

Reflection and Recurrence.

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Abstract We examine the Zermelo Fraenkel set theory with Choice (ZFC) enhanced by one of the (structural) reflection principles down to a small cardinal and/or Recurrence Axioms defined below. The strongest forms of reflection principles spotlight the three scenarios in which the size of the continuum is either \aleph_1 , or \aleph_2 , or very large, while the maximal setting of Recurrence Axioms points to the set-theoretic universe with the continuum of size \aleph_2 .

We discuss that both the Reflection Principles and Recurrence Axioms can be construed as preferable candidates of the extension of ZFC in terms of the criteria of Gödel's Program. From this view point, the maximal possible (consistent) combination of these principles and axioms, or even some natural strengthening of the combination (which we want to call "Laver-generic Maximum" (LGM)) may be considered as the ultimate extension of ZFC (of course "ultimate" only for now — because of the Incompleteness Theorems): LGM resolves the size of the continuum to be \aleph_2 and integrates practically all known statements consistent with ZFC in itself either as its consequences (as it is the case with Martin's Maximum⁺⁺) or as theorems holding in many grounds of the universe (as it is the case with Cichoń's Maximum).

アブストラクト 以下で、選択公理付きの Zermelo Fraenkel 集合論 (ZFC) を、小さな基数への (構造に関する) 反映原理 (Reflection Principles) たちのうちのひとつ、かつ/または、回帰公理 (Recurrence Axioms) たちのひとつで拡張した体系について考察する。反映原理のうちの一番強いものたちは、連続体のサイズが \aleph_1 であるか、 \aleph_2 であるか、非常に大き

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くなるかという3つのシナリオにスポットライトをあてるのに対し、Recurrence Axiomsの一番強いものたちの組合せは、連続体のサイズが \aleph_2 になる集合論の宇宙を指しめしているようである。

反映原理も回帰公理も、ゲーデルのプログラムの意味で、ZFCの望ましい拡張を与えるものの候補と看做せるので、これらの原理と公理の無矛盾なものうちの極大な組合せは、ZFCの究極の拡張と考えることができる — これを我々は“Laver-generic Maximum” (LGM) と呼びたい。もちろんこれは「究極の」、とは言っても不完全性定理のために、現在のところの究極でしかないものなのだが、LGMは、連続体問題を連続体濃度が \aleph_2 である、として解決し、現在までに知られている実質的に全てのZFCと無矛盾な命題を、(たとえば MM^{++} がそうであるように) この拡張から導かれる定理として、あるいは、(たとえばCichoń’s Maximumがそうであるように) 多くの、集合論宇宙のgroundsでの定理として、この理論に統合する。

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1 Introduction

In the following, we examine the Zermelo Fraenkel set theory with the Axiom of Choice (ZFC) enhanced by one of the (structural) reflection principles down to a small cardinal and/or Recurrence Axioms defined below. The strongest forms of reflection principles (existence of a/the \mathcal{P} -Laver-generic large cardinal — see Section 2 below) spotlight the three scenarios in which the size of the continuum is either \aleph_1 , or \aleph_2 , or very large (see Theorems 8, 9 and 10), while the maximal setting of Recurrence Axioms points to the set-theoretic universe with the continuum of size \aleph_2 (see the end of Section 3).

As we are going to discuss in sections 2 and 3, both of the Reflection Principles and Recurrence Axioms can be construed as preferable candidates of the extension of ZFC in terms of Gödel’s Program (Gödel [23], see also Bagaria [1]). From this point of view the maximal possible (consistent) combination of these principles and axioms, or even some natural strengthening of the combination, that is, either the principle LGM proposed in Section 6 or some further extension of it in the future (which we want to call “Laver-generic Maximum” (LGM), see p.30) may be considered as the ultimate extension of ZFC (of course “ultimate” only for now — because of the Incompleteness Theorems): LGM resolves the size of the continuum

to be \aleph_2 and integrates practically all known statements consistent with ZFC in itself either as its consequences (as it is the case with Martin's Maximum⁺⁺, see Theorem 5) or as theorems holding in many grounds of the universe (as it is the case with Cichoń's Maximum, Goldstern, Kellner and Shelah[24], Goldstern, Kellner, Mejía and Shelah [25], see around p.30) — for more detailed discussions about the significance of these axioms, see also the end of both of the Sections 2, and 6.

2 Reflection down to a small cardinal

The small cardinal mentioned in the title of this section may be considered as not very small by non-set-theoretic mathematicians: It is known that many “mathematical” reflection statements with reflection number less than or equal to $\kappa_{\text{refl}} := \max\{\aleph_2, 2^{\aleph_0}\}$ hold (often in some extension of ZFC). Some of them are even theorems in ZFC. For example,

Theorem 1 (1) (Dow [8]) *If X is a countably compact Hausdorff non-metrizable space then there is a subspace Y of X of cardinality $< \aleph_2$ such that Y is also non-metrizable.*

(2) *Let $L(Q)$ be a logic with new (first-order) quantifier such that “ $Qx \dots$ ” is interpreted as “there are uncountably many x such that ...”. For any structure \mathfrak{A} of countable signature, there is $\mathfrak{B} \prec_{L(Q)} \mathfrak{A}$ of size $< \aleph_2$.* \square

From very early on, it was known that, starting from a very large cardinal, we can construct models of set theory in which various strong statements on (structural) reflection down to $< \kappa_{\text{refl}}$ hold.

For example, Ben-David [5] in 1978 mentions a theorem by Shelah which states:

Theorem 2 (S. Shelah, [5]) *Suppose that κ is supercompact and $\mathbb{P} = \text{Col}(\aleph_1, \kappa)$. Then, for (V, \mathbb{P}) -generic \mathbb{G} , we have*

$V[\mathbb{G}] \models$ *for any structure \mathfrak{A} of countable signature, there is $\mathfrak{B} \prec_{L_{\text{stat}}} \mathfrak{A}$ of cardinality $< \aleph_2$.*

Here, L_{stat} denotes the stationary logic with monadic second-order variables X which run over countable subsets of the underlying set of respective structures and with the second-order quantifier $\text{stat } X$ which is to be interpreted as “there are stationarily many countable sets X ”.

The elementary submodel relation $\mathfrak{B} \prec_{L_{\text{stat}}} \mathfrak{A}$ is defined by $\mathfrak{B} \models \varphi(b_0, \dots) \Leftrightarrow \mathfrak{A} \models \varphi(b_0, \dots)$ for all L_{stat} -formula $\varphi = \varphi(x, \dots)$ without free second-order variables, and for all $b_0 \dots \in |\mathfrak{B}|$.

Today, we can understand Shelah's theorem above as a special case of the following theorem. For a class \mathcal{P} of posets, a cardinal κ is said to be **\mathcal{P} -generically supercompact**

if, for any $\lambda > \kappa$, there is a poset $\mathbb{P} \in \mathcal{P}$ such that, for (V, \mathbb{P}) -generic \mathbb{G} , there are $j, M \subseteq V[\mathbb{G}]$ such that $j : V \xrightarrow{\kappa} M$ ¹, $j(\kappa) > \lambda$, and $j''\lambda \in M$.

Theorem 3 *Suppose that \aleph_2 is \mathcal{P} -generically supercompact where \mathcal{P} is the class of all $< \aleph_1$ -closed posets. Then*

- (*1) *for any structure \mathfrak{A} of countable signature, there is $\mathfrak{B} \prec_{L_{stat}} \mathfrak{A}$ of cardinality $< \aleph_2$.*

Proof. The condition “ \aleph_2 is \mathcal{P} -generic supercompact for $\mathcal{P} =$ the class of all $< \aleph_1$ -closed posets.” is equivalent to the Game Reflection Principle (GRP) (Theorem 8 in König [31] — see Fuchino, Ottenbreit Maschio Rodrigues and Sakai [15] for a generalization of König’s theorem in [31] — note that what we call GRP here and [15] is called “the global Game Reflection Principle” in [31]). By Theorem 4.7 in [15], GRP implies (*1). □ (Theorem 3)

The downward Löwenheim-Skolem Theorem (*1) for L_{stat} is actually a strong reflection property. For example the reflection of uncountable coloring number of graphs down to $< \aleph_2$ (the following (*2)) is a consequence of (*1):

- (*2) For any graph G of uncountable coloring number, there is a subgraph H of G of size \aleph_1 with uncountable coloring number.

This implication can be proved directly but we can also see this using the terminology introduced in [15] as follows: The downward Löwenheim-Skolem Theorem (*1) for L_{stat} is equivalent to the Diagonal Reflection Principle $\text{DRP}(\text{IC}_{\aleph_0})$ down to an internally club set (Corollary 3.6 in [15]). This implies the reflection principle $\text{RP}_{\text{IU}_{\aleph_0}}$ down to an internally unbounded set of size $< \aleph_2$. This reflection principle is equivalent to Axiom R of Fleissner (Lemma 2.6 in [20]). From Axiom R, the Fodor-type Reflection Principle (FRP) follows (Corollary 2.6 in [13]). (*2) is a consequence of (actually equivalent to FRP over ZFC ([19])).

As it is mentioned in the proof of Theorem 3, the condition “ \aleph_2 is \mathcal{P} -generic supercompact for $\mathcal{P} =$ the class of all $< \aleph_1$ -closed posets.” is equivalent to Game Reflection Principle (GRP). As the name suggests, GRP is actually a reflection principle which claims the reflection of the non-existence of winning strategy of certain games of length ω_1 down to subgames of size $< \aleph_2$. A remarkable feature of this principle is that it implies CH (Theorem 8 in König [31]).

The notion of Laver-generic large cardinals was introduced in Fuchino, Ottenbreit Maschio Rodrigues and Sakai [16] in search for reflection principles which generalize GRP. The following definition of Laver-generic large cardinals is a streamlined version adopted in later papers Fuchino and Ottenbreit Maschio Rodrigues [14], Fuchino [12] etc. and slightly different from the one given in [16].

We call a non-empty class \mathcal{P} of posets *iterable* if it satisfies:

- ① $\{1\} \in \mathcal{P}$, ② \mathcal{P} is closed with respect to forcing equivalence
(i.e. if $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P} \sim \mathbb{P}'$ then $\mathbb{P}' \in \mathcal{P}$),

¹ With “ $j : V \xrightarrow{\kappa} M$ ” we denote the situation that M is transitive, j is an elementary embedding of V into M , and κ is the critical point of j .

- ③ closed with respect to restriction
(i.e. if $\mathbb{P} \in \mathcal{P}$ then $\mathbb{P} \restriction \mathbb{P} \in \mathcal{P}$ for any $\mathbb{P} \in \mathbb{P}$), and,
- ④ for any $\mathbb{P} \in \mathcal{P}$ and \mathbb{P} -name \mathbb{Q} , $\Vdash_{\mathbb{P}} \text{"}\mathbb{Q} \in \mathcal{P}\text{"}$ implies $\mathbb{P} * \mathbb{Q} \in \mathcal{P}$.

For an iterable class \mathcal{P} of posets, a cardinal κ is said to be *\mathcal{P} -Laver-generically supercompact* if, for any $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is a \mathbb{P} -name \mathbb{Q} with $\Vdash_{\mathbb{P}} \text{"}\mathbb{Q} \in \mathcal{P}\text{"}$, such that for $(V, \mathbb{P} * \mathbb{Q})$ -generic \mathbb{H} , there are $j, M \subseteq V[\mathbb{H}]$ such that $j : V \xrightarrow{\lambda}_{\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{P}, \mathbb{H}, j''\lambda \in M$.

κ is *tightly \mathcal{P} -Laver-generically supercompact* if it is \mathcal{P} -Laver-generically supercompact and \mathbb{Q} , j and M for each $\mathbb{P} \in \mathcal{P}$ in the definition of \mathcal{P} -Laver-generic supercompactness additionally satisfy that $\mathbb{P} * \mathbb{Q}$ is forcing equivalent to a poset of size $\leq j(\kappa)$.²

A cardinal κ is *(tightly) \mathcal{P} -Laver-generically superhuge*, if κ satisfies the conditions of (tightly) \mathcal{P} -Laver-generically supercompactness, with the condition $j''\lambda \in M$ replaced by $j''j(\kappa) \in M$. Clearly a (tightly) \mathcal{P} -Laver-generically superhuge cardinal is (tightly) \mathcal{P} -Laver-generically supercompact.

The name “Laver-generic large cardinal” is chosen in connection with fact that Laver-function plays central role in the construction of standard models with Laver-generic large cardinals (see Theorem 9).

Laver-genericity corresponding to other notions of large cardinals can be defined canonically: A cardinal κ is *\mathcal{P} -Laver-generically ultrahuge*, if it enjoys the definition of \mathcal{P} -Laver-generically supercompactness and that the condition “ $j''\lambda \in M$ ” in the definition of supercompactness is replaced by the stronger “ $V_{j(\lambda)}^{V[\mathbb{H}]} \in M$ ”.

That is:

κ is *(tightly) \mathcal{P} -Laver-generically ultrahuge*, if for any $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is a \mathbb{P} -name \mathbb{Q} with $\Vdash_{\mathbb{P}} \text{"}\mathbb{Q} \in \mathcal{P}\text{"}$, such that for $(V, \mathbb{P} * \mathbb{Q})$ -generic \mathbb{H} , there are $j, M \subseteq V[\mathbb{H}]$ such that $j : V \xrightarrow{\lambda}_{\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{P}, \mathbb{H}, V_{j(\lambda)}^{V[\mathbb{H}]} \in M$, (and $|\mathbb{P} * \mathbb{Q}| \leq j(\kappa)$).

κ is *\mathcal{P} -Laver-generically huge* if κ satisfies the definition obtained by replacing “ $j''\lambda \in M$ ” in the definition of \mathcal{P} -Laver-generically supercompactness by “ $j''j(\lambda) \in M$ ”.

That is:

κ is *(tightly) \mathcal{P} -Laver-generically huge*, if for any $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is a \mathbb{P} -name \mathbb{Q} with $\Vdash_{\mathbb{P}} \text{"}\mathbb{Q} \in \mathcal{P}\text{"}$, such that for $(V, \mathbb{P} * \mathbb{Q})$ -generic \mathbb{H} , there are $j, M \subseteq V[\mathbb{H}]$ such that $j : V \xrightarrow{\lambda}_{\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{P}, \mathbb{H}, j''j(\lambda) \in M$, (and $|\mathbb{P} * \mathbb{Q}| \leq j(\kappa)$).

