Reflection and Recurrence.

Sakaé Fuchino (渕野 昌)[0000−0002−8884−6005]

(February 21, 2024 01:51 version)

Contents

Abstract We examine the Zermelo Fraenkel set theory with Choice (ZFC) enhanced by one of the (structural) Reflection Principles down to a small cardinal and/or Recurrence Axioms defined below. The strongest forms of Reflection Principles spotlight the three scenarios in which the size of the continuum is either $\aleph_1$, or $\aleph_2$, or very large, while the maximal setting of Recurrence Axioms points to the set-theoretic universe with the continuum of size $\aleph_2$.

Since both the Reflection Principles and Recurrence Axioms can be interpreted as preferable candidates of the extension of ZFC in terms of the criteria of Gödel's Program, the maximal possible (consistent) combination of these Principles and Axioms, or even some natural strengthening of the combination (which we want to

Sakaé Fuchino
Graduate School of System Informatics, Kobe University, Nada, Kobe 657-8501 Japan,
e-mail: fuchino@diamond.kobe-u.ac.jp

This is an extended postprint version of the paper with the same title which will appear in a Festschrift in occasion of the 75. birthday of Professor Janos Makowsky. The most recent file of this version is downloadable as:
https://fuchino.ddo.jp/papers/reflection_and_recurrence-Janos-Festschrift-x.pdf

All additional texts not included in the published version and all major corrections after the publication are marked with "dark electric blue" foreground color.

call "Laver-generic Maximum") may be considered as the ultimate extension of ZFC (of course "ultimate" only for now — because of the Incompleteness Theorems): it will resolve the size of the continuum to be $\aleph_2$ and integrates practically all known statements consistent with ZFC in itself either as its consequences (e.g. MM^{++}) or as theorems holding in many grounds of the universe (as it is the case with Cichoń's Maximum).

アブストラクト　以下で，選択公理付きの Zermelo Fraenkel 集合論 (ZFC) を，小さな基数への (構造に関する) 反映原理 (Reflection Principles) たちのうちの一つ，かつ／または，回帰公理 (Recurrence Axioms) たちの一つで拡張した体系について考察する．反映原理のうちの一番強いものたちは，連続体のサイズが $\aleph_1$ であるか，$\aleph_2$ であるか，非常に大きくなるかという 3 つのシナリオにスポットライトをあてるのに対し，Recurrence Axioms の一番強いものたちの組合せは，連続体のサイズが $\aleph_2$ になる集合論の宇宙を指ししめしているようである．

反映原理も回帰公理も，ゲーデルのプログラムの意味で，ZFC の望ましい拡張を与えるものの候補と看做せるので，これらの原理と公理の無矛盾なもののうちの極大な組合せは，ZFC の究極の拡張と考えることができる — これを我々は "Laver-generic Maximum" と呼びたい．もちろんこれは「究極の」，とは言っても不完全性定理のために，現在のところの究極でしかないものなのだが，そのような組合せは，連続体問題を連続体濃度が $\aleph_2$ である，として解決し，現在までに知られている実質的に全ての ZFC と無矛盾な命題を，(たとえば MM^{++} がそうであるように) この拡張から導かれる定理として，あるいは，(たとえば Cichoń's Maximum がそうであるように) 多くの，集合論宇宙の grounds での定理として，この理論に統合する．

1 Introduction

In the following, we examine the Zermelo Fraenkel set theory with the Axiom of Choice (ZFC) enhanced by one of the (structural) Reflection Principles down to a small cardinal and/or Recurrence Axioms defined below. The strongest forms of Reflection Principles (existence of a/the $\mathcal{P}$-Laver-generic large cardinal — see Section 2 below) spotlight the three scenarios in which the size of the continuum is either $\aleph_1$, or $\aleph_2$, or very large (see Theorem 8), while the maximal setting of Recurrence Axioms points to the set-theoretic universe with the continuum of size $\aleph_2$ (see the end of Section 3).

Since both the Reflection Principles and Recurrence Axioms can be interpreted as preferable candidates of the extension of ZFC in terms of Gödel's Program ([23], see also [1]), the maximal possible (consistent) combination of these Principles and Axioms, or even some natural strengthening of the combination, that is, either the principle LGM_0 proposed in Section 6 or some further extension of it in the future (which we want to call "Laver-generic Maximum") may be considered as the ultimate extension of ZFC (of course "ultimate" only for now — because of the Incompleteness Theorems): it resolves the size of the continuum to be $\aleph_2$ and integrates practically all known statements consistent with ZFC in itself either as its consequences (e.g. MM^{++}) or as theorems holding in many grounds of the universe (as it is the case with Cichoń's Maximum [25], [26]) — see the discussions at the end of Sections 3, 6.

2 Reflection down to a small cardinal

The small cardinal mentioned in the title of this section may be considered as not very small by non-set-theoretic mathematicians: It is known that many "mathematical" reflection statements with reflection number $\leq \kappa_{\mathfrak{refl}} := \max\{\aleph_2, 2^{\aleph_0}\}$ hold (in some extension of ZFC). Some of them are even theorems in ZFC. For example,

Theorem 1 (1) (Dow [8]) *If X is a countably compact Hausdorff non-metrizable space then there is a subspace Y of X of cardinality $< \aleph_2$ such that Y is also non-metrizable.*

(2) *Let $L(Q)$ be a logic with new (first-order) quantifier such that "Qx ..." is interpreted as "there are uncountably many x such that ...". For any structure $\mathfrak{A}$ of countable signature, there is $\mathfrak{B} \prec_{L(Q)} \mathfrak{A}$ of size $< \aleph_2$.* □

From very early on, it was known that, starting from a very large cardinal, we can construct models of set theory in which various strong statements on (structural) reflection down to $< \kappa_{\mathfrak{refl}} := \max\{\aleph_2, 2^{\aleph_0}\}$ hold.

For example, Ben-David [5] in 1978 mentions a theorem by Shelah which states:

Theorem 2 (S. Shelah, [5]) *Suppose that κ is supercompact and $\mathbb{P} = \mathrm{Col}(\aleph_1, \kappa)$. Then, for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, we have*

$$\mathsf{V}[\mathbb{G}] \models \textit{for any structure } \mathfrak{A} \textit{ of countable signature, there is } \mathfrak{B} \prec_{L_{stat}} \mathfrak{A} \textit{ of cardinality } < \aleph_2.$$

Here, L_{stat} denotes the stationary logic with monadic second-order variables X which run over countable subsets of the underlying set of respective structures and with the second-order quantifier $stat\, X$ which is to be interpreted as "there are stationarily many countable sets X".

The elementary submodel relation $\mathfrak{B} \prec_{L_{stat}} \mathfrak{A}$ is defined by $\mathfrak{B} \models \varphi(b_0, \ldots) \Leftrightarrow \mathfrak{A} \models \varphi(b_0, \ldots)$ for all L_{stat}-formula $\varphi = \varphi(x, \ldots)$ without free second-order variables, and for all $b_0 \cdots \in |\mathfrak{B}|$.

Today, we can understand Shelah's theorem above as a special case of the following theorem. For a class $\mathcal{P}$ of posets, a cardinal κ is said to be *$\mathcal{P}$-generically supercompact* if, for any $\lambda > \kappa$, there is a poset $\mathbb{P} \in \mathcal{P}$ such that, for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are $j, M \subseteq \mathsf{V}[\mathbb{G}]$ such that $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$, and[1] $j''\lambda \in M$.

Theorem 3 *Suppose that $\aleph_2$ is $\mathcal{P}$-generically supercompact where $\mathcal{P}$ is the class of all $<\aleph_1$-closed posets. Then*

[1] With "$j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$" we denote the situation that M is transitive, j is an elementary embedding of V into M, and κ is the critical point of j.

(*1) *for any structure* $\mathfrak{A}$ *of countable signature, there is* $\mathfrak{B} \prec_{L_{stat}} \mathfrak{A}$ *of cardinality* $< \aleph_2$.

Proof. The condition "$\aleph_2$ is $\mathcal{P}$-generic supercompact for $\mathcal{P}$ = the class of all $<\aleph_1$-closed posets." is equivalent to the Game Reflection Principle (GRP) (Theorem 8 in König [30] — see [15] for a generalization of König's theorem in [30] — note that what we call GRP here and [15] is called "the global Game Reflection Principle" in [30]). By Theorem 4.7 in [15], GRP implies (*1). □ (Theorem 3)

The downward Löwenheim-Skolem Theorem (*1) for L_{stat} is actually a strong reflection property. For example the reflection of uncountable coloring number of graphs down to $< \aleph_2$ (the following (*2)) is a consequence of (*1):

(*2) For any graph G of uncountable coloring number, there is a subgraph H of G of size $\aleph_1$ with uncountable coloring number.

This implication can be proved directly but we can also see this using the terminology introduced in [15] as follows: The downward Löwenheim-Skolem Theorem (*1) for L_{stat} is equivalent to the Diagonal Reflection Principle $\mathsf{DRP}(\mathsf{IC}_{\aleph_0})$ down to an internally club set (Corollary 3.6 in [15]). This implies the reflection principle $\mathsf{RP}_{\mathsf{IU}_{\aleph_0}}$ down to an internally unbounded set of size $< \aleph_2$. This reflection principle is equivalent to Axiom R of Fleissner (Lemma 2.6 in [20]). From Axiom R, the Fodor-type Reflection Principle (FRP) follows (Corollary 2.6 in [13]). (*2) is a consequence of (actually equivalent to FRP over ZFC ([19])).

As it is mentioned in the proof of Theorem 3, the condition (*1) is equivalent to Game Reflection Principle (GRP). As the name suggests, GRP is actually a reflection principle which claims the reflection of the non-existence of winning strategy of certain games of length ω_1 down to subgames of size $< \aleph_2$. A remarkable feature of this principle is that it implies CH (Theorem 8 in [30]).

The notion of Laver-generic large cardinals was introduced in [16] in search for reflection principles which generalizes GRP. The following definition of Laver-generic large cardinals is a streamlined version adopted in later papers [14], [12] etc. and slightly different from the one given in [16].

We call a non-empty class $\mathcal{P}$ of posets *iterable* if it satisfies:

① $\{\mathbb{1}\} \in \mathcal{P}$, ② $\mathcal{P}$ is closed with respect to forcing equivalence (i.e. if $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P} \sim \mathbb{P}'$ then $\mathbb{P}' \in \mathcal{P}$),
③ closed with respect to restriction (i.e. if $\mathbb{P} \in \mathcal{P}$ then $\mathbb{P} \restriction \mathbb{p} \in \mathcal{P}$ for any $\mathbb{p} \in \mathbb{P}$), and,
④ for any $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$, $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" implies $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathcal{P}$.

For an iterable class $\mathcal{P}$ of posets, a cardinal κ is said to be *$\mathcal{P}$-Laver-generically supercompact* if, for any $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", such that for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are $j, M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{P}, \mathbb{H}, j''\lambda \in M$.

κ is *tightly $\mathcal{P}$-Laver-generically supercompact* if it is $\mathcal{P}$-Laver-generically supercompact and $\underset{\sim}{\mathbb{Q}}$, j and M for each $\mathbb{P} \in \mathcal{P}$ in the definition of $\mathcal{P}$-Laver-generic supercompactness additionally satisfy that $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a poset of size $\leq j(\kappa)$.[2]

A cardinal κ is *(tightly) $\mathcal{P}$-Laver-generically superhuge*, if κ satisfies the conditions of (tightly) $\mathcal{P}$-Laver-generically supercompactness, with the condition $j''\lambda \in M$ replaced by $j''j(\kappa) \in M$. Clearly a (tightly) $\mathcal{P}$-Laver-generically superhuge cardinal is (tightly) $\mathcal{P}$-Laver-generically supercompact.

The name "Laver-generic large cardinal" is chosen in connection with fact that Laver-function plays central role in the construction of standard models with Laver-generic large cardinals (see Theorem 9).

Laver-genericity corresponding to other notions of large cardinals can be defined canonically: A cardinal κ is *$\mathcal{P}$-Laver-generically ultrahuge*, if it enjoys the definition of $\mathcal{P}$-Laver-generically supercompactness and that the condition "$j''\lambda \in M$" in the definition of supercompactness is replaced by the stronger "${\mathsf{V}_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$".
That is:

κ is *(tightly) $\mathcal{P}$-Laver-generically ultrahuge*, if for any $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", such that for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are $j, M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \overset{\prec}{\to}_{\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{P}, \mathbb{H}, {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$, (and $|\, \mathbb{P} * \underset{\sim}{\mathbb{Q}}\, | \leq j(\kappa)$).

κ is *$\mathcal{P}$-Laver-generically hyperhuge* if κ satisfies the definition obtained by replacing "$j''\lambda \in M$" in the definition of $\mathcal{P}$-Laver-generically supercompactness by "$j''j(\lambda) \in M$".
That is:

κ is *(tightly) $\mathcal{P}$-Laver-generically hyperhuge*, if for any $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", such that for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are $j, M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \overset{\prec}{\to}_{\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{P}, \mathbb{H}, j''j(\lambda) \in M$, (and $|\, \mathbb{P} * \underset{\sim}{\mathbb{Q}}\, | \leq j(\kappa)$).

The following implications follow from the definitions:

$$\begin{gathered} \kappa \text{ is (tightly) } \mathcal{P}\text{-Laver-generically ultrahuge} \\ \Downarrow \\ \kappa \text{ is (tightly) } \mathcal{P}\text{-Laver-generically superhuge} \\ \Downarrow \\ \kappa \text{ is (tightly) } \mathcal{P}\text{-Laver-generically supercompact} \end{gathered}$$

The relationship between Laver-generic hyperhugeness and Laver-generic ultrahugeness is slightly more subtle:

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[2] In the following, we shall denote this condition simply by "$|\, \mathbb{P} * \underset{\sim}{\mathbb{Q}}\, | \leq \lambda$". More generally, we shall simply write "$|\, \mathbb{P}\, | \leq \lambda$" for a poset $\mathbb{P}$ to say that "the poset $\mathbb{P}$ is forcing equivalent to a poset of size $\leq \lambda$".

