge definable cardinal.

The Laver Generic Maximum (LGM), one of the strongest combinations of axioms in our context, integrates practically all known set-theoretic principles and axioms in itself, either as its consequences or as theorems holding in (many) grounds of the universe. For instance, double plus version of Martin's Maximum is a consequence of LGM while Cichoń's Maximum is a phenomenon in many grounds of the universe under LGM.

* Graduate School of System Informatics, Kobe University
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan
fuchino@diamond.kobe-u.ac.jp

† Faculty of Fundamental Science and Engineering, Waseda University
Okubo 3-4-1, Shinjyuku, Tokyo, 169-8555 Japan
usuba@waseda.jp

Contents

1 Introduction

The Recurrence Axiom for a class $\mathcal{P}$ of posets and a set A of parameters is an axiom scheme in the language of ZFC asserting that if a statement $\varphi(\overline{a})$ with parameters $\overline{a}$ in A is forced by a poset $\mathbb{P} \in \mathcal{P}$, then there is a ground (i.e. an inner model from which the universe V is attainable via set forcing) containing the parameters and satisfying the statement $\varphi(\overline{a})$. theintro

Recurrence Axioms can be interpreted as statements about the (eternal?) recurrence in the set generic multiverse in terms of the time-flow along with set forcing extension: everything that can happen in the near future (in form of forcing extension) actually happened already in the past (in a ground). Here, the nearness of the future is measured in terms of extent of the class $\mathcal{P}$ of posets we may consider. The extent of the set of parameters which we may use in the descriptions of "events" in the future also differentiates the strength of the axiom.

Recurrence Axioms are actually variations of known axioms and principles: they are weakenings of Maximality Principles with corresponding parameters (Proposi-

---

Date: August 10, 2023 *Last update:* June 8, 2024 (10:06 JST)

MSC2020 Mathematical Subject Classification: 03E45, 03E50, 03E55, 03E57, 03E65

Keywords: generic large cardinal, Laver-generic large cardinal, Maximality Principles, Continuum Hypothesis

The research is supported by Kakenhi Grant-in-Aid for Scientific Research (C) 20K03717.

This is an extended version of the paper with the title "On Recurrence Axioms" in preparation.

All additional details not contained in the submitted version of the paper are either typeset in dark electric blue (the color in which this paragraph is typeset) or put in a separate appendices.

The most up-to-date pdf-file of this extended version is downloadable as:

https://fuchino.ddo.jp/papers/recurrence-axioms-x.pdf tion 2.2) while they can be characterized as the set-generic versions of Sy Friedman's Inner Ground Hypothesis (Proposition 2.3). See the end of Section 2 for discussions about why we want to keep these axioms in spite of this almost identity with other known principles.

In Section 3 we show that the tight Laver-generic ultrahugeness implies Σ_2-fragment of Recurrence Axioms (Theorem 3.1), and Σ_1-fragments of Recurrence Axioms with strong enough combination of $\mathcal{P}$ and A decide the size of the continuum: in case of $\mathcal{P}$ being the class of all posets with $A = \mathcal{H}(2^{\aleph_0})$ the Continuum Hypothesis (CH) holds while the Recurrence Axiom for stationary preserving $\mathcal{P}$ with $A = \mathcal{H}(\kappa_{\mathfrak{refl}})$ implies $2^{\aleph_0} = \aleph_2$ (Theorem 3.3).

In Section 4 we introduce the notion of tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver generic hyperhuge cardinal κ and show that Recurrence Axiom for $\mathcal{P}$ and $\mathcal{H}(\kappa)$ follows from the existence of this generic large cardinal (Theorem 4.10). The consistency of the existence of this cardinal is strictly between that of hyperhuge cardinal and a 2-huge cardinal (actually the lower bound can be still raised, see Corollary 5.11). This follows from the main theorems (Theorem 5.2, Theorem 5.3) in Section 5, asserting that the minimal ground (bedrock) under a tightly $\mathcal{P}$-generic hyperhuge cardinal κ exists and that κ in the bedrock is genuinely hyperhuge, or even super $C^{(\infty)}$ hyperhuge if κ is a tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver generic hyperhuge definable cardinal.

This result strengthen the theorem on the existence of bedrock by Usuba under a hyperhuge cardinal in [42].

After examining some of the consequences of the main theorems in Section 6, we discuss in Section 7 the Laver Generic Maximum (LGM), one of the strongest axiom available in our context, which integrates practically all known set-theoretic principles and axioms in itself, either as its consequences or as theorems in (many of) the grounds of the universe. So for example, double plus version of Martin's Maximum (MM^{++})is a consequence of LGM while Cichoń's Maximum is a phenomenon in many grounds of the universe under LGM.

We tried hard to make the present paper as accessible as possible for a wide audience. The terminology and notations used here are either standard or explained fully in the text. For some basic notions nevertheless left unexplained the reader may consult [33], [36] and/or [34].

2 Recurrence Axioms

intro

In the following, $\mathcal{L}_\in$ denotes the language of ZFC consisting of single binary relation symbol $\in$.

In the language of ZFC, we always identify a class $\mathcal{P}$ with the $\mathcal{L}_\in$-formula which defines the class. Thus, if a class $\mathcal{P}$ is defined by an $\mathcal{L}_\in$-formula $\psi(x)$, with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{x} \in \mathcal{P} \ldots$" for a poset $\mathbb{P}$, we simply mean $\Vdash_{\mathbb{P}}$ "$\psi(\underset{\sim}{x}) \cdots$". We adopt model theoretic convention that (in connection with lower case letters) a letter with bar denotes a tuple of objects. Thus, $\overline{a}$ means $a_0, ..., a_{k-1}$ for some natural number k and write $\overline{a} \in X$ for $a_0,..., a_{k-1} \in X$.

We call a class $\mathcal{P}$ of posets *normal* if it satisfies (2.1) : $\{\mathbb{1}\} \in \mathcal{P}$, and x-intro-0-0
(2.2) : $\mathcal{P}$ is closed with respect to forcing equivalence (i.e. if $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P} \sim \mathbb{P}'$ x-intro-1
then $\mathbb{P}' \in \mathcal{P}$).

In the following we assume that all classes $\mathcal{P}$ of posets we consider are normal. In particular when we say that $\mathcal{P}$ is a class of posets we assume that $\mathcal{P} \neq \emptyset$ and it contains the trivial poset.

Some natural classes of posets are not closed under forcing equivalence — notably the class of σ-closed posets. For such classes we simply take the closure of the class with respect to forcing equivalence and replace the class with the closure without mentioning it.

A (normal) class of posets is *iterable* if it also satisfies (2.3) : closed with x-intro-2
respect to restriction (i.e. if $\mathbb{P} \in \mathcal{P}$ then $\mathbb{P} \restriction \mathbb{p} \in \mathcal{P}$ for any $\mathbb{p} \in \mathbb{P}$), and (2.4) :
for any $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$, $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" implies $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathcal{P}$. x-intro-3

For a class $\mathcal{P}$ of posets and a set A (of parameters), *$\mathcal{P}$-Recurrence Axiom with parameters from A* (($\mathcal{P}, A$)-RcA, for short) is the following assertion formally expressed as an axiom scheme in $\mathcal{L}_\in$:

(2.5) For any $\mathcal{L}_\in$-formula $\varphi(\overline{x})$ and $\overline{a} \in A$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$", then there is a ground M of V such that $\overline{a} \in M$ and $M \models \varphi(\overline{a})$. x-intro-4

Here, an inner model W_0 of a universe W is said to be a *ground* of W if there are a poset $\mathbb{P} \in \mathsf{W}_0$ and $(\mathsf{W}_0, \mathbb{P})$-generic $\mathbb{G} \in \mathsf{W}$ such that $\mathsf{W}_0[\mathbb{G}] = \mathsf{W}$. W_0 is a *$\mathcal{P}$-ground* of W if $\mathbb{P}$ as above can be taken such that $\mathsf{W}_0 \models$ "$\mathbb{P} \in \mathcal{P}$".

All such grounds are definable. More precisely, the following theorem holds:

Theorem 2.1 (Reitz [38], Fuchs-Hamkins-Reitz [28]) *There is an $\mathcal{L}_\in$-formula $\Phi(x, r)$ such that the following is provable in* ZFC*:* *p-intro-1*

(2.6) *for all r, $\Phi(\cdot, r) := \{x : \Phi(s, r)\}$ is a ground in* V*, and* *x-intro-10*

(2.7) *for any ground M (in* V*), there is r such that $M = \Phi(\cdot, r)$.* ❑ *x-intro-11*

In the following we use this fact often without explicitly mentioning it. As a corollary to Theorem 2.1, we immediately see that being a $\mathcal{P}$-ground for a class $\mathcal{P}$ of posets is a definable property.

The *Strong $\mathcal{P}$-Recurrence Axiom with parameters from A* ($(\mathcal{P}, A)$-RcA^+, for short) holds, if (2.5) holds with M which is a $\mathcal{P}$-ground of V.

Actually the Strong Recurrence Axiom is equivalent to an already known axiom: In the following Proposition 2.2, we show that, for any (normal) $\mathcal{P}$, $(\mathcal{P}, A)$-RcA^+ is equivalent to the Maximality Principle $\mathsf{MP}(\mathcal{P}, A)$ in the notation of [17] (see below for definition — the characterization of $\mathsf{MP}(\mathcal{P}, A)$ corresponding to this proposition as well as the statements corresponding to Proposition 2.3 and Proposition 2.4 were also observed by Barton, Caicedo, Fuchs, Hamkins, Reitz, and Schindler [2]).

For a class $\mathcal{P}$ of posets, an $\mathcal{L}_\in$-formula $\varphi(\overline{a})$ with parameters $\overline{a}$ ($\in \mathsf{V}$) is said to be a *$\mathcal{P}$-button* if there is $\mathbb{P} \in \mathcal{P}$ such that for any $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of poset with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", we have $\Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}}$ "$\varphi(\overline{a}^{\checkmark})$".

If $\varphi(\overline{a})$ is a $\mathcal{P}$-button then we call $\mathbb{P}$ as above a *push of the $\mathcal{P}$-button $\varphi(\overline{a})$*.

For a class $\mathcal{P}$ of posets and a set A (of parameters), the *Maximality Principle for $\mathcal{P}$ and A* ($\mathsf{MP}(\mathcal{P}, A)$, for short) is the following assertion which is formulated as an axiom scheme in $\mathcal{L}_\in$:

$\mathsf{MP}(\mathcal{P}, A)$: For any $\mathcal{L}_\in$-formula $\varphi(\overline{x})$ and $\overline{a} \in A$, if $\varphi(\overline{a})$ is a $\mathcal{P}$-button then $\varphi(\overline{a})$ holds.

Proposition 2.2 *Suspect that $\mathcal{P}$ is a class of posets and A a set (of parameters).* p-intro-0

(1) $(\mathcal{P}, A)$-RcA^+ *is equivalent to* $\mathsf{MP}(\mathcal{P}, A)$.

(2) $(\mathcal{P}, A)$-RcA *is equivalent to the following assertion:*

(2.8) *For any $\mathcal{L}_\in$-formula $\varphi(\overline{x})$ and $\overline{a} \in A$, if $\varphi(\overline{a})$ is a $\mathcal{P}$-button then $\varphi(\overline{a})$ holds in a ground of* V. x-intro-5-0

Proof. (1): Suppose first that $(\mathcal{P}, A)$-RcA^+ holds. We show that $\mathsf{MP}(\mathcal{P}, A)$ holds. Assume that $\mathbb{P} \in \mathcal{P}$ is a push of the $\mathcal{P}$-button $\varphi(\overline{a})$. Let $\varphi'(\overline{x})$ be the formula expressing

(2.9) for any $\mathbb{Q} \in \mathcal{P}$, $\Vdash_{\mathbb{Q}}$ "$\varphi(\overline{x}^{\checkmark})$" holds. x-intro-6

Then we have $\Vdash_{\mathbb{P}}$ "$\varphi'(\overline{a}^{\checkmark})$". By $(\mathcal{P}, A)$-RcA^+, there is a $\mathcal{P}$-ground M of V such that $\overline{a} \in M$ and $M \models \varphi'(\overline{a})$ holds. By the definition (2.9) of φ', it follows that $\mathsf{V} \models \varphi(\overline{a})$ holds.

Now suppose that $\mathsf{MP}(\mathcal{P}, A)$ holds and $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$" for $\overline{a} \in A$.

Let φ'' be a formula claiming that

(2.10) there is a $\mathcal{P}$-ground N such that $\overline{x} \in N$ and $N \models \varphi(\overline{x})$. x-intro-7

Then $\varphi''(\overline{a})$ is a $\mathcal{P}$-button and $\mathbb{P}$ is its push.

By $\mathsf{MP}(\mathcal{P}, A)$, $\varphi''(\overline{a})$ holds in V and hence there is a $\mathcal{P}$-ground M of V such that $\overline{a} \in M$ and $M \models \varphi(\overline{a})$. This shows that $(\mathcal{P}, A)$-RcA^+ holds.

(2): can be proved similarly to (1). Suppose first that $(\mathcal{P}, A)$-RcA holds. We show that (2.8) holds. Assume that $\mathbb{P} \in \mathcal{P}$ is a push of the $\mathcal{P}$-button $\varphi(\overline{a})$. Let $\varphi'(\overline{x})$ be the formula expressing

(2.11) for any $\mathbb{Q} \in \mathcal{P}$, $\Vdash_{\mathbb{Q}}$ "$\varphi(\overline{x}^{\checkmark})$" holds. x-intro-8

Then we have $\Vdash_{\mathbb{P}}$ "$\varphi'(\overline{a}^{\checkmark})$". By $(\mathcal{P}, A)$-RcA, there is a ground M of V such that $\overline{a} \in M$ and $M \models \varphi'(\overline{a})$ holds. Since $\mathcal{P} \ni \{\mathbb{1}\}$ (see the remark in the paragraph after (2.1)), it follows that $M \models \varphi(\overline{a})$.

Now suppose that (2.8) holds and $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$" for $\overline{a} \in A$. Let φ'' be a formula asserting that

(2.12) there is a $\mathcal{P}$-ground N such that $\overline{x} \in N$ and $N \models \varphi(\overline{x})$. x-intro-9

Then $\varphi''(\overline{a})$ is a $\mathcal{P}$-button and $\mathbb{P}$ is its push. Thus, By (2.8), $\varphi''(\overline{a})$ holds in a ground M of V with $\overline{a} \in M$. By the definition (2.12) of φ'', there is a $\mathcal{P}$-ground N of M such that $\overline{a} \in N$ and $N \models \varphi(\overline{a})$. Since N is also a ground of V, this shows that $(\mathcal{P}, A)$-RcA holds. ❑ (Proposition 2.2)

Recurrence Axioms are also related to the Inner Model Hypothesis introduced by Sy Friedman in [6]. *The Inner Model Hypothesis* (IMH) is the following assertion formulated in the language of second-order set theory (e.g. in the context of von Neumann-Bernays-Gödel set theory):

IMH : For any statement φ without parameters, if φ holds in an inner model of an inner extension of V then φ holds in an inner model of V.

Here we say a (not necessarily first-order definable) transitive class M an *inner model* of V if M is a model of ZF and $\mathrm{On}^M = \mathrm{On}^{\mathsf{V}}$. In the perspective from such M, we call V an *inner extension* of M.

We shall call a set-forcing version of this principle *Inner Ground Hypothesis* (IGH):

For a (definable normal) class $\mathcal{P}$ of posets and a set A (of parameters),

$\mathsf{IGH}(\mathcal{P}, A)$: For any $\mathcal{L}_{\in}$-formula $\varphi = \varphi(\overline{x})$ and $\overline{a} \in A$, if $\mathbb{P} \in \mathcal{P}$ forces "there is a ground M with $\overline{a} \in M$ satisfying $\varphi(\overline{a})$", then there is a ground W of V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$.

Proposition 2.3 *For a class $\mathcal{P}$ of posets and a set A (of parameters), $(\mathcal{P}, A)$-RcA holds if and only if $\mathsf{IGH}(\mathcal{P}, A)$ holds.* p-intro-2

Proof. Suppose that $(\mathcal{P}, A)$-RcA holds. Let $\varphi = \varphi(\overline{x})$ be an $\mathcal{L}_\in$-formula, $\overline{a} \in A$, and $\mathbb{P} \in \mathcal{P}$ be such that $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$ holds in a ground".

Let $\varphi'(\overline{x})$ be the $\mathcal{L}_\in$-formula asserting that $\varphi(\overline{x})$ holds in a ground. Then $\Vdash_{\mathbb{P}}$ "$\varphi'(\overline{a}^{\checkmark})$". By $(\mathcal{P}, A)$-RcA, it follows that there is a ground W of V such that $\mathsf{W} \models \varphi'(\overline{a}^{\checkmark})$. Since a ground of a ground is a ground, we conclude that there is a ground W_0 of V such that $\overline{a} \in M_0$ and $\mathsf{W}_0 \models \varphi(\overline{a})$. This shows that $\mathsf{IGH}(\mathcal{P}, A)$ holds.

Suppose now that $\mathsf{IGH}(\mathcal{P}, A)$ holds. Assume that $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$" for an $\mathcal{L}_\in$-formula $\varphi = \varphi(\overline{x})$, $\overline{a} \in A$, and $\mathbb{P} \in \mathcal{P}$. Then $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$ holds in a $\mathcal{P}$-ground (of the universe)" since $\Vdash_{\mathbb{P}}$ "$\{\mathbb{1}\} \in \mathcal{P}$". Thus, by $\mathsf{IGH}(\mathcal{P}, A)$, there is a ground W of V such that $\mathsf{W} \models \varphi(\overline{a})$. ❑ (Proposition 2.3)

$(\mathbb{P}, A)$-RcA^+ ($\Leftrightarrow \mathsf{MP}(\mathcal{P}, A)$ by Proposition 2.2, (1)) can be also characterized in terms of a strengthening of Inner Ground Hypothesis: For a (definable) class $\mathcal{P}$ of posets and a set A (of parameters),

$\mathsf{IGH}^+(\mathcal{P}, A)$: For any $\mathcal{L}_\in$-formula $\varphi = \varphi(\overline{x})$ and $\overline{a} \in A$ if $\mathbb{P} \in \mathcal{P}$ forces "there is a $\mathcal{P}$-ground M with $\overline{a} \in M$ satisfying $\varphi(\overline{a})$", then there is a $\mathcal{P}$-ground W of V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$.

The following proposition can be proved similarly to Proposition 2.3.

Proposition 2.4 *For a class $\mathcal{P}$ of posets and a set A (of parameters), $(\mathcal{P}, A)$-RcA^+ holds if and only if $\mathsf{IGH}^+(\mathcal{P}, A)$ holds.* p-intro-3

Proof. Suppose that $(\mathcal{P}, A)$-RcA^+ holds and assume that $\varphi = \varphi(\overline{x})$ is an $\mathcal{L}_\in$-formula, $\overline{a} \in A$, and $\mathbb{P} \in \mathcal{P}$ is such that

$$\Vdash_{\mathbb{P}} \text{"}\varphi(\overline{a}^{\checkmark}) \text{ holds in a } \mathcal{P}\text{-ground } M \text{ with } \overline{a} \in M\text{"}$$

Let $\varphi'(\overline{a})$ be the formula expressing "$\varphi(\overline{x})$ holds in a $\mathcal{P}$-ground M with $\overline{a} \in M$". Then $\mathbb{P}$ is a push of the $\mathbb{P}$-button $\varphi'(\overline{a})$. Thus, by Proposition 2.2, (1), $\varphi'(\overline{a})$ holds in V. By definition of φ', there is a $\mathcal{P}$-ground W of V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$. This shows that $\mathsf{IGH}^+(\mathcal{P}, A)$ holds.

Suppose now that $\mathsf{IGH}^+(\mathcal{P}, A)$ holds, and assume that $\varphi = \varphi(\overline{x})$ is an $\mathcal{L}_\in$-formula, $\overline{a} \in A$ and $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$" then (since $\{\mathbb{1}\} \in \mathcal{P}$) $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$ holds in a $\mathcal{P}$-ground M with $\overline{a} \in M$". By $\mathsf{IGH}^+(\mathcal{P}, A)$, it follows that there is $\mathcal{P}$-ground W of V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$. This shows that $(\mathcal{P}, A)$-RcA^+ holds.

❑ (Proposition 2.4)

In spite of these alsmost identity with other known principles, we want to keep the Recurrence Axioms as axioms on their own. One of the reasons is that the formulation of the principle emphasizes the downward absoluteness feature of the principle. Another is that we have the following monotonicity for (non plus versions) these axioms which is not valid with the Maximality Principles (i.e. the plus versions of the Recurrence Axioms).

Lemma 2.5 (Monotonicity of Recurrence Axioms) *For classes of posets $\mathcal{P}$, $\mathcal{P}'$ and sets A, A' of parameters, if $\mathcal{P} \subseteq \mathcal{P}'$ and $A \subseteq A'$, then we have* p-5

$$(\mathcal{P}', A')\text{-}\mathsf{RcA} \;\Rightarrow\; (\mathcal{P}, A)\text{-}\mathsf{RcA}. \qquad \square$$

If we decide that the Recurrence Axioms provide desirable extensions of the axioms of ZFC, then we should try to take the maximal instance of these axioms. (i.e. the one with maximal strength among the instances consistent with ZFC) By Lemma 2.5, this means we should try to take the instance of Recurrence Axioms with the maximal $\mathcal{P}$ and A (with respect to inclusion) among the consistent ones.

Theorem 3.3 in the next section suggests that the following (2.13) and (2.14) are candidates of such maximal instances.

Let $\kappa_{\mathfrak{refl}}$ denote the cardinal number $\max\{2^{\aleph_0}, \aleph_2\}$. $\kappa_{\mathfrak{refl}}$ appears often as the reflection point of a strong structural reflection principle (see [17]).