The following implications follow from the definitions:

² In the following, we shall denote this condition simply by “ $|\mathbb{P} * \mathbb{Q}| \leq \lambda$ ”. More generally, we shall simply write “ $|\mathbb{P}| \leq \lambda$ ” for a poset \mathbb{P} to say that “the poset \mathbb{P} is forcing equivalent to a poset of size $\leq \lambda$ ”.

$$\begin{array}{c}
\kappa \text{ is (tightly) } \mathcal{P}\text{-Laver-generically ultrahuge} \\
\Downarrow \\
\kappa \text{ is (tightly) } \mathcal{P}\text{-Laver-generically superhuge} \\
\Downarrow \\
\kappa \text{ is (tightly) } \mathcal{P}\text{-Laver-generically supercompact}
\end{array}$$

The relationship between Laver-generic hyperhugeness and Laver-generic ultrahugeness is slightly more subtle and at the moment, we need the tightness to obtain the expected implication:

Lemma 4. (Fuchino and Usuba [21]) *For any class \mathcal{P} of posets, if κ is tightly \mathcal{P} -Laver-generically hyperhuge then κ is tightly \mathcal{P} -Laver-generically ultrahuge.*

Proof. Suppose that κ is tightly \mathcal{P} -Laver-generically hyperhuge, and $\lambda > \kappa$. Without loss of generality, we may assume that

$$V_\lambda \prec_{\Sigma_n} V \text{ for sufficiently large } n. \quad (*3)$$

Let $\lambda^* := (|V_\lambda|^+)^V$. For $\mathbb{P} \in \mathcal{P}$, let \mathbb{Q} be a \mathbb{P} -name such that $\Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{P}\text{”}$ and, for $(V, \mathbb{P} * \mathbb{Q})$ -generic \mathbb{H} , there are $j, M \subseteq V[\mathbb{H}]$ such that

$$j : V \xrightarrow{\prec}_\kappa M, \quad j(\kappa) \geq \lambda^*, \quad (*4)$$

$$j'' j(\lambda^*), \mathbb{P}, \mathbb{H} \in M, \text{ and} \quad (*5)$$

$$|\mathbb{P} * \mathbb{Q}| \leq j(\kappa). \quad (*6)$$

Claim 4.1. *For $\alpha \leq j(\lambda)$, $V_\alpha^V \in M$.*

\vdash By induction on $\alpha \leq j(\lambda)$, we prove

$$V_\alpha^V \in M \text{ and } V_\alpha^V \subseteq V_\alpha^M. \quad (*7)$$

For $\alpha < \omega$, this is clear. Suppose we have $V_\alpha^V \in M$ and $V_\alpha^V \subseteq V_\alpha^M$. Then, since $M \models |V_\alpha^M| < j(\lambda^*)$ (by the choice of λ^* and) by elementarity, $\mathcal{P}^V(V_\alpha^V) \subseteq \mathcal{P}^M(V_\alpha^M) \subseteq M$ by (*5) and Lemma 2.5, (5) in [16]. Again by Lemma 2.5, (5) in [16], it follows that $V_{\alpha+1}^V = \mathcal{P}^V(V_\alpha^V) \in M$, and $V_{\alpha+1}^V = \mathcal{P}^V(V_\alpha^V) \subseteq \mathcal{P}^M(V_\alpha^M) = V_{\alpha+1}^M$.

If $\gamma \leq j(\lambda)$ is a limit, and $V_\alpha \in M$, $M \models V_\alpha^V \subseteq V_\alpha$ for all $\alpha < \gamma$, then $\langle V_\alpha^V : \alpha < \gamma \rangle \subseteq M$. Hence by Lemma 2.5, (5) in [16], it follows that $\langle V_\alpha^V : \alpha < \gamma \rangle \in M$. Thus $V_\gamma^V = \bigcup_{\alpha < \gamma} V_\alpha^V \in M$ and $V_\gamma^V = \bigcup_{\alpha < \gamma} V_\alpha^V \subseteq \bigcup_{\alpha < \gamma} V_\alpha^M = V_\gamma^M$. \dashv (Claim 4.1)

Now, it follows that

$$\begin{array}{c}
\text{by } (*3) \text{ and Lemma 13} \\
M \ni \underbrace{V_{j(\lambda)}[\mathbb{H}]}_{\text{by Claim 4.1 and } (*5)} = \overbrace{V_{j(\lambda)}^V[\mathbb{H}]} \\
\text{by Claim 4.1 and } (*5)
\end{array}$$

This shows that j and M taken here are as in the definition of \mathcal{P} -Laver-generically ultrahugeness. \square (Lemma 4)

The consistency strength of Laver-generic hyperhugeness can be separated from that of other notions of Laver-generic large cardinals. This is because we know that the existence of \mathcal{P} -Laver-generic hyperhuge cardinal for any \mathcal{P} (if it ever exists) is equiconsistent with that of a hyperhuge cardinal (see Theorem 33).

At first glance, it is not immediately clear if the notion of Laver-generic large cardinal is definable in the language \mathcal{L}_\in of ZFC. In [18] an abstract generic version of extender is introduced to show the definability of Laver-generic large cardinals.

Laver-generic supercompactness implies double plus versions of forcing axioms. For a class \mathcal{P} of posets and cardinals κ, μ , we denote with $\text{MA}^{+\mu}(\mathcal{P}, < \kappa)$ and $\text{MA}^{++\leq \mu}(\mathcal{P}, < \kappa)$ the following versions of Martin's Axiom:

$\text{MA}^{+\mu}(\mathcal{P}, < \kappa)$: For any $\mathbb{P} \in \mathcal{P}$, any family \mathcal{D} of dense subsets of \mathbb{P} with $|\mathcal{D}| < \kappa$ and any family \mathcal{S} of \mathbb{P} -names such that $|\mathcal{S}| \leq \mu$ and $\Vdash_{\mathbb{P}} \text{“}\dot{S} \text{ is a stationary subset of } \omega_1 \text{”}$ for all $\dot{S} \in \mathcal{S}$, there is a \mathcal{D} -generic filter \mathbb{G} over \mathbb{P} such that $\dot{S}[\mathbb{G}]$ is a stationary subset of ω_1 for all $\dot{S} \in \mathcal{S}$.

$\text{MA}^{++\leq \mu}(\mathcal{P}, < \kappa)$: For any $\mathbb{P} \in \mathcal{P}$, any family \mathcal{D} of dense subsets of \mathbb{P} with $|\mathcal{D}| < \kappa$ and any family \mathcal{S} of \mathbb{P} -names such that $|\mathcal{S}| \leq \mu$ and $\Vdash_{\mathbb{P}} \text{“}\dot{S} \text{ is a stationary subset of } \mathcal{P}_{\eta_{\dot{S}}}(\theta_{\dot{S}}) \text{”}$ for some $\omega < \eta_{\dot{S}} \leq \theta_{\dot{S}} \leq \mu$ with $\eta_{\dot{S}}$ regular, for all $\dot{S} \in \mathcal{S}$, there is a \mathcal{D} -generic filter \mathbb{G} over \mathbb{P} such that $\dot{S}[\mathbb{G}]$ is stationary in $\mathcal{P}_{\eta_{\dot{S}}}(\theta_{\dot{S}})$ for all $\dot{S} \in \mathcal{S}$.

Clearly $\text{MA}^{++< \omega_2}(\mathcal{P}, < \kappa)$ is equivalent to $\text{MA}^{+\omega_1}(\mathcal{P}, < \kappa)$.
 MM^{++} is $\text{MA}^{+\omega_1}$ (stationary preserving posets, $< \aleph_2$).

Theorem 5 (Theorem 5.7 in Fuchino, Ottenbreit Maschio Rodrigues and Sakai [16], see also Fuchino [12]) *For an iterable class \mathcal{P} of posets such that*

($\ast 8$) *the elements of \mathcal{P} preserve stationarity of subsets of $\mathcal{P}_\mu(\theta)$ for all $\mu \leq \theta < \kappa$, if $\kappa > \aleph_1$ is \mathcal{P} -Laver-generically supercompact then $\text{MA}^{++\leq \mu}(\mathcal{P}, < \kappa)$ holds for all $\mu < \kappa$. \square*

In contrast to usual generic large cardinals, a Laver-generic large cardinal if it exists, is unique and it is κ_{refl} in many cases.

Lemma 6. ([12], [21], see also Proposition 4, in [11]) (1) *If κ is \mathcal{P} -generically measurable for an ω_1 preserving iterable \mathcal{P} , then $\omega_1 < \kappa$.*

(2) *If κ is \mathcal{P} -Laver-generically supercompact for an ω_1 -preserving iterable \mathcal{P} with $\text{Col}(\omega_1, \{\omega_2\}) \in \mathcal{P}$ then $\kappa = \omega_2$.*

(3) *If κ is \mathcal{P} -Laver-generically supercompact for an iterable \mathcal{P} which contains a poset adding a new real, then $\kappa \leq 2^{\aleph_0}$.*

(4) *If κ is \mathcal{P} -generically supercompact for an iterable \mathcal{P} such that all posets in \mathcal{P} do not add any reals then $2^{\aleph_0} < \kappa$.*

(5) If κ is \mathcal{P} -Laver-generically supercompact for an iterable \mathcal{P} which contains a poset which collapses \aleph_1 then $\kappa = \aleph_1$.

Proof. We only prove (5) since it is not explicitly given in [21]. Suppose that κ is \mathcal{P} -Laver-generically supercompact and $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}} \text{“}\aleph_1^V \text{ is countable”}$. If $\kappa \neq \aleph_1$, then we have $\aleph_1 < \kappa$. Let \mathbb{Q} be a \mathbb{P} -name of a poset such that, for $(V, \mathbb{P} * \mathbb{Q})$ -generic \mathbb{H} , there are $j, M \subseteq V[\mathbb{H}]$ such that $j : V \xrightarrow{\sim}_{\kappa} M$ and $\mathbb{P}, \mathbb{H} \in M$. By $\mathbb{H} \cap \mathbb{P} \in M$ and since $\aleph_1^V < \text{crit}(j)$, we have $M \models \text{“}\aleph_1^V = j(\aleph_1^V) \text{ is countable”}$. This is a contradiction to the elementarity of j . \square (Lemma 6)

A cardinal κ is called *greatly weakly Mahlo* if κ is weakly inaccessible and there exists a non-trivial $< \kappa$ -complete normal filter \mathcal{F} over κ such that $\{\mu < \kappa : \mu \text{ is a regular cardinal}\} \in \mathcal{F}$, and \mathcal{F} is closed with respect to the Mahlo operation $M\ell$ ³ where

$$S \mapsto M\ell(S) := \{\alpha \in S : \alpha \text{ has uncountable cofinality and } S \cap \alpha \text{ is stationary in } \alpha\} \quad ([17]).$$

Note that the Mahlo operation given above is slightly different from the one in [4].

If κ is greatly weakly Mahlo then it is hyper-weakly Mahlo (Proposition 3.4 in [17]).

The tightness of the Laver-genericity can be still strengthened as follows: a cardinal κ is *tightly⁺ \mathcal{P} -Laver-generically x -large*, for a notion “ x -large” of large cardinal (e.g. “supercompact”, “superhuge” etc.) if it satisfies the definition of tightly \mathcal{P} -Laver-generically x -large cardinal with the condition “ $|\mathbb{P} * \mathbb{Q}| \leq j(\kappa)$ ” in the definition being replaced by the condition “there is a complete Boolean algebra \mathbb{B} of size $j(\kappa)$ such that \mathbb{B}^+ is forcing equivalent to $\mathbb{P} * \mathbb{Q}$ ”.

Theorem 7 (1) (Theorem 3.5 in [17]) If κ is a $\{\mathbb{P}\}$ -generically measurable for a poset \mathbb{P} with the μ -cc for some $\mu < \kappa$, then κ is greatly weakly Mahlo.

(2) (Theorem 5.8 in Fuchino, Ottenbreit Maschio Rodrigues and Sakai [16]) If κ is tightly \mathcal{P} -Laver-generically superhuge for a class \mathcal{P} of ccc posets such that at least one element of \mathcal{P} adds a real, then $\kappa = 2^{\aleph_0}$.

(3) For an iterable class \mathcal{P} of posets, if κ is tightly⁺ \mathcal{P} -Laver-generically hyperhuge, then $2^{\aleph_0} \leq \kappa$. \square

We give a sketch of the proof of Theorem 7, (3) after Corollary 38.

In Fuchino and Usuba [21], it is proved as a Corollary of Theorem 32 and Theorem 33 that for $\kappa = \aleph_1$, the ‘+’ in “tightly⁺” in Theorem 7, (3) above can be dropped.

Theorem 8 (Fuchino, Ottenbreit Maschio Rodrigues and Sakai [16], Fuchino and Usuba [21] for (Δ) and (Δ')) (A) If \mathcal{P} is the class of all $< \aleph_1$ -closed posets, and κ is \mathcal{P} -Laver-generically supercompact, then $\kappa = \aleph_2$ and CH holds.

³ Closedness here means that for any $S \in \mathcal{F}$, we have $M\ell(S) \in \mathcal{F}$.

(B) If \mathcal{P} is either the class of all proper posets or the class of all semi-proper posets, and κ is \mathcal{P} -Laver-generically supercompact, then $\kappa = 2^{\aleph_0} = \aleph_2$.

(Γ) If \mathcal{P} is the class of all ccc posets, and κ is \mathcal{P} -Laver-generically supercompact, then κ is very large and $\kappa \leq 2^{\aleph_0}$.

(Γ') If \mathcal{P} is the class of all ccc posets, and κ is tightly \mathcal{P} -Laver-generically superhuge, then κ is very large and $\kappa = 2^{\aleph_0}$.

(Δ) If \mathcal{P} is the class of all posets, and κ is \mathcal{P} -Laver-generically supercompact, then $\kappa = \aleph_1$.

(Δ') If \mathcal{P} is the class of all posets, and κ is tightly⁺ \mathcal{P} -Laver-generically supercompact, then $\kappa = \aleph_1$.

Proof. (A): By Lemma 6, (2), (4). (B): By Lemma 6, (2), (3) and Theorem 5.

(Γ): By Lemma 6, (3), and Theorem 7, (1). (Γ'): By (Γ) and Theorem 7, (2).

(Δ): By Lemma 6, (5). (Δ'): By (Δ) and Theorem 7, (3). \square (Theorem 8)

The consistency of the existence of a \mathcal{P} -Laver-generic large cardinal can be proved under the existence of corresponding genuine large cardinal.

Theorem 9 (Theorem 5.2, [16]) (A) Suppose that κ is supercompact and $\mathbb{P} = \text{Col}(\aleph_1, \kappa)$, then, in $V[\mathbb{G}]$, for any (V, \mathbb{P}) -generic \mathbb{G} , $\aleph_2^{V[\mathbb{G}]} (= \kappa)$ is tightly \mathcal{P} -closed-Laver-generically supercompact for the class \mathcal{P} of all σ -closed posets (and CH holds).

(B) Suppose that κ is superhuge with a Laver function $f : \kappa \rightarrow V_\kappa$ for superhugeness. If \mathbb{P} is the CS-iteration for forcing PFA along with f , then, in $V[\mathbb{G}]$ for any (V, \mathbb{P}) -generic \mathbb{G} , $\aleph_2^{V[\mathbb{G}]} (= \kappa)$ is tightly⁺ \mathbb{P} -Laver-generically superhuge for the class \mathcal{P} of all proper posets (and $2^{\aleph_0} = \aleph_2$ holds).

(B') Suppose that κ is superhuge with a Laver function $f : \kappa \rightarrow V_\kappa$ for superhugeness. If \mathbb{P} is the RCS-iteration for forcing MM along with f , then, in $V[\mathbb{G}]$ for any (V, \mathbb{P}) -generic \mathbb{G} , $\aleph_2^{V[\mathbb{G}]} (= \kappa)$ is tightly⁺ \mathcal{P} -Laver-generically superhuge for the class \mathcal{P} of all semi-proper posets (and $2^{\aleph_0} = \aleph_2$ holds).