Lemma 4. ([21]) *For any class $\mathcal{P}$ of posets, if κ is tightly $\mathcal{P}$-Laver-generically hyperhuge then κ is tightly $\mathcal{P}$-Laver-generically ultrahuge.*

Proof. Suppose that κ is tightly $\mathcal{P}$-Laver-generically hyperhuge, and $\lambda > \kappa$. Without loss of generality, we may assume that

$$V_\lambda \prec_{\Sigma_n} \mathsf{V} \text{ for sufficiently large } n. \tag{*3}$$

Let $\lambda^* := (|V_\lambda|^+)^{\mathsf{V}}$. For $\mathbb{P} \in \mathcal{P}$, let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that

$$j : \mathsf{V} \xrightarrow{\prec}_\kappa M,\ \ j(\kappa) \geq \lambda^*, \tag{*4}$$

$$j''j(\lambda^*),\ \mathbb{P},\ \mathbb{H} \in M,\ \text{and} \tag{*5}$$

$$|\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,| \leq j(\kappa). \tag{*6}$$

Claim 4.1. *For $\alpha \leq j(\lambda)$, $V_\alpha{}^{\mathsf{V}} \in M$.*

$\vdash$ By induction on $\alpha \leq j(\lambda)$, we prove

$$V_\alpha{}^{\mathsf{V}} \in M \text{ and } V_\alpha{}^{\mathsf{V}} \subseteq V_\alpha{}^M. \tag{*7}$$

For $\alpha < \omega$, this is clear. Suppose we have $V_\alpha{}^{\mathsf{V}} \in M$ and $V_\alpha{}^{\mathsf{V}} \subseteq V_\alpha{}^M$. Then, since $M \models |\,V_\alpha{}^M\,| < j(\lambda^*)$ (by the choice of λ^* and) by elementarity, $\mathcal{P}^{\mathsf{V}}(V_\alpha{}^{\mathsf{V}}) \subseteq \mathcal{P}^M(V_\alpha{}^M) \subseteq M$ by (*5) and Lemma 2.5, (5) in [16]. Again by Lemma 2.5, (5) in [16], it follows that $V_{\alpha+1}{}^{\mathsf{V}} = \mathcal{P}^{\mathsf{V}}(V_\alpha{}^{\mathsf{V}}) \in M$, and $V_{\alpha+1}{}^{\mathsf{V}} = \mathcal{P}^{\mathsf{V}}(V_\alpha{}^{\mathsf{V}}) \subseteq \mathcal{P}^M(V_\alpha{}^M) = V_{\alpha+1}{}^M$

If $\gamma \leq j(\lambda)$ is a limit, and $V_\alpha \in M$, $M \models V_\alpha{}^{\mathsf{V}} \subseteq V_\alpha$ for all $\alpha < \gamma$, then $\langle V_\alpha{}^{\mathsf{V}} : \alpha < \gamma \rangle \subseteq M$. Hence by Lemma 2.5, (5) in [16], it follows that $\langle V_\alpha{}^{\mathsf{V}} : \alpha < \gamma \rangle \in M$. Thus $V_\gamma{}^{\mathsf{V}} = \bigcup_{\alpha<\gamma} V_\alpha{}^{\mathsf{V}} \in M$ and $V_\gamma{}^{\mathsf{V}} = \bigcup_{\alpha<\gamma} V_\alpha{}^{\mathsf{V}} \subseteq \bigcup_{\alpha<\gamma} V_\alpha{}^M = V_\gamma{}^M$. $\dashv$ (Claim 4.1)

Now, it follows that

$$M \underbrace{\ni}_{\text{by Claim 4.1 and (*5)}} V_{j(\lambda)}[\mathbb{H}] \overbrace{=}^{\text{by (*3) and Lemma 13}} V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]}.$$

This shows that j and M taken here are as in the definition of $\mathcal{P}$-Laver-generically ultrahugeness. $\square$ (Lemma 4)

At first glance, it is not immediately clear if the notion of Laver-generic large cardinal is definable in the language $\mathcal{L}_\in$ of ZFC. In [18] an abstract generic version of extender is introduced to show the definability of Laver-generic large cardinals.

Laver-generic supercompactness implies double plus versions of forcing axioms. For a class $\mathcal{P}$ of posets and cardinals κ, μ, we denote with $\mathsf{MA}^{+\mu}(\mathcal{P}, <\kappa)$ and $\mathsf{MA}^{++\leq\mu}(\mathcal{P}, <\kappa)$ the following versions of Martin's Axiom:

$\mathsf{MA}^{+\mu}(\mathcal{P}, <\kappa)$: For any $\mathbb{P} \in \mathcal{P}$, any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ and any family $\mathcal{S}$ of $\mathbb{P}$-names such that $|\mathcal{S}| \le \mu$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{S}$ is a stationary subset of ω_1" for all $\underset{\sim}{S} \in \mathcal{S}$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ over $\mathbb{P}$ such that $\underset{\sim}{S}[\mathbb{G}]$ is a stationary subset of ω_1 for all $\underset{\sim}{S} \in \mathcal{S}$.

$\mathsf{MA}^{++\le\mu}(\mathcal{P}, <\kappa)$: For any $\mathbb{P} \in \mathcal{P}$, any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$ and any family $\mathcal{S}$ of $\mathbb{P}$-names such that $|\mathcal{S}| \le \mu$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{S}$ is a stationary subset of $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$" for some $\omega < \eta_{\underset{\sim}{S}} \le \theta_{\underset{\sim}{S}} \le \mu$ with $\eta_{\underset{\sim}{S}}$ regular, for all $\underset{\sim}{S} \in \mathcal{S}$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ over $\mathbb{P}$ such that $\underset{\sim}{S}[\mathbb{G}]$ is stationary in $\mathcal{P}_{\eta_{\underset{\sim}{S}}}(\theta_{\underset{\sim}{S}})$ for all $\underset{\sim}{S} \in \mathcal{S}$.

Clearly $\mathsf{MA}^{++<\omega_2}(\mathcal{P}, <\kappa)$ is equivalent to $\mathsf{MA}^{+\omega_1}(\mathcal{P}, <\kappa)$.
MM^{++} is $\mathsf{MA}^{+\omega_1}$(stationary preserving posets, $<\aleph_2$).

Theorem 5 (Theorem 5.7 in [16], see also [12]) *For an iterable class $\mathcal{P}$ of posets such that*

the elements of $\mathcal{P}$ preserve stationarity of subsets of $\mathcal{P}_\mu(\theta)$ for all $\mu \le \theta < \kappa$, $\quad$ (*8)

if $\kappa > \aleph_1$ is $\mathcal{P}$-Laver-generically supercompact then $\mathsf{MA}^{++\le\mu}(\mathcal{P}, <\kappa)$ holds for all $\mu < \kappa$. □

In contrast to usual generic large cardinals, a Laver-generic large cardinal if it exists, is unique and it is the size of the continuum in many cases.

Lemma 6. ([12], [21], see also Proposition 4, in [11]) (1) *If κ is $\mathcal{P}$-generically measurable for an ω_1 preserving iterable $\mathcal{P}$, then $\omega_1 < \kappa$.*

(2) *If κ is $\mathcal{P}$-Laver-generically supercompact for an ω_1-preserving iterable $\mathcal{P}$ with* $\mathrm{Col}(\omega_1, \{\omega_2\}) \in \mathcal{P}$ *then $\kappa = \omega_2$.*

(3) *If κ is $\mathcal{P}$-Laver-generically supercompact for an iterable $\mathcal{P}$ which contains a poset adding a new real, then $\kappa \le 2^{\aleph_0}$.*

(4) *If κ is $\mathcal{P}$-generically supercompact for an iterable $\mathcal{P}$ such that all posets in $\mathcal{P}$ do not add any reals then $2^{\aleph_0} < \kappa$.*

(5) *If κ is $\mathcal{P}$-Laver-generically supercompact for an iterable $\mathcal{P}$ which contains a poset which collapses $\aleph_1$ then $\kappa = \aleph_1$.*

Proof. We only prove (5) since it is not explicitly given in [21]. Suppose that κ is $\mathcal{P}$-Laver-generically supercompact and $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}}$ "${\aleph_1}^{\mathsf{V}}$ is countable". If $\kappa \ne \aleph_1$, then we have $\aleph_1 < \kappa$. Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name of a poset such that, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are $j, M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$ and $\mathbb{P}, \mathbb{H} \in M$. By $\mathbb{H} \cap \mathbb{P} \in M$ and since ${\aleph_1}^{\mathsf{V}} < crit(j)$, we have $M \models$"${\aleph_1}^{\mathsf{V}} = j({\aleph_1}^{\mathsf{V}})$ is countable". This is a contradiction to the elementarity of j. □ (Lemma 6)

A cardinal κ is called *greatly weakly Mahlo* if κ is weakly inaccessible and there exists a non-trivial $<\kappa$-complete normal filter $\mathcal{F}$ over κ such that $\{\mu < \kappa : \mu$ is a regular cardinal$\} \in \mathcal{F}$, and $\mathcal{F}$ is closed with respect to the Mahlo operation $M\ell$ [3] where

$$S \;\mapsto\; M\ell\,(S) := \{\alpha \in S \,:\, \alpha \text{ has uncountable cofinality and } S \cap \alpha \text{ is stationary in } \alpha\} \qquad ([17]).$$

Note that the Mahlo operation given above is slightly different from the one in [4].

If κ is greatly weakly Mahlo then it is hyper-weakly Mahlo (Proposition 3.4 in [17]).

The tightness of the Laver-genericity can be still strengthened as follows: a cardinal κ is *tightly$^+$ $\mathcal{P}$-Laver-generically x-large*, for a notion "x-large" of large cardinal (e.g. "supercompact", "superhuge" etc.) if it satisfies the definition of tightly $\mathcal{P}$-Laver-generically x-large cardinal with the condition "$|\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,| \;\le\; j(\kappa)$" in the definition being replaced by the condition "there is a complete Boolean algebra $\mathbb{B}$ of size $j(\kappa)$ such that $\mathbb{B}^+$ is forcing equivalent to $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$".

Theorem 7 (1) (Theorem 3.5 in [17]) *If κ is a $\{\mathbb{P}\}$-generically measurable for a poset $\mathbb{P}$ with the μ-cc for some $\mu < \kappa$, then κ is greatly weakly Mahlo.*

(2) (Theorem 5.8 in [16]) *If κ is tightly $\mathcal{P}$-Laver-generically superhuge for a class $\mathcal{P}$ of ccc posets such that at least one element of $\mathcal{P}$ adds a real, then $\kappa = 2^{\aleph_0}$.*

(3) ([21]) *For an iterable class $\mathcal{P}$ of posets, if κ is tightly$^+$ $\mathcal{P}$-Laver-generically hyperhuge, then $2^{\aleph_0} \le \kappa$.* □

We give a sketch of the proof of Theorem 7, (3) after Corollary 38 .

Theorem 8 ([16], [21] for (Δ) and (Δ′)) (A) *If $\mathcal{P}$ is the class of all $<\aleph_1$-closed posets, and κ is $\mathcal{P}$-Laver-generically supercompact, then $\kappa = \aleph_2$ and* CH *holds.*

(B) *If $\mathcal{P}$ is either the class of all proper posets or the class of all semi-proper posets, and κ is $\mathcal{P}$-Laver-generically supercompact, then $\kappa = 2^{\aleph_0} = \aleph_2$.*

(Γ) *If $\mathcal{P}$ is the class of all ccc posets, and κ is $\mathcal{P}$-Laver-generically supercompact, then κ is very large and $\kappa \le 2^{\aleph_0}$.*

(Γ′) *If $\mathcal{P}$ is the class of all ccc posets, and κ is tightly $\mathcal{P}$-Laver-generically superhuge, then κ is very large and $\kappa = 2^{\aleph_0}$.*

(Δ) *If $\mathcal{P}$ is the class of all posets, and κ is $\mathcal{P}$-Laver-generically supercompact, then $\kappa = \aleph_1$.*

(Δ′) *If $\mathcal{P}$ is the class of all posets, and κ is tightly$^+$ $\mathcal{P}$-Laver-generically supercompact, then $\kappa = \aleph_1$.*

Proof. (A): By Lemma 6, (2), (4). (B): By Lemma 6, (2), (3) and Theorem 5. (Γ): By Lemma 6, (3), and Theorem 7, (1). (Γ′): By (Γ) and Theorem 7, (2). (Δ): By Lemma 6, (5). (Δ′): By (Δ) and Theorem 7, (3). □ (Theorem 8)

The consistency of the existence of a $\mathcal{P}$-Laver-generic large cardinal can be proved under the existence of corresponding genuine large cardinal.

[3] Closedness here means that for any $S \in \mathcal{F}$, we have $M\ell\,(S) \in \mathcal{F}$.

Theorem 9 (Theorem 5.2, [16]) (A) *Suppose that κ is supercompact and $\mathbb{P} = \mathrm{Col}(\aleph_1, \kappa)$, then, in $\mathsf{V}[\mathbb{G}]$, for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, ${\aleph_2}^{\mathsf{V}[\mathbb{G}]}$ $(= \kappa)$ is tightly $\mathcal{P}$-closed-Laver-generically supercompact for the class $\mathcal{P}$ of all σ-closed posets (and CH holds).*

(B) *Suppose that κ is superhuge with a Laver function $f : \kappa \to V_\kappa$ for superhugeness. If $\mathbb{P}$ is the CS-iteration for forcing PFA along with f, then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, ${\aleph_2}^{\mathsf{V}[\mathbb{G}]}$ $(= \kappa)$ is tightly$^+$ $\mathbb{P}$-Laver-generically superhuge for the class $\mathcal{P}$ of all proper posets (and $2^{\aleph_0} = \aleph_2$ holds).*

(B') *Suppose that κ is superhuge with a Laver function $f : \kappa \to V_\kappa$ for superhugeness. If $\mathbb{P}$ is the RCS-iteration for forcing MM along with f, then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, ${\aleph_2}^{\mathsf{V}[\mathbb{G}]}$ $(= \kappa)$ is tightly$^+$ $\mathcal{P}$-Laver-generically superhuge for the class $\mathcal{P}$ of all semi-proper posets (and $2^{\aleph_0} = \aleph_2$ holds).*

(Γ) *Suppose that κ is supercompact with a Laver function $f : \kappa \to V_\kappa$ for supercompactness. If $\mathbb{P}$ is a FS-iteration for forcing MA along with f, then, in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $2^{\aleph_0}$ $(= \kappa)$ is tightly$^+$ $\mathcal{P}$-Laver-generically supercompact for the class $\mathcal{P}$ of all ccc posets(and $\kappa = 2^{\aleph_0}$ while κ still is very large).*

(Δ) *Suppose that κ is supercompact with a Laver function $f : \kappa \to V_\kappa$ for supercompactness. If $\mathbb{P}$ is a FS-iteration for forcing f where f is used to book-keep through all posets in V_κ, then in $\mathsf{V}[\mathbb{G}]$ for any $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $2^{\aleph_0}$ $(= \kappa)$ is tightly$^+$ $\mathcal{P}$-Laver-generically supercompact for the class $\mathcal{P}$ of all posets (and CH holds).* □

Theorem 9 above also holds for all other notions of large cardinal and corresponding Laver-generic version of generic large cardinal except (B) and (B') in which the supercompactness does not seem to be strong enough to show that the resulting generic extension in the proof satisfies the expected Laver-genericity.