(2.13) ZFC + $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))$-RcA for the class $\mathcal{P}$ of all stationary preserving posets. x-intro-12

(2.14) ZFC + $(\mathcal{Q}, \mathcal{H}(2^{\aleph_0}))$-RcA for the class $\mathcal{Q}$ of all posets. x-intro-13

The consistency of (2.14) follows from the consistency of ZFC + "there are stationarily many inaccessible cardinals" ([29]). The consistency of (2.13) follows from Lemma 4.5, Theorem 4.7, (B'), and Theorem 4.10.

The maximality of (2.13) and (2.14) follows from Lemma 3.3, (2') and (5') respectively.

By Lemma 3.3, (4) and (5), (2.13) implies $2^{\aleph_0} = \aleph_2$, and (2.14) implies CH. In particular, these two extensions of ZFC are not compatible. However, as we are going to discuss in Section 7, we can combine the plus version of (2.13) with the following weakening of (2.14):

(2.15) ZFC + $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))$-$\mathsf{RcA}^+$ + $(\mathcal{Q}, \mathcal{H}(\omega_1)^{\overline{\mathsf{W}}})$-$\mathsf{RcA}^+$ where $\mathcal{P}$ is the class of all semi-proper posets, $\mathcal{Q}$ the class of all posets, and $\overline{\mathsf{W}}$ the bedrock[1)] which is also assumed here to exist. x-intro-14

Note that $2^{\aleph_0} = \aleph_2$ follows from (2.15). In Section 7, we give an axiom in terms of existence of strong variants of Laver generic large cardinals from which (2.15) follows.

3 Recurrence Axioms in restricted forms and the Continuum Problem

We consider the following restricted forms of Recurrence Axiom: For an iterable class $\mathcal{P}$ of posets, a set A (of parameters), and a set Γ of $\mathcal{L}_\in$- formulas, *$\mathcal{P}$-Recurrence Axiom for formulas in Γ with parameters from A* ($(\mathcal{P}, A)_\Gamma$-RcA, for short) is the following assertion expressed as an axiom scheme in $\mathcal{L}_\in$: Lg-RcA

(3.1) For any $\varphi(\overline{x}) \in \Gamma$ and $\overline{a} \in A$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$", then there is a ground W of V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$. x-Lg-RcA-a-0

$(\mathcal{P}, A)_\Gamma$-RcA^+ corresponding to $(\mathcal{P}, A)$-RcA^+ is defined similarly.

Recall that a cardinal κ is *ultrahuge* if for any $\lambda > \kappa$, there are j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, $j(\kappa) > \lambda$, ${}^{j(\kappa)}M \subseteq M$ and $V_{j(\lambda)} \subseteq M$. ultrahuge

For an iterable class $\mathcal{P}$ of posets, a cardinal κ is said to be (*tightly*) *$\mathcal{P}$-Laver-generically ultrahuge* ((tightly) $\mathcal{P}$-Laver-gen. ultrahuge, for short), if, for any $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", such that for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are $j, M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, $j(\kappa) > \lambda$, $\mathbb{P}, \mathbb{H}, (V_{j(\lambda)})^{\mathsf{V}[\mathbb{H}]} \in M$ (and $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a poset of size $\leq j(\kappa)$).[2)]

By Theorem 6.5 in [17], any known Laver-generic large cardinal axiom (formalizable in a single formula) does not imply the full version of Recurrence Axiom (even for empty set as the set of parameters). However we have the following:

Theorem 3.1 *Suppose that κ is tightly $\mathcal{P}$-Laver-gen. ultrahuge for an iterable class $\mathcal{P}$ of posets. Then $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_2}$-$\mathsf{RcA}^+$ holds.* p-Lg-RcA-0

For the proof of Theorem 3.1, we use the following lemma which should be a well-known fact. Nevertheless, we prove the lemma since we could not find a suitable reference, and also because this lemma will be applied several times later in this article.

[1)] For the definition of the bedrock see Section 5.

[2)] In the following, we shall denote this condition simply by "$|\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,| \leq \lambda$". More generally, we shall always write "$|\,\mathbb{P}\,| \leq \lambda$" for a poset $\mathbb{P}$ to mean that "$\mathbb{P}$ is forcing equivalent to a poset of size $\leq \lambda$". fn-0

Lemma 3.2 *If α is a limit ordinal and V_α satisfies a sufficiently large finite fragment of* ZFC*, then for any $\mathbb{P} \in V_\alpha$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, we have $V_\alpha[\mathbb{G}] = {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$.* p-Lg-RcA-0-0

Proof. "$\subseteq$": This inclusion holds without the condition on the fragment of ZFC. Also the condition "$\mathbb{P} \in V_\alpha$" is irrelevant for this inclusion.

We show by induction on $\alpha \in \mathrm{On}$ that $V_\alpha[\mathbb{G}] \subseteq {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$ holds for all $\alpha \in \mathrm{On}$.

The induction steps for $\alpha = 0$ and limit ordinals α are trivial. So we assume that $V_\alpha[\mathbb{G}] \subseteq {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$ holds and show that the same inclusion holds for $\alpha+1$. Suppose $a \in V_{\alpha+1}[\mathbb{G}]$. Then $a = \underset{\sim}{a}^{\mathbb{G}}$ for a $\mathbb{P}$-name $\underset{\sim}{a} \in V_{\alpha+1}$. Since $\underset{\sim}{a} \subseteq V_\alpha$, each $\langle \underset{\sim}{b}, \mathbb{p} \rangle \in \underset{\sim}{a}$ is an element of V_α. By induction hypothesis, it follows that $\underset{\sim}{b}^{\mathbb{G}} \in {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$. It follows that $\underset{\sim}{a}^{\mathbb{G}} \subseteq {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$. Thus $a = \underset{\sim}{a}^{\mathbb{G}} \in {V_{\alpha+1}}^{\mathsf{V}[\mathbb{G}]}$.

"$\supseteq$": Suppose that $a \in {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$. Note that we can choose the "sufficiently large finite fragment of ZFC" which should hold in V_α, such that this implies that $(*)$ ${V_\alpha}^{\mathsf{V}[\mathbb{G}]}$ still satisfies a large enough fragment of ZFC, although the fragment may be different from the one V_α satisfies. In particular we find a cardinal $\lambda > |\,\mathbb{P}\,|$ in ${V_\alpha}^{\mathsf{V}[\mathbb{G}]}$ (and hence it is also a cardinal in $\mathsf{V}[\mathbb{G}]$) such that $a \in \mathcal{H}(\lambda)^{{V_\alpha}^{\mathsf{V}[\mathbb{G}]}} \subseteq \mathcal{H}(\lambda)^{\mathsf{V}[\mathbb{G}]} \subseteq {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$. [Note that $\mathcal{H}(\lambda)^{{V_\alpha}^{\mathsf{V}[\mathbb{G}]}} = \{a : |\,trcl(a)\,| < \lambda\}^{{V_\alpha}^{\mathsf{V}[\mathbb{G}]}} \subseteq \{a : |\,trcl(a)\,| < \lambda\}^{\mathsf{V}[\mathbb{G}]} = \mathcal{H}(\lambda)^{\mathsf{V}[\mathbb{G}]}$.]

Let $a^* \in \mathcal{H}(\lambda)^{\mathsf{V}[\mathbb{G}]}$ be a transitive set such that $a \in a^*$. Then a^* can be coded by a subset of λ. We can find the subset of λ in $\mathsf{V}[\mathbb{G}]$ and this subset has a nice $\mathbb{P}$-name which is an element of ${V_\alpha}^{\mathsf{V}}$ since $\mathbb{P} \in V_\alpha$. This shows that $a^* \in V_\alpha[\mathbb{G}]$ and hence also $a \in V_\alpha[\mathbb{G}]$. ❑ (Lemma 3.2)

Proof of Theorem 3.1: Assume that κ is tightly $\mathcal{P}$-Laver gen. ultrahuge for an iterable class $\mathcal{P}$ of posets.

Suppose that $\varphi = \varphi(\overline{x})$ is Σ_2 formula (in $\mathcal{L}_\in$), $\overline{a} \in \mathcal{H}(\kappa)$, and $\mathbb{P} \in \mathcal{P}$ is such that

(3.2) $\mathsf{V} \models \Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$". x-Lg-RcA-a

Let $\lambda > \kappa$ be such that $\mathbb{P} \in \mathsf{V}_\lambda$ and

(3.3) $V_\lambda \prec_{\Sigma_n} \mathsf{V}$ for a sufficiently large n. x-Lg-RcA-0

In particular, we may assume that we have chosen the n above so that a sufficiently large fragment of ZFC holds in V_λ ("sufficiently large" means here in terms of Lemma 3.2).

Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", and for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with

(3.4) $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, x-Lg-RcA-1

$$(3.5) \quad j(\kappa) > \lambda, \tag*{x-Lg-RcA-1-0}$$

$$(3.6) \quad \mathbb{P} * \underset{\sim}{\mathbb{Q}},\ \mathbb{P},\ \mathbb{H},\ {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M, \text{ and} \tag*{x-Lg-RcA-1-1}$$

$$(3.7) \quad |\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,| \le j(\kappa). \tag*{x-Lg-RcA-1-2}$$

By (3.7), we may assume that the underlying set of $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is $j(\kappa)$ and $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in {V_{j(\lambda)}}^{\mathsf{V}}$.

Let $\mathbb{G} := \mathbb{H} \cap \mathbb{P}$. Note that $\mathbb{G} \in M$ by (3.6) and we have

$$(3.8) \quad {V_{j(\lambda)}}^{M} \underbrace{=}_{\text{by (3.6)}} {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \overbrace{=}^{\substack{\text{Since } {V_{j(\lambda)}}^{M}\ (= V_{j(\lambda)}^{\mathsf{V}[\mathbb{H}]}) \text{ satisfies a sufficiently large fragment of ZFC} \\ \text{by elementarity of } j \text{, and hence the equality follows by Lemma 3.2}}} {V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{H}]. \tag*{x-Lg-RcA-2}$$

Thus, by the definability of grounds and by (3.6), we have ${V_{j(\lambda)}}^{\mathsf{V}} \in M$ and ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] \in M$.

Claim 3.1.1 ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] \models \varphi(\overline{a})$. cl-Lg-RcA-0

$\vdash$ By Lemma 3.2, ${V_\lambda}^{\mathsf{V}}[\mathbb{G}] = {V_\lambda}^{\mathsf{V}[\mathbb{G}]}$, and ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] = {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{G}]}$ by (3.3) and (3.8). By (3.3), both ${V_\lambda}^{\mathsf{V}}[\mathbb{G}]$ and $V^{\mathsf{V}}_{j(\lambda)}[\mathbb{G}]$ satisfy large enough fragment of ZFC. In particular,

$$(3.9) \quad {V_\lambda}^{\mathsf{V}}[\mathbb{G}] \prec_{\Sigma_1} {V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}]. \tag*{x-Lg-RcA-2-0}$$

By (3.2) and (3.3), we have ${V_\lambda}^{\mathsf{V}}[\mathbb{G}] \models \varphi(\overline{a})$. By (3.9) and since φ is Σ_2, it follows that ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] \models \varphi(\overline{a})$. $\dashv$ (Claim 3.1.1)

Thus we have

$$(3.10) \quad M \models \text{“there is a } \mathcal{P}\text{-ground } N \text{ of } V_{j(\lambda)} \text{ with } N \models \varphi(\overline{a})\text{”}. \tag*{x-Lg-RcA-3}$$

By the elementarity (3.4), it follows that

$$(3.11) \quad \mathsf{V} \models \text{“there is a } \mathcal{P}\text{-ground } N \text{ of } V_\lambda \text{ with } N \models \varphi(\overline{a})\text{”}. \tag*{x-Lg-RcA-4}$$

Now by (3.3), it follows that there is a $\mathcal{P}$-ground W of V such that $\mathsf{W} \models \varphi(\overline{a})$.

❑ (Theorem 3.1)

Some instances of weak forms of Recurrence Axioms decide the size of the continuum.

Theorem 3.3 *Assume that $\mathcal{P}$ is an iterable class of posets.* (1) *If $\mathcal{P}$ contains a poset which adds a real (over the universe), then $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA implies $\neg$CH.* p-Lg-RcA-1

(2) *Suppose that $\mathcal{P}$ contains a poset which forces ${\aleph_2}^{\mathsf{V}}$ to be equinumerous with ${\aleph_1}^{\mathsf{V}}$. Then $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA implies $2^{\aleph_0} \le \aleph_2$.*

(2') *If $\mathcal{P}$ contains a posets which forces ${\aleph_2}^{\mathsf{V}}$ to be equinumerous with ${\aleph_1}^{\mathsf{V}}$, then*

$(\mathcal{P}, \mathcal{H}((\aleph_2)^+))_{\Sigma_1}$-RcA *does not hold.*

(3) *If* $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA *holds then all* $\mathbb{P} \in \mathcal{P}$ *preserve* $\aleph_1$ *and they are also stationary preserving.*

(4) *If* $\mathcal{P}$ *contains a poset which adds a real as well as a poset which collapses* ${\aleph_2}^{\mathsf{V}}$, *then* $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA *implies* $2^{\aleph_0} = \aleph_2$.

(5) *If* $\mathcal{P}$ *contains a poset which collapses* ${\aleph_1}^{\mathsf{V}}$, *then* $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA *implies* CH.

(5') *If* $\mathcal{P}$ *contains a poset which collapses* ${\aleph_1}^{\mathsf{V}}$ *then* $(\mathcal{P}, \mathcal{H}((2^{\aleph_0})^+))_{\Sigma_1}$-RcA *does not hold.*

Proof. (1): Assume that $\mathcal{P}$ is an iterable class of posets containing a poset $\mathbb{P}$ adding a real and $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA holds. If CH holds, then $\mathcal{P}(\omega)^{\mathsf{V}} \in \mathcal{H}(\kappa_{\mathfrak{refl}})$. Hence "$\exists x\,(x \subseteq \omega \wedge x \notin \mathcal{P}(\omega)^{\mathsf{V}})$" is a Σ_1-formula with parameters from $\mathcal{H}(\kappa_{\mathfrak{refl}})$ and $\mathbb{P}$ forces (the formula in forcing language corresponding to) this formula.

By $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA, the formula must hold in a ground. This is a contradiction.

(2): Assume that $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA holds and $\mathbb{P} \in \mathcal{P}$ forces ${\aleph_2}^{\mathsf{V}}$ to be equinumerous with ${\aleph_1}^{\mathsf{V}}$. If $2^{\aleph_0} > \aleph_2$ then ${\aleph_1}^{\mathsf{V}}, {\aleph_2}^{\mathsf{V}} \in \mathcal{H}(2^{\aleph_0})$. Letting $\psi(x, y)$ a Σ_1-formula stating "$\exists f\,(f$ is a surjection from x to $y)$", we have $\Vdash_{\mathbb{P}}$ "$\psi(({\aleph_1}^{\mathsf{V}})^{\checkmark}, ({\aleph_2}^{\mathsf{V}})^{\checkmark})$".

By $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA, the formula $\psi({\aleph_1}^{\mathsf{V}}, {\aleph_2}^{\mathsf{V}})$ must hold in a ground. This is a contradiction.

(2'): Assume that $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}}$ "$|\,{\aleph_2}^{\mathsf{V}}\,| = |\,{\aleph_1}^{\mathsf{V}}\,|$", and $(\mathcal{P}, \mathcal{H}({\aleph_2}^+))_{\Sigma_1}$-RcA holds. Then, since $\aleph_1$, $\aleph_2 \in \mathcal{H}({\aleph_2}^+)$ and "$|\,x\,| = |\,y\,|$" is Σ_1, there is a ground W of V such that $\mathsf{W} \models |\,{\aleph_2}^{\mathsf{V}}\,| = |\,{\aleph_1}^{\mathsf{V}}\,|$. This is a contradiction.

(3): Suppose that $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}}$ "${\aleph_1}^{\mathsf{V}}$ is countable". Note that $\omega, \aleph_1 \in \mathcal{H}(\kappa_{\mathfrak{refl}})$. By $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA, it follows that there is a ground W of V such that $\mathsf{W} \models$ "${\aleph_1}^{\mathsf{V}}$ is countable". This is a contradiction.

Suppose now that $S \subseteq \omega_1$ is stationary and $\mathbb{P} \in \mathcal{P}$ destroys the stationarity of S. Note that ω_1, $S \in \mathcal{H}(\aleph_2)$. Let $\varphi = \varphi(y, z)$ be the Σ_1-formula

$$\exists x\,(x \text{ is a club subset of the ordinal } y \text{ and } z \cap x = \emptyset).$$

Then we have $\Vdash_{\mathbb{P}}$ "$\varphi(\omega_1, S)$". By $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA, it follows that there is a ground $\mathsf{W} \subseteq \mathsf{V}$ such that $S \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\omega_1, S)$. This is a contradiction to the stationarity of S.

(4): follows from (1), (2) and (3).

(5): If $\aleph_1 < 2^{\aleph_0}$, then ${\aleph_1}^{\mathsf{V}} \in \mathcal{H}(2^{\aleph_0})$.

Let $\mathbb{P} \in \mathcal{P}$ be a poset collapsing ${\aleph_1}^{\mathsf{V}}$. That is, $\Vdash_{\mathbb{P}}$ "${\aleph_1}^{\mathsf{V}}$ is countable". Since "$\cdots$ is countable" is Σ_1, there is a ground M such that $M \models$ "${\aleph_1}^{\mathsf{V}}$ is countable" by $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA. This is a contradiction.

(5'): Assume that $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}}$ "${\aleph_1}^{\mathsf{V}}$ is countable", and $(\mathcal{P}, \mathcal{H}((2^{\aleph_0})^+))$-RcA holds. Since $\aleph_1 \in \mathcal{H}((2^{\aleph_0})^+)$, it follows that there is a ground W of V such that W $\models {\aleph_1}^{\mathsf{V}}$ is countable. This is a contradiction. ❏ (Theorem 3.3)

In contrast to Theorem 3.3, $(ccc, \mathcal{H}(\kappa_{\mathfrak{refl}}))$-RcA does not decide the size of the continuum (see Theorem 4.7 and Theorem 4.10).

Recurrence Axioms can be considered as natural requirements claiming that a reflection holds from the set-generic multiverse down to the set-generic archaeology. From the standpoint that we should adopt the maximal amount of the Recurrence in the (ultimate) extension of ZFC, we arrive at either (all posets, $\mathcal{H}(2^{\aleph_0})$)-RcA or (semi-proper posets, $\mathcal{H}(\kappa_{\mathfrak{refl}})$)-RcA according to Theorem 3.3, and these axioms imply CH or $2^{\aleph_0} = \aleph_2$, respectively. In particular, these two axioms (or axiom schemes to be more precise) are not compatible to each other.

In Section 7, we shall examine an axiom(scheme) which implies the full (semi-proper posets, $\mathcal{H}(\kappa_{\mathfrak{refl}})$)-RcA$^+$ and also a large fragment of (all posets, $\mathcal{H}(\aleph_1)$-RcA$^+$ as well as MM^{++}.

4 Tightly super-$C^{(\infty)}$-Laver generic ultrahuge cardinal

In [17], it is shown that the existence of a (tightly) $\mathcal{P}$-Laver-gen. large cardinal does not imply Maximality Principle even without parameters. The proof in [17] can be modified to prove the non-implication of $(\mathcal{P}, \emptyset)_{\Pi_3}$-RcA from generic large cardinals of various sorts, and this also shows that "$(\mathcal{P}, \emptyset)_{\Sigma_2}$-RcA$^+$" in Theorem 3.1 is optimal. c-infty

In this section we show that the existence of a strong variant of $\mathcal{P}$-Laver generic large cardinal κ we are going to introduce below does imply MP$(\mathcal{P}, \mathcal{H}(\kappa))$ (see Theorem 4.10). It is essential that the variant of Laver genericity (called "the tightly super $C^{(\infty)}$-Laver gen. large cardinal" below) is formulated not in a single formula but rather in an axiom scheme.

For a natural number n, we call a cardinal κ *super-$C^{(n)}$-hyperhuge* if for any $\lambda_0 > \kappa$ there are $\lambda \geq \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, and j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, $j(\kappa) > \lambda$, ${}^{j(\lambda)}M \subseteq M$ and $V_{j(\lambda)} \prec_{\Sigma_n} \mathsf{V}$.

κ is *super-$C^{(n)}$-ultrahuge* if the condition above holds with "${}^{j(\lambda)}M \subseteq M$" replaced by "${}^{j(\kappa)}M \subseteq M$ and $V_{j(\lambda)} \subseteq M$".

If κ is super-$C^{(n)}$-hyperhuge then it is super-$C^{(n)}$-ultrahuge. This can be shown similarly to Lemma 5.1 in Section 5.

We shall also say that κ is *super-$C^{(\infty)}$-hyperhuge* (*super $C^{(\infty)}$-ultrahuge*, resp.) if it is super-$C^{(n)}$-hyperhuge (super $C^{(n)}$-ultrahuge, resp.) for all natural number n.

A similar kind of strengthening of the notions of large cardinals which we call here "super-$C^{(n)}$" appears also in Boney [3]. It is called in [3] "$C^{(n)+}$" and the notion is considered there in connection with extendibility.

For a natural number n and an iterable class $\mathcal{P}$ of posets, a cardinal κ is *super-$C^{(n)}$-$\mathcal{P}$-Laver-generically ultrahuge* (super $C^{(n)}$-$\mathcal{P}$-Laver-gen. ultrahuge, for short) if, for any $\lambda_0 > \kappa$ and for any $\mathbb{P} \in \mathcal{P}$, there are a $\lambda \geq \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, a $\mathcal{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" such that for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, $j(\kappa) > \lambda$, $\mathbb{P}$, $\mathbb{H}$, ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$ and ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$.