(Γ) Suppose that κ is supercompact with a Laver function $f : \kappa \rightarrow V_\kappa$ for supercompactness. If \mathbb{P} is a FS-iteration for forcing MA along with f , then, in $V[\mathbb{G}]$ for any (V, \mathbb{P}) -generic \mathbb{G} , $2^{\aleph_0} (= \kappa)$ is tightly⁺ \mathcal{P} -Laver-generically supercompact for the class \mathcal{P} of all ccc posets (and $\kappa = 2^{\aleph_0}$ while κ still is very large).

(Δ) Suppose that κ is supercompact with a Laver function $f : \kappa \rightarrow V_\kappa$ for supercompactness. If \mathbb{P} is a FS-iteration for forcing f where f is used to book-keep through all posets in V_κ , then in $V[\mathbb{G}]$ for any (V, \mathbb{P}) -generic \mathbb{G} , $2^{\aleph_0} (= \kappa)$ is tightly⁺ \mathcal{P} -Laver-generically supercompact for the class \mathcal{P} of all posets (and CH holds). \square

Theorem 9 above also holds for all other notions of large cardinal and corresponding Laver-generic version of generic large cardinal except (B) and (B') in which the supercompactness does not seem to be strong enough to show that the resulting generic extension in the proof satisfies the expected Laver-genericity.

In a sense, the cases treated in Theorem 9 are (almost) exhaustive. This can be seen in the following:

Theorem 10 *Suppose that \mathcal{P} is an iterable class of posets such that all $\mathbb{P} \in \mathcal{P}$ are ω_1 -preserving and \mathcal{P} contains a poset \mathbb{P}^* whose generic filter destroys a stationary subset of ω_1 .⁴ Then there is no \mathcal{P} -Laver-generically supercompact cardinal.*

Proof. Suppose, toward a contradiction, that \mathcal{P} is as above and there is \mathcal{P} -Laver-generically supercompact cardinal κ .

Let $S \subseteq \omega_1$ be stationary such that there is a poset $\mathbb{P}^* \in \mathcal{P}$ shooting a club in $\omega_1 \setminus S$. Let $\lambda > |\mathbb{P}^*|$ be large enough. By assumption, there is a \mathbb{P}^* -name $\dot{\mathbb{Q}}$ of a poset such that $\Vdash_{\mathbb{P}^*} \dot{\mathbb{Q}} \in \mathcal{P}$ and, for $(V, \mathbb{P}^* * \dot{\mathbb{Q}})$ -generic \mathbb{H} , there are j , $M \subseteq V[\mathbb{H}]$ such that

$$j : V \xrightarrow{\sim}_{\kappa} M, \quad (*)9$$

$$j(\kappa) > \lambda, \quad \text{and} \quad (*)10$$

$$j''\lambda, \mathbb{P}, \mathbb{H} \in M. \quad (*)11$$

By the choice of \mathbb{P}^* ($\leq \mathbb{P}^* * \dot{\mathbb{Q}}$), there is a nice \mathcal{P}^* -name $\dot{C} \in V$ of a club set $\subseteq \omega_1^V \setminus S$. By $(*)10$, $(*)11$ and by the choice of λ , we have $\dot{C} \in M$.

Thus $M \models$ “ S is a non-stationary subset of ω_1 ” by $(*)11$. Since $\text{crit}(j) = \kappa > \omega_1$ by Lemma 6, (1), we have $S = j(S)$. By $V \models$ “ S is stationary subset of ω_1 ”, this is a contradiction to the elementarity $(*)9$ of j . \square (Theorem 10)

\mathcal{P} -Laver genericity for stationary preserving \mathcal{P} , in particular those \mathcal{P} containing all σ -closed posets can be regarded as a strong reflection principle.

Theorem 11 *Suppose that \mathcal{P} is an iterable class of posets which are ω_1 -preserving and include all σ -closed posets. If κ is \mathcal{P} -Laver-generically supercompact then*

$$(*)1 \quad \text{for any structure } \mathfrak{A} \text{ of countable signature, there is } \mathfrak{B} \prec_{L_{\text{stat}}} \mathfrak{A} \text{ of cardinality } < \aleph_2.$$

Proof. By Theorem 5, $\text{MA}^{++}(\sigma\text{-closed})$ holds. Cox [7] proved that $\text{MA}^{++}(\sigma\text{-closed})$ implies $\text{DRP}(< \aleph_2, \text{IC}_{\aleph_0})$ (in the notation of [15]). By Lemma 3.5 in [15], this principle is equivalent to $(*)1$. \square (Theorem 11)

The existence of \mathcal{P} -Laver generic large cardinal for the class of all ccc-posets also implies a reflection statement similar to $(*)1$ (see Theorem 5.9, (3) in [16]). It is not known if the existence of \mathcal{P} -Laver generic large cardinal for the class of all ccc-posets imply FRP but it easy to see that we can force FRP and the existence of the Laver generic large cardinal starting from a model with a supercompact with the relevant large cardinal above it (Fuchino, Ottenbreit Maschio Rodrigues, and Sakai [16]).

In summary, Laver-generic large cardinal axiom (i.e. the axiom asserting the existence of a \mathcal{P} -Laver-generic large cardinal for some iterable class of posets \mathcal{P}) is

⁴ “ \mathbb{P}^* destroys a stationary subset of ω_1 ” means here that a \mathbb{P}^* -generic set codes a club subset of $\omega_1 \setminus S$ in some absolute way.

Note that, for stationary and co-stationary subset S of ω_1 , various posets are known which preserve ω_1 while shooting a club in $\omega_1 \setminus S$ (e.g. see [41]).

an assertion about the existence of a \mathcal{P} -generic large cardinal (without "Laver-" !) with the feature of resurrection property similar to the one in Resurrection Axioms studied by Hamkins and Johnstone [27], [28], and Tsaprounis [37]: for any generic extension by some $\mathbb{P} \in \mathcal{P}$ we find a further generic extension by a poset of the form $\mathbb{P} * \tilde{\mathbb{Q}} \in \mathcal{P}$ in which a previously chosen instance of generic elementary embedding corresponding to the generic largeness of the cardinal resurrects.

In spite of this similarity, it is only recent that we found there is a connection between these two similar types of axioms. First remember that, for each reasonable class \mathcal{P} consisting of proper or stationary preserving posets \mathcal{P} , \mathcal{P} -Laver generic large cardinal axiom implies the \mathcal{P} -Laver generic large cardinal is κ_{refl} (Theorem 9). In Fuchino [12] it is then proved that the tightly \mathcal{P} -Laver-generic superhuge axiom implies the boldface version Resurrection Axiom (in the sense of Hamkins and Johnstone) for parameters \mathcal{P} and $\mathcal{H}(\kappa_{\text{refl}})$ (Fuchino [12]). The tightly \mathcal{P} -Laver generic ultrahuge axiom implies the tight version of the Unbounded Resurrection Axiom of Tsaprounis for \mathcal{P} (Fuchino [12]).

\mathcal{P} -Laver generic large cardinal axioms for reasonable classes \mathcal{P} consisting of proper or stationary preserving posets imply double plus versions of Forcing Axiom (Theorem 5). By Theorem 3 (and its proof), Theorem 11 and the remarks after that most of the known structural reflection principles down to $< \kappa_{\text{refl}}$ are covered by these axioms.

\mathcal{P} -Laver generic large cardinal axioms also support the intuition shared by many set-theorists that the continuum is either \aleph_1 or \aleph_2 or very large (Theorem 8, Theorem 9 and Theorem 10). Perhaps we can also formulate this circumstance as: if we are to support one of the reasonable instances of Laver-generic large cardinal axiom then the continuum must be either \aleph_1 or \aleph_2 or extremely large ($\aleph_{2^{\aleph_0}}$ in particular, and much more).

Laver generic large cardinal axiom can be considered as a strong reflection principle with the feature of resurrection (à la Hamkins and Johnstone). Certain amount of absoluteness is also inherent in Laver generic large cardinal axiom in that it implies strong versions of forcing axioms (Theorem 5). In Section 5 we shall see that more generic absoluteness in terms of recurrence axioms (see the next section for these axioms) can be integrated into the Laver genericity by moving to the notions of large cardinals with the additional properties (in plural since they can be formulated only in an axiom scheme — or in some cases even not first-order formalizable) which we call "super $C^{(\infty)}$ " (Theorem 28) and that this is still realizable below the consistency strength of 2-huge (Theorem 29 and Theorem 30).

We can also feel more confident (or insecure in case you do not believe in large large cardinals⁵) with these stronger versions of Laver-generic large cardinal axiom since their consistency strength can be decided (see Corollaries 34, 35, 36, 37).

⁵ Note (not only) for the editor: "large large cardinals" is not a typo.

3 Recurrence, Maximality, and the solution(s) of the Continuum Problem

The Recurrence Axioms we are going to introduce below can be considered as reflection statements in set-generic multiverse down to set-generic geology. Their strengthenings can be also considered as a sort of absoluteness statements (see Proposition 16).

Adopting the terminology of set-theoretic geology (see e.g. Fuchs, Hamkins, and Reitz [9]), An inner model W of a universe U (in most of the cases U is the real universe V but sometimes it is some other universe obtained from V) is called a *ground of U* , if there is a poset $\mathbb{P} \in W$ and (W, \mathbb{P}) -generic $\mathbb{G} \in U$ such that $U = W[\mathbb{G}]$.

For a class \mathcal{P} of posets and a set A (of parameters), the *Recurrence Axiom for \mathcal{P} and A* ((\mathcal{P}, A) -RcA, for short⁶) is the following assertion formulated as an axiom scheme in the language \mathcal{L}_\in of set theory:

(\mathcal{P}, A) -RcA: For any \mathcal{L}_\in -formula $\varphi = \varphi(\bar{x})$ and $\bar{a} \in A$, if $\Vdash_{\mathbb{P}} \varphi(\bar{a}^\vee)$ for a $\mathbb{P} \in \mathcal{P}$, then there is a ground W of the universe V such that $\bar{a} \in W$ and $W \models \varphi(\bar{a})$ ⁷.

The term “Recurrence Axiom” is chosen in allusion to, but not necessarily in (full) agreement with, Nietzsche’s „ewige Wiederkehr des Gleichen“ (eternal recurrence of the same), or perhaps rather to Poincaré recurrence theorem: if we understand the relation “ N is (set) generic extension of M ” as the timeline in the set generic multiverse, we can interpret (\mathcal{P}, A) -RcA as saying that

if something (formulatable with parameters in A) happens in one of the near future universe (in terms of \mathcal{P}) then it is already happened in a not very far past universe (not very far, in the sense that the “present” is attainable from there by a set forcing).

The following is a natural strengthening of the Recurrence Axiom:

(\mathcal{P}, A) -RcA⁺: For any \mathcal{L}_\in -formula $\varphi = \varphi(\bar{x})$ and $\bar{a} \in A$, if $\Vdash_{\mathbb{P}} \varphi(\bar{a}^\vee)$ for a $\mathbb{P} \in \mathcal{P}$, then there is a \mathcal{P} -ground W of the universe V such that $\bar{a} \in W$ and $W \models \varphi(\bar{a})$.

Here an inner model W of V is called a *\mathcal{P} -ground* if there is a poset $\mathbb{P} \in W$ with $W \models \mathbb{P} \in \mathcal{P}$ and (W, \mathbb{P}) -generic $\mathbb{G} \in V$ such that $V = W[\mathbb{G}]$.

When we regard reflection principles including Laver-generic large cardinal axioms as natural axioms, the idea behind this should be that the continuum must be rich enough so that many aspects of the situations with sets of high cardinality should be reflected down to cardinality $< \kappa_{\text{refl}}$.

⁶ The notation “RcA” is chosen to avoid the collision with “RCA” which is used in Reverse Mathematics to denote “recursive comprehension axiom”.

⁷ For sets a_0, \dots, a_{n-1} and $\bar{a} := \langle a_0, \dots, a_{n-1} \rangle$, we simply write “ $\bar{a} \in A$ ” for “ $a_0, \dots, a_{n-1} \in A$ ”. For \mathbb{P} -check names (standard \mathbb{P} -names of the elements a_0, \dots, a_{n-1} in V) $\check{a}_0, \dots, \check{a}_{n-1}$ we write $\bar{a}^\vee := \langle \check{a}_0, \dots, \check{a}_{n-1} \rangle$.

Similarly we can consider Recurrence Axioms and their strengthenings as natural requirements: since our set-theoretic universe V must be highly saturated, as many aspects as possible of the situations in set generic multiverse should be reflected down to the set generic geology at which we can look down from our universe V .

We shall use the following version of Laver-Woodin Theorem often without mention. It implies in particular that (\mathcal{P}, A) -RcA and (\mathcal{P}, A) -RcA⁺ are actually formalizable as axiom schemes in \mathcal{L}_\in .

Theorem 12 (Reitz [36], Fuchs-Hamkins-Reitz [9]) *There is an \mathcal{L}_\in -formula $\Phi(x, r)$ (without any parameters) such that the following is provable in ZFC:*

- (*12) *for all r , $\Phi(\cdot, r) := \{x : \Phi(x, r)\}$ is a ground in V ,*
- (*13) *for any ground W (of V), there is r such that $W = \Phi(\cdot, r)$, and*
- (*14) *if W is a ground of V and $V = W[G]$ where G is a (W, \mathbb{P}) -generic for $\mathbb{P} \in W$, then r such that $W = \Phi(\cdot, r)$ can be chosen as an element of $\mathcal{P}((|\mathbb{P}|^+)^V)$.*

We put together here some other basic facts which will be used in the following. The next lemma was actually already used in the proof of Lemma 4.

Lemma 13. (see Fuchino and Usuba [21] for a proof) *If α is a limit ordinal and V_α satisfies a large enough fragment of ZFC, then for any $\mathbb{P} \in V_\alpha$ and (V, \mathbb{P}) -generic G we have $V_\alpha[G] = V_\alpha^{V[G]}$.*

Proof. “ \subseteq ”: This inclusion holds without the condition on the fragment of ZFC. Also the condition “ $\mathbb{P} \in V_\alpha$ ” is irrelevant for this inclusion.

We show by induction on $\alpha \in \text{On}$ that $V_\alpha[G] \subseteq V_\alpha^{V[G]}$ holds for all $\alpha \in \text{On}$.

The induction steps for $\alpha = 0$ and limit ordinals α are trivial. So we assume that $V_\alpha[G] \subseteq V_\alpha^{V[G]}$ holds and show that the same inclusion holds for $\alpha + 1$. Suppose $a \in V_{\alpha+1}[G]$. Then $a = \dot{a}^G$ for a \mathbb{P} -name $\dot{a} \in V_{\alpha+1}$. Since $\dot{a} \subseteq V_\alpha$, each $\langle \dot{b}, \dot{p} \rangle \in \dot{a}$ is an element of V_α . By induction hypothesis, it follows that $\dot{b}^G \in V_\alpha^{V[G]}$. It follows that $\dot{a}^G \subseteq V_\alpha^{V[G]}$. Thus $a = \dot{a}^G \in V_{\alpha+1}^{V[G]}$.