In a sense, the cases treated in Theorem 11 are (almost) exhaustive. This can be seen in the following:

Theorem 10 *Suppose that $\mathcal{P}$ is an iterable class of posets such that all $\mathbb{P} \in \mathcal{P}$ are ω_1-preserving and $\mathcal{P}$ contains a poset $\mathbb{P}^*$ whose generic filter destroys a stationary subset of ω_1.[4] Then there is no $\mathcal{P}$-Laver-generically supercompact cardinal.*

Proof. Suppose, toward a contradiction, that $\mathcal{P}$ is as above and there is $\mathcal{P}$-Laver-generically supercompact cardinal κ.

Let $S \subseteq \omega_1$ be stationary such that there is a poset $\mathbb{P}^* \in \mathcal{P}$ shooting a club in $\omega_1 \setminus S$. Let $\lambda > |\mathbb{P}^*|$ be large enough. By assumption, there is a $\mathbb{P}^*$-name $\underset{\sim}{\mathbb{Q}}$ of a poset such that $\Vdash_{\mathbb{P}^*}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for $(\mathsf{V}, \mathbb{P}^* * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that

$$j : \mathsf{V} \xrightarrow{\prec}_\kappa M, \tag{*9}$$

$$j(\kappa) > \lambda, \quad \text{and} \tag{*10}$$

---

[4] "$\mathbb{P}^*$ destroys a stationary subset of ω_1" means here that a $\mathbb{P}^*$-generic set codes a club subset of $\omega_1 \setminus S$ in some absolute way.

Note that, for stationary and co-stationary subset S of ω_1, various posets are known which preserve ω_1 while shooting a club in $\omega_1 \setminus S$ (e.g. see [39]).

$$j''\lambda, \mathbb{P},\ \mathbb{H} \in M. \tag{*11}$$

By the choice of $\mathbb{P}^*$ ($\lessdot \mathbb{P}^* * \underset{\sim}{\mathbb{Q}}$), there is a nice $\mathcal{P}^*$-name $\underset{\sim}{C} \in \mathsf{V}$ of a club set $\subseteq {\omega_1}^{\mathsf{V}} \setminus S$. By (*10), (*11) and by the choice of λ, we have $\underset{\sim}{C} \in M$.

Thus $M \models$ "S is a non-stationary subset of ω_1" by (*11). Since $crit(j) = \kappa > \omega_1$ by Lemma 6, (1), we have $S = j(S)$. By $\mathsf{V} \models$ "S is stationary subset of ω_1", this is a contradiction to the elementarity (*9) of j. □ (Theorem 10)

$\mathcal{P}$-Laver genericity for stationary preserving $\mathcal{P}$, in particular those $\mathcal{P}$ containing all σ-closed posets can be regarded as a strong reflection principle.

Theorem 11 *Suppose that $\mathcal{P}$ is an iterable class of posets which are ω_1-preserving and include all σ-closed posets. If κ is $\mathcal{P}$-Laver-generically supercompact then*

(*1) *for any structure $\mathfrak{A}$ of countable signature, there is $\mathfrak{B} \prec_{L_{stat}} \mathfrak{A}$ of cardinality $< \aleph_2$.*

Proof. By Theorem 5, $\mathsf{MA}^{++}(\sigma\text{-closed})$ holds. Cox [7] proved that $\mathsf{MA}^{++}(\sigma\text{-closed})$ implies $\mathsf{DRP}(<\aleph_2, \mathsf{IC}_{\aleph_0})$ (in the notation of [15]). By Lemma 3.5 in [15], this principle is equivalent to (*1). □ (Theorem 11)

The existence of $\mathcal{P}$-Laver generic large cardinal for the class of all ccc-posets also implies a reflection statement similar to (*1) (see Theorem 5.9, (3) in [16]).

3 Recurrence, Maximality, and the solution(s) of the Continuum Problem

In the following, an inner model W of a universe U (in most of the cases U is the real universe V but sometimes it is some other universe obtained from V) is called a *ground of* U, if there is a poset $\mathbb{P} \in \mathsf{W}$ and $(\mathsf{W}, \mathbb{P})$-generic $\mathbb{G} \in \mathsf{U}$ such that $\mathsf{U} = \mathsf{W}[\mathbb{G}]$.

For a class $\mathcal{P}$ of posets and a set A (of parameters), the *Recurrence Axiom for $\mathcal{P}$ and A* (($\mathcal{P}, A$)-RcA, for short[5]) is the following assertion formulated as an axiom scheme in the language $\mathcal{L}_\in$ of set theory:

($\mathcal{P}, A$)-RcA: For any $\mathcal{L}_\in$-formula $\varphi = \varphi(\overline{x})$ and $\overline{a} \in A$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$" for a $\mathbb{P} \in \mathcal{P}$, then there is a ground W of the universe V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$.

The term "Recurrence Axiom" is chosen in allusion to, but not necessarily in (full) agreement with, Nietzsche's „ewige Wiederkehr des Gleichen‟ (eternal recurrence of the same), or perhaps rather to Poincaré recurrence theorem: if we understand the relation "N is (set) generic extension of M" as the timeline in the set generic multiverse, we can interpret ($\mathcal{P}, A$)-RcA as saying that

[5] The notation "RcA" is chosen to avoid the collision with "RCA" which is used in Reverse Mathematics to denote "recursive comprehension axiom".

if something (formulatable with parameters in A) happens in one of the near future universe (in terms of $\mathcal{P}$) then is is already happened in a not very far past universe (not very far, in the sense that the "present" is attainable from there by a set forcing).

The following is a natural strengthening of the Recurrence Axiom:

$(\mathcal{P}, A)$-RcA^+: For any $\mathcal{L}_\in$-formula $\varphi = \varphi(\overline{x})$ and $\overline{a} \in A$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^\checkmark)$" for a $\mathbb{P} \in \mathcal{P}$, then there is a $\mathcal{P}$-ground W of the universe V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$.

Here an inner model W of V is called a *$\mathcal{P}$-ground* if there is a poset $\mathbb{P} \in \mathsf{W}$ with $\mathsf{W} \models \mathbb{P} \in \mathcal{P}$ and $(\mathsf{W}, \mathbb{P})$-generic $\mathbb{G} \in \mathsf{V}$ such that $\mathsf{V} = \mathsf{W}[\mathbb{G}]$.

We shall use the following version of Laver-Woodin Theorem often without mention. It implies in particular that $(\mathcal{P}, A)$-RcA and $(\mathcal{P}, A)$-RcA^+ are actually formalizable as axiom schemes in $\mathcal{L}_\in$.

Theorem 12 (Reitz [35], Fuchs-Hamkins-Reitz [9]) *There is an $\mathcal{L}_\in$-formula $\Phi(x, r)$ such that the following is provable in* ZFC*:*

(*12) *for all r, $\Phi(\cdot, r) := \{x : \Phi(s, r)\}$ is a ground in* V*,*

(*13) *for any ground W (of V), there is r such that $\mathsf{W} = \Phi(\cdot, r)$, and*

(*14) *if W is a ground of V and $\mathsf{V} = \mathsf{W}[\mathbb{G}]$ where $\mathbb{G}$ is a $(\mathsf{W}, \mathbb{P})$-generic for $\mathbb{P} \in \mathsf{W}$, then r such that $\mathsf{W} = \Phi(\cdot, r)$ can be chosen as an element of $\mathcal{P}((|\mathbb{P}|^+)^{\mathsf{V}})$.*

We put together here some other basic facts which will be used in the following. The next lemma was actually already used in the proof of Lemma 4.

Lemma 13. (see [21] for a proof) *If α is a limit ordinal and V_α satisfies a large enough fragment of* ZFC*, then for any $\mathbb{P} \in V_\alpha$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ we have $V_\alpha[\mathbb{G}] = {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$.*

Proof. "$\subseteq$": This inclusion holds without the condition on the fragment of ZFC. Also the condition "$\mathbb{P} \in V_\alpha$" is irrelevant for this inclusion.

We show by induction on $\alpha \in \mathsf{On}$ that $V_\alpha[\mathbb{G}] \subseteq {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$ holds for all $\alpha \in \mathsf{On}$.

The induction steps for $\alpha = 0$ and limit ordinals α are trivial. So we assume that $V_\alpha[\mathbb{G}] \subseteq {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$ holds and show that the same inclusion holds for $\alpha + 1$. Suppose $a \in V_{\alpha+1}[\mathbb{G}]$. Then $a = \underset{\sim}{a}^{\mathbb{G}}$ for a $\mathbb{P}$-name $\underset{\sim}{a} \in V_{\alpha+1}$. Since $\underset{\sim}{a} \subseteq V_\alpha$, each $\langle \underset{\sim}{b}, \mathbb{p} \rangle \in \underset{\sim}{a}$ is an element of V_α. By induction hypothesis, it follows that $\underset{\sim}{b}^{\mathbb{G}} \in {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$. It follows that $\underset{\sim}{a}^{\mathbb{G}} \subseteq {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$. Thus $a = \underset{\sim}{a}^{\mathbb{G}} \in {V_{\alpha+1}}^{\mathsf{V}[\mathbb{G}]}$.

"$\supseteq$": Suppose that $a \in {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$. Note that we can choose the "large enough fragment of ZFC" which should be satisfied in V_α such that $(*)$ ${V_\alpha}^{\mathsf{V}[\mathbb{G}]}$ still satisfies a large enough fragment of ZFC, although the fragment may be different from the one V_α satisfies. In particular we find a cardinal $\lambda > |\mathbb{P}|$ in ${V_\alpha}^{\mathsf{V}[\mathbb{G}]}$ (and hence also in $\mathsf{V}[\mathbb{G}]$) such that $a \in \mathcal{H}(\lambda)^{{V_\alpha}^{\mathsf{V}[\mathbb{G}]}} \subseteq \mathcal{H}(\lambda)^{\mathsf{V}[\mathbb{G}]} \subseteq {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$. Note that
$\mathcal{H}(\lambda)^{{V_\alpha}^{\mathsf{V}[\mathbb{G}]}} = \{a : |trcl(a)| < \lambda\}^{{V_\alpha}^{\mathsf{V}[\mathbb{G}]}} \subseteq \{a : |trcl(a)| < \lambda\}^{\mathsf{V}[\mathbb{G}]} = \mathcal{H}(\lambda)^{\mathsf{V}[\mathbb{G}]}$.

Let $a^* \in \mathcal{H}(\lambda)^{\mathsf{V}[\mathbb{G}]}$ be a transitive set such that $a \in a^*$. Then a^* can be coded by a subset of λ. We can find the subset of λ in $\mathsf{V}[\mathbb{G}]$ and this subset has a nice $\mathbb{P}$-name which is an element of $V_\alpha{}^{\mathsf{V}}$ since $\mathbb{P} \in V_\alpha$. This shows that $a \in V_\alpha[\mathbb{G}]$. □ (Lemma 13)

Lemma 14. *For any $n \in \mathbb{N} \setminus 1$, there is a Σ_n-formula $\varphi_n = \varphi_n(\overline{x}, y)$ (for each finite sequence $\overline{x}$ of variables[6]) such that, if U is a transitive models of large enough fragment of* ZFC*, then for any Σ_n formula $\psi = \psi(\overline{x})$ there is $p \in \mathcal{H}(\omega)$ such that $\mathsf{U} \models \psi(\overline{a})$ if and only if $\mathsf{U} \models \varphi_n(\overline{a}, p)$ for all $a \in \mathsf{U}$.*

Proof. For $n = 1$, $\varphi_1(\overline{x}, y)$ can be chosen as a Σ_1-formula saying

($\aleph$1) $\exists M$ (M is transitive, $\overline{x} \in M$, $M \models$" large enough fragment of ZFC"
y is a code of a Σ_1-formula and $M \models \llcorner y \lrcorner(\overline{x})$).

If $\varphi_n(\overline{x}, y)$ is defined for $\overline{x}$ of various lengths we can define $\varphi_{n+1}(\overline{x})$ as $\exists x \neg\varphi_n(\overline{x}, x, y)$. □ (Lemma 14)

Lemma 15. (1) *For any $n^* \in \mathbb{N}$ there is $n > n^*$ such that, if $V_\alpha \prec_{\Sigma_n} \mathsf{V}$, then $V_\alpha[\mathbb{G}] \prec_{\Sigma_{n^*}} \mathsf{V}[\mathbb{G}]$ for any $\mathbb{P} \in V_\alpha$ and $(V, \mathbb{P})$-generic $\mathbb{G}$.*

(2) *For a natural number n, there is $n' > n$ such that, for any $\alpha \in$ On, if $V_\alpha[\mathbb{G}] \prec_{\Sigma_{n'}} \mathsf{V}[\mathbb{G}]$ for a poset $\mathbb{P} \in V_\alpha$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, then we have $V_\alpha \prec_{\Sigma_n} \mathsf{V}$.*

Proof. (1): Suppose that $n > n^*$ is sufficiently large, $V_\lambda \prec \mathsf{V}$, $\overline{a} \in V_\lambda[\mathbb{G}]$, and $\varphi = \varphi(\overline{x})$ is a Σ_{n^*}-formula. There are $\mathbb{P}$-names $\underset{\sim}{\overline{a}} \in V_\lambda$ such that $\overline{a} = \underset{\sim}{\overline{a}}[\mathbb{G}]$.

If $V_\lambda[\mathbb{G}] \models \varphi(\overline{a})$, there is $\mathbb{p} \in \mathbb{G}$ such that $V_\lambda \models \mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{\overline{a}})$". By the choice of n it follows that $\mathsf{V} \models \mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{\overline{a}})$". Thus $\mathsf{V}[\mathbb{G}] \models \varphi(\overline{a})$.

The same argument also applies to $\neg\varphi$.