A super-$C^{(n)}$-$\mathcal{P}$-Laver-generically ultrahuge cardinal κ is *tightly super-$C^{(n)}$-$\mathcal{P}$-Laver-generically ultrahuge* (tightly super-$C^{(n)}$-$\mathcal{P}$-Laver-gen. ultrahuge, for short), if $|\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,| \leq j(\kappa)$ (see Footnote 2)).

super-$C^{(\infty)}$- $\mathcal{P}$-Laver-gen. ultrahugeness and *tightly super-$C^{(\infty)}$- $\mathcal{P}$-Laver gen. ultrahugeness* are defined similarly to super-$C^{(\infty)}$-ultrahugeness.

Note that, in general, super-$C^{(\infty)}$-hyperhugeness and super-$C^{(\infty)}$-ultrahugeness are notions unformalizable in the language of ZFC without introducing a new constant symbol for κ since we need infinitely many $\mathcal{L}_\in$-formulas to formulate them. Exceptions are when we are talking about a cardinal in a set model being with one of these properties like in Lemma 4.5 below (and in such a case "natural number n" actually refers to "$n \in \omega$"), or when we are talking about a cardinal definable in V having these properties in an inner model like in Theorem 5.8 or Corollary 5.9. In the latter case, the situation is formalizable with infinitely may $\mathcal{L}_\in$-sentences.

In contrast, the super-$C^{(\infty)}$-$\mathcal{P}$-Laver gen. ultrahugeness of κ is expressible in infinitely many $\mathcal{L}_\in$-sentences. This is because a $\mathcal{P}$-Laver gen. large cardinal κ for relevant classes $\mathcal{P}$ of posets is uniquely determined as $\kappa_{\mathfrak{refl}}$ or $2^{\aleph_0}$ (see e.g. [21] or [17]).

Lemma 4.1 *Suppose that κ is super-$C^{(n)}$-ultrahuge cardinal. Then we have $V_\kappa \prec_{\Sigma_{n+1}} \mathsf{V}$. In particular, in a context in which we can express that κ is super-$C^{(\infty)}$-ultrahuge cardinal in a (set or class) model W, we have ${V_\kappa}^{\mathsf{W}} \prec \mathsf{W}$.* p-Lg-RcA-1-a

Proof. We check that V_κ passes Vaught's test. Let $\varphi(x, \overline{y})$ be a Π_n-formula. Suppose that $\overline{b} \in V_\kappa$ and $a \in \mathsf{V}$ are such that $\mathsf{V} \models \varphi(a, \overline{b})$. We want to show that there is $a' \in V_\kappa$ such that $\mathsf{V} \models \varphi(a', \overline{b})$.

Let λ be such that

(4.1) $a \in V_\lambda$, x-Lg-RcA-4-a-0

(4.2) $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, x-Lg-RcA-4-a-1

(4.3) there are $j, M \subseteq \mathsf{V}$ such that x-Lg-RcA-4-a-2

(4.3a) $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$, (4.3b) $j(\kappa) > \lambda$, (4.3c) ${}^{j(\kappa)}M \subseteq M$, {x-Lg-RcA-4-a-2}{a}

(4.3d) $V_{j(\lambda)} \subseteq M$, and (4.3e) $V_{j(\lambda)} \prec_{\Sigma_n} \mathsf{V}$. {x-Lg-RcA-4-a-2}{d}

By (4.3e), we have $V_{j(\lambda)} \models \varphi(a, \bar{b})$. By (4.3d), it follows that $M \models$ "$V_{j(\lambda)} \models \varphi(a, \bar{b})$". Noting that $\bar{b} = j(\bar{b})$, it follows that

$$M \models \text{“ there is } x \in V_{j(\kappa)} \text{ such that } V_{j(\lambda)} \models \varphi(x, j(\bar{b}))\text{”}.$$

By the elementarity (4.3a), it follows

$$\mathsf{V} \models \text{“ there is } x \in V_\kappa \text{ such that } V_\lambda \models \varphi(x, \bar{b})\text{”}.$$

Let $a' \in V_\kappa$ be a witness of this. Then by (4.2) we have $\mathsf{V} \models \varphi(a', \bar{b})$. ❏ (Lemma 4.1)

Corollary 4.2 *Suppose that κ is a super-$C^{(\infty)}$-ultrahuge (-hyperhuge, resp.) cardinal in a (set or class) model W. Then for each $n \in \mathbb{N}$, $\mathsf{W} \models$* " there are stationarily many super-$C^{(n)}$-ultrahuge (-hyperhuge, resp.) cardinals"*.* p-Lg-RcA-1-a-0

Proof. Suppose that κ is a super-$C^{(\infty)}$-ultrahuge cardinal in W. By Lemma 4.1, we have ${V_\kappa}^{\mathsf{W}} \prec \mathsf{W}$. Suppose that $\varphi = \varphi(x, y)$ is an $\mathcal{L}_\in$-formula and $a \in {V_\kappa}^{\mathsf{W}}$ such that ${V_\kappa}^{\mathsf{W}} \models$ "$\varphi(\cdot, b)$ is a club $\subseteq$ On". Then, by elementarity, $\mathsf{W} \models$ "$\varphi(\cdot, b)$ is a club $\subseteq$ On" and $\mathsf{W} \models \varphi(\kappa, b)$. Thus $\mathsf{W} \models$ " there is a super-$C^{(n)}$-ultrahuge cardinal μ such that $\varphi(\mu, b)$". By elementarity, it follows that ${V_\kappa}^{\mathsf{W}} \models$ " there is a super-$C^{(n)}$-ultrahuge cardinal μ such that $\varphi(\mu, b)$".

Since b was arbitrary, it follows that ${V_\kappa}^{\mathsf{W}} \models$ "$\forall y$ (if $\varphi(\cdot, y)$ is a club in On, then there is a super-$C^{(n)}$-ultrahuge cardinal μ such that $\varphi(\mu, y)$)". By elementarity, the same statement also holds in W. ❏ (Corollary 4.2)

An ultrafilter $U \subseteq \mathcal{P}(\mathcal{P}(\lambda^*))$ is said to be normal if

(4.4) $\{x \in \mathcal{P}(\lambda^*) : \alpha \in x\} \in U$ for all $\alpha \in \lambda^*$, and x-Lg-RcA-4-0

(4.5) for any $\langle X_\alpha : \alpha < \lambda^* \rangle \in {}^{\lambda^*}U$, $\triangle_{\alpha < \lambda^*} X_\alpha \in U$. x-Lg-RcA-4-1

Under (4.4), the condition (4.5) is equivalent to

(4.6) For any $X \in U$ and any regressive[3] $f : X \to \lambda^*$, there is $X' \in U$ such that $X' \subseteq X$ and f is constant on X'. x-Lg-RcA-4-2

---

[3] $f : X \to \lambda^*$ is *regressive* if $f(x) \in x$ for all $x \in X$.

Lemma 4.3 *For $\kappa < \lambda < \kappa^* < \lambda^*$, the following* (A) *and* (B) *are equivalent.* p-Lg-RcA-1-0

(A) *there are j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, $j(\kappa) = \kappa^*$, $j(\lambda) = \lambda^*$, ${}^{j(\lambda)}M \subseteq M$.*

(B) *there is a κ-complete normal ultrafilter $U \subseteq \mathcal{P}(\mathcal{P}(\lambda^*))$ such that*

(4.7) $X^* := \{x \in \mathcal{P}(\lambda^*) \,:\, x \cap \kappa \in \kappa,\ otp(x \cap \kappa^*) = \kappa,\ otp(x) = \lambda\} \in U.$ x-Lg-RcA-4-3

A Sketch of the Proof: (A) $\Rightarrow$ (B): For j as in (A), the ultrafilter U defined by

(4.8) $U := \{X \in \mathcal{P}(\mathcal{P}(\lambda^*)) \,:\, j''\lambda^* \in j(X)\}$ x-Lg-RcA-4-3-0

satisfies (B).

(B) $\Rightarrow$ (A): For U as in (B), the elementary embedding j_U defined by

(4.9) $j_U : \mathsf{V} \xrightarrow{\preceq}_\kappa N := mcol({}^{\mathcal{P}(\lambda^*)}\mathsf{V}/U);\ x \mapsto [c_x]$ x-Lg-RcA-4-4

satisfies the conditions in (A).

Here, *mcol* denotes the Mostowski collapse, $[f]$ the element of N which corresponds to the equivalence class of $f \in {}^{\mathcal{P}(\lambda^*)}\mathsf{V}$ and c_x the function on $\mathcal{P}(\lambda^*)$ whose value is constantly x. ❑ (Lemma 4.3)

Similarly to the case of measurable cardinals or supercompact cardinals we can define another function k_j associated with j by:

(4.10) $k_j : N \to M;\ [f] \mapsto j(f)(j''\lambda^*).$ x-Lg-RcA-4-5

.

Lemma A 4.1 *Suppose that $\kappa < \lambda < \kappa^* < \lambda^*$ and $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M \subseteq \mathsf{V}$ are as in* Lemma 4.3, (A). *Let U be defined by* (4.8). *Then U satisfies the condition* (4.8) *in* Lemma 4.3, (B). p-Lg-RcA-1-0-0

Let j_U and k_j be defined as above. Then

(1) *k_j is well-defined.*

(2) *k_j is an elementary embedding.*

(3) *$k_j \circ j_U = j$.*

(4) *$k_j \restriction \mathcal{H}((\lambda^*)^+) = id_{\mathcal{H}((\lambda^*)^+)}$.*

(5) *$j \restriction \mathcal{H}(\lambda^+) = j_U \restriction \mathcal{H}(\lambda^+)$.*

In the following we shall use a (local) notation "$(\kappa, \lambda)^{(\kappa^*, \lambda^*)}$" to denote the condition (B) in Lemma 4.3. Clearly we have

Lemma 4.4 *For a cardinal κ and a natural number n, κ is a super-$C^{(n)}$-hyperhuge cardinal $\Leftrightarrow$ for any $\lambda_0 > \kappa$ there are $\lambda^* > \kappa^* > \lambda \geq \lambda_0$ such that* p-Lg-RcA-1-1

(C) $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, $V_{\lambda^*} \prec_{\Sigma_n} \mathsf{V}$, *and* $(\kappa, \lambda)^{(\kappa^*, \lambda^*)}$. ❏

We shall denote the condition (C) by $[\kappa, \lambda]^{[\kappa^*, \lambda^*, n]}$. This is also merely a local notation.

Lemma 4.5 *If κ is 2-huge with the 2-huge elementary embedding j, that is, there is $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \xrightarrow{\prec}_\kappa M \subseteq \mathsf{V}$, and* p-Lg-RcA-2

(4.11) $\quad {}^{j^2(\kappa)}M \subseteq M,$ x-Lg-RcA-5

then $V_{j(\kappa)} \models$ "κ is super-$C^{(\infty)}$-hyperhuge cardinal", and for each $n \in \omega$, $V_{j(\kappa)} \models$ "there are stationarily many super-$C^{(n)}$-hyperhuge cardinals".

Proof. Suppose that j is as above and $n \in \omega$.

By elementarity of j and since $\kappa = crit(j)$, we have $V_\kappa \prec {V_{j(\kappa)}}^M$. By (4.11), it follows that

(4.12) $\quad V_\kappa \prec V_{j(\kappa)}.$ x-Lg-RcA-6

By elementarity of j, this implies $M \models$ "$V_{j(\kappa)} \prec V_{j(j(\kappa))}$". By the closedness (4.11) of M (and since $j(j(\kappa))$ is inaccessible), it follows that

(4.13) $\quad V_{j(\kappa)} \prec V_{j(j(\kappa))}.$ x-Lg-RcA-6-a

For a λ_0 with $\kappa < \lambda_0 < j(\kappa)$, let $\lambda_0 \leq \lambda < j(\kappa)$ be such that

(4.14) $\quad V_\lambda \prec_{\Sigma_n} V_{j(\kappa)}.$ x-Lg-RcA-6-0

By (4.13), it follows that

(4.15) $\quad V_\lambda \prec_{\Sigma_n} V_{j(j(\kappa))}.$ x-Lg-RcA-6-1

We also have $M \models$ "$V_{j(\lambda)} \prec_{\Sigma_n} V_{j(j(\kappa))}$" by (4.14) and by elementarity of j. By the closedness (4.11) (and since $j(j(\kappa))$ is inaccessible), it follows that

(4.16) $\quad V_{j(\lambda)} \prec_{\Sigma_n} V_{j(j(\kappa))}.$ x-Lg-RcA-6-0-0

Let

(4.17) $\quad U := \{X \subseteq \mathcal{P}(j(\lambda)) \,:\, j''j(\lambda) \in j(X)\}.$ x-Lg-RcA-7

The following is easy to check:

Claim 4.5.1 *In $V_{j(j(\kappa))}$, U witnesses $(\kappa,\lambda)^{(j(\kappa),j(\lambda))}$. Hence we have $V_{j(j(\kappa))} \models$ "$[\kappa,\lambda]^{[j(\kappa),j(\lambda),n]}$" for all $n \in \omega$.*

$\vdash$ Clearly U is a ultrafilter.

U is κ-complete: Suppose that $\langle X_\xi : \xi < \mu\rangle \in {}^{\mu}U$ for some $\mu < \kappa$. Then $j(\langle X_\xi : \xi < \mu\rangle) = \langle j(X_\xi) : \xi < \mu\rangle$ by the elementarity of j and $\mu < crit(j)$. Since $X_\xi \in U$, we have $j''\lambda \in j(X_\xi)$ for all ξ by the definition (4.17) of U. It follows that $j''j(\lambda) \in \bigcap_{\xi<\mu} j(X_\xi) = j(\bigcap_{\xi<\mu} X_\xi)$. Thus $\bigcup_{\xi<\mu} X_\xi \in U$.

$U \models$(4.4): For $\alpha < j(\lambda)$, $j(\{x \in \mathcal{P}(j(\lambda)) : \alpha \in x\}) = \{x \in \mathcal{P}(j(j(\lambda))) : j(\alpha) \in x\} \ni j''j(\lambda)$. Hence $\{x \in \mathcal{P}(j(\lambda)) : \alpha \in x\} \in U$.

$U \models$(4.5): Suppose that $\vec{X} := \langle X_\alpha : \alpha < j(\lambda)\rangle \in {}^{j(\lambda)}U$. Then, by elementarity of j, $j(\triangle_{\alpha<j(\lambda)} X_\alpha) = \{x \in \mathcal{P}(j(j(\lambda))) : \forall \alpha \in x\,(x \in (j(\vec{X}))_\alpha)\}$. For $\alpha \in j''j(\lambda)$, $j''j(\lambda) \in j(X_\alpha) = (j(\vec{X}))_\alpha$ by the definition (4.17) of U. It follows that $j''j(\lambda) \in j(\triangle_{\alpha<j(\lambda)} X_\alpha)$. This means $\triangle_{\alpha<j(\lambda)} X_\alpha \in U$ by the definition (4.17) of U.

$X^* \in U$: Note that

$$j(X^*) = \{x \in \mathcal{P}(j(j(\lambda))) : x \cap j(\kappa) \in j(\kappa),\ otp(x \cap j(j(\kappa))) = j(\kappa),\ otp(x) = j(\lambda)\}$$

by elementarity of j. Hence $j''j(\lambda) \in j(X^*)$ and thus $X^* \in U$.

This shows $V_{j(j(\kappa))} \models$ "$(\kappa,\lambda)^{(j(\kappa),j(\lambda))}$". By (4.15) and (4.16), it follows that $V_{j(j(\kappa))} \models$ "$[\kappa,\lambda]^{[j(\kappa),j(\lambda),n]}$" $\dashv$ (Claim 4.5.1)

By Claim 4.5.1, $V_{j(j(\kappa))} \models \exists x \exists y\,([\kappa,\lambda]^{[x,y,n]})$ for all n. By (4.13), it follows that $V_{j(\kappa)} \models \exists x \exists y\,([\kappa,\lambda]^{[x,y,n]})$. Since $n \in \omega$ and $\kappa < \lambda_0 < j(\kappa)$ were arbitrary, it follows that $V_{j(\kappa)} \models$ "κ is super-$C^{(\infty)}$-hyperhuge".

For a fixed n and club $C \subseteq j(\kappa)$, we have $V_{j(j(\kappa))} \models$ "$\kappa \in j(C)$ and κ is super-$C^{(n)}$-hyperhuge". Thus $V_{j(j(\kappa))} \models$ "$\exists x\,(x \in j(C)$ and x is super-$C^{(n)}$-hyperhuge)". It follows that $V_{j(\kappa)} \models$ "$\exists x\,(x \in C$ and x is super-$C^{(n)}$-hyperhuge)". This shows that $V_{j(\kappa)} \models$ "there are stationarily many super-$C^{(n)}$-hyperhuge cardinals". ❏ (Lemma 4.5)

Theorem 4.6 (Laver function for a super-$C^{(n)}$-hyperhuge cardinal) *Suppose that μ is an inaccessible cardinal and κ is super-$C^{(\infty)}$-hyperhuge in V_μ. Then there is $f : \kappa \to V_\kappa$ such that for any $n \in \omega$, $a \in V_\mu$, and $\lambda_0 \geq \kappa$, there are $\lambda_0 < \lambda < \kappa^* < \lambda^* < \mu$ such that $\kappa < \lambda$, $|\,trcl(a)\,| \leq \lambda^*$,* p-Lg-RcA-3

(4.18) $\quad V_\mu \models [\kappa,\lambda]^{[\kappa^*,\lambda^*,n]}$, x-Lg-RcA-7-a

and there is a ultrafilter $U \subseteq \mathcal{P}(\mathcal{P}(\lambda^*))$ witnessing (4.18) such that $j_U(f)(\kappa) = a$ where j_U is given by (4.9).[4)]

Proof. The proof is just an adaptation of the proof of Theorem 20.21 in [33].

Suppose toward a contradiction, that the Theorem does not hold. Then for each $f : \kappa \to V_\kappa$, there are a_f, $n_f \in \omega$ and $\kappa < \lambda_f < \kappa^*_f < \lambda^*_f < \mu$ such that

(4.19) $\quad |\, trcl(a_f)\,| \leq \lambda^*_f \ (\Leftrightarrow\ a_f \in \mathcal{H}((\lambda^*_f)^+))$; x-Lg-RcA-7-a-0

(4.20) $\quad V_\mu \models [\kappa, \lambda_f]^{[\kappa^*_f, \lambda^*_f, n_f]}$; but x-Lg-RcA-7-a-1

(4.21) $\quad$ there is no witness U of (4.20) with $j_U(f)(\kappa) = a_f$. x-Lg-RcA-7-a-2

We assume that, for each $f : \kappa \to V_\kappa$, $\langle \lambda_f, \kappa^*_f, \lambda^*_f, n_f \rangle$ is chosen to be the minimal possible (with respect to the lexicographical ordering) among those which satisfy (4.19) $\sim$ (4.21) together with f and some a_f.

For a $\alpha \in \mathrm{On}$, and $g : \alpha \to V_\alpha$, let $(*)_{\alpha,g}$ be the assertion

(4.22) $\quad \alpha$ is a cardinal and there are $\alpha < \delta < \alpha^* < \delta^* < \mu$, $n \in \omega$ and a such that x-Lg-RcA-7-a-3

(4.22a) $\quad V_\mu \models [\alpha, \delta]^{[\alpha^*, \delta^*, n]}, \qquad a \in \mathcal{H}((\delta^*)^+)$ {x-Lg-RcA-7-a-3}{a}

but there is no witness U of (4.22a) such that $j_U(g)(\alpha) = a$.

Let $f^\dagger : \kappa \to V_\kappa$ be defined recursively such that for $\alpha < \kappa$, if $(*)_{\alpha, f^\dagger \restriction \alpha}$ holds, then $f^\dagger(\alpha)$ witnesses $(*)_{\alpha, f^\dagger \restriction \alpha}$ as a in (4.22) together with $\langle \delta, \alpha^*, \delta^*, n \rangle$ which is chosen to be minimal possible (with respect to the lexicographic ordering). Otherwise, we let $f(\alpha) = \emptyset$.

Let

(4.23) $\quad a^\dagger := j_{U_{f^\dagger}}(f^\dagger)(\kappa)$ x-Lg-RcA-7-a-4

where $U_{f^\dagger}$ is a witness of $[\kappa, \lambda_{f^\dagger}]^{[\kappa^*_{f^\dagger}, \lambda^*_{f^\dagger}, n_{f^\dagger}]}$. By definition of $f^\dagger$, elementarity and the assumption of this indirect proof, $a^\dagger$ together with $\kappa^*_{f^\dagger}, \lambda^*_{f^\dagger}, n_{f^\dagger}$ is a witness of $(*)_{\alpha, \underbrace{j_{U_{f^\dagger}}(f^\dagger) \restriction \kappa}_{= f^\dagger}}$. But the existence of the ultrafilter $U_{f^\dagger}$ in (4.23) is a contradiction to this. ❑ (Theorem 4.6)

In [21] and further in [22], [23], [25], we studied in connection with Laver genericity, only classes $\mathcal{P}$ of posets which are all stationary preserving. This is because the existence of a Laver-gen. large cardinal for one of such classes of posets naturally extends the known reflection properties down to $< \kappa_{\mathfrak{refl}}$, and strong versions of forcing axioms.

However, we can also consider the class of all posets in connection with Laver genericity. (5) in the following Theorem 4.7 is such an instance of Laver genericity. We shall go more into this scenario in Section 6.

---

4) In particular, we have $j_U(\kappa) = \kappa^*$, $j_U(\lambda) = \lambda^*$, $V_\lambda \prec_{\Sigma_n} \mathsf{V}$ and V_{λ^*} $(= V_{j(\lambda)}) \prec_{\Sigma_n} \mathsf{V}$.