“ \supseteq ”: Suppose that $a \in V_\alpha^{V[G]}$. Note that we can choose the “large enough fragment of ZFC” which should be satisfied in V_α such that (*) $V_\alpha^{V[G]}$ still satisfies a large enough fragment of ZFC, although the fragment may be different from the one V_α satisfies. In particular we find a cardinal $\lambda > |\mathbb{P}|$ in $V_\alpha^{V[G]}$ (and hence also in $V[G]$) such that $a \in \mathcal{H}(\lambda)^{V_\alpha^{V[G]}} \subseteq \mathcal{H}(\lambda)^{V[G]} \subseteq V_\alpha^{V[G]}$. Note that $\mathcal{H}(\lambda)^{V_\alpha^{V[G]}} = \{a : |\text{trcl}(a)| < \lambda\}^{V_\alpha^{V[G]}} \subseteq \{a : |\text{trcl}(a)| < \lambda\}^{V[G]} = \mathcal{H}(\lambda)^{V[G]}$.

Let $a^* \in \mathcal{H}(\lambda)^{V[G]}$ be a transitive set such that $a \in a^*$. Then a^* can be coded by a subset of λ . We can find the subset of λ in $V[G]$ and this subset has a nice \mathbb{P} -name which is an element of V_α^V since $\mathbb{P} \in V_\alpha$. This shows that $a \in V_\alpha[G]$. \square (Lemma 13)

Lemma 14. *For any $n \in \mathbb{N} \setminus 1$, there is a Σ_n -formula $\varphi_n = \varphi_n(\vec{x}, y)$ (for each finite sequence \vec{x} of variables⁸) such that, if U is a transitive models of large enough*

fragment of ZFC, then for any Σ_n formula $\psi = \psi(\bar{x})$ there is $p \in \mathcal{H}(\omega)$ such that $\mathbb{U} \models \psi(\bar{a})$ if and only if $\mathbb{U} \models \varphi_n(\bar{a}, p)$ for all $a \in \mathbb{U}$.

Proof. For $n = 1$, $\varphi_1(\bar{x}, y)$ can be chosen as a Σ_1 -formula saying

$$(\aleph_1) \quad \exists M (M \text{ is transitive, } \bar{x} \in M, M \models \text{“large enough fragment of ZFC”} \\ y \text{ is a code of a } \Sigma_1\text{-formula and } M \models \ulcorner y \urcorner(\bar{x})).$$

If $\varphi_n(\bar{x}, y)$ is defined for \bar{x} of various lengths we can define $\varphi_{n+1}(\bar{x})$ as $\exists x \neg \varphi_n(\bar{x}, x, y)$. □ (Lemma 14)

Lemma 15. (1) For any $n^* \in \mathbb{N}$ there is $n > n^*$ such that, if $V_\alpha \prec_{\Sigma_n} V$, then $V_\alpha[\mathbb{G}] \prec_{\Sigma_{n^*}} V[\mathbb{G}]$ for any $\mathbb{P} \in V_\alpha$ and (V, \mathbb{P}) -generic \mathbb{G} .

(2) For a natural number n , there is $n' > n$ such that, for any $\alpha \in \text{On}$, if $V_\alpha[\mathbb{G}] \prec_{\Sigma_{n'}} V[\mathbb{G}]$ for a poset $\mathbb{P} \in V_\alpha$ and (V, \mathbb{P}) -generic \mathbb{G} , then we have $V_\alpha \prec_{\Sigma_n} V$.

Proof. (1): Suppose that $n > n^*$ is sufficiently large, $V_\lambda \prec V$, $\bar{a} \in V_\lambda[\mathbb{G}]$, and $\varphi = \varphi(\bar{x})$ is a Σ_{n^*} -formula. There are \mathbb{P} -names $\bar{a} \in V_\lambda$ such that $\bar{a} = \bar{a}[\mathbb{G}]$.

If $V_\lambda[\mathbb{G}] \models \varphi(\bar{a})$, there is $\mathbb{P} \in \mathbb{G}$ such that $V_\lambda \models \mathbb{P} \Vdash \ulcorner \varphi(\bar{a}) \urcorner$. By the choice of n it follows that $V \models \mathbb{P} \Vdash \ulcorner \varphi(\bar{a}) \urcorner$. Thus $V[\mathbb{G}] \models \varphi(\bar{a})$.

The same argument also applies to $\neg \varphi$.

(2): We use the \mathcal{L}_ϵ -formula $\Phi(x, y)$ of Lemma 12. By assumption, there is $r \in V_\alpha[\mathbb{G}]$ such that

$$V_\alpha = \Phi(\cdot, r)^{V_\alpha[\mathbb{G}]} = \Phi(\cdot, r)^V \cap V_\alpha[\mathbb{G}] \subseteq \Phi(\cdot, r)^V.$$

For any Σ_n -formula $\varphi(\bar{x})$ and $\bar{a} \in \Phi(\cdot, r)^{V_\alpha[\mathbb{G}]}$. Since $\varphi^{\Phi(\cdot, r)}$ is a $\Sigma_{n'}$ -formula (by the choice of n'), we have

$$\begin{aligned} V_\alpha \models \varphi(\bar{a}) &\Leftrightarrow V_\alpha[\mathbb{G}] \models \varphi^{\Phi(\cdot, r)}(\bar{a}) \quad \xrightarrow{\text{by assumption}} \quad V[\mathbb{G}] \models \varphi^{\Phi(\cdot, r)}(\bar{a}) \\ &\Leftrightarrow V \models \varphi(\bar{a}). \end{aligned}$$

This shows that $V_\alpha \prec_{\Sigma_n} V$. □ (Lemma 15)

RcA and RcA^+ are actually (almost) identical with (certain variations of) already well-known axioms and principles.

For a class \mathcal{P} of posets, an \mathcal{L}_ϵ -formula $\varphi(\bar{a})$ with parameters $\bar{a} (\in V)$ is said to be a **\mathcal{P} -button** if there is $\mathbb{P} \in \mathcal{P}$ such that, for any \mathbb{P} -name \bar{a} of poset with $\mathbb{P} \Vdash \ulcorner \bar{a} \in \mathcal{P} \urcorner$, we have $\mathbb{P} \Vdash \ulcorner \varphi(\bar{a}) \urcorner$.

If $\varphi(\bar{a})$ is a \mathcal{P} -button then we call \mathbb{P} as above a **push of the button $\varphi(\bar{a})$** .

For a class \mathcal{P} of posets and a set A (of parameters), the **Maximality Principle for \mathcal{P} and A** (**MP(\mathcal{P}, A)**, for short) introduced in Hamkins [26] is the following assertion formulated in an axioms scheme in \mathcal{L}_ϵ :

MP(\mathcal{P}, A): For any \mathcal{L}_ϵ -formula $\varphi(\bar{x})$ and $\bar{a} \in A$, if $\varphi(\bar{a})$ is a \mathcal{P} -button then $\varphi(\bar{a})$ holds.

⁸ Similarly to the convention of some computer languages we consider here that we have distinct $\varphi_n(\bar{x}, y)$ for each length of sequence \bar{x} of variables.

Proposition 16. (Barton, Caicedo, Fuchs, Hamkins, Reitz, and Schindler [3]) Suppose that \mathcal{P} is an *iterable* class of posets and A a set (of parameters). (1) (\mathcal{P}, A) - RcA^+ is equivalent to $\text{MP}(\mathcal{P}, A)$.

(2) (\mathcal{P}, A) - RcA is equivalent to the following assertion:

(*15) For any \mathcal{L}_\in -formula $\varphi(\bar{x})$ and $\bar{a} \in A$, if $\varphi(\bar{a})$ is a \mathcal{P} -button then $\varphi(\bar{a})$ holds in a ground of \mathbb{V} .

Proof. (1): Suppose first that (\mathcal{P}, A) - RcA^+ holds. We show that $\text{MP}(\mathcal{P}, A)$ holds. Suppose that $\mathbb{P} \in \mathcal{P}$ is a push of the \mathcal{P} -button $\varphi(\bar{a})$. Let $\varphi'(\bar{x})$ be the formula expressing

$$\text{for any } \mathbb{Q} \in \mathcal{P}, \Vdash_{\mathbb{Q}} \varphi(\bar{x}^\vee) \text{ holds.} \quad (*16)$$

Then we have $\Vdash_{\mathbb{P}} \varphi'(\bar{a}^\vee)$. By (\mathcal{P}, A) - RcA^+ , there is a \mathcal{P} -ground M of \mathbb{V} such that $\bar{a} \in M$ and $M \models \varphi'(\bar{a})$ holds. By the definition (*16) of φ' , it follows that $\mathbb{V} \models \varphi(\bar{a})$ holds.

Now suppose that $\text{MP}(\mathcal{P}, A)$ holds and $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}} \varphi(\bar{a})$ for $\bar{a} \in A$.

Let φ' be a formula claiming that

$$\text{there is a } \mathcal{P}\text{-ground } N \text{ such that } \bar{x} \in N \text{ and } N \models \varphi(\bar{x}). \quad (*17)$$

Then $\varphi'(\bar{a})$ is a \mathcal{P} -button and \mathbb{P} is its push.

By $\text{MP}(\mathcal{P}, A)$, $\varphi'(\bar{a})$ holds in \mathbb{V} and hence there is a \mathcal{P} -ground M of \mathbb{V} such that $\bar{a} \in M$ and $M \models \varphi'(\bar{a})$. This shows that (\mathcal{P}, A) - RcA^+ holds.

(2): can be proved similarly to (1). Suppose first that (\mathcal{P}, A) - RcA holds. We show that (*15) holds. Suppose that $\mathbb{P} \in \mathcal{P}$ is a push of the \mathcal{P} -button $\varphi(\bar{a})$. Let $\varphi'(\bar{x})$ be the formula expressing

$$\text{for any } \mathbb{Q} \in \mathcal{P}, \Vdash_{\mathbb{Q}} \varphi(\bar{x}^\vee) \text{ holds.} \quad (*18)$$

Then we have $\Vdash_{\mathbb{P}} \varphi'(\bar{a}^\vee)$. By (\mathcal{P}, A) - RcA , there is a ground M of \mathbb{V} such that $\bar{a} \in M$ and $M \models \varphi'(\bar{a})$ holds. Since $\mathcal{P} \ni \{\mathbb{1}\}$, it follows that $M \models \varphi(\bar{a})$.

Now suppose that (*15) holds and $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}} \varphi(\bar{a}^\vee)$ for $\bar{a} \in A$.

Let φ'' be a formula asserting that

$$\text{there is a } \mathcal{P}\text{-ground } N \text{ such that } \bar{x} \in N \text{ and } N \models \varphi(\bar{x}). \quad (*19)$$

Then $\varphi''(\bar{a})$ is a \mathcal{P} -button and \mathbb{P} is its push. Thus, By (*15), $\varphi''(\bar{a})$ holds in a ground M of \mathbb{V} with $\bar{a} \in M$. By the definition (*19) of φ'' , there is a \mathcal{P} -ground N of M such that $\bar{a} \in N$ and $N \models \varphi(\bar{a})$. Since N is also a ground of \mathbb{V} , this shows that (\mathcal{P}, A) - RcA holds. \square (Theorem 16)

Recurrence Axioms are also related to the Inner Model Hypothesis introduced by Sy Friedman in [22]. *The Inner Model Hypothesis (IMH)* is the following assertion formulated in the language of second-order set theory (e.g. in the context of von Neumann-Bernays-Gödel set theory):

IMH : For any statement φ without parameters, if φ holds in an inner model of an inner extension of V then φ holds in an inner model of V .

Here we say a (not necessarily first-order definable) transitive class M an *inner model* of V if M is a model of ZF and $\text{On}^M = \text{On}^V$. In the perspective from such M , we call V an *inner extension* of M .

We call a set-forcing version of this principle *Inner Ground Hypothesis* (IGH):

For a (definable) class \mathcal{P} of posets and a set A (of parameters),

IGH(\mathcal{P}, A) : For any \mathcal{L}_ϵ -formula $\varphi = \varphi(\bar{x})$ and $\bar{a} \in A$, if $\mathbb{P} \in \mathcal{P}$ forces “there is a ground M with $\bar{a} \in M$ satisfying $\varphi(\bar{a})$ ”, then there is a ground W of V such that $\bar{a} \in W$ and $W \models \varphi(\bar{a})$.

Proposition 17. (Barton, Caicedo, Fuchs, Hamkins, Reitz, and Schindler [3]) *For a class \mathcal{P} of posets with $\{1\} \in \mathcal{P}$ and a set A (of parameters), (\mathcal{P}, A) -RcA holds if and only if IGH(\mathcal{P}, A) holds.*

Proof. Suppose that (\mathcal{P}, A) -RcA holds. Let $\varphi = \varphi(\bar{x})$ be an \mathcal{L}_ϵ -formula, $\bar{a} \in A$, and $\mathbb{P} \in \mathcal{P}$ be such that $\Vdash_{\mathbb{P}} \text{“}\varphi(\bar{a}') \text{ holds in a ground”}$.

Let $\varphi'(\bar{x})$ be the \mathcal{L}_ϵ -formula asserting that $\varphi(\bar{x})$ holds in a ground. Then $\Vdash_{\mathbb{P}} \text{“}\varphi'(\bar{a}') \text{”}$. By (\mathcal{P}, A) -RcA, it follows that there is a ground W of V such that $W \models \varphi'(\bar{a}')$. Since a ground of a ground is a ground, we conclude that there is a ground W_0 of V such that $\bar{a} \in W_0$ and $W_0 \models \varphi(\bar{a})$. This shows that IGH(\mathcal{P}, A) holds.

Suppose now that IGH(\mathcal{P}, A) holds. Assume that $\Vdash_{\mathbb{P}} \text{“}\varphi(\bar{a}') \text{”}$ for an \mathcal{L}_ϵ -formula $\varphi = \varphi(\bar{x})$, $\bar{a} \in A$, and $\mathbb{P} \in \mathcal{P}$. Then $\Vdash_{\mathbb{P}} \text{“}\varphi(\bar{a}') \text{ holds in a } \mathcal{P}\text{-ground (of the universe)”}$ since $\Vdash_{\mathbb{P}} \text{“}\{1\} \in \mathcal{P} \text{”}$. Thus, by IGH(\mathcal{P}, A), there is a ground W of V such that $W \models \varphi(\bar{a})$. □ (Proposition 17)

(\mathcal{P}, A) -RcA⁺ (\Leftrightarrow MP(\mathcal{P}, A) for an iterable \mathcal{P}) can be also characterized in terms of a strengthening of Inner Ground Hypothesis: For a (definable) class \mathcal{P} of posets and a set A (of parameters),

IGH⁺(\mathcal{P}, A) : For any \mathcal{L}_ϵ -formula $\varphi = \varphi(\bar{x})$ and $\bar{a} \in A$ if $\mathbb{P} \in \mathcal{P}$ forces “there is a \mathcal{P} -ground M with $\bar{a} \in M$ satisfying $\varphi(\bar{a})$ ”, then there is a \mathcal{P} -ground W of V such that $\bar{a} \in W$ and $W \models \varphi(\bar{a})$.

The following proposition can be proved similarly to Proposition 17.