(2): We use the $\mathcal{L}_\in$-formula $\Phi(x, y)$ of Lemma 12. By assumption, there is $r \in V_\alpha[\mathbb{G}]$ such that

$$V_\alpha = \Phi(\cdot, r)^{V_\alpha[\mathbb{G}]} = \Phi(\cdot, r)^{\mathsf{V}} \cap V_\alpha[\mathbb{G}] \subseteq \Phi(\cdot, r)^{\mathsf{V}}.$$

For any Σ_n-formula $\varphi(\overline{x})$ and $\overline{a} \in \Phi(\cdot, r)^{V_\alpha[\mathbb{G}]}$. Since $\varphi^{\Phi(\cdot,r)}$ is a $\Sigma_{n'}$-formula (by the choice of n'), we have

$$V_\alpha \models \varphi(\overline{a}) \iff V_\alpha[\mathbb{G}] \models \varphi^{\Phi(\cdot,r)}(\overline{a}) \overbrace{\iff}^{\text{by assumption}} \mathsf{V}[\mathbb{G}] \models \varphi^{\Phi(\cdot,r)}(\overline{a}) \iff V \models \varphi(\overline{a}).$$

This shows that $V_\alpha \prec_{\Sigma_n} \mathsf{V}$. □ (Lemma 15)

RcA and RcA$^+$ are actually (almost) identical with (certain variations of) already well-known axioms and principles.

For a class $\mathcal{P}$ of posets, an $\mathcal{L}_\in$-formula $\varphi(\overline{a})$ with parameters $\overline{a}$ ($\in \mathsf{V}$) is said to be a *$\mathcal{P}$-button* if there is $\mathbb{P} \in \mathcal{P}$ such that, for any $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of poset with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", we have $\Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}}$ "$\varphi(\overline{a}^{\checkmark})$".

[6] Similarly to the convention of some computer languages we consider here that we have distinct $\varphi_n(\overline{x}, y)$ for each length of sequence $\overline{x}$ of variables.

If $\varphi(\overline{a})$ is a $\mathcal{P}$-button then we all $\mathbb{P}$ as above a *push of the button $\varphi(\overline{a})$*.

For a class $\mathbb{P}$ of posetsand a set A (of parameters), the *Maximality Principle for $\mathcal{P}$ and A* ($\mathsf{MP}(\mathcal{P}, A)$, for short) introduced in Hamkins [27] is the following assertion formulated in an axioms scheme in $\mathcal{L}_\in$:

$\mathsf{MP}(\mathcal{P}, A)$: For any $\mathcal{L}_\in$-formula $\varphi(\overline{x})$ and $\overline{a} \in A$, if $\varphi(\overline{a})$ is a $\mathcal{P}$-button then $\varphi(\overline{a})$ holds.

Proposition 16. (Barton, Caicedo, Fuchs, Hamkins, Reitz, and Schindler [3]) *Suppose that $\mathcal{P}$ is an iterable class of posets and A a set (of parameters).* (1) $(\mathcal{P}, A)$-RcA^+ *is equivalent to* $\mathsf{MP}(\mathcal{P}, A)$.

(2) $(\mathcal{P}, A)$-RcA *is equivalent to the following assertion:*

(*15) *For any $\mathcal{L}_\in$-formula $\varphi(\overline{x})$ and $\overline{a} \in A$, if $\varphi(\overline{a})$ is a $\mathcal{P}$-button then $\varphi(\overline{a})$ holds in a ground of* V.

Proof. (1): Suppose first that $(\mathcal{P}, A)$-RcA^+ holds. We show that $\mathsf{MP}(\mathcal{P}, A)$ holds. Suppose that $\mathbb{P} \in \mathcal{P}$ is a push of the $\mathcal{P}$-button $\varphi(\overline{a})$. Let $\varphi'(\overline{x})$ be the formula expressing

$$\text{for any } \mathbb{Q} \in \mathcal{P},\ \Vdash_{\mathbb{Q}} \text{“}\varphi(\overline{x}^{\checkmark})\text{” holds.} \tag{*16}$$

Then we have $\Vdash_{\mathbb{P}}$ “$\varphi'(\overline{a}^{\checkmark})$”. By $(\mathcal{P}, A)$-RcA^+, there is a $\mathcal{P}$-ground M of V such that $\overline{a} \in M$ and $M \models \varphi'(\overline{a})$ holds. By the definition (*16) of φ', it follows that $\mathsf{V} \models \varphi(\overline{a})$ holds.

Now suppose that $\mathsf{MP}(\mathcal{P}, A)$ holds and $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}}$ “$\varphi(\overline{a})$” for $\overline{a} \in A$.

Let φ' be a formula claiming that

$$\text{there is a } \mathcal{P}\text{-ground } N \text{ such that } \overline{x} \in N \text{ and } N \models \varphi(\overline{x}). \tag{*17}$$

Then $\varphi'(\overline{a})$ is a $\mathcal{P}$-button and $\mathbb{P}$ is its push.

By $\mathsf{MP}(\mathcal{P}, A)$, $\varphi'(\overline{a})$ holds in V and hence there is a $\mathcal{P}$-ground M of V such that $\overline{a} \in M$ and $M \models \varphi(\overline{a})$. This shows that $(\mathcal{P}, A)$-RcA^+ holds.

(2): can be proved similarly to (1). Suppose first that $(\mathcal{P}, A)$-RcA holds. We show that (*15) holds. Suppose that $\mathbb{P} \in \mathcal{P}$ is a push of the $\mathcal{P}$-button $\varphi(\overline{a})$. Let $\varphi'(\overline{x})$ be the formula expressing

$$\text{for any } \mathbb{Q} \in \mathcal{P},\ \Vdash_{\mathbb{Q}} \text{“}\varphi(\overline{x}^{\checkmark})\text{” holds.} \tag{*18}$$

Then we have $\Vdash_{\mathbb{P}}$ “$\varphi'(\overline{a}^{\checkmark}$”. By $(\mathcal{P}, A)$-RcA, there is a ground M of V such that $\overline{a} \in M$ and $M \models \varphi'(\overline{a})$ holds. Since $\mathcal{P} \ni \{\mathbb{1}\}$, it follows that $M \models \varphi(\overline{a})$.

Now suppose that (*15) holds and $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}}$ “$\varphi(\overline{a}^{\checkmark})$” for $\overline{a} \in A$.

Let φ'' be a formula asserting that

$$\text{there is a } \mathcal{P}\text{-ground } N \text{ such that } \overline{x} \in N \text{ and } N \models \varphi(\overline{x}). \tag{*19}$$

Then $\varphi''(\overline{a})$ is a $\mathcal{P}$-button and $\mathbb{P}$ is its push. Thus, By (*15), $\varphi''(\overline{a})$ holds in a ground M of V with $\overline{a} \in M$. By the definition (*19) of φ'', there is a $\mathcal{P}$-ground N of M such that $\overline{a} \in N$ and $N \models \varphi(\overline{a})$. Since N is also a ground of V, this shows that $(\mathcal{P}, A)$-RcA holds. $\square$ (Theorem 16)

Recurrence Axioms are also related to the Inner Model Hypothesis introduced by Sy Friedman in [22]. *The Inner Model Hypothesis* (IMH) is the following assertion formulated in the language of second-order set theory (e.g. in the context of von Neumann-Bernays-Gödel set theory):

IMH : For any statement φ without parameters, if φ holds in an inner model of an inner extension of V then φ holds in an inner model of V.

Here we say a (not necessarily first-order definable) transitive class M an *inner model* of V if M is a model of ZF and $\mathsf{On}^M = \mathsf{On}^{\mathsf{V}}$. In the perspective from such M, we call V an *inner extension* of M.

We call a set-forcing version of this principle *Inner Ground Hypothesis* (IGH): For a (definable) class $\mathcal{P}$ of posets and a set A (of parameters),

$\mathsf{IGH}(\mathcal{P}, A)$: For any $\mathcal{L}_\in$-formula $\varphi = \varphi(\overline{x})$ and $\overline{a} \in A$, if $\mathbb{P} \in \mathcal{P}$ forces "there is a ground M with $\overline{a} \in M$ satisfying $\varphi(\overline{a})$", then there is a ground W of V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$.

Proposition 17. (Barton, Caicedo, Fuchs, Hamkins, Reitz, and Schindler [3]) *For a class $\mathcal{P}$ of posets with $\{\mathbb{1}\} \in \mathcal{P}$ and a set A (of parameters), $(\mathcal{P}, A)$-RcA holds if and only if $\mathsf{IGH}(\mathcal{P}, A)$ holds.*

Proof. Suppose that $(\mathcal{P}, A)$-RcA holds. Let $\varphi = \varphi(\overline{x})$ be an $\mathcal{L}_\in$-formula, $\overline{a} \in A$, and $\mathbb{P} \in \mathcal{P}$ be such that $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$ holds in a ground".

Let $\varphi'(\overline{x})$ be the $\mathcal{L}_\in$-formula asserting that $\varphi(\overline{x})$ holds in a ground. Then $\Vdash_{\mathbb{P}}$ "$\varphi'(\overline{a}^{\checkmark})$". By $(\mathcal{P}, A)$-RcA, it follows that there is a ground W of V such that $\mathsf{W} \models \varphi'(\overline{a}^{\checkmark})$. Since a ground of a ground is a ground, we conclude that there is a ground W_0 of V such that $\overline{a} \in M_0$ and $\mathsf{W}_0 \models \varphi(\overline{a})$. This shows that $\mathsf{IGH}(\mathcal{P}, A)$ holds.

Suppose now that $\mathsf{IGH}(\mathcal{P}, A)$ holds. Assume that $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$" for an $\mathcal{L}_\in$-formula $\varphi = \varphi(\overline{x})$, $\overline{a} \in A$, and $\mathbb{P} \in \mathcal{P}$. Then $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$ holds in a $\mathcal{P}$-ground (of the universe)" since $\Vdash_{\mathbb{P}}$ "$\{\mathbb{1}\} \in \mathcal{P}$". Thus, by $\mathsf{IGH}(\mathcal{P}, A)$, there is a ground W of V such that $\mathsf{W} \models \varphi(\overline{a})$. $\square$ (Proposition 17)

$(\mathcal{P}, A)$-RcA^+ ($\Leftrightarrow \mathsf{MP}(\mathcal{P}, A)$ for an iterable $\mathcal{P}$) can be also characterized in terms of a strengthening of Inner Ground Hypothesis: For a (definable) class $\mathcal{P}$ of posets and a set A (of parameters),

$\mathsf{IGH}^+(\mathcal{P}, A)$: For any $\mathcal{L}_\in$-formula $\varphi = \varphi(\overline{x})$ and $\overline{a} \in A$ if $\mathbb{P} \in \mathcal{P}$ forces "there is a $\mathcal{P}$-ground M with $\overline{a} \in M$ satisfying $\varphi(\overline{a})$", then there is a $\mathcal{P}$-ground W of V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$.

The following proposition can be proved similarly to Proposition 17.

Proposition 18. *For an iterable class $\mathcal{P}$ of posets and a set A (of parameters), $(\mathcal{P}, A)$-RcA^+ holds if and only if $\mathsf{IGH}^+(\mathcal{P}, A)$ holds.*

Proof. Suppose that $(\mathcal{P}, A)$-RcA^+ holds and assume that $\varphi = \varphi(\overline{x})$ is an $\mathcal{L}_\in$-formula, $\overline{a} \in A$, and $\mathbb{P} \in \mathcal{P}$ is such that

$$\Vdash_{\mathbb{P}} \text{“}\,\varphi(\overline{a}^{\checkmark})\text{ holds in a }\mathcal{P}\text{-ground }M\text{ with }\overline{a} \in M\,\text{”}$$

Let $\varphi'(\overline{a})$ be the formula expressing “$\varphi(\overline{x})$ holds in a $\mathcal{P}$-ground M with $\overline{a} \in M$”. Then $\mathbb{P}$ is a push of the switch $\varphi'(\overline{a})$. Thus, by Proposition 16, (1), $\varphi'(\overline{a})$ holds in V. By definition of φ', there is a $\mathcal{P}$-ground W of V such that $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$. This shows that $\mathsf{IGH}^+(\mathcal{P}, A)$ holds.

Suppose now that $\mathsf{IGH}^+(\mathcal{P}, A)$ holds, and assume that $\varphi = \varphi(\overline{x})$ is an $\mathcal{L}_\in$-formula, $\overline{a} \in A$ and $\Vdash_{\mathbb{P}}$ “$\varphi(\overline{a}^{\checkmark})$” then (since $\{\mathbb{1}\} \in \mathcal{P}$) $\Vdash_{\mathbb{P}}$ “$\varphi(\overline{a}^{\checkmark})$ holds in a $\mathcal{P}$-ground M with $\overline{a} \in M$”. By $\mathsf{IGH}^+(\mathcal{P}, A)$, it follows that there is $\mathcal{P}$-ground W_0 of $\mathcal{P}$-ground of V such that $\overline{a} \in \mathsf{W}_0$ and $\mathsf{W}_0 \models \varphi(\overline{a})$. Since $\mathcal{P}$ is iterable, W_0 is a $\mathcal{P}$-ground of V. This shows that $(\mathcal{P}, A)$-RcA^+ holds. □ (Proposition 18)

In spite of these characterizations and near characterizations, we want to keep the Recurrence Axioms as autarchic axioms. The reason is that we have the following monotonicity which does not hold e.g. for Maximality Principles.

Lemma 19. (Monotonicity of Recurrence Axioms) *For classes of posets $\mathcal{P}$, $\mathcal{P}'$ and sets A, A' of parameters, if $\mathcal{P} \subseteq \mathcal{P}'$ and $A \subseteq A'$, then we have*

$$(\mathcal{P}', A')\text{-}\mathsf{RcA} \;\Rightarrow\; (\mathcal{P}, A)\text{-}\mathsf{RcA}. \qquad \square$$

If we decide that the Recurrence Axioms are desirable extensions of the axioms of ZFC, then we should adopt the maximal instance of these axioms. (i.e. the one with maximal strength among the instances consistent with ZFC) By Lemma 19, this means we should try to take the instance of Recurrence Axioms with the maximal $\mathcal{P}$ and A (with respect to inclusion) among the consistent ones.

Lemma 20 in the next section suggests that the following two as candidates of such maximal instances:

(E) $\mathsf{ZFC} + (\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))$-RcA for the class $\mathcal{P}$ of all stationary preserving posets.