Theorem 4.7 (1) *Suppose that μ is inaccessible and $\kappa < \mu$ is super-$C^{(\infty)}$-ultrahuge in V_μ. Let $\mathbb{P} = \mathrm{Col}(\aleph_1, \kappa)$. Then, in $V_\mu[\mathbb{G}]$, for any $V_\mu, \mathbb{P}$-generic $\mathbb{G}$, $\aleph_2^{V_\mu[\mathbb{G}]}$ $(= \kappa)$ is tightly super-$C^{(\infty)}$-σ-closed-Laver-gen. ultrahuge and* CH *holds.* p-Lg-RcA-4

(2) *Suppose that μ is inaccessible and $\kappa < \mu$ is super-$C^{(\infty)}$-ultrahuge with a Laver function $f : \kappa \to V_\kappa$ for super-$C^{(\infty)}$-ultrahugeness in V_μ. If $\mathbb{P}$ is the CS-iteration of length κ for forcing* PFA *along with f, then, in $V_\mu[\mathbb{G}]$ for any $(V_\mu, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{V_\mu[\mathbb{G}]}$ $(= \kappa)$ is tightly super-$C^{(\infty)}$-proper-Laver-gen. ultrahuge and $2^{\aleph_0} = \aleph_2$ holds.*

(2′) *Suppose that μ is inaccessible and $\kappa < \mu$ is super-$C^{(\infty)}$-ultrahuge with a Laver function $f : \kappa \to V_\kappa$ for super-$C^{(\infty)}$-ultrahugeness in V_μ. If $\mathbb{P}$ is the RCS-iteration of length κ for forcing* MM *along with f, then, in $V_\mu[\mathbb{G}]$ for any $(V_\mu, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{V_\mu[\mathbb{G}]}$ $(= \kappa)$ is tightly super-$C^{(\infty)}$-semi-proper-Laver-gen. ultrahuge and $2^{\aleph_0} = \aleph_2$ holds.*

(3) *Suppose that μ is inaccessible and κ is super-$C^{(\infty)}$-ultrahuge with a Laver function $f : \kappa \to V_\kappa$ for super-$C^{(\infty)}$-ultrahugeness in V_μ. If $\mathbb{P}$ is a FS-iteration of length κ for forcing* MA *along with f, then, in $V_\mu[\mathbb{G}]$ for any $(V_\mu, \mathbb{P})$-generic $\mathbb{G}$, $2^{\aleph_0}$ $(= \kappa)$ is tightly super-$C^{(\infty)}$-c.c.c.-Laver-gen. ultrahuge, and $2^{\aleph_0}$ is very large in $V_\mu[\mathbb{G}]$.*

(4) *Suppose that μ is inaccessible and κ is super-$C^{(\infty)}$-ultrahuge with a Laver function $f : \kappa \to V_\kappa$ for super-$C^{(\infty)}$-ultrahugeness in V_μ. If $\mathbb{P}$ is a FS-iteration of length κ along with f enumerating "all" posets, then, in $V_\mu[\mathbb{G}]$ for any $(V_\mu, \mathbb{P})$-generic $\mathbb{G}$, $2^{\aleph_0}$ $(= \aleph_1)$ is tightly super-$C^{(\infty)}$-all posets-Laver-gen. ultrahuge, and* CH *holds.*[5)]

Proof. The proof can be done similarly to that of Theorem 5.2 in [21] using Lemma 4.8 below. In the following we shall only check the case (4).

Suppose that $f : \kappa \to V_\kappa$ is a super-$C^{(\infty)}$-hyperhuge Laver function as in Theorem 4.6.

Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be a FS-iteration defined by

$$(4.24) \quad \underset{\sim}{\mathbb{Q}}_\beta = \begin{cases} f(\alpha), & \text{if } f(\alpha) \text{ is a } \mathbb{P}_\beta\text{-name of a poset;} \\ \{\mathbb{1}\}, & \text{otherwise} \end{cases}$$ p-Lg-RcA-4-a-0

for $\beta < \kappa$.

Let $\mathbb{G}$ be a $(V_\mu, \mathbb{P}_\kappa)$-generic filter. Clearly $V_\mu[\mathbb{G}] \models$ "$2^{\aleph_0} = \kappa = \aleph_1$". We show that κ is tightly super-$C^{(\infty)}$-all posets-Laver-gen. ultrahuge in $V_\mu[\mathbb{G}]$.

Suppose that $\mathbb{P}$ is a poset in $V_\mu[\mathbb{G}]$, $\kappa < \lambda_0$ and $n \in \omega$. Let $n' > n$ be sufficiently large and let $\underset{\sim}{\mathbb{P}}$ be a $\mathbb{P}_\kappa$-name of $\mathbb{P}$.

---

5) Actually we can obtain a model with the desired property simply by Levy collapsing κ to ω_1. We just chose this narrative to emphasize the parallelism to the cases (2), (2') and (3).

Working in V_μ, we can find

$$(4.25)\quad |\,\mathbb{P}\,| < \lambda < \kappa^* < \lambda^* \text{ and } j,\ M \subseteq \mathsf{V}$$ x-Lg-RcA-6-1-a

such that

$$(4.26)\quad j : \mathsf{V} \xrightarrow{\preceq}_\kappa M,$$ x-Lg-RcA-6-1-0

$$(4.27)\quad j(\kappa) = \kappa^*,\ j(\lambda) = \lambda^*,$$ x-Lg-RcA-6-1-1

$$(4.28)\quad {}^{\lambda^*}M \subseteq M,$$ x-Lg-RcA-6-1-2

$$(4.29)\quad V_\lambda \prec_{\Sigma_{n'}} \mathsf{V},\ V_{\lambda^*} \prec_{\Sigma_{n'}} \mathsf{V}, \text{ and}$$ x-Lg-RcA-6-1-3

$$(4.30)\quad j(f)(\kappa) = \underset{\sim}{\mathbb{P}}$$ x-Lg-RcA-6-1-4

by definition of f.

By elementarity (and by the definition (4.24) of $\mathbb{P}_\kappa$),

$$(4.31)\quad j(\mathbb{P}_\kappa)\ \sim_{\mathbb{P}_\kappa}\ (\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}}) * \underset{\sim}{\mathbb{R}}$$ x-Lg-RcA-6-2

for a $(\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}})$-name $\underset{\sim}{\mathbb{R}}$ of a poset. Note that $(\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}} * \underset{\sim}{\mathbb{R}})/\mathbb{G}$ corresponds to a poset of the form $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$.

Let $\mathbb{H}^*$ be $(\mathsf{V}, (\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}}) * \underset{\sim}{\mathbb{R}})$-generic filter with $\mathbb{G} \subseteq \mathbb{H}^*$. $\mathbb{H}^*$ corresponds to a $(\mathsf{V}, j(\mathbb{P}_\kappa))$-generic filter $\mathbb{H} \supseteq \mathbb{G}$ via the equivalence (4.31).

Let $\tilde{j}$ be defined by

$$(4.32)\quad \tilde{j} : \mathsf{V}[\mathbb{G}] \to M[\mathbb{H}];\quad \underset{\sim}{a}^{\mathbb{G}} \mapsto j(\underset{\sim}{a})^{\mathbb{H}}$$ x-Lg-RcA-6-3

for all $\mathbb{P}_\kappa$-names $\underset{\sim}{a}$.

A standard proof shows that f is well-defined, and $j : \mathsf{V}[\mathbb{G}] \xrightarrow{\preceq}_\kappa M[\mathbb{H}]$. By (4.27) and (4.28), we have $\tilde{j}\,''\tilde{j}(\lambda) = j\,''j(\lambda) \in M[\mathbb{H}]$. Since $\mathbb{H} \in M[\mathbb{H}]$, the $(\mathsf{V}[\mathbb{G}], \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic filter corresponding to $\mathbb{H}$ is also in $M[\mathbb{H}]$.

By (4.25), (4.29), by the choice of n', and by Lemma 4.8, (1), we have
${V_\lambda}^{\mathsf{V}[\mathbb{G}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{G}]$ and ${V_{\tilde{j}(\lambda)}}^{\mathsf{V}[\mathbb{H}]} = {V_{\lambda^*}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$.

Since $\mathbb{P}$ and n were arbitrary, this shows that κ is tightly super-$C^{(\infty)}$-all posets-Laver-gen. ultrahuge in $V_\mu[\mathbb{G}]$. ❑ (Theorem 4.7)

Lemma 4.8 (1) *For a natural number n, if $n' > n$ is sufficiently large and* p-Lg-RcA-4-0

(4.33) : $V_\lambda \prec_{\Sigma_{n'}} \mathsf{V}$, x-Lg-RcA-7-0

then we have $V_\lambda[\mathbb{G}] \prec_{\Sigma_n} \mathsf{V}[\mathbb{G}]$ for any poset $\mathbb{P} \in V_\lambda$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$.

(2) *For a natural number n, if $n' > n$ is sufficiently large and*

(4.34) : $V_\lambda[\mathbb{G}] \prec_{\Sigma_{n'}} \mathsf{V}[\mathbb{G}]$ *for some poset $\mathbb{P} \in V_\lambda$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$,* x-Lg-RcA-7-1

then we have $V_\lambda \prec_{\Sigma_n} \mathsf{V}$.

Proof. (1): Suppose that $\overline{a} \in V_\lambda[\mathbb{G}]$ and $\varphi = \varphi(\overline{x})$ is a Σ_n-formula. There are $\mathbb{P}$-names $\underset{\sim}{\overline{a}} \in V_\lambda$ such that $\overline{a} = \underset{\sim}{\overline{a}}[\mathbb{G}]$.

If $V_\lambda[\mathbb{G}] \models \varphi(\overline{a})$, there is $\mathbb{p} \in \mathbb{G}$ such that $V_\lambda \models \mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{\overline{a}})$". By (4.33) it follows that $\mathsf{V} \models \mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{\overline{a}})$". Thus $\mathsf{V}[\mathbb{G}] \models \varphi(\overline{a})$.

The same argument also applies to $\neg\varphi$.

(2): We use the $\mathcal{L}_\in$-formula $\Phi(x, y)$ of Theorem 2.1. By (4.34), there is $r \in V_\lambda[\mathbb{G}]$ such that

$$V_\lambda = \Phi(\cdot, r)^{V_\lambda[\mathbb{G}]} = \Phi(\cdot, r)^{\mathsf{V}[\mathbb{G}]} \cap V_\lambda[\mathbb{G}] \subseteq \Phi(\cdot, r)^{\mathsf{V}[\mathbb{G}]}.$$

For any Σ_n-formula $\varphi(\overline{x})$ and $\overline{a} \in \Phi(\cdot, r)^{V_\lambda[\mathbb{G}]}$. Since $\varphi^{\Phi(\cdot,r)}$ is a $\Sigma_{n'}$-formula (by the choice of n'), we have

$$V_\lambda \models \varphi(\overline{a}) \;\Leftrightarrow\; V_\lambda[\mathbb{G}] \models \varphi^{\Phi(\cdot,r)}(\overline{a}) \;\overbrace{\Leftrightarrow}^{\text{by (4.34)}}\; \mathsf{V}[\mathbb{G}] \models \varphi^{\Phi(\cdot,r)}(\overline{a}) \;\Leftrightarrow\; V \models \varphi(\overline{a}).$$

This shows that $V_\lambda \prec_{\Sigma_n} \mathsf{V}$. ❏ (Lemma 4.8)

In Theorem 4.7, (4), the model constructed in the proof satisfies CH. The next lemma suggests that this does not depend on the specific construction of the model given there.

In the following, we consider a strengthening of tightness of Laver genericity: we say a cardinal κ is tightly$^+$ $\mathcal{P}$-Laver-gen. hyperhuge/ultrahuge if the definition of the tightly $\mathcal{P}$-Laver gen. hyperhugeness/ultrahugeness of κ holds with the clause "$|\,\mathbb{P}*\underset{\sim}{\mathbb{Q}}\,| \leq j(\kappa)$" replaced by "there is a complete Boolean algebra $\mathbb{B}$ of size $j(\kappa)$ such that $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \sim \mathbb{B}^+$". Note that this condition is equivalent to $2^{<\kappa} = \kappa$, $|\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,| \leq \kappa$ and $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ has the $j(\kappa)$-cc. Note also that all models constructed in the proof of Theorem 4.7 actually satisfy this stronger notion of tightness of the (super-$C^{(n)}$) $\mathcal{P}$-Laver-gen. hyperhugeness or ultrahugeness of κ. Later with much more work we show in Corollary 6.3 that this stronger version of tightness can be dropped from the following Lemma.

Lemma 4.9 *If $\mathcal{P}$ is the class of all posets and κ is tightly$^+$ $\mathcal{P}$-Laver-gen. hyperhuge cardinal then $\kappa = 2^{\aleph_0} = \aleph_1$.* p-Lg-RcA-4-1

Proof. $\kappa \leq 2^{\aleph_0}$ holds even without tightness: see e.g. Lemma 3.7 in [17].

To show $2^{\aleph_0} \leq \kappa$, let λ and $\mathbb{Q}$ be such that

(4.35) $\lambda > 2^{\aleph_0}$, κ and λ is large enough such that SCH holds above some $\mu < \lambda$ (this is possible by Corollary 5.4 , (2) in Section 5, and we need here the Laver gen. hyperhugeness of κ), x-Lg-RcA-8-a-0

(4.36) $\mathbb{Q}$ is positive elements of a complete Boolean algebra, and, x-Lg-RcA-8-a-1

(4.37) for $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that (1) $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$, (2) $j(\kappa) > \lambda$, (3) $|\,\mathbb{Q}\,| \le j(\kappa)$, and (4) ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \subseteq M$. x-Lg-RcA-8-a-2

By (4.36), each $\mathbb{Q}$-name $\underset{\sim}{r}$ of a real corresponds to a mapping $f : \omega \to \mathbb{Q}$. By (4.35) and by (4.37), (3), there are at most $j(\kappa)$ many such mappings. Thus we have $\mathsf{V}[\mathbb{H}] \models$ "$2^{\aleph_0} \le j(\kappa)$", By (4.37), (4), it follows $M \models$ "$2^{\aleph_0} \le j(\kappa)$". By elementarity, it follows that $\mathsf{V} \models$ "$2^{\aleph_0} \le \kappa$".

$\kappa = \aleph_1$: Otherwise $\aleph_1 < \kappa$. In V, let $\mathbb{P}$ be the standard poset collapsing $\aleph_1$ to be countable. Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name such that for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$ there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$, and $\mathbb{P}$, $\mathbb{H} \in M$.

By $\mathbb{H} \cap \mathbb{P} \in M$, we have $M \models$ "${\aleph_1}^{\mathsf{V}} = j({\aleph_1}^{\mathsf{V}})$ is countable". This is a contradiction to the elementarity of j. ❑ (Lemma 4.9)

Recall that, for an iterable $\mathcal{P}$, $(\mathcal{P}, \mathcal{H}(\kappa))$-$\mathsf{RcA}^+$ holds if and only if $\mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa))$ holds (Proposition 2.2, (1)).

Theorem 4.10 *Suppose that $\mathcal{P}$ is an iterable class of posets and κ is super-$C^{(\infty)}$-$\mathcal{P}$-Laver-gen. ultrahuge. Then $(\mathcal{P}, \mathcal{H}(\kappa))$-$\mathsf{RcA}^+$ holds.* p-Lg-RcA-5

Proof. A modification of the proof of Theorem 3.1 works.

Suppose that κ is super-$C^{(\infty)}$-$\mathcal{P}$-Laver-gen. ultrahuge, $\mathbb{P} \in \mathcal{P}$, and $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$" for an $\mathcal{L}_\in$-formula φ and $\overline{a} \in \mathcal{H}(\kappa)$. We want to show that $\varphi(\overline{a})$ holds in some $\mathcal{P}$-ground of V.

Let n be a sufficiently large natural number such that the following arguments go through. In particular, we assume that ${V_\alpha}^{\mathsf{V}} \prec_{\Sigma_n} \mathsf{V}$ implies that "$\varphi(\overline{x})$" and "$\Vdash_{\cdot}$ "$\varphi(\overline{x}^{\checkmark})$"" are absolute between ${V_\alpha}^{\mathsf{V}}$ and V, and ${V_\alpha}^{\mathsf{V}} \prec_{\Sigma_n} \mathsf{V}$ also implies that a sufficiently large fragment of ZFC holds in V_α.

Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are a $\lambda > \kappa$ with

(4.38) $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, x-Lg-RcA-9

and j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$, $j(\kappa) > \lambda$, $\mathbb{P}$, $\mathbb{H}$, ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$ and ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$.

By the choice of n, we have $V_\lambda \models \Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$". $j({V_\lambda}^{\mathsf{V}}) = {V_{j(\lambda)}}^M \prec_{\Sigma_n} M$ by elementarity of j, and ${V_{j(\lambda)}}^M = {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]}$ by the closedness of M. Since $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, we have $V_\lambda[\mathbb{H}] \prec_{\Sigma_{n_0}} \mathsf{V}[\mathbb{H}]$ for a still large enough $n_0 \le n$ by Lemma 4.8, (1). Since ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$, it follows that ${V_\lambda}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_{n_0}} {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]}$. Thus

(4.39) ${V_\lambda}^{\mathsf{V}} \prec_{\Sigma_{n_1}} {V_{j(\lambda)}}^{\mathsf{V}}$ x-Lg-RcA-10 for a still large enough $n_1 \leq n_0$ by Lemma 4.8, (2).

In particular, we have ${V_{j(\lambda)}}^{\mathsf{V}} \models \Vdash_{\mathbb{P}}$ " $\varphi(\overline{a}^{\checkmark})$ ", and hence $V_{j(\lambda)}[\mathbb{G}] \models \varphi(\overline{a})$ where $\mathbb{G}$ is the $\mathbb{P}$-part of $\mathbb{H}$. Note that by (4.38) and (4.39), $V_{j(\lambda)}$ satisfies a sufficiently large fragment of ZFC.

Thus we have $V_{j(\lambda)}[\mathbb{H}] \models$ " there is a $\mathcal{P}$-ground satisfying $\varphi(\overline{a})$", and hence

$$ {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \models \text{" there is a } \mathcal{P}\text{-ground satisfying } \varphi(\overline{a})\text{"} $$

by Lemma 3.2. By elementarity, it follows that

$$ V_\lambda \models \text{" there is a } \mathcal{P}\text{-ground satisfying } \varphi(\overline{a})\text{"}. $$

Finally, this implies $\mathsf{V} \models$ " there is a $\mathcal{P}$-ground satisfying $\varphi(\overline{a})$" by (4.38).

□ (Theorem 4.10)

5 Bedrock of a tightly generic hyperhuge cardinal

Recall that a cardinal κ is *hyperhuge*, if for every $\lambda > \kappa$, there is $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M \subseteq \mathsf{V}$ such that $\lambda < j(\kappa)$ and ${}^{j(\lambda)}M \subseteq M$. By Lemma 4.3, a hyperhuge cardinal κ can be characterized in terms of existence of κ-complete normal ultrafilters with the property (4.7). bedrock

For a class $\mathcal{P}$ of posets, a cardinal κ is *tightly $\mathcal{P}$-generic hyperhuge* (tightly $\mathcal{P}$-gen. hyperhuge, for short) if for any $\lambda > \kappa$, there is $\mathbb{Q} \in \mathcal{P}$ such that for a $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, $\lambda < j(\kappa)$, $|\,\mathbb{Q}\,| \leq j(\kappa)$, and $j''j(\lambda), \mathbb{H} \in M$.

For a class $\mathcal{P}$ of posets, a cardinal κ is *tightly $\mathcal{P}$-Laver-generically hyperhuge* (tightly $\mathcal{P}$-Laver-gen. hyperhuge, for short) if for any $\lambda > \kappa$, and $\mathbb{P} \in \mathcal{P}$ there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ " $\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$ " such that for a $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, $\lambda < j(\kappa)$, $|\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,| \leq j(\kappa)$, and $j''j(\lambda), \mathbb{H} \in M$.

The following Lemma is easy to prove.

Lemma 5.1 (1) *κ is hyperhuge if and only if the following holds:* p-bedrock-0

(5.1) *for every $\lambda > \kappa$, there is $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M \subseteq \mathsf{V}$ such that $\lambda < j(\kappa)$, $V_{j(\lambda)} \in M$, and ${}^{j(\lambda)}M \subseteq M$.* x-bedrock-0

(2) *If κ is hyperhuge then it is ultrahuge.*

(3) *Suppose that $\mathcal{P}$ is a class of posets such that the trivial poset $\{\mathbb{1}\}$ is in $\mathcal{P}$. If κ is hyperhuge then κ is tightly $\mathcal{P}$-gen. hyperhuge.*

(4) *For a class $\mathcal{P}$ of posets, κ is tightly $\mathcal{P}$-gen. hyperhuge if and only if the following holds:*

(5.2) *for any $\lambda > \kappa$, there is $\mathbb{Q} \in \mathcal{P}$ such that for a $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \overset{\preceq}{\rightarrow}_\kappa M$, $\lambda < j(\kappa)$, $|\,\mathbb{Q}\,| \leq j(\kappa)$, and ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]}$, $j''j(\lambda), \mathbb{H} \in M$.* x-bedrock-1

(5) *For a class $\mathcal{P}$ of posets, κ is tightly $\mathcal{P}$-Laver-gen. hyperhuge if and only if the following holds:*

(5.3) *for any $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" such that for a $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \overset{\preceq}{\rightarrow}_\kappa M$, $\lambda < j(\kappa)$, $|\,\mathbb{Q}\,| \leq j(\kappa)$, and ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]}$, $j''j(\lambda), \mathbb{H} \in M$.* x-bedrock-1-0

(6) *For an iterable class $\mathcal{P}$ of posets, a tightly $\mathcal{P}$-Laver generically hyperhuge cardinal is tightly $\mathcal{P}$-gen. hyperhuge.*

(7) *For a class $\mathcal{P}$ of posets, if κ is (tightly resp.) $\mathcal{P}$-Laver-gen. hyperhuge, then it is (tightly resp.) $\mathcal{P}$-Laver-gen. ultrahuge.*

Proof. (1): It is clear that (5.1) implies that κ is hyperhuge.