Proposition 18. *For an iterable class \mathcal{P} of posets and a set A (of parameters), (\mathcal{P}, A) -RcA⁺ holds if and only if IGH⁺(\mathcal{P}, A) holds.*

Proof. Suppose that (\mathcal{P}, A) -RcA⁺ holds and assume that $\varphi = \varphi(\bar{x})$ is an \mathcal{L}_ϵ -formula, $\bar{a} \in A$, and $\mathbb{P} \in \mathcal{P}$ is such that

$$\Vdash_{\mathbb{P}} \text{“}\varphi(\bar{a}') \text{ holds in a } \mathcal{P}\text{-ground } M \text{ with } \bar{a} \in M \text{”}$$

Let $\varphi'(\bar{a})$ be the formula expressing “ $\varphi(\bar{x})$ holds in a \mathcal{P} -ground M with $\bar{a} \in M$ ”. Then \mathbb{P} is a push of the switch $\varphi'(\bar{a})$. Thus, by Proposition 16, (1), $\varphi'(\bar{a})$ holds in V . By definition of φ' , there is a \mathcal{P} -ground W of V such that $\bar{a} \in W$ and $W \models \varphi(\bar{a})$. This shows that IGH⁺(\mathcal{P}, A) holds.

Suppose now that $\text{IGH}^+(\mathcal{P}, A)$ holds, and assume that $\varphi = \varphi(\bar{x})$ is an \mathcal{L}_\in -formula, $\bar{a} \in A$ and $\Vdash_{\mathbb{P}} \varphi(\bar{a}^\vee)$ then (since $\{1\} \in \mathcal{P}$) $\Vdash_{\mathbb{P}} \varphi(\bar{a}^\vee)$ holds in a \mathcal{P} -ground M with $\bar{a} \in M$. By $\text{IGH}^+(\mathcal{P}, A)$, it follows that there is \mathcal{P} -ground W_0 of \mathcal{P} -ground of V such that $\bar{a} \in W_0$ and $W_0 \models \varphi(\bar{a})$. Since \mathcal{P} is iterable, W_0 is a \mathcal{P} -ground of V . This shows that $(\mathcal{P}, A)\text{-RcA}^+$ holds. \square (Proposition 18)

In spite of these characterizations and near characterizations, we want to keep the Recurrence Axioms as autarchic axioms. One of the reasons is that we want to retain the narration that is is a reflection principle with reflection from the set-theoretic multiverse to the geology; another reason is that we have the following monotonicity which does not hold e.g. for Maximality Principles.

Lemma 19. (Monotonicity of Recurrence Axioms) *For classes of posets \mathcal{P} , \mathcal{P}' and sets A , A' of parameters, if $\mathcal{P} \subseteq \mathcal{P}'$ and $A \subseteq A'$, then we have*

$$(\mathcal{P}', A')\text{-RcA} \Rightarrow (\mathcal{P}, A)\text{-RcA}.$$

\square

If we decide that the Recurrence Axioms are desirable extensions of the axioms of ZFC, then we should adopt the maximal instance of these axioms. (i.e. the one with maximal strength among the instances consistent with ZFC) By Lemma 19, this means we should try to take the instance of Recurrence Axioms with the maximal \mathcal{P} and A (with respect to inclusion) among the consistent ones.

Theorem 20 in the next section suggests that the following two as candidates of such maximal instances:

(E) $\text{ZFC} + (\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))\text{-RcA}$ for the class \mathcal{P} of all stationary preserving posets.

(Z) $\text{ZFC} + (Q, \mathcal{H}(2^{\aleph_0}))\text{-RcA}$ for the class Q of all posets.

The consistency of (Z) follows from the consistency of $\text{ZFC} +$ “there are stationarily many inaccessible cardinals” ([26]). The consistency of (E) follows from Lemma 29, Theorem 30, (B’), and Theorem 28.

The maximality of (E) and (Z) follows from Lemma 20, (2’) and (5’) respectively.

By Lemma 20, (4) and (5), (E) implies $2^{\aleph_0} = \aleph_2$, and (Z) implies CH. In particular, these two extensions of ZFC are not compatible. However, as we are going to discuss in Section 7, we can combine (E) with a reasonable weakening of (Z).

(Z⁺) $\text{ZFC} + (\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))\text{-RcA}^+ + (Q, \mathcal{H}(\omega_1)^{\bar{W}})\text{-RcA}^+$ where \mathcal{P} is the class of all proper posets, Q the class of all posets, and \bar{W} the bedrock⁹ which is also assumed here to exist.

(Z⁺) implies $2^{\aleph_0} = \aleph_2$ (see Lemma 20, (4) in the next section). This is what I meant when I wrote “the maximal setting of Recurrence Axioms points to the universe with the continuum of size \aleph_2 ” in the introduction. We shall further discuss about (Z⁺) in Section 6.

⁹ For the definition of the bedrock see Section 5.

4 Restricted Recurrence Axioms

The following restricted forms of Recurrence Axioms are enough to decide many interesting aspects including the cardinal arithmetic around the continuum.

For an iterable class \mathcal{P} of posets, a set A (of parameters), and a set Γ of \mathcal{L}_\in -formulas, **$(\mathcal{P}, A)_\Gamma$ -RcA**, for short) is the following assertion expressed as an axiom scheme in \mathcal{L}_\in :

$(\mathcal{P}, A)_\Gamma$ -RcA: For any $\varphi(\bar{x}) \in \Gamma$ and $\bar{a} \in A$, if $\Vdash_{\mathbb{P}} \varphi(\bar{a}^\vee)$, then there is a ground W of V such that $\bar{a} \in W$ and $W \models \varphi(\bar{a})$.

$(\mathcal{P}, A)_\Gamma$ -RcA⁺ corresponding to (\mathcal{P}, A) -RcA⁺ is defined similarly.

$(\mathcal{P}, A)_\Gamma$ -RcA⁺: For any $\varphi(\bar{x}) \in \Gamma$ and $\bar{a} \in A$, if $\Vdash_{\mathbb{P}} \varphi(\bar{a}^\vee)$, then there is a \mathcal{P} -ground W of V such that $\bar{a} \in W$ and $W \models \varphi(\bar{a})$.

Lemma 20. (Fuchino and Usuba [21]) Assume that \mathcal{P} is an *iterable* class of posets.

(1) If \mathcal{P} contains a poset which adds a real (over the universe), then $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_1}$ -RcA implies $\neg\text{CH}$.

(2) Suppose that \mathcal{P} contains a poset which forces \aleph_2^V to be equinumerous with \aleph_1^V . Then $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$ -RcA implies $2^{\aleph_0} \leq \aleph_2$.

(2') If \mathcal{P} contains a posets which forces \aleph_2^V to be equinumerous with \aleph_1^V , then $(\mathcal{P}, \mathcal{H}((\aleph_2)^+))_{\Sigma_1}$ -RcA does not hold.

(3) If $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_1}$ -RcA holds then all $\mathbb{P} \in \mathcal{P}$ preserve \aleph_1 and they are also stationary preserving.

(4) If \mathcal{P} contains a poset which adds a real as well as a poset which collapses \aleph_2^V , then $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_1}$ -RcA implies $2^{\aleph_0} = \aleph_2$.

(5) If \mathcal{P} contains a poset which collapses \aleph_1^V , then $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$ -RcA implies CH.

(5') If \mathcal{P} contains a poset which collapses \aleph_1^V then $(\mathcal{P}, \mathcal{H}((2^{\aleph_0})^+))_{\Sigma_1}$ -RcA does not hold.

Proof. (1): Assume that \mathcal{P} is an iterable class of posets containing a poset \mathbb{P} adding a real and $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_1}$ -RcA holds. If CH holds, then $\mathcal{P}(\omega)^V \in \mathcal{H}(\kappa_{\text{refl}})$. Hence

(\aleph_2) “ $\exists x (x \subseteq \omega \wedge x \notin \mathcal{P}(\omega)^V)$ ”

is a Σ_1 -formula with parameters from $\mathcal{H}(\kappa_{\text{refl}})$ and \mathbb{P} forces the formula in the forcing language corresponding to this formula: “ $\exists x (x \subseteq \check{\omega} \wedge x \notin (\mathcal{P}(\omega)^V)^\vee)$ ”.

By $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_1}$ -RcA, the formula (2) must hold in a ground. This is a contradiction.

(2): Assume that $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$ -RcA holds and $\mathbb{P} \in \mathcal{P}$ forces \aleph_2^V to be equinumerous with \aleph_1^V . If $2^{\aleph_0} > \aleph_2$ then $\aleph_1^V, \aleph_2^V \in \mathcal{H}(2^{\aleph_0})$. Letting $\psi(x, y)$ a Σ_1 -formula saying “ $\exists f (f \text{ is a surjection from } x \text{ to } y)$ ”, we have $\Vdash_{\mathbb{P}} \psi((\aleph_1^V)^\vee, (\aleph_2^V)^\vee)$.

By $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$ -RcA, the formula $\psi(\aleph_1^V, \aleph_2^V)$ must hold in a ground. This is a contradiction.

(2'): Assume that $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}} "|(\aleph_2^V)^\vee| = |(\aleph_1^V)^\vee|"$, and $(\mathcal{P}, \mathcal{H}(\aleph_2^+))_{\Sigma_1}$ -RcA holds. Then, since $\aleph_1, \aleph_2 \in \mathcal{H}(\aleph_2^+)$ and " $|x| = |y|$ " is Σ_1 , there is a ground W of V such that $W \models |\aleph_2^V| = |\aleph_1^V|$. This is a contradiction.

(3): Suppose that $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}} "\aleph_1^V$ is countable". Note that $\omega, \aleph_1 \in \mathcal{H}(\kappa_{\text{refl}})$. By $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_1}$ -RcA, it follows that there is a ground W of V such that $W \models "\aleph_1^V$ is countable". This is a contradiction.

Suppose now that $S \subseteq \omega_1$ is stationary and $\mathbb{P} \in \mathcal{P}$ destroys the stationarity of S . Note that $\omega_1, S \in \mathcal{H}(\aleph_2)$. Let $\varphi = \varphi(y, z)$ be the Σ_1 -formula

$\exists x (x \text{ is a club subset of the ordinal } y \text{ and } z \cap x = \emptyset)$.

Then we have $\Vdash_{\mathbb{P}} "\varphi(\omega_1, S)"$. By $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Sigma_1}$ -RcA, it follows that there is a ground $W \subseteq V$ such that $S \in W$ and $W \models \varphi(\omega_1, S)$. This is a contradiction to the stationarity of S .

(4): follows from (1), (2) and (3).

(5): If $\aleph_1 < 2^{\aleph_0}$, then $\aleph_1^V \in \mathcal{H}(2^{\aleph_0})$.

Let $\mathbb{P} \in \mathcal{P}$ be a poset collapsing \aleph_1^V . That is, $\Vdash_{\mathbb{P}} "\aleph_1^V$ is countable". Since " \dots is countable" is Σ_1 , there is a ground M such that $M \models "\aleph_1^V$ is countable" by $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$ -RcA. This is a contradiction.

(5'): Assume that $\mathbb{P} \in \mathcal{P}$ is such that $\Vdash_{\mathbb{P}} "\aleph_1^V$ is countable", and $(\mathcal{P}, \mathcal{H}((2^{\aleph_0})^+))$ -RcA holds. Since $\aleph_1 \in \mathcal{H}((2^{\aleph_0})^+)$, it follows that there is a ground W of V such that $W \models \aleph_1^V$ is countable. This is a contradiction. \square (Lemma 20)

Corollary A1. (E) (stationary preserving, $\mathcal{H}(\kappa_{\text{refl}})$)-RcA is maximal among Recurrence Axioms with similar pair of parameters, and it implies $2^{\aleph_0} = \aleph_2$.

(Z) (all posets, $\mathcal{H}(2^{\aleph_0})$)-RcA is maximal among Recurrence Axioms with similar pair of parameters, and it implies CH.

Proof. (E): By Lemma 20, (3), "stationary preserving" cannot be replaced by a larger class of posets. By Lemma 20, (4), (stationary preserving, $\mathcal{H}(2^{\aleph_0})$)-RcA implies $2^{\aleph_0} = \aleph_2$. Thus $\mathcal{H}(\kappa_{\text{refl}}) = \mathcal{H}(\aleph_2)$ in this case and, by Lemma 20, (2'), this cannot be replaced by $\mathcal{H}(\mathcal{H}(\aleph_3))$.

(Z): CH holds by Lemma 20, (5), and, by Lemma 20, (5'), $\mathcal{H}(2^{\aleph_0})$ cannot be replaced by $\mathcal{H}((2^{\aleph_0})^+)$. \square (Corollary A1)

Laver-genericity implies the plus version of Recurrence Axiom (\Leftrightarrow Maximality Principle) restricted to Σ_2 .

Theorem 21 Suppose that κ is *tightly \mathcal{P} -Laver-generically ultrahuge* for an *iterable class \mathcal{P} of posets*. Then $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_2}$ -RcA⁺ holds.

Proof. Assume that κ is tightly \mathcal{P} -Laver-generically ultrahuge for an iterable class \mathcal{P} of posets.

Suppose that $\varphi = \varphi(\bar{x})$ is Σ_2 formula (in \mathcal{L}_ϵ), $\bar{a} \in \mathcal{H}(\kappa)$, and $\mathbb{P} \in \mathcal{P}$ is such that

$$V \models \Vdash_{\mathbb{P}} "\varphi(\bar{a}^\vee)". \quad (*20)$$

Let $\lambda > \kappa$ be such that $\mathbb{P} \in V_\lambda$ and

$$V_\lambda \prec_{\Sigma_n} V \text{ for a sufficiently large } n. \quad (*21)$$

In particular, we may assume that we have chosen the n above so that a sufficiently large fragment of ZFC holds in V_λ (“sufficiently large” means here, in particular, in terms of Lemma 13 and that the argument at the end of this proof is possible).

Let \mathbb{Q} be a \mathbb{P} -name such that $\Vdash_{\mathbb{P}} \mathbb{Q} \in \mathcal{P}$, and for $(V, \mathbb{P} * \mathbb{Q})$ -generic \mathbb{H} , there are $j, M \subseteq V[\mathbb{H}]$ with

$$j : V \xrightarrow{\sim}_\kappa M, \quad (*22)$$

$$j(\kappa) > \lambda, \quad (*23)$$

$$\mathbb{P} * \mathbb{Q}, \mathbb{P}, \mathbb{H}, V_{j(\lambda)}^{V[\mathbb{H}]} \in M, \text{ and} \quad (*24)$$

$$|\mathbb{P} * \mathbb{Q}| \leq j(\kappa). \quad (*25)$$

By (*25), we may assume that the underlying set of $\mathbb{P} * \mathbb{Q}$ is $j(\kappa)$ and $\mathbb{P} * \mathbb{Q} \in V_{j(\lambda)}^V$.

Let $\mathbb{G} := \mathbb{H} \cap \mathbb{P}$. Note that $\mathbb{G} \in M$ by (*24) and we have

$$\begin{aligned} & \text{Since } V_{j(\lambda)}^M (= V_{j(\lambda)}^{V[\mathbb{H}]}) \text{ satisfies a sufficiently large fragment of ZFC} \\ & \text{by elementarity of } j, \text{ and hence the equality follows by Lemma 13} \\ & \underbrace{V_{j(\lambda)}^M}_{\text{by (*24)}} = V_{j(\lambda)}^{V[\mathbb{H}]} = \overbrace{V_{j(\lambda)}^V[\mathbb{H}]}^{V[\mathbb{H}]} = V_{j(\lambda)}^V[\mathbb{H}]. \end{aligned} \quad (*26)$$

Thus, by (*24) and by the definability of grounds, we have $V_{j(\lambda)}^V \in M$ and $V_{j(\lambda)}^V[\mathbb{G}] \in M$.