(Z) $\mathsf{ZFC} + (\mathcal{Q}, \mathcal{H}(2^{\aleph_0}))$-RcA for the class $\mathcal{Q}$ of all posets.

The consistency of (Z) follows from the consistency of ZFC + “there are stationarily many inaccessible cardinals” ([27]). The consistency of (E) follows from Lemma 29, Theorem 30, (B'), and Theorem 28.

The maximality of (E) and (Z) follows from Lemma 20, (2’) and (5’) respectively.

By Lemma 20, (4) and (5), (E) implies $2^{\aleph_0} = \aleph_2$, and (Z) implies CH. In particular, these two extensions of ZFC are not compatible. However, as we are going to discuss in Section 7, we can combine (E) with a reasonable weakening of (Z).

(Z^+) $\mathsf{ZFC} + (\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))$-$\mathsf{RcA}^+ + (\mathcal{Q}, \mathcal{H}(\omega_1)^{\overline{\mathsf{W}}})$-$\mathsf{RcA}^+$ where $\mathcal{P}$ is the class of all proper posets, $\mathcal{Q}$ the class of all posets, and $\overline{\mathsf{W}}$ the bedrock[7] which is also assumed here to exist.

[7] For the definition of the bedrock see Section 5.

Clearly (Z^+) implies $2^{\aleph_0} = \aleph_2$. This is what I meant when I wrote "the maximal setting of Recurrence Axioms points to the universe with the continuum of size $\aleph_2$" in the introduction. We shall further discuss about (Z^+) in Section 6.

4 Restricted Recurrence Axioms

The following restricted forms of Recurrence Axioms are enough to decide many interesting aspects including the cardinal arithmetic around the continuum.

For an iterable class $\mathcal{P}$ of posets, a set A (of parameters), and a set Γ of $\mathcal{L}_\in$-formulas, *$\mathcal{P}$-Recurrence Axiom for formulas in Γ with parameters from A* ($(\mathcal{P}, A)_\Gamma$-RcA, for short) is the following assertion expressed as an axiom scheme in $\mathcal{L}_\in$:

$(\mathcal{P}, A)_\Gamma$-RcA: For any $\varphi(\overline{x}) \in \Gamma$ and $\overline{a} \in A$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$", then there is a ground W of V such that $\overline{a} \in$ W and W $\models \varphi(\overline{a})$.

$(\mathcal{P}, A)_\Gamma$-RcA$^+$ corresponding to $(\mathcal{P}, A)$-RcA$^+$ is defined similarly.

$(\mathcal{P}, A)_\Gamma$-RcA$^+$:For any $\varphi(\overline{x}) \in \Gamma$ and $\overline{a} \in A$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$", then there is a $\mathcal{P}$-ground W of V such that $\overline{a} \in$ W and W $\models \varphi(\overline{a})$.

Lemma 20. ([21]) *Assume that $\mathcal{P}$ is an iterable class of posets.* (1) *If $\mathcal{P}$ contains a poset which adds a real (over the universe), then $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA implies $\neg$CH.*

(2) *Suppose that $\mathcal{P}$ contains a poset which forces $\aleph_2{}^{\mathsf{V}}$ to be equinumerous with $\aleph_1{}^{\mathsf{V}}$. Then $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA implies $2^{\aleph_0} \le \aleph_2$.*

(2') *If $\mathcal{P}$ contains a posets which forces $\aleph_2{}^{\mathsf{V}}$ to be equinumerous with $\aleph_1{}^{\mathsf{V}}$, then $(\mathcal{P}, \mathcal{H}((\aleph_2)^+))_{\Sigma_1}$-RcA does not hold.*

(3) *If $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA holds then all $\mathbb{P} \in \mathcal{P}$ preserve $\aleph_1$ and they are also stationary preserving.*

(4) *If $\mathcal{P}$ contains a poset which adds a real as well as a poset which collapses $\aleph_2{}^{\mathsf{V}}$, then $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA implies $2^{\aleph_0} = \aleph_2$.*

(5) *If $\mathcal{P}$ contains a poset which collapses $\aleph_1{}^{\mathsf{V}}$, then $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA implies* CH.

(5') *If $\mathcal{P}$ contains a poset which collapses $\aleph_1{}^{\mathsf{V}}$ then $(\mathcal{P}, \mathcal{H}((2^{\aleph_0})^+))_{\Sigma_1}$-RcA does not hold.*

Proof. (1): Assume that $\mathcal{P}$ is an iterable class of posets containing a poset $\mathbb{P}$ adding a real and $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA holds. If CH holds, then $\mathcal{P}(\omega)^{\mathsf{V}} \in \mathcal{H}(\kappa_{\mathfrak{refl}})$. Hence

$$\text{``}\exists x\,(x \subseteq \omega \wedge x \notin \mathcal{P}(\omega)^{\mathsf{V}})\text{''} \tag{$^{\aleph}$2}$$

is a Σ_1-formula with parameters from $\mathcal{H}(\kappa_{\mathfrak{refl}})$ and $\mathbb{P}$ forces the formula in the forcing language corresponding to this formula: "$\exists x\,(s \subseteq \check{\omega} \wedge x \notin (\mathcal{P}(\omega)^{\mathsf{V}})^{\checkmark})$".

By $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA, the formula (2) must hold in a ground. This is a contradiction.

(2): Assume that $(\mathcal{P}, \mathcal{H}(2^{\aleph_0})_{\Sigma_1})$-RcA holds and $\mathbb{P} \in \mathcal{P}$ forces ${\aleph_2}^{\mathsf{V}}$ to be equinumerous with ${\aleph_1}^{\mathsf{V}}$. If $2^{\aleph_0} > \aleph_2$ then ${\aleph_1}^{\mathsf{V}}, {\aleph_2}^{\mathsf{V}} \in \mathcal{H}(2^{\aleph_0})$. Letting $\psi(x, y)$ a Σ_1-formula saying "$\exists f$ (f is a surjection from x to y)", we have $\Vdash_{\mathbb{P}}$ "$\psi(({\aleph_1}^{\mathsf{V}})^{\checkmark}, ({\aleph_2}^{\mathsf{V}})^{\checkmark})$".

By $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA, the formula $\psi({\aleph_1}^{\mathsf{V}}, {\aleph_2}^{\mathsf{V}})$ must hold in a ground. This is a contradiction.

(2'): Assume that $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}}$ "$|({\aleph_2}^{\mathsf{V}})^{\checkmark}| = |({\aleph_1}^{\mathsf{V}})^{\checkmark}|$", and $(\mathcal{P}, \mathcal{H}({\aleph_2}^+))_{\Sigma_1}$-RcA holds. Then, since $\aleph_1, \aleph_2 \in \mathcal{H}({\aleph_2}^+)$ and "$|x| = |y|$" is Σ_1, there is a ground W of V such that $\mathsf{W} \models |{\aleph_2}^{\mathsf{V}}| = |{\aleph_1}^{\mathsf{V}}|$. This is a contradiction.

(3): Suppose that $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}}$ "${\aleph_1}^{\mathsf{V}}$ is countable". Note that $\omega, \aleph_1 \in \mathcal{H}(\kappa_{\mathfrak{refl}})$. By $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA, it follows that there is a ground W of V such that $\mathsf{W} \models$ "${\aleph_1}^{\mathsf{V}}$ is countable". This is a contradiction.

Suppose now that $S \subseteq \omega_1$ is stationary and $\mathbb{P} \in \mathcal{P}$ destroys the stationarity of S. Note that $\omega_1, S \in \mathcal{H}(\aleph_2)$. Let $\varphi = \varphi(y, z)$ be the Σ_1-formula

$\exists x$ (x is a club subset of the ordinal y and $z \cap x = \emptyset$).

Then we have $\Vdash_{\mathbb{P}}$ "$\varphi(\omega_1, S)$". By $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA, it follows that there is a ground $\mathsf{W} \subseteq \mathsf{V}$ such that $S \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\omega_1, S)$. This is a contradiction to the stationarity of S.

(4): follows from (1), (2) and (3).

(5): If $\aleph_1 < 2^{\aleph_0}$, then ${\aleph_1}^{\mathsf{V}} \in \mathcal{H}(2^{\aleph_0})$.

Let $\mathbb{P} \in \mathcal{P}$ be a poset collapsing ${\aleph_1}^{\mathsf{V}}$. That is, $\Vdash_{\mathbb{P}}$ "${\aleph_1}^{\mathsf{V}}$ is countable". Since "$\cdots$ is countable" is Σ_1, there is a ground M such that $M \models$ "${\aleph_1}^{\mathsf{V}}$ is countable" by $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA. This is a contradiction.

(5'): Assume that $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}}$ "${\aleph_1}^{\mathsf{V}}$ is countable", and $(\mathcal{P}, \mathcal{H}((2^{\aleph_0})^+))$-RcA holds. Since $\aleph_1 \in \mathcal{H}((2^{\aleph_0})^+)$, it follows that there is a ground W of V such that $\mathsf{W} \models {\aleph_1}^{\mathsf{V}}$ is countable. This is a contradiction. □ (Lemma 20)

Corollary A1. (E) *(stationary preserving, $\mathcal{H}(\kappa_{\mathfrak{refl}})$)-RcA is maximal among Recurrence Axioms with similar pair of parameters, and it implies $2^{\aleph_0} = \aleph_2$.*

(Z) *(all posets, $\mathcal{H}(2^{\aleph_0})$)-RcA is maximal among Recurrence Axioms with similar pair of parmeters, and it implies CH.*

Proof. (E): By Lemma 20, (3), "stationary preserving" cannot be replaced by a larger class of posets. By Lemma 20, (4), (stationary preserving, $\mathcal{H}(2^{\aleph_0})$)-RcA implies $2^{\aleph_0} = \aleph_2$. Thus $\mathcal{H}(\kappa_{\mathfrak{refl}}) = \mathcal{H}(\aleph_2)$ in this case and, by Lemma 20, (2'), this cannot be replaced by $\mathcal{H}(\mathcal{H}(\aleph_3))$.

(Z): CH holds by Lemma 20, (5), and, by By Lemma 20, (5'), $\mathcal{H}(2^{\aleph_0})$ cannot be replaced by $\mathcal{H}((2^{\aleph_0})^+)$. □ (Corollary A1)

Laver-genericity implies the plus version of Recurrence Axiom ($\Leftrightarrow$ Maximality Principle) restricted to Σ_2.

Theorem 21 *Suppose that κ is tightly $\mathcal{P}$-Laver-generically ultrahuge for an iterable class $\mathcal{P}$ of posets. Then $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_2}$-RcA$^+$ holds.*

Proof. Assume that κ is tightly $\mathcal{P}$-Laver generically ultrahuge for an iterable class $\mathcal{P}$ of posets.

Suppose that $\varphi = \varphi(\overline{x})$ is Σ_2 formula (in $\mathcal{L}_\in$), $\overline{a} \in \mathcal{H}(\kappa)$, and $\mathbb{P} \in \mathcal{P}$ is such that

$$\mathsf{V} \models\ \Vdash_{\mathbb{P}} \text{“} \varphi(\overline{a}^{\checkmark}) \text{”}. \tag{*20}$$

Let $\lambda > \kappa$ be such that $\mathbb{P} \in \mathsf{V}_\lambda$ and

$$V_\lambda \prec_{\Sigma_n} \mathsf{V} \text{ for a sufficiently large } n. \tag{*21}$$

In particular, we may assume that we have chosen the n above so that a sufficiently large fragment of ZFC holds in V_λ (“sufficiently large” means here, in particular, in terms of Lemma 13 and that the argument at the end of this proof is possible).

Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \text{“} \underset{\sim}{\mathbb{Q}} \in \mathcal{P} \text{”}$, and for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with

$$j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M, \tag{*22}$$

$$j(\kappa) > \lambda, \tag{*23}$$

$$\mathbb{P} * \underset{\sim}{\mathbb{Q}},\ \mathbb{P},\ \mathbb{H},\ {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M, \text{ and} \tag{*24}$$

$$|\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,| \leq j(\kappa). \tag{*25}$$

By (*25), we may assume that the underlying set of $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is $j(\kappa)$ and $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in {V_{j(\lambda)}}^{\mathsf{V}}$.

Let $\mathbb{G} := \mathbb{H} \cap \mathbb{P}$. Note that $\mathbb{G} \in M$ by (*24) and we have

$${V_{j(\lambda)}}^{M} \underbrace{=}_{\text{by (*24)}} {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \overbrace{=}^{\substack{\text{Since } {V_{j(\lambda)}}^{M}\ (= V^{\mathsf{V}[\mathbb{H}]}_{j(\lambda)}) \text{ satisfies a sufficiently large fragment of } \mathsf{ZFC} \\ \text{by elementarity of } j \text{, and hence the equality follows by Lemma 13}}} {V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{H}]. \tag{*26}$$

Thus, by (*24) and by the definability of grounds, we have ${V_{j(\lambda)}}^{\mathsf{V}} \in M$ and ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] \in M$.

Claim 21.1. ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] \models \varphi(\overline{a})$.

$\vdash$ By Lemma 13, ${V_\lambda}^{\mathsf{V}}[\mathbb{G}] = {V_\lambda}^{\mathsf{V}[\mathbb{G}]}$, and ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] = {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{G}]}$. By (*21), both ${V_\lambda}^{\mathsf{V}}[\mathbb{G}]$ and $V^{\mathsf{V}}_{j(\lambda)}[\mathbb{G}]$ satisfy large enough fragment of ZFC. Thus

$${V_\lambda}^{\mathsf{V}}[\mathbb{G}] \prec_{\Sigma_1} {V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}]. \tag{*27}$$

By (*20) and (*21), we have ${V_\lambda}^{\mathsf{V}}[\mathbb{G}] \models \varphi(\overline{a})$. By (*27) and since φ is Σ_2, it follows that ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] \models \varphi(\overline{a})$. $\dashv$ (Claim 21.1)

Thus we have

$$M \models \text{“there is a } \mathcal{P}\text{-ground } N \text{ of } V_{j(\lambda)} \text{ with } N \models \varphi(\overline{a})\text{”}. \tag{*28}$$

By the elementarity (*22), it follows that

$$\mathsf{V} \models \text{“there is a } \mathcal{P}\text{-ground } N \text{ of } V_\lambda \text{ with } N \models \varphi(\overline{a})\text{”}. \tag{*29}$$

Now by (*21), it follows that there is a $\mathcal{P}$-ground W of V such that $\mathsf{W} \models \varphi(\overline{a})$.
□ (Theorem 21)

5 Recurrence, Laver-generic large cardinal, and beyond

Laver-genericity does not imply full Recurrence Axiom (Theorem 26 and Corollary 27 below).