Suppose that κ is hyperhuge. We show that (5.1) holds. For $\lambda > \kappa$, let $\lambda^* := |\,V_\lambda\,|^+$ and $j : \mathsf{V} \overset{\preceq}{\rightarrow}_\kappa M$ be such that $j(\kappa) > \lambda^*$ and

(5.4) $\quad {}^{j(\lambda^*)}M \subseteq M.$ x-bedrock-2

By elementarity, we have

(5.5) $\quad M \models j(\lambda^*) = |\,{V_{j(\lambda)}}^M\,|^+.$ x-bedrock-3

Claim 5.1.1 $M \ni {V_{j(\lambda)}}^M = V_{j(\lambda)}$. cl-bedrock-0

$\vdash$ We prove ${V_\alpha}^M = V_\alpha$ for all $\omega \leq \alpha \leq j(\lambda)$.

For $\alpha = \omega$, this clearly holds.

Suppose that ${V_\alpha}^M = V_\alpha$ for $\omega \leq \alpha < j(\lambda)$. Then $j(\lambda^*) > |\,{V_\alpha}^M\,| = |\,V_\alpha\,|$ by (5.5). By (5.4), we have $V_{\alpha+1} = \mathcal{P}(V_\alpha) \in M$: $\mathcal{P}(V_\alpha) \subseteq M$ by (5.4). Since $M \models \mathsf{ZFC}$ by elementarity of j, it follows that $\mathcal{P}(V_\alpha) = \mathcal{P}(V_\alpha)^M \in M$.

If $\alpha \leq j(\lambda)$ is a limit and the Claim holds for $\beta < \alpha$, then $\langle V_\beta : \beta < \alpha \rangle \in {}^{\alpha}M \subseteq M$. Thus $V_\alpha = \bigcup_{\beta<\alpha} V_\beta \in M$. $\dashv$ (Claim 5.1.1)

${}^{j(\lambda)}M \subseteq M$ by $j(\lambda) < j(\lambda^*)$ and (5.4). Thus j witnesses (5.1) for λ.

(2): follows from (1). (3): is trivial.

(4): If κ satisfies (5.2) then κ is clearly tightly $\mathcal{P}$-gen. hyperhuge.

Suppose that κ is tightly $\mathcal{P}$-gen. hyperhuge. For $\lambda > \kappa$, let λ^* be such that

(5.6) $\lambda^* \geq |\,V_\lambda\,|^+$ and x-bedrock-4

(5.7) $V_{\lambda^*} \prec_{\Sigma_n} \mathsf{V}$ for a sufficiently large natural number n. x-bedrock-5

By (5.2), there is $\mathbb{Q} \in \mathcal{P}$ such that for $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$ there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that

(5.8) $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, x-bedrock-5-0

(5.9) $\lambda^* < j(\kappa)$, x-bedrock-6

(5.10) $|\,\mathbb{Q}\,| \leq j(\kappa)$, and x-bedrock-7

(5.11) $j''j(\lambda^*)$, $\mathbb{H} \in M$. x-bedrock-8

By (5.11), and Lemma 2.5, (5) in [21], we have $V_{j(\lambda^*)}{}^{\mathsf{V}} \in M$. Hence $V_{j(\lambda^*)}{}^{\mathsf{V}}[\mathbb{H}] \in M$ by (5.11). By (5.7) and (5.11), $V_{j(\lambda^*)}{}^{\mathsf{V}}[\mathbb{H}] = V_{j(\lambda^*)}{}^{\mathsf{V}[\mathbb{H}]}$, and thus $V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \in M$.

This shows that our j and M witnesses (5.2) for λ.

(5): Similarly to (4).

(6): Since $\mathcal{P}$ is iterable, the poset $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ in the definition of $\mathcal{P}$-Laver-gen. hyperhugeness is an element of $\mathcal{P}$.

(7): By (5). ❑ (Lemma 5.1)

For a cardinal κ, a ground W of the universe V is called a *$\leq\kappa$-ground* if there is a poset $\mathbb{P} \in \mathsf{W}$ of cardinality $\leq\kappa$ (in the sense of V) and $(\mathsf{W}, \mathbb{P})$-generic filter $\mathbb{G}$ such that $\mathsf{V} = \mathsf{W}[\mathbb{G}]$. Note that this definition of $\leq\kappa$-ground diverges from the convention of "$\mathcal{P}$-ground" in that "of cardinality $\leq\kappa$" is meant here not in the sense of W but rather of V. In Proposition 6.1 we will show that in our context, this actually implies "$\leq\kappa$-ground" in line with the definition of $\mathcal{P}$-ground.

Let

(5.12) $\overline{\mathsf{W}} := \bigcap\{\mathsf{W} : \mathsf{W} \text{ is a } \leq\kappa\text{-ground}\}$. x-bedrock-9

Since there are only set many $\leq\kappa$-grounds, $\overline{\mathsf{W}}$ contains a ground by Theorem 1.3 in [42]. We shall call $\overline{\mathsf{W}}$ the *$\leq\kappa$-mantle of* V.

By Lemma 5.1, (3), the following theorem generalizes Theorem 1.6 in [42].

Theorem 5.2 *Suppose that $\mathcal{P}$ is a class of posets. If κ is a tightly $\mathcal{P}$-gen. hyperhuge cardinal, then the $\leq\kappa$-mantle is the smallest ground of* V *(i.e. it is the bedrock of* V*) and it is also a $\leq\kappa$-ground.* p-bedrock-1

Proof. Suppose that κ is tightly $\mathcal{P}$-gen. hyperhuge and let $\overline{\mathsf{W}}$ be the $\leq\kappa$-mantle for this κ.

By Theorem 1.3 in [42] mentioned above, it is enough to show that, for any ground $\mathsf{W} \subseteq \overline{\mathsf{W}}$ is actually a $\leq\kappa$-ground and hence $\mathsf{W} = \overline{\mathsf{W}}$ holds.

Let $\mathsf{W} \subseteq \overline{\mathsf{W}}$ be a ground. Let μ be the cardinality (in the sense of V) of a poset $\mathbb{S} \in \mathsf{W}$ such that there is a $(\mathsf{W}, \mathbb{S})$-generic $\mathbb{F}$ such that $\mathsf{V} = \mathsf{W}[\mathbb{F}]$. Without loss of generality, $\mu \geq \kappa$.

By Theorem 2.1, there is $r \in \mathsf{V}$ such that $\mathsf{W} = \Phi(\cdot, r)^{\mathsf{V}}$.

Let $\theta \geq \mu$ be such that $r \in V_\theta$, and for a sufficiently large natural number n, we have

$$(5.13) \quad {V_\theta}^{\mathsf{V}} \prec_{\Sigma_n} \mathsf{V}. \qquad \text{x-bedrock-9-1}$$

By the choice of θ, we have

$$(5.14) \quad \Phi(\cdot, r)^{{V_\theta}^{\mathsf{V}}} = \Phi(\cdot, r)^{\mathsf{V}} \cap {V_\theta}^{\mathsf{V}} = \mathsf{W} \cap {V_\theta}^{\mathsf{V}} = {V_\theta}^{\mathsf{W}}. \qquad \text{x-bedrock-9-2}$$

There is a $\mathbb{Q} \in \mathcal{P}$ such that for $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with

$$(5.15) \quad j : \mathsf{V} \xrightarrow{\prec}_\kappa M,\ \theta < j(\kappa), \qquad \text{x-bedrock-10}$$

$$(5.16) \quad |\,\mathbb{Q}\,| \leq j(\kappa), \qquad \text{x-bedrock-11}$$

$$(5.17) \quad {V_{j(\theta)}}^{\mathsf{V}[\mathbb{H}]} \subseteq M, \text{ and} \qquad \text{x-bedrock-12}$$

$$(5.18) \quad \mathbb{H},\ j''j(\theta) \in M \qquad \text{x-bedrock-13}$$

by Lemma 5.1, (4).

Since ${V_\theta}^{\mathsf{V}} \equiv {V_{j(\theta)}}^M$ by elementarity, (5.13) implies that ${V_{j(\theta)}}^M$ $(= {V_{j(\theta)}}^{\mathsf{V}[\mathbb{H}]})$ satisfies a sufficiently large fragment of ZFC. Thus it follows that

(5.19) ${V_{j(\theta)}}^{\mathsf{V}}$ satisfies a sufficiently large fragment of ZFC and also ${V_{j(\theta)}}^{\mathsf{W}}$ satisfies still a sufficiently large fragment of ZFC. x-bedrock-16

Thus we have

$$(5.20) \quad {V_{j(\theta)}}^M \underbrace{=}_{\text{by (5.17)}} {V_{j(\theta)}}^{\mathsf{V}[\mathbb{H}]} \overbrace{=}^{\text{by Lemma 3.2 and the remark above}} {V_{j(\theta)}}^{\mathsf{V}}[\mathbb{H}] \underbrace{=}_{\text{by Lemma 3.2 and the remark above}} {V_{j(\theta)}}^{\mathsf{W}}[\mathbb{F}][\mathbb{H}]. \qquad \text{x-bedrock-14}$$

Since

$$(5.21) \quad {V_{j(\theta)}}^M \prec_{\Sigma_n} M \qquad \text{x-bedrock-16-0}$$

by elementarity and V is a ground of M by Grigorieff's Theorem, it follows from Lemma 4.8, (2) that

$$(5.22) \quad {V_{j(\theta)}}^{\mathsf{V}} \prec_{\Sigma_{n_0}} \mathsf{V} \text{ for a still large enough } n_0 \leq n. \qquad \text{x-bedrock-15}$$

Claim 5.2.1 (1) *${V_{j(\theta)}}^{\mathsf{V}}$ is a generic extension of ${V_{j(\theta)}}^{\mathsf{W}}$ by a poset of size $\leq j(\kappa)$.* *cl-bedrock-1*

(2) *$\mathsf{V}[\mathbb{H}]$ is a generic extension of W by a poset of size $\leq j(\kappa)$.*

(3) ${V_{j(\theta)}}^M$ *is a generic extension of* ${V_{j(\theta)}}^{\mathsf{W}}$ *by a poset of size* $\leq j(\kappa)$.

(4) M *is a generic extension of* $j(W) := \Phi(\cdot, j(r))^M$ *by a poset of size* $\leq j(\mu)$.

$\vdash$ (1): V is a generic extension of W by a poset of size $\mu < \lambda < j(\kappa)$. Thus the assertion follows from (5.22).

(2): We have $\mathsf{V}[\mathbb{H}] = \mathsf{W}[\mathbb{F}][\mathbb{H}]$ and $|\mathbb{S} * \mathbb{Q}| \leq j(\kappa)$ by $|\mathbb{S}| \leq \mu < \lambda < j(\kappa)$ and by (5.16).

(3): By (5.20) and (1).

(4): By definition of μ and elementarity of j. $\dashv$ (Claim 5.2.1)

Claim 5.2.2 ${V_{j(\theta)}}^{j(\mathsf{W})} = j({V_\theta}^{\mathsf{W}}) \subseteq j({V_\theta}^{\overline{\mathsf{W}}}) \subseteq {V_{j(\theta)}}^{\mathsf{W}} \subseteq {V_{j(\theta)}}^{\mathsf{V}}$. *cl-bedrock-1-0*

$\vdash$ By (5.17), we have ${V_{j(\theta)}}^M = {V_{j(\theta)}}^{\mathsf{V}[\mathbb{H}]}$. By Claim 5.2.1, (3), it follows that

(5.23) ${V_{j(\theta)}}^{\mathsf{W}}$ is a $\leq j(\kappa)$-ground of ${V_{j(\theta)}}^M$. x-bedrock-16-1

By elementarity of j and (5.21),

(5.24) $j({V_\theta}^{\overline{\mathsf{W}}})$ is a $\leq j(\kappa)$-mantle of $V^M_{j(\theta)}$. x-bedrock-18

Thus we have

$${V_{j(\theta)}}^{j(\mathsf{W})} = j({V_\theta}^{\mathsf{W}}) \subseteq j({V_\theta}^{\overline{\mathsf{W}}}) \overbrace{\subseteq}^{\text{by (5.23), (5.24) and by the definition of } \leq j(\kappa)\text{-mantle}} {V_{j(\theta)}}^{\mathsf{W}} \subseteq {V_{j(\theta)}}^{\mathsf{V}}.$$

$\dashv$ (Claim 5.2.2)

Claim 5.2.3 $j''\lambda \in j(\mathsf{W})$ *for every* $\lambda < j(\theta)$. *cl-bedrock-2*

$\vdash$ Since θ is a limit cardinal, it is enough to show the claim for all regular $\lambda < j(\theta)$ with $\lambda > j(\mu)$. Let λ be one of such cardinals, and let $\langle S_\alpha : \alpha < \lambda \rangle$ be a partition of $(E^\lambda_\omega)^{\mathsf{W}}$ into stationary sets in W. Since $|\mathbb{S}| \leq \mu < \lambda$ each S_α is stationary subset of λ in V. Since $\mathsf{V}[\mathbb{H}]$ is a generic extension of V by $\mathbb{Q}$ of size $\leq j(\kappa)$, and $\lambda > j(\mu) \geq j(\kappa)$, each S_α, $\alpha < \lambda$ is a stationary subset of λ in $\mathsf{V}[\mathbb{H}]$. Let

(5.25) $\langle S'_\alpha : \alpha < j(\lambda) \rangle := j(\langle S_\alpha : \alpha < \lambda \rangle) \in j(\mathsf{W})$. x-bedrock-17

Subclaim 5.2.3.1 *sub-bedrock-a*

$j''\lambda = \{\beta < \sup(j''\lambda) : S'_\beta \cap \sup(j''\lambda)$ *is stationary subset of* $\sup(j''\lambda)$ *in* $M\}$.

$\vdash$ Suppose $\alpha \in \lambda$. To show that $S'_{j(\alpha)} \cap \sup(j''\lambda) = j(S_\alpha) \cap \sup(j''\lambda)$ is a stationary subset of $\sup(j''\lambda)$ in M, suppose $C \subseteq \sup(j''\lambda)$ is a club subset of $\sup(j''\lambda)$ in M. We have to show that C intersects with $j(S_\alpha)$.

Since $|\mathbb{Q}| \leq j(\kappa) \leq j(\mu) < \lambda$ by the choice of λ, and $M \subseteq \mathsf{V}[\mathbb{H}]$, there is an unbounded $D \subseteq \lambda$, $D \in \mathsf{W}$ such that for any $\xi_0, \xi_1 \in D$ with $\xi_0 < \xi_1$, $[j(\xi_0), j(\xi_1)] \cap$

$C \neq \emptyset$. Since S_α is stationary, there is $\eta \in S_\alpha \cap \lim(D)$. Since $cf(\eta) = \omega$, $j(S_\alpha) \ni j(\eta) = \sup(j''\eta)$ the right-most side of this is an element of C by definition of D. This shows that $j(S_\alpha) \cap C \neq \emptyset$.

Suppose now that $S'_\beta \cap \sup(j''\lambda)$ is stationary subset of $\sup(j''\lambda)$ in M for some $\beta < \sup(j''\lambda)$. We want to show that $\beta = j(\alpha)$ for some $\alpha < \lambda$.

Since $j''\lambda = j''j(\theta) \cap \sup(j''\lambda) \in M$ by (5.18), and thus, also $\{\langle \alpha, j(\alpha)\rangle : \alpha < \lambda\} \in M$, we have $M \models cf(j''\lambda) = \lambda$, and hence also $\mathsf{V}[\mathbb{H}] \models cf(j''\lambda) = \lambda$.

By (5.17), $\mathcal{P}(\lambda)^M = \mathcal{P}(\lambda)^{\mathsf{V}[\mathbb{H}]}$. Hence stationarity of a subset of $\sup(j''\lambda)$ is absolute between M and $V[\mathbb{H}]$. Thus, our assumption implies that $S'_\beta \cap \sup(j''\lambda)$ is stationary subset of $\sup(j''\lambda)$ in $\mathsf{V}[\mathbb{H}]$.

By (5.13) and elementarity of j, $j(V_\theta{}^{\mathsf{W}})$ is a $\leq j(\kappa)$-mantle of $V_{j(\theta)}{}^M$ $(= V_{j(\theta)}{}^{V[\mathbb{H}]}$ by (5.17)).

Subsubclaim 5.2.3.1.1 *For every $\eta < \lambda$, if $cf^{j(\mathsf{W})}(\eta) = \omega$ then $cf^{\mathsf{V}}(\eta) = \omega$.* subsub-bedrock-0

$\vdash$ By Claim 5.2.2. $\dashv$ (Subsubclaim 5.2.3.1.1)

Subsubclaim 5.2.3.1.2 *There is an unbounded $E \subseteq \lambda$ such that $j \restriction E \in j(\mathsf{W})$.* subsub-bedrock-1

$\vdash$ By Claim 5.2.1, (4) and $j(\mu) < \lambda$ by choice of λ, M is a generic extension of $j(\mathsf{W})$ by the λ-c.c. poset $j(\mathbb{S})$.[6] Since $j''\lambda \in M$, there is a $j(\mathbb{S})$-name $\underset{\sim}{j}$ for $j \restriction \lambda$.

Working in $j(\mathsf{W})$, suppose that $\gamma < \lambda$. By the λ-c.c. of $j(\mathbb{S})$, we can find an increasing sequences $\langle \alpha_n : n < \omega\rangle$, $\langle \beta_n : n \in \omega\rangle \in j(\mathsf{W})$ such that $\gamma < \alpha_0$, $\alpha_n < \lambda$ for all $n \in \omega$ and $\Vdash_{j(\mathbb{S})}$ "$\underset{\sim}{j}(\alpha_n) < \beta_n < \underset{\sim}{j}(\alpha_{n+1})$" for all $n \in \omega$.

Let $\alpha := \sup_{n\in\omega} \alpha_n$ and $\beta := \sup_{n\in\omega} \beta_n$. Then $cf^{j(\mathsf{W})}(\alpha) = \omega$ and hence $cf^{\mathsf{V}}(\alpha) = \omega$ by Subsubclaim 5.2.3.1.1. Thus $\Vdash_{j(\mathbb{S})}$ "$\underset{\sim}{j}(\alpha) = \sup_{n\in\omega} \underset{\sim}{j}(\alpha_n) = \sup_{n\in\omega} \beta_n = \beta$".

This shows that

$$E = \{\alpha < \lambda : \text{there is } \beta \text{ such that } \Vdash_{j(\mathbb{S})} \text{“}\underset{\sim}{j}(\check{\alpha}) = \check{\beta}\text{”}\} \in j(\mathsf{W})$$

is a cofinal subset of λ and $j \restriction E \in j(\mathsf{W})$. $\dashv$ (Subsubclaim 5.2.3.1.2)

Now returning to the proof of the second-half of the Subclaim 5.2.3.1, since $S'_\beta \cap \sup(j''\lambda)$ is stationary and $\lim(j''E)$ is club in $\sup(j''\lambda)$, there is $\eta \in S'_\beta \cap \lim(j''E)$. Then $cf^{j(\mathsf{W})}(\eta) = \omega = cf^{\mathsf{V}}(\eta)$ (the last equality by Subsubclaim 5.2.3.1.1). Let $\zeta < \lambda$ be minimal with $\eta \leq j(\zeta)$.

We have $\sup(j''\zeta) = \sup(j''(\zeta \cap E)) = \eta \leq j(\zeta)$. Since the cofinality of η is ω and $j \restriction (\eta \cap E)$ is increasing and $\eta \cap E$ is cofinal in η, we have $j(\zeta) = \sup(j''\zeta) = \eta$,

[6] Since $|j(\mathbb{S})| \leq j(\mu) < \lambda$ in M, $j(\mathbb{S})$ has the λ-c.c. in M and hence it has the λ-c.c. also in $j(\mathsf{W}) \subseteq M$.

and hence $j(\zeta) \in S'_\beta$. By elementarity, it follows that $\zeta \in S_\alpha$ for some $\alpha < \lambda$. Then $j(\zeta) \in S'_{j(\alpha)} \cap S'_\beta$. Since S'_ξ's are pairwise disjoint, it follows that $j(\alpha) = \beta$. Thus $\beta \in j''\lambda$ as desired. ⊣ (Subclaim 5.2.3.1)

By Subclaim 5.2.3.1 we have

$$j''\lambda = \{\beta < \sup(j''\lambda) \,:\, S'_\beta \cap \sup(j''\lambda) \text{ is stationary subset of } \sup(j''\lambda) \text{ in } M\}.$$

By Claim 5.2.1, (4), and since $j(\mu) < \lambda$, it follows that

$$j''\lambda = \{\beta < \sup(j''\lambda) \,:\, S'_\beta \cap \sup(j''\lambda) \text{ is stationary subset of } \sup(j''\lambda) \text{ in } j(\mathsf{W})\}.$$

Apparently the right side of the equality is a definition of an element of $j(\mathsf{W})$.

⊣ (Claim 5.2.3)

Claim 5.2.4 $V_{j(\theta)}{}^{\mathsf{W}} = V_{j(\theta)}{}^{j(\mathsf{W})}$. *cl-bedrock-3*

⊢ "$\supseteq$": By Claim 5.2.2.