Claim 21.1. $V_{j(\lambda)}^V[\mathbb{G}] \models \varphi(\bar{a})$.

\vdash By Lemma 13, $V_\lambda^V[\mathbb{G}] = V_\lambda^{V[\mathbb{G}]}$, and $V_{j(\lambda)}^V[\mathbb{G}] = V_{j(\lambda)}^{V[\mathbb{G}]}$. By (*21), both $V_\lambda^V[\mathbb{G}]$ and $V_{j(\lambda)}^V[\mathbb{G}]$ satisfy large enough fragment of ZFC. Thus

$$V_\lambda^V[\mathbb{G}] \prec_{\Sigma_1} V_{j(\lambda)}^V[\mathbb{G}]. \quad (*27)$$

By (*20) and (*21), we have $V_\lambda^V[\mathbb{G}] \models \varphi(\bar{a})$. By (*27) and since φ is Σ_2 , it follows that $V_{j(\lambda)}^V[\mathbb{G}] \models \varphi(\bar{a})$. \dashv (Claim 21.1)

Thus we have

$$M \models \text{“there is a } \mathcal{P}\text{-ground } N \text{ of } V_{j(\lambda)} \text{ with } N \models \varphi(\bar{a})\text{”}. \quad (*28)$$

By the elementarity (*22), it follows that

$$V \models \text{“there is a } \mathcal{P}\text{-ground } N \text{ of } V_\lambda \text{ with } N \models \varphi(\bar{a})\text{”}. \quad (*29)$$

Now by (*21), it follows that there is a \mathcal{P} -ground W of V such that $W \models \varphi(\bar{a})$.

□ (Theorem 21)

5 Recurrence, Laver-generic large cardinal, and beyond

Laver-genericity does not imply full Recurrence Axiom (Theorem 26 and Corollary 27 below).

For an \mathcal{L}_\in -formula $\psi = \psi(\bar{x})$, a (large) cardinal κ is ψ -absolute if the formula ψ is absolute between V_κ and V (i.e. $(\forall \bar{x} \in V_\kappa)(\psi^{V_\kappa}(\bar{x}) \leftrightarrow \psi(\bar{x}))$ holds for $y = \kappa$).

Lemma 22. *For any $n \in \mathbb{N}$, there is an \mathcal{L}_\in -formula ψ_n^* such that, for any inaccessible κ , κ is ψ_n^* -absolute if and only if*

(*30) *for any ground W of V such that $V = W[\mathbb{G}]$ for a poset $\mathbb{P} \in V_\kappa^W$ and (W, \mathbb{P}) -generic \mathbb{G} ,¹⁰ we have that all Σ_n -formulas are absolute between V_κ^W and W .*

Proof. By Theorem 12 and Lemma 14.

□ (Lemma 22)

Lemma 23. *Let ψ_n^* be as in Lemma 22. ψ_n^* -absolute inaccessible cardinals are not resurrectable. That is, if a cardinal λ satisfies*

$$\Vdash_{\mathbb{P}} \text{“}\check{\lambda} \text{ is } \psi_n^*\text{-absolute inaccessible”} \quad (*31)$$

for some poset \mathbb{P} , then λ is really ψ_n^ -absolute inaccessible.*

Proof. (*31) implies that λ is inaccessible. By the definition (*30) of ψ_n^* , if ψ_n^* is absolute between $V_\lambda^{V[\mathbb{G}]}$ and $V[\mathbb{G}]$ for some (V, \mathbb{P}) -generic \mathbb{G} then, it is absolute between V_λ^V and V .

□ (Lemma 23)

Lemma 24. *Suppose that there are stationarily many inaccessible cardinals.¹¹ Then, for each $n \in \mathbb{N}$, there are stationarily many ψ_n^* -absolute inaccessible cardinals.*

Proof. For $n \in \mathbb{N}$ let $n^+ \geq n$ be such that ψ_n^* is Σ_{n^+} . For any club $C \subseteq \text{On}$, $C \cap C^{(n^+)} = \{\alpha \in C : V_\alpha \prec_{\Sigma_{n^+}} V\}$ is a club in On (Lévy-Montague Reflection Theorem), there is an inaccessible cardinal $\mu \in C \cap C^{(n^+)}$. By the choice of n^+ , such μ is a ψ_n^* -absolute inaccessible cardinal.

□ (Lemma 24)

Theorem 25 *Suppose that (\mathcal{P}, \emptyset) -RcA holds, where \mathcal{P} is a class of posets such that either (a) \mathcal{P} contains posets collapsing arbitrary large cardinals to a small cardinality (less than the first inaccessible if there are inaccessible at all), or (b) \mathcal{P} contains posets adding arbitrarily many reals.*

¹⁰ Note that this includes the case that $\mathbb{P} = \{1\}$ and $V = W$.

¹¹ “There are stationarily many inaccessible cardinals” is the statement formalizable in an axiom scheme claiming, for each \mathcal{L}_\in -formula $\varphi = \varphi(x)$, that “if $\varphi(x)$ defines a club subclass of On then there is an inaccessible μ with $\varphi(\mu)$ ”.

If there is a ψ_n^ -absolute inaccessible cardinal for some $n \in \mathbb{N}$, then there are cofinally many ψ_n^* -absolute inaccessible cardinals.*

Proof. Assume that (\mathcal{P}, \emptyset) -RcA holds for \mathcal{P} as above and there is a cardinal λ such that there are some ψ_n^* -absolute inaccessible cardinals but all of them are below λ .

Let \mathbb{P} be a poset which either collapses λ to small cardinality or add at least λ many reals. Then, by Lemma 23, we have $\Vdash_{\mathbb{P}}$ “there is no ψ_n^* -absolute inaccessible cardinal”.

By (\mathcal{P}, \emptyset) -RcA, it follows that there is a ground W of V such that $W \models$ “there is no ψ_n^* -absolute inaccessible cardinal”. Again by Lemma 23, this is a contradiction.

□ (Theorem 25)

Theorem 26 *Suppose that λ is an inaccessible cardinal, $\kappa < \lambda$ is such that $V_\lambda \models$ “ κ is x -large cardinal”, where “ x -large cardinal” is a notion of large cardinal, for which a Laver function exists. Assume also that $\{\mu < \lambda : \mu \text{ is inaccessible}\}$ is stationary in λ .*

Then, for each of the classes \mathcal{P} of posets considered in Theorem 9, there are λ_0 with $\lambda > \lambda_0 > \kappa$, and $\mathbb{P} \in \mathcal{P}$ with $\mathbb{P} \subseteq V_\kappa$ such that, for a (V_λ, \mathbb{P}) -generic \mathbb{G} , we have

$$V_{\lambda_0}[\mathbb{G}] \models \text{“}\kappa \text{ is a tightly}^+ \mathcal{P}\text{-Laver-generically } x\text{-large cardinal and } \neg(\mathcal{P}, \emptyset)\text{-RcA”}.$$

Proof. Suppose that \mathcal{P} is one of the classes of posets considered in Theorem 9. Note that then, (a) or (b) of Theorem 25 holds. Let $n \in \mathbb{N}$ be such that the formula “ κ is an x -large cardinal” is Σ_n . By the assumption, there is an inaccessible cardinal μ with $\lambda > \mu > \kappa$ such that $V_\lambda \succ V_\mu$. Let $\lambda > \lambda_0 > \mu$ be the minimal cardinal such that $V_\lambda \models \lambda_0$ is ψ_n^* -absolute inaccessible cardinal — such λ_0 exists by Lemma 24. Then we have

$$V_{\lambda_0} \models \text{“}\kappa \text{ is an } x\text{-large cardinal”} \quad (*32)$$

In V_{λ_0} , let \mathbb{P} be the limit of κ -iteration with appropriate support as described in Theorem 9 which forces that κ is tightly⁺ \mathcal{P} -Laver generically x -large cardinal in the generic extension of V_{λ_0} . Let \mathbb{G} be (V_λ, \mathbb{P}) -generic filter. Then we have $V_{\lambda_0}[\mathbb{G}] \models$ “ κ is a tightly⁺ \mathcal{P} -Laver generically x -large cardinal” and $V_\lambda[\mathbb{G}] \succ V_\mu[\mathbb{G}]$ by Lemma 15, (1). In particular, by Lemma 23, we have $V_{\lambda_0}[\mathbb{G}] \models \mu$ is the largest ψ_n^* -absolute inaccessible cardinal. By Theorem 25, it follows that $V_{\lambda_0}[\mathbb{G}] \models \neg(\mathcal{P}, \emptyset)\text{-RcA}$.

□ (Theorem 26)

The conditions of Theorem 26 are satisfied by practically all large cardinal notions. For example, under the consistency of the existence of a 2-huge cardinal, the conditions of Theorem 26 are satisfied by x -large cardinal = hyperhuge cardinal (see Lemma 29 below). Thus we obtain the following:

Corollary 27. *Under the assumption of the consistency of the existence of a 2-huge cardinal, the existence of a tightly⁺ \mathcal{P} -Laver generically hyperhuge cardinal does not imply (\mathcal{P}, \emptyset) -RcA for any class \mathcal{P} of posets as in Theorem 9.* □

A natural strengthening of Laver-genericity does imply the full Maximality Principle (hence also the full Recurrence Axiom). As the proof of Theorem 25 suggests, such property must be formulated not in a single formula but as an axiom scheme.

For a natural number n , we call a cardinal

κ *super $C^{(n)}$ -hyperhuge* if for any $\lambda_0 > \kappa$ there are $\lambda \geq \lambda_0$ with $V_\lambda \prec_{\Sigma_n} V$, and j , $M \subseteq V$ such that $j : V \xrightarrow{\prec}_\kappa M$, $j(\kappa) > \lambda$, $j^{(\lambda)} M \subseteq M$ and $V_{j(\lambda)} \prec_{\Sigma_n} V$.

κ is *super $C^{(n)}$ -ultrahuge* if the condition above holds with “ $j^{(\lambda)} M \subseteq M$ ” replaced by “ $j^{(\kappa)} M \subseteq M$ and $V_{j(\lambda)} \subseteq M$ ”.

If κ is super $C^{(n)}$ -hyperhuge then it is super $C^{(n)}$ -ultrahuge. This can be shown similarly to Lemma 4.

We shall also say that κ is *super $C^{(\infty)}$ -hyperhuge* (*super $C^{(\infty)}$ -ultrahuge*, resp.) if it is super $C^{(n)}$ -hyperhuge (super $C^{(n)}$ -ultrahuge, resp.) for all natural number n .

A similar kind of strengthening of the notions of large cardinals which we call here “super $C^{(n)}$ ” appears also in Boney [6]. It is called in [6] “ $C^{(n)+}$ ” and the notion is considered there in connection with extendibility.

For a natural number n and an iterable class \mathcal{P} of posets, a cardinal

κ is *super $C^{(n)}$ - \mathcal{P} -Laver-generically ultrahuge* if, for any $\lambda_0 > \kappa$ and for any $\mathbb{P} \in \mathcal{P}$, there are a $\lambda \geq \lambda_0$ with $V_\lambda \prec_{\Sigma_n} V$, a \mathcal{P} -name \mathbb{Q} with $\Vdash_{\mathbb{P}} \mathbb{Q} \in \mathcal{P}$, such that, for $(V, \mathbb{P} * \mathbb{Q})$ -generic \mathbb{H} , there are j , $M \subseteq V[\mathbb{H}]$ with $j : V \xrightarrow{\prec}_\kappa M$, $j(\kappa) > \lambda$, $\mathbb{P}, \mathbb{H}, V_{j(\lambda)}^{V[\mathbb{H}]} \in M$ and $V_{j(\lambda)}^{V[\mathbb{H}]} \prec_{\Sigma_n} V[\mathbb{H}]$.

A super $C^{(n)}$ - \mathcal{P} -Laver-generically ultrahuge cardinal

κ is *tightly super $C^{(n)}$ - \mathcal{P} -Laver-generically ultrahuge*, if additionally $|\mathbb{P} * \mathbb{Q}| \leq j(\kappa)$ (see Footnote 2) holds in the definition above.

Super $C^{(\infty)}$ - \mathcal{P} -Laver-generically hyperhugeness and *tightly super $C^{(\infty)}$ - \mathcal{P} -Laver generically hyperhugeness* are defined similarly to super $C^{(\infty)}$ -ultrahugeness.

Note that, in general, super $C^{(\infty)}$ -hyperhugeness and super $C^{(\infty)}$ -ultrahugeness are notions not formalizable in the language of ZFC without introducing a new constant symbol for κ since we need infinitely many \mathcal{L}_\in -formulas to formulate them. Exceptions are when we are talking about a cardinal in a set model being with one of these properties like in Lemma 29 below (and in such a case “natural number n ” actually refers to “ $n \in \omega$ ”), or when we are talking about a cardinal definable in V having these properties in an inner model like in Corollary 36 or Corollary 37. In the latter case, the situation is formalizable with infinitely many \mathcal{L}_\in -sentences.

In contrast, the super $C^{(\infty)}$ - \mathcal{P} -Laver generically ultrahugeness of κ is expressible in infinitely many \mathcal{L}_\in -sentences. This is because a \mathcal{P} -Laver generic large cardinal κ for relevant classes \mathcal{P} of posets is uniquely determined as κ_{refl} or 2^{\aleph_0} (see e.g. Theorem 7 and Theorem 8).

A modification of the proof of Theorem 21 shows the following:

Theorem 28 (Fuchino and Usuba [21]) *Suppose that \mathcal{P} is an iterable class of posets and κ is super $C^{(\infty)}$ - \mathcal{P} -Laver-generically ultrahuge. Then $(\mathcal{P}, \mathcal{H}(\kappa))\text{-RcA}^+$ holds.*

Proof. Suppose that κ is super $C^{(\infty)}$ - \mathcal{P} -Laver-gen. ultrahuge, $\mathbb{P} \in \mathcal{P}$, and $\Vdash_{\mathbb{P}} \ulcorner \varphi(\bar{a}^\vee) \urcorner$ for an \mathcal{L}_\in -formula φ and $\bar{a} \in \mathcal{H}(\kappa)$. We want to show that $\varphi(\bar{a})$ holds in some \mathcal{P} -ground of V .

Let n be a sufficiently large natural number such that the following arguments go through. In particular, we assume that $V_\alpha^V \prec_{\Sigma_n} V$ implies that “ $\varphi(\bar{x})$ ” and “ $\Vdash_{\mathbb{P}} \ulcorner \varphi(\bar{x}^\vee) \urcorner$ ” are absolute between V_α^V and V , and $V_\alpha^V \prec_{\Sigma_n} V$ also implies that a sufficiently large fragment of ZFC holds in V_α .

Let \mathbb{Q} be a \mathbb{P} -name such that $\Vdash_{\mathbb{P}} \ulcorner \mathbb{Q} \in \mathcal{P} \urcorner$ and, for $(V, \mathbb{P} * \mathbb{Q})$ -generic \mathbb{H} , there are a $\lambda > \kappa$ with

$$(\aleph 3) \quad V_\lambda \prec_{\Sigma_n} V,$$

and $j, M \subseteq V[\mathbb{H}]$ such that $j : V \xrightarrow{\prec_\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{P}, \mathbb{H}, V_{j(\lambda)}^{V[\mathbb{H}]} \in M$ and $V_{j(\lambda)}^{V[\mathbb{H}]} \prec_{\Sigma_n} V[\mathbb{H}]$.