For an $\mathcal{L}_\in$-formula $\psi = \psi(\overline{x})$, a (large) cardinal κ is *ψ-absolute* if the formula ψ is absolute between V_κ and V (i.e. $(\forall \overline{x} \in V_y)(\psi^{V_y}(\overline{x}) \leftrightarrow \psi(\overline{x}))$ holds for $y = \kappa$).

Lemma 22. *For any $n \in \mathbb{N}$, there is an $\mathcal{L}_\in$-formula ψ^*_n such that, for any inaccessible κ, κ is ψ^*_n-absolute if and only if*

(*30) *for any ground W of V such that $\mathsf{V} = \mathsf{W}[\mathbb{G}]$ for a poset $\mathbb{P} \in {V_\kappa}^{\mathsf{W}}$ and $(\mathsf{W}, \mathbb{P})$-generic $\mathbb{G}$,[8] we have that all Σ_n-formulas are absolute between ${V_\kappa}^{\mathsf{W}}$ and W.*

Proof. By Theorem 12 and Lemma 14. □ (Lemma 22)

Lemma 23. *Let ψ^*_n be as in Lemma 22. ψ^*_n-absolute inaccessible cardinals are not resurrectable. That is, if a cardinal λ satisfies*

$$\Vdash_{\mathbb{P}} \text{“}\check{\lambda} \text{ is } \psi^*_n\text{-absolute inaccessible”} \tag{*31}$$

for some poset $\mathbb{P}$, then λ is really ψ^*_n-absolute inaccessible.

Proof. (*31) implies that λ is inaccessible. By the definition (*30) of ψ^*_n, if ψ^*_n is absolute between ${V_\lambda}^{\mathsf{V}[\mathbb{G}]}$ and $\mathsf{V}[\mathbb{G}]$ for some $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ then, it is absolute between ${V_\lambda}^{\mathsf{V}}$ and V. □ (Lemma 23)

Lemma 24. *Suppose that there are stationarily many inaccessible cardinals.[9] Then, for each $n \in \mathbb{N}$, there are stationarily many ϕ^*_n-absolute inaccessible cardinals.*

Proof. For $n \in \mathbb{N}$ let $n^+ \geq n$ be such that ψ^*_n is Σ_{n^+}. For any club $C \subseteq \mathsf{On}$, $C \cap C^{(n^+)} = \{\alpha \in C : V_\alpha \prec_{\Sigma_{n^+}} \mathsf{V}\}$ is a club in On (Lévy-Montague Reflection Theorem), there is an inaccessible cardinal $\mu \in C \cap C^{(n^+)}$. By the choice of n^+, such μ is a ψ^*_n-absolute inaccessible cardinal. □ (Lemma 24)

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[8] Note that this includes the case that $\mathbb{P} = \{\mathbb{1}\}$ and $\mathsf{V} = \mathsf{W}$.

[9] “There are stationarily many inaccessible cardinals” is the statement formalizable in an axiom scheme claiming, for each $\mathcal{L}_\in$-formula $\varphi = \varphi(x)$, that “if $\varphi(x)$ defines a club subclass of On then there is an inaccessible μ with $\varphi(\mu)$”.

Theorem 25 *Suppose that* $(\mathcal{P}, \emptyset)$-RcA *holds, where* $\mathcal{P}$ *is a class of posets such that either* (a) $\mathcal{P}$ *contains posets collapsing arbitrary large cardinals to a small cardinality (less than the first inaccessible if there are inaccessibles at all), or* (b) $\mathcal{P}$ *contains posets adding arbitrarily many reals.*
If there is a ψ^*_n*-absolute inaccessible cardinal for some* $n \in \mathbb{N}$*, then there are cofinally many* ψ^*_n*-absolute inaccessible cardinals.*

Proof. Assume that $(\mathcal{P}, \emptyset)$-RcA holds for $\mathcal{P}$ as above and there is a cardinal λ such that there are some ψ^*_n-absolute inaccessible cardinals but all of them are below λ.

Let $\mathbb{P}$ be a poset which either collapses λ to small cardinality or add at least λ many reals. Then, by Lemma 23, we have $\Vdash_{\mathbb{P}}$ "there is no ψ^*_n-absolute inaccessible cardinal".

By $(\mathcal{P}, \emptyset)$-RcA, it follows that there is a ground W of V such that $\mathsf{W} \models$ "there is no ψ^*_n-absolute inaccessible cardinal". Again by Lemma 23, this is a contradiction.

□ (Theorem 25)

Theorem 26 *Suppose that* λ *is an inaccessible cardinal,* $\kappa < \lambda$ *is such that* $V_\lambda \models$ "κ is x-large cardinal"*, where "x-large cardinal" is a notion of large cardinal, for which a Laver function exists. Assume also that* $\{\mu < \lambda : \mu$ *is inaccessible*$\}$ *is stationary in* λ*.*
Then, for each of the classes $\mathcal{P}$ *of posets considered in Theorem 9, there are* λ_0 *with* $\lambda > \lambda_0 > \kappa$*, and* $\mathbb{P} \in \mathcal{P}$ *with* $\mathbb{P} \subseteq V_\kappa$ *such that, for a* $(V_\lambda, \mathbb{P})$*-generic* $\mathbb{G}$*, we have*

$$V_{\lambda_0}[\mathbb{G}] \models \text{"}\kappa \textit{ is a tightly}^{+}\ \mathcal{P}\textit{-Laver-generically x-large cardinal and } \neg(\mathcal{P}, \emptyset)\text{-}\mathsf{RcA}\text{"}.$$

Proof. Suppose that $\mathcal{P}$ is one of the classes of posets considered in Theorem 9. Note that then, (a) or (b) of Theorem 25 holds. Let $n \in \mathbb{N}$ be such that the formula "κ is an x-large cardinal" is Σ_n. By the assumption, there is an inaccessible cardinal μ with $\lambda > \mu > \kappa$ such that $V_\lambda \succ V_\mu$. Let $\lambda > \lambda_0 > \mu$ be the minimal cardinal such that $V_\lambda \models \lambda_0$ is ψ^*_n-absolute inaccessible cardinal — such λ_0 exists by Lemma 24. Then we have

$$V_{\lambda_0} \models \text{"}\kappa \text{ is an x-large cardinal"} \tag{*32}$$

In V_{λ_0}, let $\mathbb{P}$ be the limit of κ-iteration with appropriate support as described in Theorem 9 which forces that κ is tightly$^+$ $\mathcal{P}$-Laver generically x-large cardinal in the generic extension of V_{λ_0}. Let $\mathbb{G}$ be $(V_\lambda, \mathbb{P})$-generic filter. Then we have $V_{\lambda_0}[\mathbb{G}] \models$ "κ is a tightly$^+$ $\mathcal{P}$-Laver generically x-large cardinal" and $V_\lambda[\mathbb{G}] \succ V_\mu[\mathbb{G}]$ by Lemma 15, (1). In particular, by Lemma 23, we have $V_{\lambda_0}[\mathbb{G}] \models \mu$ is the largest ψ^*_n-absolute inaccessible cardinal. By Theorem 25, it follows that $V_{\lambda_0}[\mathbb{G}] \models$ "$\neg(\mathcal{P}, \emptyset)$-$\mathsf{RcA}$".

□ (Theorem 26)

The conditions of Theorem 26 are satisfied by practically all large cardinal notions. For example, under the consistency of the existence of a 2-huge cardinal, the conditions of Theorem 26 are satisfied by x-large cardinal = hyperhuge cardinal (see Lemma 29 below). Thus we obtain the following:

Corollary 27. *Under the assumption of the consistency of the existence of a 2-huge cardinal, the existence of a tightly$^+$ $\mathcal{P}$-Laver generically hyperhuge cardinal does not imply $(\mathcal{P}, \emptyset)$-RcA for any class $\mathcal{P}$ of posets as in Theorem 9.* □

A natural strengthening of Laver-genericity does imply the full Maximality Principle (hence also the full Recurrence Axiom). As the proof of Theorem 25 suggests, such property must be formulated not in a single formula but as an axiom scheme.

For a natural number n, we call a cardinal

κ *super $C^{(n)}$-hyperhuge* if for any $\lambda_0 > \kappa$ there are $\lambda \geq \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, and j, $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$, $j(\kappa) > \lambda$, ${}^{j(\lambda)}M \subseteq M$ and $V_{j(\lambda)} \prec_{\Sigma_n} \mathsf{V}$.

κ is *super $C^{(n)}$-ultrahuge* if the condition above holds with "${}^{j(\lambda)}M \subseteq M$" replaced by "${}^{j(\kappa)}M \subseteq M$ and $V_{j(\lambda)} \subseteq M$".

If κ is super $C^{(n)}$-hyperhuge then it is super $C^{(n)}$-ultrahuge. This can be shown similarly to Lemma 4.

We shall also say that κ is *super $C^{(\infty)}$-hyperhuge* (*super $C^{(\infty)}$-ultrahuge*, resp.) if it is super $C^{(n)}$-hyperhuge (super $C^{(n)}$-ultrahuge, resp.) for all natural number n.

A similar kind of strengthening of the notions of large cardinals which we call here "super $C^{(n)}$" appears also in Boney [6]. It is called in [6] "$C^{(n)+}$" and the notion is considered there in connection with extendibility.

For a natural number n and an iterable class $\mathcal{P}$ of posets, a cardinal

κ is *super $C^{(n)}$- $\mathcal{P}$-Laver-generically ultrahuge* if, for any $\lambda_0 > \kappa$ and for any $\mathbb{P} \in \mathcal{P}$, there are a $\lambda \geq \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, a $\mathcal{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", such that, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$, $j(\kappa) > \lambda$, $\mathbb{P}$, $\mathbb{H}$, ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$ and ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$.

A super $C^{(n)}$- $\mathcal{P}$-Laver-generically ultrahuge cardinal

κ is *tightly super $C^{(n)}$- $\mathcal{P}$-Laver-generically ultrahuge*, if additionally $|\mathbb{P} * \underset{\sim}{\mathbb{Q}}| \leq j(\kappa)$ (see Footnote 2) holds in the definition above.

Super $C^{(\infty)}$- $\mathcal{P}$-Laver-generically hyperhugeness and *tightly super $C^{(\infty)}$- $\mathcal{P}$-Laver generically hyperhugeness* are defined similarly to super $C^{(\infty)}$-ultrahugeness.

Note that, in general, super $C^{(\infty)}$-hyperhugeness and super $C^{(\infty)}$-ultrahugeness are notions not formalizable in the language of ZFC without introducing a new constant symbol for κ since we need infinitely many $\mathcal{L}_\in$-formulas to formulate them. Exceptions are when we are talking about a cardinal in a set model being with one of these properties like in Lemma 29 below (and in such a case "natural number n" actually refers to "$n \in \omega$"), or when we are talking about a cardinal definable in V having these properties in an inner model like in Corollary 36 or Corollary 37. In the latter case, the situation is formalizable with infinitely may $\mathcal{L}_\in$-sentences.

In contrast, the super $C^{(\infty)}$-$\mathcal{P}$-Laver generically ultrahugeness of κ is expressible in infinitely many $\mathcal{L}_\in$-sentences. This is because a $\mathcal{P}$-Laver generic large cardinal

κ for relevant classes $\mathcal{P}$ of posets is uniquely determined as $\kappa_{\mathfrak{refl}}$ or $2^{\aleph_0}$ (see e.g. Theorem 7 and Theorem 8).

A modification of the proof of Theorem 21 shows the following:

Theorem 28 ([21]) *Suppose that $\mathcal{P}$ is an iterable class of posets and κ is super $C^{(\infty)}$-$\mathcal{P}$-Laver-generically ultrahuge. Then $(\mathcal{P}, \mathcal{H}(\kappa))$-$\mathsf{RcA}^+$ holds.*

Proof. Suppose that κ is super $C^{(\infty)}$-$\mathcal{P}$-Laver-gen. ultrahuge, $\mathbb{P} \in \mathcal{P}$, and $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$" for an $\mathcal{L}_\in$-formula φ and $\overline{a} \in \mathcal{H}(\kappa)$. We want to show that $\varphi(\overline{a})$ holds in some $\mathcal{P}$-ground of V.

Let n be a sufficiently large natural number such that the following arguments go through. In particular, we assume that ${V_\alpha}^{\mathsf{V}} \prec_{\Sigma_n} \mathsf{V}$ implies that "$\varphi(\overline{x})$" and "$\Vdash_{\cdot}$ "$\varphi(\overline{x}^{\checkmark})$"" are absolute between ${V_\alpha}^{\mathsf{V}}$ and V, and ${V_\alpha}^{\mathsf{V}} \prec_{\Sigma_n} \mathsf{V}$ also implies that a sufficiently large fragment of ZFC holds in V_α.

Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are a $\lambda > \kappa$ with

$$V_\lambda \prec_{\Sigma_n} \mathsf{V}, \tag{$\aleph$3}$$

and j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$, $j(\kappa) > \lambda$, $\mathbb{P}$, $\mathbb{H}$, ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$ and ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$.

By the choice of n, we have $V_\lambda \models \Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$". $j({V_\lambda}^{\mathsf{V}}) = {V_{j(\lambda)}}^M \prec_{\Sigma_n} M$ by elementarity of j, and ${V_{j(\lambda)}}^M = {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]}$ by the closedness of M. Since $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, we have $V_\lambda[\mathbb{H}] \prec_{\Sigma_{n_0}} \mathsf{V}[\mathbb{H}]$ for a still large enough $n_0 \leq n$ by Lemma 15, (1). Since ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$, it follows that ${V_\lambda}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_{n_0}} {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]}$. Thus

$${V_\lambda}^{\mathsf{V}} \prec_{\Sigma_{n_1}} {V_{j(\lambda)}}^{\mathsf{V}} \tag{$\aleph$4}$$

for a still large enough $n_1 \leq n_0$ by Lemma 15, (2).

In particular, we have ${V_{j(\lambda)}}^{\mathsf{V}} \models \Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$", and hence $V_{j(\lambda)}[\mathbb{G}] \models \varphi(\overline{a})$ where $\mathbb{G}$ is the $\mathbb{P}$-part of $\mathbb{H}$. Note that by (3) and (4), $V_{j(\lambda)}$ satisfies a sufficiently large fragment of ZFC.