"$\subseteq$": For $\lambda < j(\theta)$, if $X \in \mathcal{P}(\lambda) \cap \mathsf{W}$, then $j(X) \in j(\mathsf{W})$, and $j''\lambda \in j(\mathsf{W})$ by Claim 5.2.3. Thus $j''X = j(X) \cap j''\lambda \in j(\mathsf{W})$. Since we also have $j \restriction \lambda \in j(\mathsf{W})$, it follows that $X \in j(\mathsf{W})$.

Thus, by induction on $\alpha \leq j(\theta)$, we can prove that $V_\alpha{}^{\mathsf{W}} \subseteq j(\mathsf{W})$. ⊣ (Claim 5.2.4)

Now, we have

$$j(V_\theta{}^{\mathsf{W}}) \underbrace{\subseteq}_{\text{by } V_\theta{}^{\mathsf{W}} \subseteq V_\theta{}^{\overline{\mathsf{W}}} \text{ and by elementarity of } j} j(V_\theta{}^{\overline{\mathsf{W}}}) \overbrace{\subseteq}^{\text{by (5.24) and Claim 5.2.1, (3)}} V_{j(\theta)}{}^{\mathsf{W}} \underbrace{=}_{\text{by Claim 5.2.4}} V_{j(\theta)}{}^{j(\mathsf{W})} = j(V_\theta{}^{\mathsf{W}}).$$

It follows $j(V_\theta{}^{\mathsf{W}}) = V_{j(\theta)}{}^{\mathsf{W}}$. Since $j(V_\theta{}^{\mathsf{V}}) = V_{j(\theta)}{}^{M}$ is a generic extension of $V_{j(\theta)} = j(V_\theta{}^{\mathsf{W}})$ by a poset of size $\leq j(\kappa)$ (by Claim 5.2.1, (3)). Thus, by elementarity of j, $V_\theta{}^{\mathsf{V}}$ is a generic extension of $V_\theta{}^{\mathsf{W}}$ by a poset of size $\leq \kappa$.

By definition of $\overline{\mathsf{W}}$ and by (5.13), it follows that $V_\theta{}^{\overline{\mathsf{W}}} \subseteq V_\theta{}^{\mathsf{W}}$. Since θ was arbitrary, this implies that $\overline{\mathsf{W}} \subseteq \mathsf{W}$ and hence $\overline{\mathsf{W}} = \mathsf{W}$ as desired. ❑ (Theorem 5.2)

As already noticed Theorem 5.2 is a generalization of Theorem 1.6 in [42] which states that there is the bedrock (of the universe V) if there is a hyperhuge cardinal. In Theorem 1.3 in [43] the assumption of hyperhuge cardinal in this theorem is weakened to the existence of an extendible cardinal.

Though our proof of Theorem 5.2 has a global structure quite similar to that of the proof of Theorem 1.3 in [43], it seems to be necessary here that $\overline{\mathsf{W}}$ in our proof is the $\leq\kappa$-mantle while, in [43], a similar proof could be done with $\overline{\mathsf{W}}$ which is the $<\kappa$-mantle, and that difference seems to force us to assume the stronger consistency strength of a hyperhuge cardinal.

Theorem 5.3 *Suppose that $\mathcal{P}$ is a class of posets. If κ is a tightly $\mathcal{P}$-gen. hyperhuge cardinal, then κ is a hyperhuge cardinal in the bedrock $\overline{\mathsf{W}}$ of V.* p-bedrock-2

Proof. In V, let $\lambda > \kappa$ be arbitrary. Let $\theta > \lambda$ be sufficiently large such that it satisfies $V_\theta{}^{\mathsf{V}} \prec_{\Sigma_n} \mathsf{V}$ for sufficiently large natural number n (this is just the condition (5.13) in the proof of Theorem 5.2). Let $\mathbb{Q} \in \mathcal{P}$ be such that for a $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\preceq}_\kappa M$, $\theta < j(\kappa)$, $j''j(\theta) \in M$, and $V_{j(\theta)}{}^{\mathsf{V}[\mathbb{H}]} = V_{j(\theta)}{}^{M}$. Note that W in the proof of Theorem 5.2 coincides with the bedrock $\overline{\mathsf{W}}$ of V.

Thus we have (5.26) : $j''j(\lambda^*) \in j(\overline{\mathsf{W}})$ for all $\lambda^* < j(\theta)$ by Claim 5.2.3, x-bedrock-19
and (5.27) : $j(V_\theta{}^{\overline{\mathsf{W}}}) = V_{j(\theta)}{}^{\overline{\mathsf{W}}}$ by Claim 5.2.4. x-bedrock-20

Let $\langle X_\alpha : \alpha < \gamma\rangle$ be an enumeration of $\mathcal{P}(\mathcal{P}(j(\lambda)))^{\overline{\mathsf{W}}}$ in $\overline{\mathsf{W}}$. Note that $\gamma < j(\theta)$ by the choice of θ.

By (5.27), we have $\langle X_\alpha : \alpha < \gamma\rangle \in j(\overline{\mathsf{W}})$ and $j''j(\lambda) \in j(\overline{\mathsf{W}})$ by (5.26).

Also for any $\lambda^* < j(\theta)$ and $A \in \mathcal{H}((\lambda^*)^+)$ we have $j \restriction A \in j(\overline{\mathsf{W}})$ by (5.26) and Lemma 2.5 in [21]. It follows that

$$I := \{\alpha < \lambda : j''j(\lambda) \in j(X_\alpha)\} \in j(\overline{\mathsf{W}}). \tag{5.28}$$ x-bedrock-21

Let $U = \{X_\alpha : \alpha \in I\}$. Then $U \in j(\overline{\mathsf{W}})$. Furthermore we have

$$U \in V_{j(\theta)}{}^{j(\overline{\mathsf{W}})} = j(V_\theta{}^{\overline{\mathsf{W}}}) \overbrace{=}^{\text{by (5.27)}} V_{j(\theta)}{}^{\overline{\mathsf{W}}} \subseteq \overline{\mathsf{W}}. \tag{5.29}$$

It is a routine to check that U is a κ-complete normal ultrafilter over $\mathcal{P}(j(\lambda))^{\overline{\mathsf{W}}}$ and $\{x \in \mathcal{P}(j(\lambda)) : x \cap \kappa \in \kappa, otp(x \cap j(\kappa)) = \kappa, otp(x) = \lambda\} \in U$ holds.

Since λ was arbitrary, this implies that κ is hyperhuge in $\overline{\mathsf{W}}$ by Lemma 4.3.

❑ (Theorem 5.3)

Corollary 5.4 *Suppose that $\mathcal{P}$ is an arbitrary class of posets and κ is a tightly $\mathcal{P}$-gen. hyperhuge cardinal. Then (1) there are cofinally many huge cardinals.* p-bedrock-3
(2) SCH holds above some cardinal.

Proof. Suppose that κ is a tightly $\mathcal{P}$-gen. hyperhuge cardinal. By Theorem 5.2 there is the bedrock $\overline{\mathsf{W}}$ and κ is hyperhuge cardinal in $\overline{\mathsf{W}}$.

(1): Since the existence of a hyperhuge cardinal implies the existence of cofinally many huge cardinals (it is easy to show that the target $j(\kappa)$ of hyperhuge embedding for a sufficiently large inaccessible λ is a huge cardinal), there are cofinally many huge cardinals in $\overline{\mathsf{W}}$. Since V is attained by a set forcing starting from $\overline{\mathsf{W}}$, a final segment of these huge cardinals survive in V.

(2): By Theorem 20.8 in [33], SCH holds above κ in $\overline{\mathsf{W}}$. Since V is a set generic extension of $\overline{\mathsf{W}}$. SCH should hold above some cardinal $\mu \geq \kappa$. ❑ (Corollary 5.4)

Compare Corollary 5.4, (1) with Corollary 4.2 and Corollary 6.7.

For iterable stationary preserving $\mathcal{P}$ containing all proper posets, Theorem 5.4, (2) holds already under the $\mathcal{P}$-Laver-gen. supercompactness of κ. The reason is that in such case PFA holds (see Theorem 5.7 in [21]), and by Viale [44], SCH follows from it.

In the following Corollary, we adopt the notation of Ikegami-Trang in [32] on their version of Maximality Principle.

Corollary 5.5 *Suppose that $\mathcal{P}$ is the class of semi-proper posets. If κ is a tightly $\mathcal{P}$-Laver gen. hyperhuge cardinal, then $\mathsf{MP}_{\Pi_2}(\omega_1$, all posets$)$ holds.* *p-bedrock-4*

Proof. Theorem 1.8, (1) in [32] states that the conclusion of the present corollary holds under MM^{++} and proper class many Woodin cardinals.

If κ is a $\mathcal{P}$-Laver gen. supercompact cardinal, then $\kappa = \aleph_2$ and MM^{++} holds (see [21]). By Corollary 5.4, tightly $\mathcal{P}$-Laver gen. hyperhugeness implies that there are proper class many Woodin cardinals. ❑ (Corollary 5.5)

Corollary 5.6 *Suppose that $\mathcal{P}$ is the class of all posets. Then the following theories are equiconsistent:* *p-bedrock-5*

(a) ZFC + *"there is a hyperhuge cardinal".*
(b) ZFC + *"there is a tightly $\mathcal{P}$-Laver gen. hyperhuge cardinal".*
(c) ZFC + *"there is a tightly $\mathcal{P}$-gen. hyperhuge cardinal".*
(d) ZFC + *"bedrock $\overline{\mathsf{W}}$ exists and ω_1 is a hyperhuge cardinal in $\overline{\mathsf{W}}$".* ❑

Corollary 5.7 *Suppose that $\mathcal{P}$ is one of the following classes of posets: all semi-proper posets; all proper posets; all ccc posets; all σ-closed posets. Then the following theories are equiconsistent:* *p-bedrock-6*

(a) ZFC + *"there is a hyperhuge cardinal".*
(b) ZFC + *"there is a tightly $\mathcal{P}$-Laver gen. hyperhuge cardinal".*
(c) ZFC + *"there is a tightly $\mathcal{P}$-gen. hyperhuge cardinal".*
(d) ZFC + *"bedrock $\overline{\mathsf{W}}$ exists and $\kappa_{\mathfrak{refl}}$ is a hyperhuge cardinal in $\overline{\mathsf{W}}$".* ❑

The proof of Theorem 5.3 can be modified to obtain the following:

Theorem 5.8 *Suppose that $\mathcal{P}$ is a class of posets. If a definable cardinal κ is a tightly super-$C^{(\infty)}$-$\mathcal{P}$-gen. hyperhuge cardinal, then κ is super-$C^{(\infty)}$-hyperhuge in the bedrock $\overline{\mathsf{W}}$ of V.* ❏ p-bedrock-2-0

The definability of the cardinal κ (e.g. as ω_1, $2^{\aleph_0}$ etc.) in Theorem 5.8 actually is in needed so that the conclusion of the theorem is formalizable (in infinitely many formulas).

Corollary 5.9 *Suppose that $\mathcal{P}$ is one of the following classes of posets: all semi-proper posets; all proper posets; all ccc posets; all σ-closed posets. Then the following theories are equiconsistent:* p-bedrock-8

(a) ZFC + *"c is a super-$C^{(\infty)}$ hyperhuge cardinal" where c is a new constant symbol but "... is super-$C^{(\infty)}$ hyperhuge ..." is formulated in an infinite collection of formulas in $\mathcal{L}_\in$.*

(b) ZFC + *"there is a tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver gen. hyperhuge cardinal".*

(c) ZFC + *"bedrock $\overline{\mathsf{W}}$ exists and $\kappa_{\mathfrak{refl}}{}^{\mathsf{V}}$ is a super $C^{(\infty)}$-hyperhuge cardinal in $\overline{\mathsf{W}}$".* ❏

Theorem 5.10 *Suppose that κ is tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver gen. hyperhuge for an iterable $\mathcal{P}$. Then, for each $n \in \mathbb{N}$, there are stationarily many super-$C^{(n)}$-hyperhuge cardinals.* p-bedrock-8-0

Proof. Let $n \in \mathbb{N}$. By a modification of Theorem 5.3, we can show that κ is super-$C^{(\infty)}$-hyperhuge in the bedrock $\overline{\mathsf{W}}$. Thus there are stationarily many super-$C^{(n)}$-hyperhuge cardinals in $\overline{\mathsf{W}}$ by Corollary 4.2. Since V is a set generic extension of $\overline{\mathsf{W}}$, the stationarity of these classes is preserved by the generic extension. ❏ (Theorem 5.10)

Corollary 5.11 *The consistency strength of the existence of a tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver gen. hyperhuge cardinal for one of the iterable $\mathcal{P}$'s in Theorem 4.7 is strictly between that of the existence of a super $C^{(n)}$-hyperhuge cardinal, and that of the existence of a 2-huge cardinal.* p-bedrock-8-1

Proof. Suppose that Suppose that κ is tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver gen. hyperhuge for an iterable $\mathcal{P}$. Then by Theorem 5.10, there is a super-$C^{(n)}$-hyperhuge $\lambda > \kappa$ with $V_\lambda \prec_{\Sigma_{n'}}$ for sufficiently large $n' > n$. any super-$C^{(n)}$-hyperhuge $\lambda_0 < \lambda$ is super-$C^{(n)}$-hyperhuge in V_λ by elementarity. Thus V_λ is a model of ZFC + "there is a super- $C^{(n)}$-hyperhuge cardinal".

If κ is 2-huge then by Lemma 4.5 and Theorem 4.7, there is a set model with a tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver gen. hyperhuge cardinal. ❏ (Corollary 5.11)

6 Bedrock and Laver genericity

Proposition 6.1 *Suppose that $\mathcal{P}$ is a class of posets and κ is tightly $\mathcal{P}$-gen. ultra-huge cardinal. By Theorem 5.2 there is the bedrock $\overline{\mathsf{W}}$ of V.* edrock-Lg p-bedrock-9

We have $(\kappa^+)^{\overline{\mathsf{W}}} = (\kappa^+)^{\mathsf{V}}$ and V is a set generic extension $\overline{\mathsf{W}}$ by some poset $\mathbb{P} \in \overline{\mathsf{W}}$ such that $\overline{\mathsf{W}} \models |\,\mathbb{P}\,| \leq \kappa$.

Proof. Let $\mathbb{P}_0 \in \overline{\mathsf{W}}$ be such that there is a $(\overline{\mathsf{W}}, \mathbb{P}_0)$-generic $\mathbb{G}_0$ such that $\mathsf{V} = \overline{\mathsf{W}}[\mathbb{G}_0]$. By the proof of Theorem 5.2, $\mathbb{P}_0$ can be chosen such that (6.1) : $\mathsf{V} \models |\,\mathbb{P}_0\,| \leq \kappa$. x-bedrock-21-0

Let (6.2) : $\theta > \kappa + |\,\mathbb{P}_0\,|$ (in V) be large enough, and such that it satisfies (5.13) in the proof of Theorem 5.2 for sufficiently large natural number n. Let $\mathbb{Q} \in \mathcal{P}$ be such that for $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$ there are j, $N \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$, $j(\kappa) > \theta$, (6.3) : $|\,\mathbb{Q}\,| \leq j(\kappa)$, $\mathbb{H}$, $j''j(\theta) \in M$, and $V_{j(\theta)}{}^{\mathsf{V}[\mathbb{H}]} \subseteq M$. x-bedrock-22 x-bedrock-22-0

Since $|\,\mathbb{P}_0\,| < \theta < j(\kappa)$ we have

$$(6.4) \qquad \overline{\mathsf{W}} \models |\,\mathbb{P}_0\,| < j(\kappa).$$ x-bedrock-22-1

Thus

$$(6.5) \qquad (j(\kappa)^+)^{j(\overline{\mathsf{W}})} = (j(\kappa)^+)^{V_{j(\theta)}{}^{j(\overline{\mathsf{W}})}} \underbrace{=}_{\text{by Claim 5.2.4}} (j(\kappa)^+)^{V_{j(\theta)}{}^{\overline{\mathsf{W}}}} \overbrace{=}^{\text{by (6.4)}} (j(\kappa)^+)^{V_{j(\theta)}{}^{\mathsf{V}}} \overbrace{=}^{\text{by (6.3)}} (j(\kappa)^+)^{V_{j(\theta)}{}^{\mathsf{V}[\mathbb{H}]}}.$$ x-bedrock-23

By elementarity it follows that $(\kappa^+)^{\mathsf{W}} = (\kappa^+)^{\mathsf{V}}$. By (6.1) this implies that $\overline{\mathsf{W}} \models |\,\mathbb{P}_0\,| \leq \kappa$. ❑ (Proposition 6.1)

Proposition 6.2 *Suppose that κ is tightly $\mathcal{P}$-gen. hyperhuge for a class $\mathcal{P}$ of posets. By Proposition 6.1, there is a poset $\mathbb{P}$ in the bedrock $\overline{\mathsf{W}}$ with $\overline{\mathsf{W}} \models |\,\mathbb{P}\,| \leq \kappa$ such that $\mathsf{V} = \overline{\mathsf{W}}[\mathbb{G}]$ for a $(\overline{\mathsf{W}}, \mathbb{P})$-generic $\mathbb{G}$.* p-bedrock-10

For any bounded $b \subseteq \kappa$ in V, there is $\mathbb{P}_b \in \overline{\mathsf{W}}$ with $\overline{\mathsf{W}} \models |\,\mathbb{P}_b\,| < \kappa$, $\mathbb{P}_b \subseteq \mathbb{P}$, such that $\mathbb{G} \cap \mathbb{P}_b$ is $(\overline{\mathsf{W}}, \mathbb{P}_b)$-generic and $\overline{\mathsf{W}}[\mathbb{G} \cap \mathbb{P}_b] \ni b$.

Proof. Since $\overline{\mathsf{W}} \models$ "$|\,\mathbb{P}\,| \leq \kappa$" we may assume, without loss of generality, that the underlying set of $\mathbb{P}$ is κ. Let $\theta > |\,\mathbb{P}\,|$ be large enough and such that $V_\theta \prec_{\Sigma_n} \mathsf{V}$ for a large enough n in accordance with Lemma 3.2.

Let $\mathbb{Q} \in \mathcal{P}$ be such that for $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$, $\theta < j(\kappa)$, $|\,\mathbb{Q}\,| \leq j(\kappa)$, $V_{j(\theta)}{}^{\mathsf{V}[\mathbb{H}]} \subseteq M$ and $j''j(\theta) \in M$ (see Lemma 5.1, (5)).

By Claim 5.2.4, we have (6.6) : $j(V_\theta{}^{\overline{\mathsf{W}}})$ $(= V_{j(\theta)}{}^{j(\overline{\mathsf{W}})}) = V_{j(\theta)}{}^{\overline{\mathsf{W}}}$. By the choice of θ, we have (6.7) : $V_{j(\theta)}{}^{\mathsf{V}} = V_{j(\theta)}{}^{\overline{\mathsf{W}}}[\mathbb{G}]$. x-bedrock-24 x-bedrock-25

Since the underlying set of $\mathbb{P}$ is κ we have $j(\mathbb{P}) \cap \kappa = \mathbb{P}$ and $j(\mathbb{G}) \cap \kappa = \mathbb{G}$. By $b = j(b) \in {V_{j(\theta)}}^{\mathsf{V}} \overbrace{=}^{\text{by (6.6) and (6.7)}} j({V_\theta}^{\overline{\mathsf{W}}})[\mathbb{G}] = j({V_\theta}^{\overline{\mathsf{W}}})[j(\mathbb{G} \cap \kappa)]$, and since $|\,\mathbb{P}\,| < \theta < j(\kappa)$, we have

$$M \models \text{“ there is } P \in j(\overline{\mathsf{W}}) \text{ with } P \subseteq j(\mathbb{P}) \text{ and } j(\overline{\mathsf{W}}) \models \text{“}|\,P\,| < j(\kappa)\text{”} \\ \text{such that } j(\overline{\mathsf{W}})[j(\mathbb{G}) \cap P] \ni j(b)\text{”}.$$

By elementarity, it follows that

$$\mathsf{V} \models \text{“ there is } P \in \overline{\mathsf{W}} \text{ and } G \subseteq P \text{ with } P \subseteq \mathbb{P} \text{ and } \overline{\mathsf{W}} \models \text{“}|\,P\,| < \kappa\text{”} \\ \text{such that } G \text{ is } (\overline{\mathsf{W}}, P)\text{-generic filter and } \overline{\mathsf{W}}[\mathbb{G} \cap P] \ni j(b)\text{”}$$

as desired. ❑ (Proposition 6.2)

Corollary 6.3 *Suppose κ is a tightly $\mathcal{P}$-gen. hyperhuge cardinal for a class $\mathcal{P}$ of posets. Then we have $2^{<\kappa} = \kappa$.* p-bedrock-11

Proof. κ is hyperhuge in the bedrock $\overline{\mathsf{W}}$ of V by Theorem 5.3. Thus, $\overline{\mathsf{W}} \models 2^{<\kappa} = \kappa$. It follows that essentially there are at most κ many (nice) $\mathbb{P}$-names of elements of $[\kappa]^{<\kappa}$ for posets $\mathbb{P}$ of size $<\kappa$. Since all elements of $([\kappa]^{<\kappa})^{\mathsf{V}}$ are realizations of names of this kind by Proposition 6.2, it follows that $\mathsf{V} \models 2^{<\kappa} = \kappa$. ❑ (Corollary 6.3)

Hamkins [29] proved that if $\mathcal{P}$ contains $\mathrm{Col}(\omega, \lambda)$ for every λ, and $(\mathcal{P}, \mathcal{H}(\aleph_1))$-RcA holds, then we have $L_\kappa \prec L$ for $\kappa = \omega_1$. Practically the same proof concludes that the mantle W (the intersection of all grounds which is shown to be a model of ZFC in [42]) also satisfies ${V_\kappa}^{\mathsf{W}} \prec \mathsf{W}$ for $\kappa = {\omega_1}^{\mathsf{V}}$. Note that, since grounds are downward directed, the mantle of the universe V is also the mantle of any ground W of V.