By the choice of n , we have $V_\lambda \models \Vdash_{\mathbb{P}} \ulcorner \varphi(\bar{a}^\vee) \urcorner$. $j(V_\lambda^V) = V_{j(\lambda)}^M \prec_{\Sigma_n} M$ by elementarity of j , and $V_{j(\lambda)}^M = V_{j(\lambda)}^{V[\mathbb{H}]}$ by the closedness of M . Since $V_\lambda \prec_{\Sigma_n} V$, we have $V_\lambda[\mathbb{H}] \prec_{\Sigma_{n_0}} V[\mathbb{H}]$ for a still large enough $n_0 \leq n$ by Lemma 15, (1). Since $V_{j(\lambda)}^{V[\mathbb{H}]} \prec_{\Sigma_n} V[\mathbb{H}]$, it follows that $V_\lambda^{V[\mathbb{H}]} \prec_{\Sigma_{n_0}} V_{j(\lambda)}^{V[\mathbb{H}]}$. Thus

$$(\aleph 4) \quad V_\lambda^V \prec_{\Sigma_{n_1}} V_{j(\lambda)}^V$$

for a still large enough $n_1 \leq n_0$ by Lemma 15, (2).

In particular, we have $V_{j(\lambda)}^V \models \Vdash_{\mathbb{P}} \ulcorner \varphi(\bar{a}^\vee) \urcorner$, and hence $V_{j(\lambda)}[\mathbb{G}] \models \varphi(\bar{a})$ where \mathbb{G} is the \mathbb{P} -part of \mathbb{H} . Note that by (3) and (4), $V_{j(\lambda)}$ satisfies a sufficiently large fragment of ZFC.

Thus we have $V_{j(\lambda)}[\mathbb{H}] \models \ulcorner \text{there is a } \mathcal{P}\text{-ground satisfying } \varphi(\bar{a}) \urcorner$, and hence

$$V_{j(\lambda)}^{V[\mathbb{H}]} \models \ulcorner \text{there is a } \mathcal{P}\text{-ground satisfying } \varphi(\bar{a}) \urcorner$$

by Lemma 13. By elementarity, it follows that

$$V_\lambda \models \ulcorner \text{there is a } \mathcal{P}\text{-ground satisfying } \varphi(\bar{a}) \urcorner.$$

Finally, this implies $V \models \ulcorner \text{there is a } \mathcal{P}\text{-ground satisfying } \varphi(\bar{a}) \urcorner$ by (3).

□ (Theorem 28)

The following Lemma can be proved similarly to Theorem 5c in Barbanel-DiPrisco-Tan [2] (see also Theorem 24.13 in Kanamori [30]).

Lemma 29. (Fuchino and Usaba [21]) *If κ is 2-huge with the 2-huge elementary embedding j , that is, there is $M \subseteq V$ such that $j : V \xrightarrow{\prec_\kappa} M \subseteq V$, and*

$$j^2(\kappa) M \subseteq M, \tag{*33}$$

then $V_{j(\kappa)} \models \ulcorner \kappa \text{ is super } C^{(\infty)}\text{-hyperhuge cardinal} \urcorner$, and for each $n \in \omega$, $V_{j(\kappa)} \models \ulcorner \text{there are stationarily many super } C^{(n)}\text{-hyperhuge cardinals} \urcorner$. □

The proof of the existence of Laver-function for a supercompact cardinal can be modified to show that super $C^{(\infty)}$ -hyperhuge cardinal in V_μ has a Laver function for super $C^{(\infty)}$ -hyperhugeness ([21]). Similarly to Theorem 9 we obtain the following:

Theorem 30 (Fuchino and Usuba [21]) (A) Suppose that μ is inaccessible and $\kappa < \mu$ is super $C^{(\infty)}$ -hyperhuge in V_μ . Let $\mathbb{P} = \text{Col}(\aleph_1, \kappa)$. Then, in $V_\mu[\mathbb{G}]$, for any V_μ, \mathbb{P} -generic \mathbb{G} , $\aleph_2^{V_\mu[\mathbb{G}]} (= \kappa)$ is tightly super $C^{(\infty)}$ - σ -closed-Laver-generically hyperhuge and CH holds.

(B) Suppose that μ is inaccessible and $\kappa < \mu$ is super $C^{(\infty)}$ -hyperhuge with a Laver function $f : \kappa \rightarrow V_\kappa$ for super $C^{(\infty)}$ -hyperhugeness in V_μ . If \mathbb{P} is the CS-iteration of length κ for forcing PFA along with f , then, in $V_\mu[\mathbb{G}]$ for any (V_μ, \mathbb{P}) -generic \mathbb{G} , $\aleph_2^{V_\mu[\mathbb{G}]} (= \kappa)$ is tightly super $C^{(\infty)}$ -proper-Laver-generically hyperhuge and $2^{\aleph_0} = \aleph_2$ holds.

(B') Suppose that μ is inaccessible and $\kappa < \mu$ is super $C^{(\infty)}$ -hyperhuge with a Laver function $f : \kappa \rightarrow V_\kappa$ for super $C^{(\infty)}$ -hyperhugeness in V_μ . If \mathbb{P} is the RCS-iteration of length κ for forcing MM along with f , then, in $V_\mu[\mathbb{G}]$ for any (V_μ, \mathbb{P}) -generic \mathbb{G} , $\aleph_2^{V_\mu[\mathbb{G}]} (= \kappa)$ is tightly super $C^{(\infty)}$ -stationary_preserving-Laver-generically hyperhuge and $2^{\aleph_0} = \aleph_2$ holds.

(Γ) Suppose that μ is inaccessible and κ is super $C^{(\infty)}$ -hyperhuge with a Laver function $f : \kappa \rightarrow V_\kappa$ for super $C^{(\infty)}$ -hyperhugeness in V_μ . If \mathbb{P} is a FS-iteration of length κ for forcing MA along with f , then, in $V_\mu[\mathbb{G}]$ for any (V_μ, \mathbb{P}) -generic \mathbb{G} , $2^{\aleph_0} (= \kappa)$ is tightly super $C^{(\infty)}$ -c.c.c.-Laver-generically hyperhuge, and 2^{\aleph_0} is very large in $V_\mu[\mathbb{G}]$.

(Δ) Suppose that μ is inaccessible and κ is super $C^{(\infty)}$ -hyperhuge with a Laver function $f : \kappa \rightarrow V_\kappa$ for super $C^{(\infty)}$ -hyperhugeness in V_μ . If \mathbb{P} is a FS-iteration of length κ along with f enumerating “all” posets, then, in $V_\mu[\mathbb{G}]$ for any (V_μ, \mathbb{P}) -generic \mathbb{G} , $2^{\aleph_0} (= \aleph_1)$ is tightly super $C^{(\infty)}$ -all posets-Laver-generically hyperhuge, and CH holds.

Proof. The proof can be done similarly to that of Theorem 5.2 in [16]. In the following we shall only check the case (Δ).

Suppose that $f : \kappa \rightarrow V_\kappa$ is a super $C^{(\infty)}$ -hyperhuge Laver function.

Let $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be a FS-iteration defined by

$$(\aleph 5) \quad \mathbb{Q}_\beta = \begin{cases} f(\alpha), & \text{if } f(\alpha) \text{ is a } \mathbb{P}_\beta\text{-name of a poset;} \\ \{\mathbb{1}\}, & \text{otherwise} \end{cases}$$

for $\beta < \kappa$.

Let \mathbb{G} be a $(V_\mu, \mathbb{P}_\kappa)$ -generic filter. Clearly $V_\mu[\mathbb{G}] \models “2^{\aleph_0} = \kappa = \aleph_1”$. We show that κ is tightly super $C^{(\infty)}$ -all posets-Laver-generically ultrahuge in $V_\mu[\mathbb{G}]$.

Suppose that \mathbb{P} is a poset in $V_\mu[\mathbb{G}]$, $\kappa < \lambda_0$ and $n \in \omega$. Let $n' > n$ be sufficiently large and let \mathbb{P} be a \mathbb{P}_κ -name of \mathbb{P} .

Working in V_μ , we can find

$$(\aleph 6) \quad |\mathbb{P}| < \lambda < \kappa^* < \lambda^* \text{ and } j, M \subseteq V$$

such that

$$(\aleph 7) \quad j : V \xrightarrow{\prec_\kappa} M,$$

$$(\aleph 8) \quad j(\kappa) = \kappa^*, j(\lambda) = \lambda^*,$$

$$(\aleph 9) \quad \lambda^* M \subseteq M,$$

$$(\aleph 10) \quad V_\lambda \prec_{\Sigma_n} V, V_{\lambda^*} \prec_{\Sigma_{n'}} V, \text{ and}$$

$$(\aleph 11) \quad j(f)(\kappa) = \mathbb{P}$$

by definition of f .

By elementarity (and by the definition (5) of \mathbb{P}_κ),

$$(\aleph 12) \quad j(\mathbb{P}_\kappa) \sim_{\mathbb{P}_\kappa} (\mathbb{P}_\kappa * \mathbb{P}) * \mathbb{R}$$

for a $(\mathbb{P}_\kappa * \mathbb{P})$ -name \mathbb{R} of a poset. Note that $(\mathbb{P}_\kappa * \mathbb{P} * \mathbb{R})/\mathbb{G}$ corresponds to a poset of the form $\mathbb{P} * \mathbb{Q}$.

Let \mathbb{H}^* be $(V, (\mathbb{P}_\kappa * \mathbb{P}) * \mathbb{R})$ -generic filter with $\mathbb{G} \subseteq \mathbb{H}^*$. \mathbb{H}^* corresponds to a $(V, j(\mathbb{P}_\kappa))$ -generic filter $\mathbb{H} \supseteq \mathbb{G}$ via the equivalence (12).

Let \tilde{j} be defined by

$$(\aleph 13) \quad \tilde{j} : V[\mathbb{G}] \rightarrow M[\mathbb{H}]; \quad \check{a}^\mathbb{G} \mapsto j(\check{a})^\mathbb{H}$$

for all \mathbb{P}_κ -names \check{a} .

A standard proof shows that f is well-defined, and $j : V[\mathbb{G}] \xrightarrow{\sim}_\kappa M[\mathbb{H}]$. By (8) and (9), we have $\tilde{j}''\tilde{j}(\lambda) = j''j(\lambda) \in M[\mathbb{H}]$. Since $\mathbb{H} \in M[\mathbb{H}]$, the $(V[\mathbb{G}], \mathbb{P} * \mathbb{Q})$ -generic filter corresponding to \mathbb{H} is also in $M[\mathbb{H}]$.

By (6), (10), by the choice of n' , and by Lemma 15, (1), we have $V_\lambda^{V[\mathbb{G}]} \prec_{\Sigma_n} V[\mathbb{G}]$ and $V_{\tilde{j}(\lambda)}^{V[\mathbb{H}]} = V_{\lambda^*}^{V[\mathbb{H}]} \prec_{\Sigma_n} V[\mathbb{H}]$.

Since \mathbb{P} and n were arbitrary, this shows that κ is tightly super $C^{(\infty)}$ -all posets-Laver-generically ultrahuge in $V_\mu[\mathbb{G}]$. □ (Theorem 30)

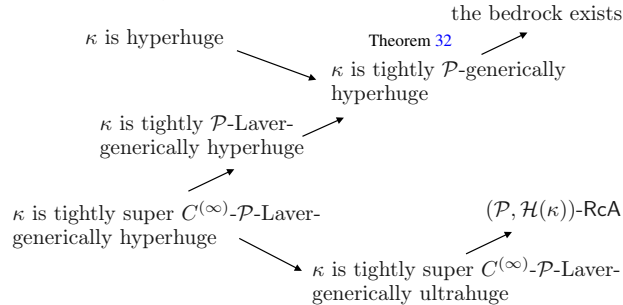
6 Toward the Laver-generic Maximum

Besides Theorem 21 and Theorem 28, , we also have some other advantages of assuming the existence of \mathcal{P} -Laver-generically hyperhuge cardinal or even its “super $C^{(\infty)}$ ” version. One of them is that they imply the existence of the (set-generic) bedrock (see below for definition); another is that we know the exact consistency strength of these principles.

For a class \mathcal{P} of posets, a cardinal κ is *tightly \mathcal{P} -generic hyperhuge* if for any $\lambda > \kappa$, there is $\mathbb{Q} \in \mathcal{P}$ such that for a (V, \mathbb{Q}) -generic \mathbb{H} , there are $j, M \subseteq V[\mathbb{H}]$ such that $j : V \xrightarrow{\sim}_\kappa M$, $\lambda < j(\kappa)$, $|\mathbb{Q}| \leq j(\kappa)$, and $j''j(\lambda), \mathbb{H} \in M$.

Note that, for any class \mathcal{P} of posets with $\{1\} \in \mathcal{P}$, the hyperhugeness of a cardinal κ implies its tightly \mathcal{P} -generically hyperhuge. Likewise, if \mathcal{P} is iterable then, the tightly \mathcal{P} -Laver-generic hyperhugeness of κ implies its tightly \mathcal{P} -generically hyperhugeness.

For an iterable \mathcal{P} :



Theorem 28

Usuba [38] proved that the grounds of V are downward directed (with respect to subclass relation) for class many grounds (this is formalizable by virtue of Theorem 12). More concretely

Theorem 31 (Theorem 1.3 in Usuba [38]) *For any collection of grounds of V , indexed by a set of parameters (in the sense of Theorem 12), there is a ground which is included in all grounds in the collection.* \square

From this theorem, it follows that the *mantle*, i.e., the intersection of all grounds is a model of ZFC. In [38], it is proved that the mantle is a ground and hence it is the *bedrock*, i.e., the smallest ground of V provided that there exists a hyperhuge cardinal (Theorem 1.6 in [38]). Later the assumption of the existence of a hyperhuge cardinal in this theorem is weakened to the existence of an extendible cardinal (Theorem 1.3 in Usuba [39]).

In [21], we obtained the following generalization of Theorem 1.6 in [38]. In the following theorems, tightness of a \mathcal{P} -generic Large cardinal is defined similarly to the tightness of \mathcal{P} -Laver generic large cardinal. Note that the (in many cases unique (Theorem 8)) tightly \mathcal{P} -Laver generic large cardinal as well as corresponding genuine large cardinals are tightly \mathcal{P} -generic large cardinals by definition.

Theorem 32 (Fuchino and Usuba [21]) *If there is a tightly \mathcal{P} -generically hyperhuge cardinal κ , then the mantle is a ground of V . In particular it is the bedrock.*

A sketch of the proof. The overall structure of the structure of the proof is just the same as that of Theorem 1.6 in [38] or Theorem 1.3 in [39].

We call a ground W of V a $\leq \kappa$ -ground if there is $\mathbb{P} \in W$ with $|\mathbb{P}|^V \leq \kappa$ and a (W, \mathbb{P}) -generic \mathbb{G} such that $V = W[\mathbb{G}]$. Let

$$\overline{W} = \bigcap \{W : W \text{ is a } \leq \kappa\text{-ground}\}. \quad (*34)$$

By Theorem 31, there is a ground $W \subseteq \overline{W}$. For such W it is enough to show that actually $\overline{W} \subseteq W$ holds.