Thus we have $V_{j(\lambda)}[\mathbb{H}] \models$" there is a $\mathcal{P}$-ground satisfying $\varphi(\overline{a})$", and hence

$${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \models \text{" there is a } \mathcal{P}\text{-ground satisfying } \varphi(\overline{a})\text{"}$$

by Lemma 13. By elementarity, it follows that

$$V_\lambda \models \text{" there is a } \mathcal{P}\text{-ground satisfying } \varphi(\overline{a})\text{".}$$

Finally, this implies $\mathsf{V} \models$" there is a $\mathcal{P}$-ground satisfying $\varphi(\overline{a})$" by (3).

□ (Theorem 28)

The following Lemma can be proved similarly to Theorem 5c in Barbanel-DiPrisco-Tan [2] (see also Theorem 24.13 in Kanamori [29]).

Lemma 29. ([21]) *If κ is 2-huge with the 2-huge elementary embedding j, that is, there is $M \subseteq \mathsf{V}$ such that $j : \mathsf{V} \xrightarrow{\prec}_\kappa M \subseteq \mathsf{V}$, and*

$${}^{j^2(\kappa)}M \subseteq M, \tag{*33}$$

then $V_{j(\kappa)} \models$"κ *is super* $C^{(\infty)}$*-hyperhuge cardinal*", *and for each* $n \in \omega$, $V_{j(\kappa)} \models$"*there are stationarily many super* $C^{(n)}$*-hyperhuge cardinals*". □

The proof of the existence of Laver-function for a supercompact cardinal can be modified to show that super $C^{(\infty)}$-hyperhuge cardinal in V_μ has a Laver function for super $C^{(\infty)}$-hyperhugeness ([21]). Similarly to Theorem 9 we obtain the following:

Theorem 30 ([21]) (A) *Suppose that* μ *is inaccessible and* $\kappa < \mu$ *is super* $C^{(\infty)}$*-hyperhuge in* V_μ. *Let* $\mathbb{P} = \mathrm{Col}(\aleph_1, \kappa)$. *Then, in* $V_\mu[\mathbb{G}]$, *for any* $V_\mu, \mathbb{P}$*-generic* $\mathbb{G}$, $\aleph_2^{V_\mu[\mathbb{G}]}$ $(= \kappa)$ *is tightly super* $C^{(\infty)}$*-*σ*-closed-Laver-generically hyperhuge and* CH *holds.*

(B) *Suppose that* μ *is inaccessible and* $\kappa < \mu$ *is super* $C^{(\infty)}$*-hyperhuge with a Laver function* $f : \kappa \to V_\kappa$ *for super* $C^{(\infty)}$*-hyperhugeness in* V_μ. *If* $\mathbb{P}$ *is the CS-iteration of length* κ *for forcing* PFA *along with* f, *then, in* $V_\mu[\mathbb{G}]$ *for any* $(V_\mu, \mathbb{P})$*-generic* $\mathbb{G}$, $\aleph_2^{V_\mu[\mathbb{G}]}$ $(= \kappa)$ *is tightly super* $C^{(\infty)}$*-proper-Laver-generically hyperhuge and* $2^{\aleph_0} = \aleph_2$ *holds.*

(B') *Suppose that* μ *is inaccessible and* $\kappa < \mu$ *is super* $C^{(\infty)}$*-hyperhuge with a Laver function* $f : \kappa \to V_\kappa$ *for super* $C^{(\infty)}$*-hyperhugeness in* V_μ. *If* $\mathbb{P}$ *is the RCS-iteration of length* κ *for forcing* MM *along with* f, *then, in* $V_\mu[\mathbb{G}]$ *for any* $(V_\mu, \mathbb{P})$*-generic* $\mathbb{G}$, $\aleph_2^{V_\mu[\mathbb{G}]}$ $(= \kappa)$ *is tightly super* $C^{(\infty)}$*-stationary_preserving-Laver-generically hyperhuge and* $2^{\aleph_0} = \aleph_2$ *holds.*

(Γ) *Suppose that* μ *is inaccessible and* κ *is super* $C^{(\infty)}$*-hyperhuge with a Laver function* $f : \kappa \to V_\kappa$ *for super* $C^{(\infty)}$*-hyperhugeness in* V_μ. *If* $\mathbb{P}$ *is a FS-iteration of length* κ *for forcing* MA *along with* f, *then, in* $V_\mu[\mathbb{G}]$ *for any* $(V_\mu, \mathbb{P})$*-generic* $\mathbb{G}$, $2^{\aleph_0}$ $(= \kappa)$ *is tightly super* $C^{(\infty)}$*-c.c.c.-Laver-generically hyperhuge, and* $2^{\aleph_0}$ *is very large in* $V_\mu[\mathbb{G}]$.

(Δ) *Suppose that* μ *is inaccessible and* κ *is super* $C^{(\infty)}$*-hyperhuge with a Laver function* $f : \kappa \to V_\kappa$ *for super* $C^{(\infty)}$*-hyperhugeness in* V_μ. *If* $\mathbb{P}$ *is a FS-iteration of length* κ *along with* f *enumerating "all" posets, then, in* $V_\mu[\mathbb{G}]$ *for any* $(V_\mu, \mathbb{P})$*-generic* $\mathbb{G}$, $2^{\aleph_0}$ $(= \aleph_1)$ *is tightly super* $C^{(\infty)}$*-all posets-Laver-generically hyperhuge, and* CH *holds.*

Proof. The proof can be done similarly to that of Theorem 5.2 in [16]. In the following we shall only check the case (Δ).

Suppose that $f : \kappa \to V_\kappa$ is a super $C^{(\infty)}$-hyperhuge Laver function.

Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be a FS-iteration defined by

$$(^{\aleph}5)\qquad \underset{\sim}{\mathbb{Q}}_\beta = \begin{cases} f(\alpha), & \text{if } f(\alpha) \text{ is a } \mathbb{P}_\beta\text{-name of a poset;} \\ \{\mathbb{1}\}, & \text{otherwise} \end{cases}$$

for $\beta < \kappa$.

Let $\mathbb{G}$ be a $(V_\mu, \mathbb{P}_\kappa)$-generic filter. Clearly $V_\mu[\mathbb{G}] \models$"$2^{\aleph_0} = \kappa = \aleph_1$". We show that κ is tightly super $C^{(\infty)}$-all posets-Laver-generically ultrahuge in $V_\mu[\mathbb{G}]$.

Suppose that $\mathbb{P}$ is a poset in $V_\mu[\mathbb{G}]$, $\kappa < \lambda_0$ and $n \in \omega$. Let $n' > n$ be sufficiently large and let $\underset{\sim}{\mathbb{P}}$ be a $\mathbb{P}_\kappa$-name of $\mathbb{P}$.

Working in V_μ, we can find

($\aleph$6) $\quad |\,\mathbb{P}\,| < \lambda < \kappa^* < \lambda^*$ and j, $M \subseteq \mathsf{V}$

such that

($\aleph$7) $\quad j : \mathsf{V} \xrightarrow{\prec}_\kappa M$,

($\aleph$8) $\quad j(\kappa) = \kappa^*$, $j(\lambda) = \lambda^*$,

($\aleph$9) $\quad {}^{\lambda^*}M \subseteq M$,

($\aleph$10) $\quad V_\lambda \prec_{\Sigma_{n'}} \mathsf{V}$, $V_{\lambda^*} \prec_{\Sigma_{n'}} \mathsf{V}$, and

($\aleph$11) $\quad j(f)(\kappa) = \underset{\sim}{\mathbb{P}}$

by definition of f.

By elementarity (and by the definition (5) of $\mathbb{P}_\kappa$),

($\aleph$12) $\quad j(\mathbb{P}_\kappa) \sim_{\mathbb{P}_\kappa} (\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}}) * \underset{\sim}{\mathbb{R}}$

for a $(\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}})$-name $\underset{\sim}{\mathbb{R}}$ of a poset. Note that $(\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}} * \underset{\sim}{\mathbb{R}})/\mathbb{G}$ corresponds to a poset of the form $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$.

Let $\mathbb{H}^*$ be $(\mathsf{V}, (\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}}) * \underset{\sim}{\mathbb{R}})$-generic filter with $\mathbb{G} \subseteq \mathbb{H}^*$. $\mathbb{H}^*$ corresponds to a $(\mathsf{V}, j(\mathbb{P}_\kappa))$-generic filter $\mathbb{H} \supseteq \mathbb{G}$ via the equivalence (12).

Let $\tilde{j}$ be defined by

($\aleph$13) $\quad \tilde{j} : \mathsf{V}[\mathbb{G}] \to M[\mathbb{H}];\quad \underset{\sim}{a}^{\mathbb{G}} \mapsto j(\underset{\sim}{a})^{\mathbb{H}}$

for all $\mathbb{P}_\kappa$-names $\underset{\sim}{a}$.

A standard proof shows that f is well-defined, and $j : \mathsf{V}[\mathbb{G}] \xrightarrow{\prec}_\kappa M[\mathbb{H}]$. By (8) and (9), we have $\tilde{j}\,''\tilde{j}(\lambda) = j\,''j(\lambda) \in M[\mathbb{H}]$. Since $\mathbb{H} \in M[\mathbb{H}]$, the $(\mathsf{V}[\mathbb{G}], \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic filter corresponding to $\mathbb{H}$ is also in $M[\mathbb{H}]$.

By (6), (10), by the choice of n', and by Lemma 15, (1), we have ${V_\lambda}^{\mathsf{V}[\mathbb{G}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{G}]$ and ${V_{\tilde{j}(\lambda)}}^{\mathsf{V}[\mathbb{H}]} = {V_{\lambda^*}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$.

Since $\mathbb{P}$ and n were arbitrary, this shows that κ is tightly super $C^{(\infty)}$-all posets-Laver-generically ultrahuge in $V_\mu[\mathbb{G}]$. $\square$ (Theorem 30)

6 Toward the Laver-generic Maximum

Besides Theorem 21 and Theorem 28, , we also have some other advantages of assuming the existence of $\mathcal{P}$-Laver-generically hyperhuge cardinal or even its "super $C^{(\infty)}$" version. One of them is that they imply the existence of the (set-generic) bedrock (see below for definition); another is that we know the exact consistency strength of these principles.

For a class $\mathcal{P}$ of posets, a cardinal κ is *tightly $\mathcal{P}$-generic hyperhuge* if for any $\lambda > \kappa$, there is $\mathbb{Q} \in \mathcal{P}$ such that for a $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$, $\lambda < j(\kappa)$, $|\,\mathbb{Q}\,| \le j(\kappa)$, and $j\,''j(\lambda), \mathbb{H} \in M$.

Note that, for any class $\mathcal{P}$ of posets with $\{\mathbb{1}\} \in \mathcal{P}$, the hyperhugeness of a cardinal κ implies its tightly $\mathcal{P}$-generically hyperhugeness. Likewise, if $\mathcal{P}$ is iterable then, the tightly $\mathcal{P}$-Laver-generic hyperhugeness of κ implies its tightly $\mathcal{P}$-generically hyperhugeness.

For an iterable $\mathcal{P}$:

$$\begin{tikzcd}
 & & \text{the bedrock exists} \\
\kappa \text{ is hyperhuge} \arrow[r] & \kappa \text{ is tightly } \mathcal{P}\text{-generically hyperhuge} \arrow[ur, "\text{Theorem 32}"] & \\
\kappa \text{ is tightly } \mathcal{P}\text{-Laver-generically hyperhuge} \arrow[ur] & & \\
\kappa \text{ is tightly super } C^{(\infty)}\text{-}\mathcal{P}\text{-Laver-generically hyperhuge} \arrow[u] \arrow[r] & \kappa \text{ is tightly super } C^{(\infty)}\text{-}\mathcal{P}\text{-Laver-generically ultrahuge} \arrow[r, "\text{Theorem 28}"] & (\mathcal{P}, \mathcal{H}(\kappa))\text{-}\mathsf{RcA}
\end{tikzcd}$$

Usuba [36] proved that the grounds of V are downward directed (with respect to subclass relation) for class many grounds (this is formalizable by virtue of Theorem 12). More concretely

Theorem 31 (Theorem 1.3 in Usuba [36]) *For any collection of grounds of* V*, indexed by a set of parameters (in the sense of Theorem 12), there is a ground which is included in all grounds in the collection.* □

From this theorem, it follows that the *mantle*, i.e., the intersection of all grounds is a model of ZFC. In [36], it is proved that the mantle is a ground and hence it is the *bedrock*, i.e., the smallest ground of V provided that there exists a hyperhuge cardinal(Theorem 1.6 in [36]). Later the assumption of the existence of a hyperhuge cardinal in this theorem is weakened to the existence of an extendible cardinal (Theorem 1.3 in Usuba [37]).

In [21], we obtained the following generalization of Theorem 1.6 in [36]:

Theorem 32 ([21]) *If there is a tightly* $\mathcal{P}$*-generically hyperhuge cardinal* κ*, then the mantle is a ground of* V*. In particular it is the bedrock.*

A sketch of the proof. The overall structure of the structure of the proof is just the same as that of Theorem 1.6 in [36] or Theorem 1.3 in [37].

We call a ground W of V a $\leq\kappa$-ground if there is $\mathbb{P} \in W$ with $|\,\mathbb{P}\,|^{\mathsf{V}} \leq \kappa$ and a $(\mathsf{W}, \mathbb{P})$-generic $\mathbb{G}$ such that $\mathsf{V} = \mathsf{W}[\mathbb{G}]$. Let

$$\overline{W} = \bigcap \{W \ : \ W \text{ is a } \leq\kappa\text{-ground}\}. \tag{*34}$$

By Theorem 31, there is a ground $\mathsf{W} \subseteq \overline{\mathsf{W}}$. For such W it is enough to show that actually $\overline{\mathsf{W}} \subseteq \mathsf{W}$ holds.

Let $\mathbb{S} \in \mathsf{W}$ be a poset with cardinality μ (in V) such that there is a $(\mathsf{W}, \mathbb{S})$-generic $\mathbb{F} \in \mathsf{V}$ with $\mathsf{V} = \mathsf{W}[\mathbb{F}]$. Without loss of generality, $\mu \geq \kappa$.

By Theorem 12, there is $r \in \mathsf{V}$ such that $\mathsf{W} = \Phi(\cdot, r)^{\mathsf{V}}$.