Actually, we can say a little bit more.

Lemma 6.4 *Let W^* be the mantle of V. (1) Suppose that $(\mathcal{P}, \mathcal{H}(2^{\aleph_0})^{\mathsf{V}})$-RcA holds for a class $\mathcal{P}$ of posets such that either $\mathcal{P}$ contains posets collapsing λ to be countable for cofinally many cardinals λ, or it contains posets forcing $2^{\aleph_0}$ arbitrary large adding reals without collapsing cardinals below the number of the reals added.* p-bedrock-12

Then ${V_\alpha}^{\mathsf{W}^*}$ for all $\alpha < 2^{\aleph_0}$ is of cardinality $< 2^{\aleph_0}$ in V.

(2) Suppose that $(\mathcal{P}, \mathcal{H}(\omega_1)^{\mathsf{V}})$-RcA holds for a class $\mathcal{P}$ of posets such that $\mathcal{P}$ contains posets collapsing λ to be countable for cofinally many cardinals λ.

Then ${V_\alpha}^{\mathsf{W}^*}$ for all $\alpha < (\omega_1)^{\mathsf{V}}$ is countable in V.

Proof. (1): For $\alpha < 2^{\aleph_0}$, “${V_\alpha}^{W^*}$ is of cardinality $< 2^{\aleph_0}$” can be formulated as an $\mathcal{L}_\in$-formula with the the parameter $\alpha \in \mathcal{H}(2^{\aleph_0})$, and it is forcable by a poset in

$\mathcal{P}$. By $(\mathcal{P}, \mathcal{H}(2^{\aleph_0})^{\mathsf{V}})$-RcA, there is a ground W of V ($\mathsf{W}^* \subseteq \mathsf{W} \subseteq \mathsf{V}$) such that the statement above holds in W. Then $\mathsf{W} \models$ "${V_\alpha}^{W^*}$ is of cardinality $< 2^{\aleph_0}$", and hence $\mathsf{V} \models$ "${V_\alpha}^{W^*}$ is of cardinality $< 2^{\aleph_0}$".

(2): A proof similar to that of (1) will do.

For $\alpha < \omega_1$, "${V_\alpha}^{W^*}$ is of cardinality $< \aleph_1$" is a statement represented as an $\mathcal{L}_\in$-formula with the parameter $\alpha \in \mathcal{H}(\aleph_1)$, and it is forcable by a poset in $\mathcal{P}$. By $(\mathcal{P}, \mathcal{H}(\omega_1)^{\mathsf{V}})$-RcA, there is a ground W of V ($\mathsf{W}^* \subseteq \mathsf{W} \subseteq \mathsf{V}$) such that the statement above holds in W. Then $\mathsf{W} \models$ "${V_\alpha}^{W^*}$ is countable". and hence $\mathsf{V} \models$ "${V_\alpha}^{W^*}$ is countable". ❑ (Lemma 6.4)

Compare the following proposition with Lemma 4.1:

Proposition 6.5 *Let W^* be the mantle of V. (1) Suppose that $(\mathcal{P}, \mathcal{H}(2^{\aleph_0})^{\mathsf{V}})$-RcA holds for $\mathcal{P}$ as in Lemma 6.4,(1). That is, either $\mathcal{P}$ contains posets collapsing λ to be countable for cofinally many cardinals λ, or it contains posets forcing $2^{\aleph_0}$ arbitrary large adding reals without collapsing small cardinals.* p-bedrock-13

Then we have ${V_\kappa}^{\mathsf{W}^*} \prec \mathsf{W}^*$ for $\kappa = (2^{\aleph_0})^{\mathsf{V}}$.

(2) Suppose that $(\mathcal{P}, \mathcal{H}((\aleph_1)^{\mathsf{V}}))$-RcA holds for a class $\mathcal{P}$ of posets such that $\mathcal{P}$ contains posets collapsing λ to make it countable for cofinally many cardinals λ in On.

Then we have ${V_\kappa}^{\mathsf{W}^*} \prec \mathsf{W}^*$ for $\kappa = {\omega_1}^{\mathsf{V}}$.

Proof. (1): We show that $(V_{(2^{\aleph_0})^{\mathsf{V}}})^{\mathsf{W}^*}$ in W^* passes the Vaught's test. Suppose $\mathsf{W}^* \models \varphi(\overline{a}, b)$ where $\overline{a} \in (V_{(2^{\aleph_0})^{\mathsf{V}}})^{\mathsf{W}^*}$ and $b \in \mathsf{W}^*$. We have $\overline{a} \in \mathcal{H}(2^{\aleph_0})^{\mathsf{V}}$ by Lemma 6.4, (1). The statement

$$\psi := \exists y (y \in \mathsf{W}^* \wedge |\, trcl(y) \,| < 2^{\aleph_0} \wedge \mathsf{W}^* \models \varphi(\overline{a}, y))$$

is an $\mathcal{L}_\in$-formula the parameters $\overline{a} \in \mathcal{H}(2^{\aleph_0})^{\mathsf{V}}$ and forcable by a poset in $\mathcal{P}$ (just by collapsing a large enough cardinal to countable). By $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))$-RcA, it follows that there is a ground W of V (so $\mathsf{W}^* \subseteq \mathsf{W} \subseteq \mathsf{V}$) such that (6.8) : $\mathsf{W} \models \psi$. x-bedrock-26

Let $b' \in \mathsf{W}$ be a witness for (6.8). Then we have (6.9) : $\mathsf{W} \models |\, trcl(b') \,| < 2^{\aleph_0}$, x-bedrock-27
and $\mathsf{W}^* \models \varphi(\overline{a}, b')$.

Since $(2^{\aleph_0})^{\mathsf{W}} \leq (2^{\aleph_0})^{\mathsf{V}}$, it follows that $\mathsf{W}^* \models |\, trcl(b') \,| < (2^{\aleph_0})^{\mathsf{V}}$ by (6.9), and thus $b' \in (V_{(2^{\aleph_0})^{\mathsf{V}}})^{\mathsf{W}^*}$.

(2): can be proved similarly to (1).

We show that $(V_{{\omega_1}^{\mathsf{V}}})^{\mathsf{W}^*}$ in W^* passes the Vaught's test. Suppose $\mathsf{W}^* \models \varphi(\overline{a}, b)$ where $\overline{a} \in (V_{{\omega_1}^{\mathsf{V}}})^{\mathsf{W}^*}$ and $b \in \mathsf{W}^*$. We have $\overline{a} \in \mathcal{H}(\aleph_1)^{\mathsf{V}}$ by Lemma 6.4, (2). The statement

$$\psi := \exists y (y \in \mathsf{W}^* \wedge |\, trcl(y) \,| < \aleph_1 \wedge \mathsf{W}^* \models \varphi(\overline{a}, y))$$

is $\mathcal{L}_\in$-formula with the parameters $\overline{a} \in \mathcal{H}(\aleph_1)^{\mathsf{V}}$ and forcable by a poset in $\mathcal{P}$. By $(\mathcal{P}, \mathcal{H}(\aleph_1)^{\mathsf{V}})$-RcA, it follows that there is a ground W of V (so $\mathsf{W}^* \subseteq \mathsf{W} \subseteq \mathsf{V}$) such that (ℵ6.1) : $\mathsf{W} \models \psi$. x-bedrock-26-0

Let $b' \in \mathsf{W}$ be a witness for (ℵ6.1). Then we have (ℵ6.2) : $\mathsf{V} \models |\,trcl(b')\,| < \aleph_1$, and $\mathsf{W}^* \models \varphi(\overline{a}, b')$. x-bedrock-27-0

Since $\omega_1{}^{\mathsf{W}} \leq \omega_1{}^{\mathsf{V}}$, it follows that $\mathsf{W}^* \models |\,trcl(b')\,| < (\aleph_1)^{\mathsf{V}}$ by (ℵ6.2), and thus $b' \in (V_{\omega_1{}^{\mathsf{V}}})^{\mathsf{W}^*}$. ❑ (Proposition 6.5)

Theorem 6.6 *Suppose that $\mathcal{P}$ is a class of posets and $\kappa := (\omega_1)^{\mathsf{V}}$ is tightly $\mathcal{P}$-gen. hyperhuge. Then the following are equivalent:* p-bedrock-16

(a) *(all posets, $\mathcal{H}(\kappa)$)-RcA holds.*

(b) *(all posets, $\mathcal{H}(\kappa)$)-RcA$^+$ holds.*

(c) $V_\kappa{}^{\overline{\mathsf{W}}} \prec \overline{\mathsf{W}}$ *where $\overline{\mathsf{W}}$ is the bedrock of* V.

Proof. (a) ⇔ (b): is trivial. (a) ⇒ (c): By Proposition 6.5, (2).

(c) ⇒ (b): Assume (6.10) : $V_\kappa{}^{\overline{\mathsf{W}}} \prec \overline{\mathsf{W}}$, and $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$", for a poset $\mathbb{P}$, an $\mathcal{L}_\in$-formula $\varphi = \varphi(\overline{x})$, and $\overline{a} \in \mathcal{H}(\kappa)^{\mathsf{V}}$. x-bedrock-29

By Proposition 6.2, there are poset $\mathbb{Q} \in \overline{\mathsf{W}}$ with $\overline{\mathsf{W}} \models |\,\mathbb{Q}\,| < \kappa$, and $(\overline{\mathsf{W}}, \mathbb{Q})$-generic $\mathbb{G} \in \mathsf{V}$ such that $\overline{a} \in \overline{W}[\mathbb{G}]$. Let $\underset{\sim}{\overline{a}}$ be a $\mathbb{Q}$-name of $\overline{a}$.

By the choice of φ and $\overline{a}$, we have

$$\overline{\mathsf{W}} \models\ \Vdash_{\mathbb{Q}} \text{"there is a poset } P \text{ which forces } \varphi(\underset{\sim}{\overline{a}}^{\checkmark})\text{"}.$$

By the elementarity (6.10), it follows that $V_\kappa{}^{\overline{\mathsf{W}}} \models\ \Vdash_{\mathbb{Q}}$ "there is a poset P which forces $\varphi(\underset{\sim}{\overline{a}}^{\checkmark})$". Thus, there is a $\mathbb{Q}$-name $\underset{\sim}{\mathbb{R}} \in V_\kappa{}^{\overline{\mathsf{W}}}$ such that $\Vdash_{\mathbb{Q}*\underset{\sim}{\mathbb{R}}}$ "$\varphi((\underset{\sim}{\overline{a}}^{\checkmark})$".

We have $\mathbb{Q} * \underset{\sim}{\mathbb{R}}, \mathcal{P}(\mathbb{Q} * \underset{\sim}{\mathbb{R}})^{\overline{\mathsf{W}}} \in V_\kappa{}^{\overline{\mathsf{W}}} = \mathcal{H}(\kappa)^{\overline{\mathsf{W}}} \subseteq \mathcal{H}(\kappa)^{\mathsf{V}}$, and κ is ω_1 in V. Hence, we can construct a $(\overline{\mathsf{W}}[\mathbb{G}], \underset{\sim}{\mathbb{R}}[\mathbb{G}])$-generic $\mathbb{G}' \in \mathsf{V}$ in ω steps in V. $\overline{\mathsf{W}}[\mathbb{G}][\mathbb{G}']$ is a ground in V and $\overline{\mathsf{W}}[\mathbb{G}][\mathbb{G}'] \models \varphi(\overline{a})$. ❑ (Theorem 6.6)

The following corollary is proved similarly to Corollary 4.2.

Corollary 6.7 *Suppose that $\mathcal{P}$ is a class of posets and ω_1 is tightly $\mathcal{P}$-gen. hyperhuge. If (all posets, $\mathcal{H}(\aleph_1)$)-RcA holds then there are stationarily many hyperhuge cardinals. More precisely, under this condition, for any club subclass $\mathcal{C}$ of* On *defined with a parameter, there is a hyperhuge cardinal in $\mathcal{C}$. In particular, there are class may hyperhuge cardinals.* p-bedrock-17

Proof. Since V is a set generic extension of the bedrock $\overline{\mathsf{W}}$, it is enough to show that there are stationarily many hyperhuge cardinals in $\overline{\mathsf{W}}$ (in the same sense as in the statement of the corollary).

Suppose this is not the case. Then there is an $\mathcal{L}_\in$-formula $\Phi = \Phi(x, y)$ such that

$$\overline{\mathsf{W}} \models \text{“}\exists y\,(\,\Phi(\cdot,y) \text{ is a club in On} \\ \text{but } \Phi(\cdot,y) \text{ does not contain any hyperhuge cardinal)”}.$$

Since we have $V_{(\omega_1)^{\mathsf{V}}}{}^{\overline{\mathsf{W}}} \prec \overline{\mathsf{W}}$ by Theorem 6.6, it follows that

$$(\aleph 6.3)\quad V_{(\omega_1)^{\mathsf{V}}}{}^{\overline{\mathsf{W}}} \models \text{“}\exists y\,(\,\Phi(\cdot,y) \text{ is a club in On} \\ \text{but } \Phi(\cdot,y) \text{ does not contain any hyperhuge cardinal)”}.$$ x-bedrock-30

Let $b \in V_{(\omega_1)^{\mathsf{V}}}{}^{\overline{\mathsf{W}}}$ be a witness of ($\aleph$6.3). Then we have $(\omega_1)^{\mathsf{V}} \in \Phi(\cdot,b)$ by the closedness of $\Phi(\cdot,b)$. But this is a contradiction since $(\omega_1)^{\mathsf{V}}$ is a hyperhuge cardinal by Theorem 5.4. ❑ (Corollary 6.7)

The following proposition is a variation of Theorem 5.7 in [21].

Proposition 6.8 *Suppose that κ is tightly $\mathcal{P}$-Laver gen. supercompact for an iterable class $\mathcal{P}$ of posets. Then we have $\mathsf{MA}(\mathcal{P},<\kappa)$.* p-bedrock-14

Proof. Suppose that $\mathbb{P} \in \mathcal{P}$ and $\mathcal{D}$ is a family of dense subsets of $\mathbb{P}$ with $|\,\mathcal{D}\,| < \kappa$. Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name of a poset such that, for $(\mathsf{V},\mathbb{P}*\underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j:\mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, $\mathcal{P}$, $\mathbb{H} \in M$, $j''\lambda \in M$, and $|\,\mathbb{P}*\mathbb{Q}\,| \leq j(\kappa)$.

Note that $j(\mathcal{D}) = \{j(D) : D \in \mathcal{D}\}$. Let $\mathbb{G}$ be the $\mathbb{P}$ part of $\mathbb{H}$. We have $\mathbb{G} \in M$. Thus

$$\mathbb{G}^* = \{\mathbb{p} \in j(\mathbb{P}) : j(\mathbb{q}) \leq_{j(\mathbb{P})} \mathbb{p} \text{ for some } \mathbb{q} \in \mathbb{G}\}$$

is an element in M. Since $\mathbb{G}^*$ is $j(\mathcal{D})$-generic filter over $j(\mathbb{P})$,

$$M \models \exists G\,(G \text{ is a } j(\mathcal{D})\text{-generic filter over } j(\mathbb{P})).$$

By elementarity, it follows that

$$\mathsf{V} \models \exists G\,(G \text{ is a } \mathcal{D}\text{-generic filter over } \mathbb{P}).$$ ❑ (Proposition 6.8)

Lemma 6.9 *Suppose that κ is a tightly $\mathcal{P}$-gen. hyperhuge cardinal for a class $\mathcal{P}$ of posets, and $a \in \mathcal{H}(\kappa)$ is such that $\mathsf{V} \models \psi(a)$ for some $\mathcal{L}_\in$-formula $\psi = \psi(x)$. Let $\overline{\mathsf{W}}$ be the bedrock of V.* p-bedrock-15

Then there is $\mathbb{P}^* \in V_\kappa{}^{\overline{\mathsf{W}}}$ with $\overline{\mathsf{W}} \models |\,\mathbb{P}^*\,| < \kappa$, and $(\overline{\mathsf{W}},\mathbb{P}^*)$-generic $\mathbb{G}^* \in \mathsf{V}$ such that $a \in \overline{\mathsf{W}}[\mathbb{G}^*]$, $\overline{\mathsf{W}}[\mathbb{G}^*] \models$ “$\psi(a)$”, and $\overline{\mathsf{W}}[\mathbb{G}^*]$ is a $\mathcal{P}$-ground of V.

Proof. Assume that $\mathsf{V} = \overline{\mathsf{W}}[\mathbb{G}]$ where $\mathbb{G}$ is a $(\overline{\mathsf{W}},\mathbb{P})$-generic filter over a $\mathbb{P} \in \overline{\mathsf{W}}$ with $\overline{\mathsf{W}} \models$ “$|\,\mathbb{P}\,| \leq \kappa$” (Proposition 6.1). Without loss of generality, we shall assume that the underlying set of $\mathbb{P}$ is κ.

Let $\lambda > \kappa$ be such that (6.11) : $V_\lambda \prec_{\Sigma_n} \mathsf{V}$ for a sufficiently large n.[7)] x-bedrock-27-1

Let $\mathbb{Q} \in \mathcal{P}$ be such that, for $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$, $j(\kappa) > \lambda$, (6.12) : ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{H}] \in M$, and $|\,\mathbb{Q}\,| = j(\kappa)$ (see Lemma 5.1, (5)). x-bedrock-28

Note that we have ${V_{j(\lambda)}}^M \prec_{\Sigma_n} M$ by elementarity. By Lemma 3.2, it follows that $M \models V_{j(\lambda)} = {V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{H}]$. Also, by Lemma 3.2 and (6.12), we have ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} = {V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{H}] = {V_{j(\lambda)}}^M$.

Thus, for $\delta < \kappa$, noting $j(\delta) = \delta$, we have

$$M \models \text{“}V_{j(\lambda)} \text{ is a } \mathcal{P}\text{-generic extension of a ground } W \text{ which is a model of } \psi(j(a)), \text{and } W \text{ is a } P\text{-generic extension of } {V_{j(\lambda)}}^{\overline{\mathsf{W}}} \text{ for some poset } P \text{ of size } < j(\kappa).\text{”}$$

By elementarity, it follows that

$$\mathsf{V} \models \text{“}V_\lambda \text{ is a } \mathcal{P}\text{-generic extension of a ground } W \text{ which is a model of } \psi(a), \text{and } W \text{ is a } P\text{-generic extension of } {V_\lambda}^{\overline{\mathsf{W}}} \text{ for some poset } P \text{ of size } < \kappa.\text{”}$$

Now by the choice (6.11) of λ, it follows that

> V is a $\mathcal{P}$-generic extension of a ground W which is a model of $\psi(a)$, and W is a $\mathbb{P}^*$-generic extension of $\overline{\mathsf{W}}$ for a poset $\mathbb{P}^*$ of size $< \kappa$. ❑ (Lemma 6.9)

A poset $\mathbb{P}$ has pre-caliber κ if for any $A \in [\mathbb{P}]^{\geq\kappa}$ there is $B \in [A]^{\geq\kappa}$ such that B is centered (i.e. b has a lower bound in $\mathbb{P}$ for and $b \in [B]^{<\aleph_0}$).

Lemma A 6.1 (Lemma III, 3.35, in [36]) $\mathsf{MA}(\aleph_1)$ *implies that all ccc poset* $\mathbb{P}$ *has pre-caliber* $\aleph_1$*.* p-bedrock-18

Proof. Suppose $\mathbb{p}_\alpha \in \mathbb{P}$ for $\alpha < \omega_1$. It is enough to find a filter containing uncountably many of $\mathbb{p}_\alpha$'s.

For each $\alpha < \omega_1$, let $D_\alpha = \{\mathbb{q} \in \mathbb{P} : \mathbb{q} \leq_{\mathbb{P}} \mathbb{p}_\beta \text{ for some } \beta \geq \alpha\}$.

If there is no $\mathbb{s} \in \mathbb{P}$ such that uncountably many D_α's are dense below *posets*, we can construct an strictly increasing sequence $\langle \beta_\alpha : \alpha < \omega_1 \rangle$ in ω_1 such that $\mathbb{p}_{\beta_\alpha}$ is incompatible with all $\mathbb{p}_{\beta_\gamma}$, $\gamma < \alpha$. But then t $\{\mathbb{p}_{\beta_\alpha} : \alpha < \omega_1\}$ is a pairwise incompatible uncountable set which is a contradiction to the ccc of $\mathbb{P}$.

Thus, there is $\mathbb{s} \in \mathbb{P}$ such that there are uncountably many α's such that $\mathcal{D}_\alpha$ is dense below $\mathbb{s}$. Then since D_α, $\alpha < \omega_1$ build a decreasing sequence, D_α is dense below $\mathbb{s}$ for all $\alpha < \omega_1$. Let $\mathcal{D} := \{D_\alpha : \alpha < \omega_1\}$ and let $\mathbb{G}$ be a $\mathcal{D}$-generic filter over $\mathbb{P}$. Then $\mathbb{p}_\alpha \in \mathbb{G}$ for all $\alpha \in I$ as desired. ❑ (Lemma A 6.1)

---

7) Here “sufficiently large n” refers, among other things, largeness in terms of Lemma 3.2, and the absoluteness of “$\mathbb{P} \in \mathcal{P}$”.