Let $\mathbb{S} \in W$ be a poset with cardinality μ (in V) such that there is a (W, \mathbb{S}) -generic $\mathbb{F} \in V$ with $V = W[\mathbb{F}]$. Without loss of generality, $\mu \geq \kappa$.

By Theorem 12, there is $r \in V$ such that $W = \Phi(\cdot, r)^V$.

Let $\theta \geq \mu$ be such that $r \in V_\theta$, and for a sufficiently large natural number n , we have $V_\theta^V \prec_{\Sigma_n} V$. By the choice of θ , $\Phi(\cdot, r)^{V_\theta^V} = \Phi(\cdot, r)^V \cap V_\theta^V = W \cap V_\theta^V = V_\theta^W$. Let $\mathbb{Q} \in \mathcal{P}$ such that for (V, \mathbb{Q}) -generic \mathbb{H} , there are $j, M \subseteq V[\mathbb{H}]$ with $j : V \xrightarrow{\prec}_\kappa M$, $\theta < j(\kappa)$, $|\mathbb{Q}| \leq j(\kappa)$, $V_{j(\theta)}^{V[\mathbb{H}]} \subseteq M$, and $\mathbb{H}, j''j(\theta) \in M$.

Using this j we can show that $V_\theta^{\overline{W}} \subseteq V_\theta^W$ holds (this part of the proof is quite involved, for the details, the reader is referred to [21]). Since θ can be arbitrary large, It follows that $\overline{W} \subseteq W$. \square (Theorem 32)

Analyzing the details of the proof of Theorem 32 which we omitted from our present exposition, we also obtain the following result with many surprising consequences:

Theorem 33 (Fuchino and Usuba [21]) *Suppose that \mathcal{P} is any class of posets. If κ is a tightly \mathcal{P} -generically hyperhuge cardinal, then κ is a hyperhuge cardinal in the bedrock \overline{W} of V .* \square

The following equiconsistency results are immediate consequences of the theorem above:

Corollary 34. *Suppose that \mathcal{P} is the class of all posets. Then the following theories are equiconsistent:*

- (a) ZFC + “there is a hyperhuge cardinal”.
- (b) ZFC + “there is a tightly \mathcal{P} -Laver generically hyperhuge cardinal”.
- (c) ZFC + “there is a tightly \mathcal{P} -generically hyperhuge cardinal”.
- (d) ZFC + “the bedrock \overline{W} exists and ω_1 is a hyperhuge cardinal in \overline{W} ”. \square

Corollary 35. *Suppose that \mathcal{P} is one of the following classes of posets: all semi-proper posets; all proper posets; all ccc posets; all σ -closed posets. Then the following theories are equiconsistent:*

- (a) ZFC + “there is a hyperhuge cardinal”.
- (b) ZFC + “there is a tightly \mathcal{P} -Laver generically hyperhuge cardinal”.
- (c) ZFC + “there is a tightly \mathcal{P} -generically hyperhuge cardinal”.
- (d) ZFC + “the bedrock \overline{W} exists and κ_{refl} is a hyperhuge cardinal in \overline{W} ”. \square

These equiconsistency results are quite remarkable when we remember that equiconsistency of axioms like PFA, MM, MM^{++} etc. are unknown at the moment.

The tightness of generic large cardinals was originally thought as a technical condition when it was introduced to overcome the difficulty in the proof of Theorem 7, (2). The (proofs of) theorems 32 and 33 and their consequences, some of which we are presenting here as their corollaries, suggest that this notion is much more intrinsic than merely a technicality.

A slight modification of the proofs of the theorems above also show the following. Note that as we already noticed, $\text{super-}C^{(\infty)}$ -large cardinal is not formalizable in the language of ZFC. However, the assertions (a) and (b) in the following Corollary 36 and Corollary 36 can be formulated as schemes of sentences in \mathcal{L}_ϵ .

Corollary 36. *Suppose that \mathcal{P} is the class of all posets. Then the following theories are equiconsistent:*

- (a) ZFC + “ c is a super $C^{(\infty)}$ hyperhuge cardinal” where c is a new constant symbol but “... is super $C^{(\infty)}$ hyperhuge ...” is formulated in an infinite collection of formulas in \mathcal{L}_ϵ .
- (b) ZFC + “there is a tightly super $C^{(\infty)}$ - \mathcal{P} -Laver generically hyperhuge cardinal”.
- (c) ZFC + “the bedrock \overline{W} exists and ω_1^V is a super $C^{(\infty)}$ -hyperhuge cardinal in \overline{W} ”. \square

Corollary 37. *Suppose that \mathcal{P} is one of the following classes of posets: all semi-proper posets; all proper posets; all ccc posets; all σ -closed posets. Then the following theories are equiconsistent:*

- (a) $\text{ZFC} + \text{“}c \text{ is a super } C^{(\infty)} \text{ hyperhuge cardinal”}$ where c is a new constant symbol but “... is super $C^{(\infty)}$ hyperhuge ...” is formulated in an infinite collection of formulas in \mathcal{L}_\in .
- (b) $\text{ZFC} + \text{“there is a tightly super } C^{(\infty)}\text{-}\mathcal{P}\text{-Laver generically hyperhuge cardinal”}$.
- (c) $\text{ZFC} + \text{“the bedrock } \overline{W} \text{ exists and } \kappa_{\text{refl}}^V \text{ is a super } C^{(\infty)}\text{-hyperhuge cardinal in } \overline{W}\text{”}$. □

Finally, we move to the promised proof of Theorem 7, (3).

Corollary 38. *Suppose that \mathcal{P} is an arbitrary class of posets and κ is a tightly \mathcal{P} -generically hyperhuge cardinal. Then*

- (1) *there are cofinally many huge cardinals.*
- (2) *SCH holds above some cardinal.*

Proof. Suppose that κ is a tightly \mathcal{P} -generically hyperhuge cardinal. By Theorem 32, there is the bedrock \overline{W} and κ is hyperhuge cardinal in \overline{W} .

(1): Since the existence of a hyperhuge cardinal implies the existence of cofinally many huge cardinals (it is easy to show that the target $j(\kappa)$ of hyperhuge embedding for a sufficiently large inaccessible λ is a huge cardinal), there are cofinally many huge cardinals in \overline{W} . Since V is attained by a set forcing starting from \overline{W} , a final segment of these huge cardinals survive in V .

(2): By Theorem 20.8 in [29], SCH holds above κ in \overline{W} . Since V is a set generic extension of \overline{W} , SCH should hold above some cardinal $\mu \geq \kappa$. □ (Corollary 38)

For iterable stationary preserving \mathcal{P} containing all proper posets, Corollary 38, (2) holds already under the \mathcal{P} -Laver-generic supercompactness of κ . The reason is that in such case PFA holds by Theorem 5, and by Viale [40], SCH follows from it.

Proof of Theorem 7, (3). Let λ and \mathbb{Q} be such that

- (*35) $\lambda > 2^{\aleph_0}$, κ and λ is large enough such that SCH holds above some $\mu < \lambda$ (this is possible by Corollary 38, (2), and it is here that we need a strong property like the Laver generic hyperhugeness of κ),
- (*36) \mathbb{Q} is positive elements of a complete Boolean algebra, and,
- (*37) for (V, \mathbb{Q}) -generic \mathbb{H} , there are $j, M \subseteq V[\mathbb{H}]$ such that (1) $j : V \xrightarrow{\lambda} M$, (2) $j(\kappa) > \lambda$, (3) $|\mathbb{Q}| \leq j(\kappa)$, and (4) $V_{j(\lambda)}^{V[\mathbb{H}]} \subseteq M$.

By (*36), each \mathbb{Q} -name \dot{r} of a real corresponds to a mapping $f : \omega \rightarrow \mathbb{Q}$. By (*35) and by (*37), (3), there are at most $j(\kappa)$ many such mappings. Thus we have $V[\mathbb{H}] \models “2^{\aleph_0} \leq j(\kappa)”$. By (*37), (4), it follows $M \models “2^{\aleph_0} \leq j(\kappa)”$. By elementarity, it follows that $V \models “2^{\aleph_0} \leq \kappa”$. □ (Theorem 7, (3))

Returning to (E) and (Z) at the end of Section 3, we now know that the existence of a super $C^{(\infty)}$ -stationary preserving-Laver-generically hyperhuge cardinal implies (E) (actually it even implies (stationary preserving posets, $\mathcal{H}(\kappa_{\text{refl}})$)- RcA^+), and that the existence of a super $C^{(\infty)}$ -all posets-Laver-generically hyperhuge cardinals implies (Z) (actually it even implies (all posets, $\mathcal{H}(2^{\aleph_0})$)- RcA^+). These two scenarios are not compatible since the former implies $2^{\aleph_0} = \aleph_2$ while the latter implies CH.

However, with the following axiom, (E) is reconciled with a fragment of (Z):

LGM: the continuum is tightly super $C^{(\infty)}$ -stationary preserving-Laver generically hyperhuge and there is a ground W of V such that the continuum is tightly super $C^{(\infty)}$ -all posets-Laver generically hyperhuge in W .

This combination of Laver-genericity and “ground”¹² Laver-genericity above implies (Z^+) mentioned at the end of Section 3 (by Theorem 28). As (Z^+) represents in a sense the maximal amount of available strengthening of recurrence, I would like to choose the name Laver-Generic Maximum (LGM) for it.

If we admit that Recurrence Axioms, Maximal Principles and Laver-genericity are natural requirements, we should be also ready to accept $\text{ZFC} + \text{LGM}$ as a natural candidate of the extension of ZFC .

By Theorem 5, LGM implies the double plus version of Martin’s Maximum (MM^{++}) and hence all the consequences of it including $2^{\aleph_0} = \aleph_2$.

By Theorem 32, LGM implies that there is the bedrock. So by Theorem 28, LGM implies (Z^+) on page 17. (Z^+) implies that if some statement φ is forcable by a stationary preserving poset, then for any $A \in \mathcal{H}(\aleph_2)$, there is a semi-proper-ground W of V such that $A \in W$ and $W \models \varphi$. In particular, Cichoń’s Maximum [24], [25] is a phenomena in many semi-proper-grounds in this sense. Note that, by Corollary 38, (1) the forcing argument for φ may even utilize class many huge cardinals (e.g. the proof in [24] uses four strongly compact cardinals).¹³

Even in the case that the forcing to show the consistency of φ is not stationary preserving, we can still find some ground W of V which satisfies φ .

Thus $\text{ZFC} + \text{LGM}$ is a very strong axiom system which integrates practically all statements into itself as far as these statements can be proved to be consistent by way of forcing and/or methods of inner models or some combination of them. Against this backdrop, we want to call the axiom system LGM (or possibly some further extension of it in the future) the *Laver Generic Maximum*.

The consistency and equiconsistency of LGM is easily established: we start from a model with two super $C^{(\infty)}$ hyperhuge cardinals $\kappa_0 < \kappa_1$. We force κ_0 to be tightly super $C^{(\infty)}$ -all posets-Laver generically hyperhuge (Theorem 30, (Δ)). We then force make κ_1 to be tightly super $C^{(\infty)}$ -stationary preserving-Laver generically hyperhuge (Theorem 30, (B')).

¹² I am using the word “ground” here as an adjective contrasting with the word “generic” in “generic large cardinal”.

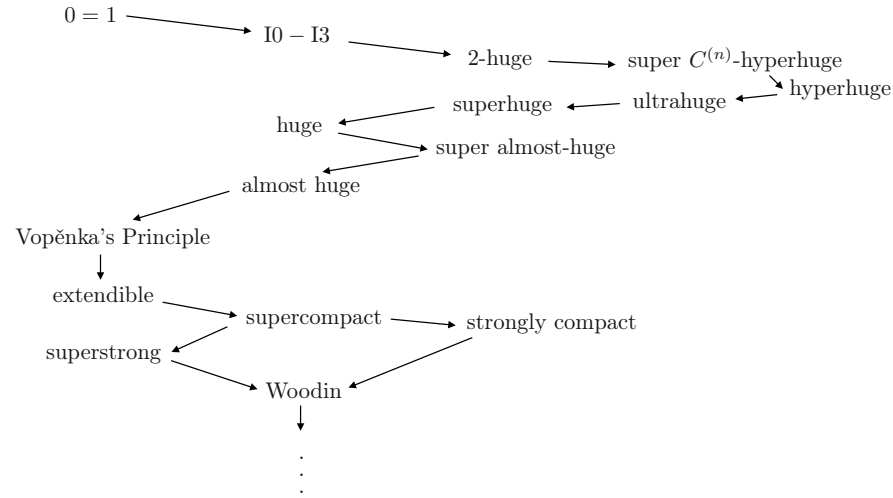
¹³ In [21], we even show that LGM implies that there are stationarily many super- $C^{(\infty)}$ -hyperhuge cardinals.

By Theorem 33 the consistency strength of LGM can be proved to be equivalent with that of two super $C^{(\infty)}$ hyperhuge cardinals (which may be formulated by using two new constant symbols).

7 More about consistency strength

When I was a student of Janos Makowsky in the early 1980s at the Free University of Berlin, one of the papers he was preparing then was [32] in which the effects of Vopěnka Principle on properties of model theoretic logics is studied. I remember that Janos gave a talk on this subject in the (West) Berliner Logic Colloquium. Back then, I was still living in a set theory of consistency strength way below a measurable cardinal, and could not begin with the material of his paper at all because of the vertiginous consistency strength of the Vopěnka Principle.

Janos left Berlin before I wrote up my diploma thesis on abstract elementary classes which was the subject Janos gave me; all the assertions I proved in the thesis remained in the consistency strength of ZFC.



The consistency strength of my set theoretic world view reached the realm of one supercompact cardinal when I wrote [10] in the early 1990s in which some consequences of $MA^+(\sigma\text{-closed})$ were discussed. However, it is only quite recently that I caught up with Janos definitively (at least in terms of consistency strength) when I considered in [21] the super $C^{(\infty)}$ - \mathcal{P} -Laver-generically hyperhuge cardinals whose consistency strength is (demonstrably — see Section 6) strictly between hyperhuge and 2-huge.

In the meantime, active research on abstract model classes is resumed and Janos's [32] begins to attract the attention of young logicians. For example, Janos' [32] was recently cited in Boney [6] which was already mentioned in Section 6. Boney cites

the main theorem of [32] as Fact 3.12 in his paper and comments in allusion to Aki Kanamori’s comment on Kunen’s inconsistency proof in [30] that Vopěnka’s Principle “ ‘rallies at least to force a veritable Götterdämmerung’ for compactness cardinals for logics.” The gap between Vopěnka’s Principle and a huge cardinal Boney mentions in connection with this „götterdämmerigen“ statement seems to have some resemblance to the discrepancy between usual Laver-generic large cardinal axioms and the super $C^{(\infty)}$ -Laver-generic large cardinals.

Now one of the urgent items in my to-do-list is to check Janos’s [32] as well as [33], [34], [35] more carefully to find out further possible connections of his results to the context I described above.

Is this also an instance of (the eternal?) recurrence?

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