Let $\theta \geq \mu$ be such that $r \in V_\theta$, and for a sufficiently large natural number n, we have ${V_\theta}^{\mathsf{V}} \prec_{\Sigma_n} \mathsf{V}$. By the choice of θ, $\Phi(\cdot, r)^{{V_\theta}^{\mathsf{V}}} = \Phi(\cdot, r)^{\mathsf{V}} \cap {V_\theta}^{\mathsf{V}} = \mathsf{W} \cap {V_\theta}^{\mathsf{V}} = {V_\theta}^{\mathsf{W}}$.

Let $\mathbb{Q} \in \mathcal{P}$ such that for $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are $j, M \subseteq \mathsf{V}[\mathbb{H}]$ with $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$, $\theta < j(\kappa)$, $|\,\mathbb{Q}\,| \leq j(\kappa)$, ${V_{j(\theta)}}^{\mathsf{V}[\mathbb{H}]} \subseteq M$, and $\mathbb{H}$, $j''j(\theta) \in M$.

Using this j we can show that ${V_\theta}^{\overline{\mathsf{W}}} \subseteq {V_\theta}^{\mathsf{W}}$ holds (this part of the proof is quite involved, for the details, the reader is referred to [21]). Since θ can be arbitrary large, It follows that $\overline{\mathsf{W}} \subseteq \mathsf{W}$. □ (Theorem 32)

Analyzing the details of the proof of Theorem 32 we dropped in our present exposition, we also obtain the following surprising result:

Theorem 33 ([21]) *Suppose that $\mathcal{P}$ is any class of posets. If κ is a tightly $\mathcal{P}$-generically hyperhuge cardinal, then κ is a hyperhuge cardinal in the bedrock $\overline{W}$ of* V. □

The following equiconsistency results are immediate consequences of the theorem above:

Corollary 34. *Suppose that $\mathcal{P}$ is the class of all posets. Then the following theories are equiconsistent:*

(a) ZFC + *"there is a hyperhuge cardinal".*
(b) ZFC + *"there is a tightly $\mathcal{P}$-Laver generically hyperhuge cardinal".*
(c) ZFC + *"there is a tightly $\mathcal{P}$-generically hyperhuge cardinal".*
(d) ZFC + *"the bedrock $\overline{\mathsf{W}}$ exists and ω_1 is a hyperhuge cardinal in $\overline{\mathsf{W}}$".* □

Corollary 35. *Suppose that $\mathcal{P}$ is one of the following classes of posets: all semi-proper posets; all proper posets; all ccc posets; all σ-closed posets. Then the following theories are equiconsistent:*

(a) ZFC + *"there is a hyperhuge cardinal".*
(b) ZFC + *"there is a tightly $\mathcal{P}$-Laver generically hyperhuge cardinal".*
(c) ZFC + *"there is a tightly $\mathcal{P}$-generically hyperhuge cardinal".*
(d) ZFC + *"the bedrock $\overline{\mathsf{W}}$ exists and $\kappa_{\mathfrak{refl}}$ is a hyperhuge cardinal in $\overline{\mathsf{W}}$".* □

A slight modification of the proofs of the theorems above also show the following. Note that as we already noticed, super-$C^{(\infty)}$-large cardinal is not formalizable in the language of ZFC. However, the assertions (a) and (b) in the following Corollary 36 and Corollary 36 can be formulated as schemes of sentences in $\mathcal{L}_\in$.

Corollary 36. *Suppose that $\mathcal{P}$ is the class of all posets. Then the following theories are equiconsistent:*

(a) ZFC + *"c is a super $C^{(\infty)}$ hyperhuge cardinal" where c is a new constant symbol but "... is super $C^{(\infty)}$ hyperhuge ..." is formulated in an infinite collection of formulas in $\mathcal{L}_\in$.*

(b) ZFC + *"there is a tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver generically hyperhuge cardinal".*

(c) ZFC + *"the bedrock $\overline{\mathsf{W}}$ exists and ω_1^{V} is a super $C^{(\infty)}$-hyperhuge cardinal in $\overline{\mathsf{W}}$".* □

Corollary 37. *Suppose that $\mathcal{P}$ is one of the following classes of posets: all semi-proper posets; all proper posets; all ccc posets; all σ-closed posets. Then the following theories are equiconsistent:*

(a) ZFC + *"c is a super $C^{(\infty)}$ hyperhuge cardinal" where c is a new constant symbol but "... is super $C^{(\infty)}$ hyperhuge ..." is formulated in an infinite collection of formulas in $\mathcal{L}_\in$.*

(b) ZFC + *"there is a tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver generically hyperhuge cardinal".*

(c) ZFC + *"the bedrock $\overline{\mathsf{W}}$ exists and $\kappa_{\mathfrak{refl}}{}^{\mathsf{V}}$ is a super $C^{(\infty)}$-hyperhuge cardinal in $\overline{\mathsf{W}}$".* □

Finally, we move to the promised proof of Theorem 7, (3).

Corollary 38. *Suppose that $\mathcal{P}$ is an arbitrary class of posets and κ is a tightly $\mathcal{P}$-generically hyperhuge cardinal. Then*

(1) *there are cofinally many huge cardinals.*

(2) SCH *holds above some cardinal.*

Proof. Suppose that κ is a tightly $\mathcal{P}$-generically hyperhuge cardinal. By Theorem 32, there is the bedrock $\overline{\mathsf{W}}$ and κ is hyperhuge cardinal in $\overline{\mathsf{W}}$.

(1): Since the existence of a hyperhuge cardinal implies the existence of cofinally many huge cardinals (it is easy to show that the target $j(\kappa)$ of hyperhuge embedding for a sufficiently large inaccessible λ is a huge cardinal), there are cofinally many huge cardinals in $\overline{\mathsf{W}}$. Since V is attained by a set forcing starting from $\overline{\mathsf{W}}$, a final segment of these huge cardinals survive in V.

(2): By Theorem 20.8 in [28], SCH holds above κ in $\overline{\mathsf{W}}$. Since V is a set generic extension of $\overline{\mathsf{W}}$. SCH should hold above some cardinal $\mu \geq \kappa$. □ (Corollary 0)

For iterable stationary preserving $\mathcal{P}$ containing all proper posets, Theorem 38, (2) holds already under the $\mathcal{P}$-Laver-generic supercompactness of κ. The reason is that in such case PFA holds by Theorem 5, and by Viale [38], SCH follows from it.

Proof of Theorem 7, (3). let λ and $\mathbb{Q}$ be such that

(*35) $\lambda > 2^{\aleph_0}$, κ and λ is large enough such that SCH holds above some $\mu < \lambda$ (this is possible by Corollary 38 , (2), and it is here the place that we need the Laver generic hyperhugeness of κ),

(*36) $\mathbb{Q}$ is positive elements of a complete Boolean algebra, and,

(*37) for $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ such that (1) $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, (2) $j(\kappa) > \lambda$, (3) $|\mathbb{Q}| \leq j(\kappa)$, and (4) $V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \subseteq M$.

By (*36), each $\mathbb{Q}$-name $\underset{\sim}{r}$ of a real corresponds to a mapping $f : \omega \rightarrow \mathbb{Q}$. By (*35) and by (*37), (3), there are at most $j(\kappa)$ many such mappings. Thus we have $\mathsf{V}[\mathbb{H}] \models$"$2^{\aleph_0} \leq j(\kappa)$", By (*37), (4), it follows $M \models$"$2^{\aleph_0} \leq j(\kappa)$". By elementarity, it follows that $\mathsf{V} \models$"$2^{\aleph_0} \leq \kappa$". □ (Theorem 7, (3))

Returning to (E) and (Z) at the end of Section 3, we now know that the existence of a super $C^{(\infty)}$-stationary preserving-Laver-generically hyperhuge cardinals implies

(E) (actually it even implies (stationary preserving posets, $\mathcal{H}(\kappa_{\mathfrak{refl}})$)-$\mathsf{RcA}^+$), and that the existence of a super $C^{(\infty)}$-all posets-Laver-generically hyperhuge cardinals implies (Z) (actually it even implies (all posets, $\mathcal{H}(2^{\aleph_0})$)-$\mathsf{RcA}^+$). These two scenarios are not compatible since the former implies $2^{\aleph_0} = \aleph_2$ while the latter implies CH.

However, with the following axiom, (E) is reconciled with a meaningful fragment of (Z):

LGM_0: the continuum is tightly super $C^{(\infty)}$-stationary preserving-Laver generically hyperhuge and there is a ground W of V such that the continuum is tightly super $C^{(\infty)}$-all posets-Laver generically hyperhuge in W.

If we admit that Recurrence Axioms, Maximal Principles and Laver-genericity are natural requirements, we should be also ready to accept $\mathsf{ZFC} + \mathsf{LGM}_0$ as a natural candidate of the extension of ZFC.

By Theorem 5, LGM_0 implies the double plus version of Martin's Maximum (MM^{++}) and hence all the consequences of it including $2^{\aleph_0} = \aleph_2$.

By Theorem 32, LGM_0 implies that there is the bedrock. So by Theorem 28, LGM_0 implies (Z^+) on page 15. (Z^+) implies that if some statement φ is forcable by a stationary preserving poset, then for any $A \in \mathcal{H}(\aleph_2)$, there is a semi-proper-ground W of V such that $A \in \mathsf{W}$ and $\mathsf{W} \models \varphi$. In particular, Cichoń's Maximum [25], [26] is a phenomena in many semi-proper-grounds in this sense. Note that, by Corollary 38, (1) the forcing argument for φ may even utilize class many huge cardinals (e.g. the proof in [25] uses four strongly compact cardinals). [10]

Even in the case that the forcing to show the consistency of φ is not stationary preserving, we can still find some ground W of V which satisfies φ.

Thus $\mathsf{ZFC} + \mathsf{LGM}_0$ is a very strong axiom system which integrates practically all known statements into itself which can be proved to be consistent by way of forcing and/or methods of inner models. Against this backdrop, we want to call the axiom system LGM_0(or possibly some further extension of it in the future) the *Laver Generic Maximum*.

The consistency and equiconsistency of LGM_0 is easily established: we start from a model with two super $C^{(\infty)}$ hyperhuge cardinals $\kappa_0 < \kappa_1$. We force κ_0 to be tightly super $C^{(\infty)}$-all postsLaver generically hyperhuge (Theorem 30, (Δ)). We then force make κ_1 to be tightly super $C^{(\infty)}$-stationary preserving-Laver generically hyperhuge (Theorem 30, (B')).

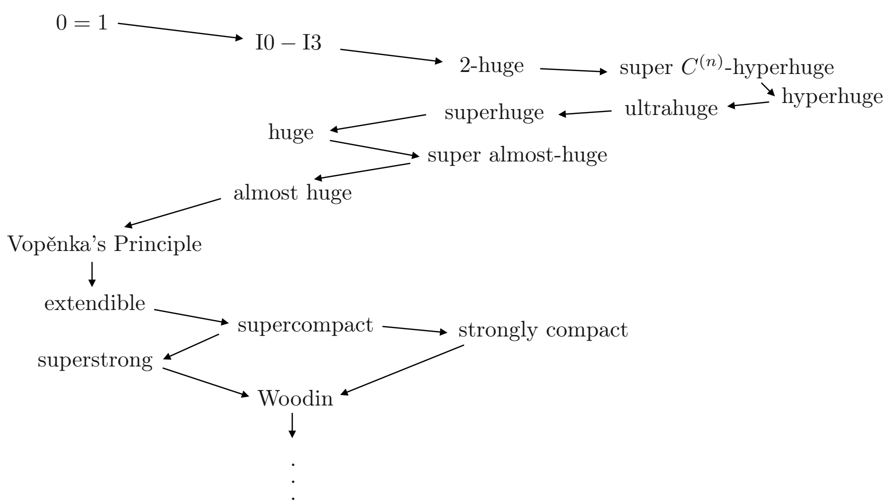
By Theorem 33 the consistency strength of LGM_0 can be proved to be equivalent with that of two super $C^{(\infty)}$ hyperhuge cardinals (which may be formulated by using two new constant symbols).

[10] In [21], we even show that LGM_0 implies that there are stationarily many super-$C^{(\infty)}$-hyperhuge cardinals.

7 More about Consistency Strength

When I was a student of Janos Makowsky in the early 1980s at the Free University of Berlin, one of the papers he was preparing then was [31] in which the effects of Vopěnka Principle on properties of model theoretic logics is studied. I remember that Janos gave a talk on this subject in the (West) Berliner Logic Colloquium. Back then, I was still living in a set theory of consistency strength way below a measurable cardinal, and could not begin with the material of his paper at all because of the vertiginous consistency strength of the Vopěnka Principle.

Janos left Berlin before I wrote up my diploma thesis on abstract elementary classes which was the subject Janos gave me; all the assertions I proved in the thesis remained in the consistency strength of ZFC.



The consistency strength of my set theoretic world view reached the realm of one supercompact cardinal when I wrote [10] in the early 1990s in which some consequences of $\mathsf{MA}^+(\sigma\text{-closed})$ were discussed. However, it is only quite recently that I caught up Janos definitively (at least in terms of consistency strength) when I considered in [21] the super $C^{(\infty)}$-$\mathcal{P}$-Laver-generically hyperhuge cardinals whose consistency strength is (demonstrably — see Section 6) strictly between hyperhuge and 2-huge.

In the meantime, active research on abstract model classes is resumed and Janos's [31] begins to attract the attention of young logicians. For example, the paper [31] was recently cited in Boney [6] which was already mentioned in Section 6. Boney cites the main theorem of [31] as Fact 3.12 in his paper and comments in allusion to Aki Kanamori's comment on Kunen's inconsistency proof in [29] that Vopěnka's Principle " 'rallies at least to force a veritable Götterdämmerung' for compactness cardinals for logics." The gap between Vopěnka's Principle and a huge cardinal Boney mentions in connection with this „götterdämmerigen" statement seems to have some resemblance to the discrepancy between usual Laver-generic large cardinal axioms and the super $C^{(\infty)}$-Laver-generic large cardinals.

Now one of the urgent items in my to-do-list is to check Janos's [31] as well as [32], [33], [34] more carefully to find out further possible connections of his results to the context I described above.

Is this also an instance of (the eternal?) recurrence?

Acknowledgements A part of the material of the chapter was presented at the RIMS set theory workshop 2023 in Kyoto. The author would like to thank Gunter Fuchs and Joan Bagaria for valuable comments at the talk.

Competing Interests The author's research was supported by Kakenhi Grant-in-Aid for Scientific Research (C) 20K03717.

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