Corollary A 6.2 $\mathsf{MA}(\aleph_1)$ *implies that, for all ccc posets* $\mathbb{P}$, $\mathbb{Q}$, *the product* $\mathbb{P}\times\mathbb{Q}$ *has the ccc.* ❑ p-bedrock-19

Proof. It is enough to show that $\mathbb{P}\times\mathbb{Q}$ has pre-caliber $\aleph_1$ by Lemma A 6.1. Suppose that $\langle \mathbb{p}_\alpha, \mathbb{q}_\alpha\rangle \in \mathbb{P}\times\mathbb{Q}$ for $\alpha < \omega_1$. Since $\mathbb{P}$ has pre-caliber $\aleph_1$, there is $I_0 \in [\omega_1]^{\aleph_1}$ such that $\{\mathbb{p}_\alpha : \alpha \in I_0\}$ is centered in $\mathbb{P}$. Since $\mathbb{Q}$ also has pre-caliber $\aleph_1$ by Lemma A 6.1, there is $I_1 \subseteq [I_0]^{\aleph_1}$ such that $\{\mathbb{q}_\alpha : \alpha \in I_1\}$ is centered in $\mathbb{Q}$. $\{\langle \mathbb{p}_\alpha, \mathbb{q}_\alpha\rangle : \alpha \in I_1\}$ is centered inc $\mathbb{P}\times\mathbb{Q}$. ❑ (Corollary A 6.2)

The following Lemma is classical:

Lemma 6.10 (1) *Suppose that* $\mathsf{V} \models$ "$\mathbb{P}\times\mathbb{Q}$ *is ccc*" *and* $\mathbb{G}$ *is a* $(\mathsf{V},\mathbb{P})$*-generic filter, then* $\mathsf{V}[\mathbb{G}] \models$ "$\mathbb{Q}$ *is ccc*". p-bedrock-20

(2) *Suppose that* $\mathsf{V} \models \mathsf{MA}(\aleph_1)$. *If* $\mathsf{V} \models$ "$\mathbb{P}$ *and* $\mathbb{Q}$ *are ccc*" *and* $\mathbb{G}$ *is a* $(\mathsf{V},\mathbb{P})$*-generic filter, then* $\mathsf{V}[\mathbb{G}] \models$ "$\mathbb{Q}$ *is ccc*".

Proof. (1): Suppose otherwise and there is a $f \in \mathsf{V}[\mathbb{G}]$ such that $V[\mathbb{G}] \models$ "$f : \omega_1 \to \mathbb{Q}$ is such that, $f(\alpha), f(\beta)$ for distinct $\alpha, \beta < \omega_1$ are pairwise incompatible".

Let $\underset{\sim}{f}$ be a $\mathbb{P}$-name of f and let $\mathbb{p}_\alpha \in \mathbb{P}$, $\mathbb{q}_\alpha \in \mathbb{Q}$ for $\alpha < \omega_1$ are such that $\mathbb{p}_\alpha \Vdash_{\mathbb{P}}$ "$\underset{\sim}{f}(\alpha) = \check{\mathbb{q}}_0$". Then $\{\langle \mathbb{p}_\alpha, \mathbb{q}_\alpha\rangle : \alpha < \omega_1\}$ is pairwise incompatible in $\mathbb{P}\times\mathbb{Q}$. This is a contradiction to the assumption.

(2): By (1) and Corollary A 6.2. ❑ (Lemma 6.10)

Theorem 6.11 *Suppose that* κ *is tightly ccc-Laver-gen. hyperhuge. Then the following are equivalent:* p-bedrock-21

(a) $(ccc, \mathcal{H}(\kappa))$-$\mathsf{RcA}$ *holds.*

(b) $(ccc, \mathcal{H}(\kappa))$-$\mathsf{RcA}^+$ *holds.*

(c) $V_\kappa^{\overline{\mathsf{W}}} \prec \overline{\mathsf{W}}$ *where* $\overline{\mathsf{W}}$ *is the bedrock of* V.

Proof. Note that $\kappa = 2^{\aleph_0}$ by Theorem 5.8 in [21]. Thus, by Proposition 6.8, MA holds.

(b) $\Rightarrow$ (a): is trivial. (a) $\Rightarrow$ (c): By Proposition 6.5, (1).

(c) $\Rightarrow$ (b): Suppose (in V) that (6.13) : $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$" for a ccc poset $\mathbb{P}$, $\mathcal{L}_\in$-formula $\varphi = \varphi(\overline{x})$, and $\overline{a} \in \mathcal{H}(\kappa)$. x-bedrock-31

By Lemma 6.9 there is $\mathbb{P}^* \in \overline{\mathsf{W}}$ with $\overline{\mathsf{W}} \models |\,\mathbb{P}^*\,| < \kappa$ and $(\overline{\mathsf{W}}, \mathbb{P}^*)$-generic $\mathbb{G}^* \in \mathsf{V}$ such that (6.14) : $\overline{a} \in \overline{\mathsf{W}}[\mathbb{G}^*]$, (6.15) : $\overline{\mathsf{W}}[\mathbb{G}^*] \models \mathsf{MA}$, and (6.16) : $\overline{\mathsf{W}}$ is a ccc-ground of V. x-bedrock-32 x-bedrock-33 x-bedrock-34

By (6.13) and (6.16), we have

$$\overline{\mathsf{W}}[\mathbb{G}^*] \models \text{"there is a ccc poset } P \text{ such that } \Vdash_P \text{"}\varphi(\overline{a})\text{""}.$$

By the assumption (c), e have (6.17) : ${V_\kappa}^{\overline{\mathsf{W}}[\mathbb{G}^*]} \prec \overline{\mathsf{W}}[\mathbb{G}^*]$. Hence x-bedrock-35

(6.18) ${V_\kappa}^{\overline{\mathsf{W}}[\mathbb{G}^*]} \models$ "there is a ccc poset P such that $\Vdash_P$ "$\varphi(\overline{a})$ "". x-bedrock-36

Let $\mathbb{Q}^*$ be a witness of (6.18). Then $\overline{\mathsf{W}}[\mathbb{G}^*] \models |\,\mathbb{Q}^*\,| < \kappa$ and $\overline{\mathsf{W}}[\mathbb{G}^*] \models$ "$\mathbb{Q}^*$ is ccc" by (6.17). By (6.16) and Lemma 6.10, we have $\mathsf{V} \models$ "$\mathbb{Q}^*$ is ccc". Since $\mathcal{P}(\mathbb{Q}^*)^{\overline{\mathsf{W}}[\mathbb{G}^*]} \in {V_\kappa}^{\overline{\mathsf{W}}[\mathbb{G}^*]}$ and $\kappa = 2^{\aleph_0}$ in V, MA (in V) implies that there is a $(\overline{\mathsf{W}}, \mathbb{Q}^*)$-generic $\mathbb{H}^* \in \mathsf{V}$.

$\overline{\mathsf{W}}[\mathbb{G}^*][\mathbb{H}^*] \models \varphi(\overline{a})$ since $\mathbb{Q}^*$ is a witness of (6.18), and $\overline{\mathsf{W}}[\mathbb{G}^*][\mathbb{H}^*]$ is a ccc-ground of V by Lemma 6.10. ❑ (Theorem 6.11)

7 The Laver-Generic Maximum

Suppose that $\kappa_0 < \kappa_1 < \kappa^*$ are regular cardinals such that $V_{\kappa^*} \models \mathsf{ZFC}$ and κ_0, κ_1 are super-$C^{(\infty)}$-hyperhuge cardinals in V_{κ^*} . Note that existence of a 2-huge cardinal implies the consistency of this constellation (see Lemma 4.5). LGM

Consider the following construction:

> first make κ_0 a tightly super-$C^{(\infty)}$-all posets-Laver gen. hyperhuge by a poset of size κ_0 as in Theorem 4.7, (4). Then we force κ_1 to be a tightly super- $C^{(\infty)}$-semi-proper-Laver gen. hyperhuge by a poset of size κ_1 as in Theorem 4.7, (2').

The resulting model satisfies:

(7.1) ZFC x-bedrock-37

+ "$\omega_2 = 2^{\aleph_0}$ is the tightly super-$C^{(\infty)}$-semi-proper-Laver gen. hyperhuge cardinal"

+ "There is a semi-proper ground W of the universe V such that $(2^{\aleph_0})^{\mathsf{W}} = \omega_1 = {\omega_1}^{\mathsf{V}}$ is the tightly super-$C^{(\infty)}$-all-posets-Laver gen. hyperhuge cardinal in W".

We want to call (7.1) Laver Generic Maximum (LGM), of cause not because of the maximality of possible consistency strength among similar assertions (this is not true since we can also switch in some other notion of large cardinal stronger than the hyperhugeness) but rather because this combination of the properties implies that practically all set-theoretic assertions known to be consistent with ZFC are realized either as consequences of (7.1) or as theorems in (many of) grounds of V or some other inner models of V.

So we have under the LGM (7.1) that

– The bedrock $\overline{\mathsf{W}}$ exists (Theorem 5.2).

- ${\omega_1}^{\mathsf{V}}$ and ${\omega_2}^{\mathsf{V}}$ are super-$C^{(\infty)}$-hyperhuge cardinals in $\overline{\mathsf{W}}$ (Theorem 5.8).
- ${V_{({\omega_1})^{\mathsf{V}}}}^{\overline{\mathsf{W}}} \prec \overline{\mathsf{W}}$. (Theorem 4.1).
- ${V_{({\omega_2})^{\mathsf{V}}}}^{\overline{\mathsf{W}}} \prec \overline{\mathsf{W}}$. (Theorem 4.1).
- (semi-proper, $\mathcal{H}(2^{\aleph_0})$)-$\mathsf{RcA}^+$ holds (Theorem 4.10).
- (all posets, $\mathcal{H}(\aleph_1)^{\overline{\mathsf{W}}}$)-$\mathsf{RcA}^+$ holds (Theorem 4.10, and since RcA^+ for all posets with the same parameters is preserved by generic extensions).
- For each natural number n there are stationarily many super-$C^{(n)}$-hyperhuge cardinals (this holds in $\overline{\mathsf{W}}$ by Corollary 4.2, and since the statement is preserved by set forcing, it also holds in V).
- For any natural number n and any $a \in \mathcal{H}(2^{\aleph_0})$, there is a semi-proper ground W with $a \in \mathsf{W}$ such that $\mathsf{W} \models$ "$2^{\aleph_0}$ is the tightly super-$C^{(n)}$-ccc-Laver gen. hyperhuge cardinal". (there is a super- $C^{(n)}$-hyperhuge cardinal

  by the previous item and hence we can force the statement by a ccc forcing (by a variation of Theorem 4.7, (3)). (semi-proper, $\mathcal{H}(2^{\aleph_0})$)-$\mathsf{RcA}^+$ now implies the existence of a semi-proper ground W with $a \in \mathsf{W}$ satisfying the statement).
- Unbounded Resurrection Axiom of Tsaprounis in [40] for semi-proper (see [17]).
- MM^{++} (by Theorem 5.7 in [21]).
- $2^{\aleph_0} = \omega_2$ (either by MM^{++} or by Theorem 3.3, (4)).
- For any known instance CM of Cichoń's Maximum (even one of those in which some mild large cardinals are involved) and any $a \in \mathcal{H}(\aleph_2)$ there is a semi-proper ground W with $a \in \mathsf{W}$ such that $\mathsf{W} \models \mathsf{CM}$ (by (semi-proper, $\mathcal{H}(2^{\aleph_0})$)-$\mathsf{RcA}^+$).
- . . .

If we can accept the tightly super-$C^{(\infty)}$-semi-proper-Laver gen. hyperhuge continuum as a natural and/or even desirable set-theoretic assumption, (7.1) may be considered as a sort of the final solution to the continuum problem (and actually much more) in terms of the properties listed above — for discussions about arguments supporting naturalness of the tightly super-$C^{(\infty)}$-semi-proper-Laver gen. hyperhuge continuum, see also section 2 of [18].

On the other hand, it also seems that Theorem 6.6 and Theorem 6.11, together with the intuition that the universe of set theory should accommodate as many prominent grounds as possible, suggest that each of the combinations below are reasonable one:

(7.2) ZFC x-bedrock-37-0

+ "$\omega_1 = 2^{\aleph_0}$ is the tightly super-$C^{(\infty)}$-all posets-Laver gen. hyperhuge cardinal".

(7.3) ZFC x-bedrock-38

+ "$2^{\aleph_0}$ is the tightly super-$C^{(\infty)}$-ccc Laver gen. hyperhuge cardinal"

+ "There is a ccc-ground W of the universe V such that $(2^{\aleph_0})^{\mathsf{W}} = {\omega_2}^{\mathsf{W}}$ is the tightly semi-proper-Laver gen. hyperhuge cardinal in W"

+ "There is a semi-proper-ground W' of the universe V such that $(2^{\aleph_0})^{\mathsf{W}'} = \omega_1 = {\omega_1}^{\mathsf{V}}$ is the tightly super-$C^{(\infty)}$-all-posets-Laver gen. hyperhuge cardinal in W'".

The combinations of axioms (7.2) and (7.3) will be also examined further in the subsequent papers. Note that (7.3) implies that the continuum is extremely large (weakly Mahlo and much more, see [21], [26]), and that the Fodor-type Reflection Principle (FRP) holds (FRP holds in the ccc-ground W with the tightly semi-proper-Laver gen. hyperhuge continuum, since this implies MM^{++} in W (by Theorem 5.7 in [21]). Since FRP is preserved by ccc generic extension (Theorem 3.4 in [19]), V also satisfies it).

References

[1] Uri Abraham, and Saharon Shelah, Forcing closed unbounded sets, Journal of Symbolic Logic, Vol.48(3), (1983), 643–657. abraham-shelah

[2] Neil Barton, Andrés Eduardo Caicedo, Gunter Fuchs, Joel David Hamkins, Jonas Reitz, and Ralf Schindler, Inner-Model Reflection Principles, Studia Logica, 108 (2020), 573–595. 5 5a

[3] Will Boney, Model Theoretic Characterizations of Large Cardinals, Israel Journal of Mathematics, 236, (2020), 133–181. 14 boney

[4] Sean Cox, The diagonal reflection principle, Proceedings of the American Mathematical Society, Vol.140, No.8 (2012), 2893–2902. cox

[5] Alan Dow, Franklin D. Tall and W.A.R. Weiss, New proofs of the consistency of the normal Moore space conjecture I, II, Topology and its Applications, 37 (1990) 33–51, 115-129. dtw

[6] Sy-David Friedman, Internal Consistency and the Inner Model Hypothesis, The Bulletin of Symbolic Logic, Vol. 12, No. 4 (Dec., 2006), 591–600. 6 friedman-sy

[7] Sakaé Fuchino, On potential embedding and versions of Martin's axiom, Notre Dame Journal of Logic, Vol.33, No.4, (1992), 481–492. potential

[8] ____________, Left-separated topological spaces under Fodor-type Reflection Principle, RIMS Kôkyûroku, No.1619, (2008), 32–42. fuchino-left

[9] ____________, Fodor-type Reflection Principle and Balogh's reflection theorems, RIMS Kôkyûroku, No.1686, (2010), 41–58. balogh

[10] ____________, Rado's Conjecture implies the Fodor-type Reflection Principle, Note, https://fuchino.ddo.jp/notes/RCimpliesFRP2.pdf sf-note

[11] ____________, On reflection numbers under large continuum, RIMS Kôkyûroku, No.1988, 1–16, (2016). RIMS15

[12] ____________, Pre-Hilbert spaces without orthonormal bases, preprint. pre-hilbert

[13] ____________, A reflection principle as a reverse-mathematical fixed point over the base theory ZFC, Annals of the Japan Association for the Philosophy of Science, Vol.25, (2017), 67–77. fuchino-reverse

[14] ____________, Images of the white board of a seminar talk given at Nagoya University on May 31, 2019, https://fuchino.ddo.jp/talks/talk-nagoya-2019-05-31.pdf nagoya

[15] ____________, Resurrection and Maximality under a/the tightly Laver-generically ultrahuge cardinal, slides of a series zoom talks given at May 29, and June 5, 2023 at Kobe Set Theory Seminar. https://fuchino.ddo.jp/slides/kobe2023-05-29-pf.pdf slides

[16] ____________, Resurrection and Maximality under a/the tightly Laver-generically ultrahuge cardinal — additional slides, additional slides of the zoom talks given at June 5, 2023 at Kobe Set Theory Seminar. https://fuchino.ddo.jp/slides/kobe2023-06-05a-pf.pdf slides-x

[17] ____________, Maximality Principles and Resurrection Axioms in light of a Laver-generic large cardinal, Pre-preprint. https://fuchino.ddo.jp/papers/RIMS2022-RA-MP-x.pdf 5, 8, 9, 13, 14, 22, 42 future

[18] ____________, Reflection and Recurrence, submitted, to appear in the festschrift on the occasion of the 75. birthday of Professor Janos Makowsky. https://fuchino.ddo.jp/papers/reflection_and_recurrence-Janos-Festschriftx.pdf 42 janos

[19] Sakaé Fuchino, István Juhász, Lajos Soukup, Zoltán Szentmiklóssy, and Toshimichi Usuba, Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness, Topology and its Applications, Vol.157, 8 (2010), 1415–1429. 43 fjetal

[20] Sakaé Fuchino, André Ottenbreit Maschio Rodrigues and Hiroshi Sakai, Strong Löwenheim-Skolem theorems for stationary logics, I, Archive for Mathematical Logic, Volume 60, issue 1-2, (2021), 17–47. https://fuchino.ddo.jp/papers/SDLS-x.pdf sfetal-I

[21] ________________________________, Strong Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum, Archive for Mathematical Logic, Volume 60, issue 3-4, (2021), 495–523. https://fuchino.ddo.jp/papers/SDLS-II-x.pdf 14, 19, 20, 26, 31, 32, 38, 40, 42, 43 sfetal-II

[22] ________________________________, Strong downward Löwenheim-Skolem theorems for stationary logics, III — mixed support iteration, to appear in the Proceedings of the Asian Logic Conference 2019. https://fuchino.ddo.jp/papers/SDLS-III-xx.pdf 19 sfetal-III

[23] Sakaé Fuchino, and André Ottenbreit Maschio Rodrigues, Reflection principles, generic large cardinals, and the Continuum Problem, to appear in the Proceedings of the Symposium on Advances in Mathematical Logic 2018. https://fuchino.ddo.jp/papers/refl_principles_gen_large_cardinals_continuum_problem-x.pdf 19 FuOt

[24] Sakaé Fuchino and Assaf Rinot, Openly generated Boolean algebras and the Fodor-type Reflection Principle, Fundamenta Mathematicae 212, (2011), 261-283. rinot

[25] Fuchino, Sakaé, and Sakai, Hiroshi, The first-order definability of generic large cardinals, submitted. Extended version of the paper: https://fuchino.ddo.jp/papers/definability-of-glc-x.pdf 19 fuchino-sakai

[26] Sakaé Fuchino, and Hiroshi Sakai, Generically and Laver-generically supercompact cardinals by forcing with chain conditions, RIMS Kôkûroku, No.2213, (2022), 94–111. https://repository.kulib.kyoto-u.ac.jp/dspace/bitstream/2433/275453/1/2213-08.pdf 43 fuchino-sakai-2

[27] Gunter Fuchs Closed Maximality Principles: Implications, Separations and Combinations, The Journal of Symbolic Logic, Vol. 73, No. 1, (2008), 276–308. fuchs

[28] Gunter Fuchs, Joel David Hamkins, and Jonas Reitz, Set-theoretic geology, Annals of Pure and Applied Logic 166(4), (2015), 464–501. 4 fhr

[29] Joel David Hamkins, A simple maximality principle, The Journal of Symbolic Logic Vol.68, no.7, (2003), 527–550. 8, 35 hamkins

[30] Joel David Hamkins, and Thomas A. Johnstone, Resurrection axioms and uplifting cardinals, Archive for Mathematical Logic, Vol.53, Iss.3-4, (2014), 463–485. hamkins-johnstone

[31] Joel David Hamkins, and Thomas A. Johnstone, Strongly uplifting cardinals and the boldface resurrection axioms, Archive for Mathematical Logic Vol. 56, (2017), 1115–1133. hamkins-johnstone2

[32] Daisuke Ikegami and Nam Trang, On a class of maximality principles, Stcive for Mathematical Logic, Vol. 57, (2018), 713–725. 32 ikegami-trang

[33] Thomas Jech, Set Theory, The Third Millennium Edition, Springer (2001/2006). 3, 19, 32 millennium-book

[34] Akihiro Kanamori, The Higher Infinite, Springer–Verlag (1994/2003). 3 higher-inf

[35] Bernhard König, Generic compactness reformulated, Archive for Mathematical Logic 43, (2004), 311–326. koenig

[36] Kenneth Kunen, Set Theory, College Publications (2011). 3, 39 kunen-2011

[37] Kaethe Minden, Combining resurrection and maximality, The Journal of Symbolic Logic, Vol. 86, No. 1, (2021), 397–414. minden

[38] Jonas Reitz, The Ground Axiom, The Journal of Symbolic Logic, Vol. 72, No. 4 (2007), 1299–1317. 4 reitz

[39] Stevo Todorčević, Conjectures of Rado and Chang and cardinal arithmetic, In: Sauer, N.W., Woodrow, R.E., Sands, B. (eds.) Finite and Infinite Combinatorics in Sets and Logic. NATO ASI Series, vol 411, (1993), 385 – 398. stevo1993

[40] Konstantinos Tsaprounis, On resurrection axioms, The Journal of Symbolic Logic, Vol.80, No.2, (2015), 587–608. 42 tsaprounis1

[41] ______________________, Ultrahuge cardinals, Mathematical Logic Quarterly, Vol.62, No.1-2, (2016), 1–2. tsaprounis2

[42] Toshimichi Usuba, The downward directed grounds hypothesis and very large cardinals, Journal of Mathematical Logic, Vol. 17(2) (2017), 1–24. 3, 26, 30, 35 usuba

[43] ________________, Extendible cardinals and the mantle, Archive of Mathematical Logic, Vol.58, (2019), 71-75. 30, 31 usuba2

[44] Matteo Viale, THE Proper Forcing Axiom and the Singular Cardinal Hypothesis, The Journal of Symbolic Logic, Vol.71, (2), (2006), 473–479. 32 viale

Additional references cited in the extended version