Some remarks on openly generated Boolean algebras

Sakaé Fuchino[1]

July 30,1992

Revised on: December 5, 1993

Abstract

A Boolean algebra B is said to be *openly generated* if $\{ A : A \leq_{\mathrm{rc}} B, | A | = \aleph_0 \}$ includes a club subset of $[B]^{\aleph_0}$. We show:

($V = L$) For any cardinal κ there exists an $\mathcal{L}_{\infty\kappa}$-free Boolean algebra which is not openly generated (Proposition 4.1).

($\mathrm{MA}^+(\sigma\text{-closed})$) Every $\mathcal{L}_{\infty\aleph_2}$-free Boolean algebra is openly generated (Theorem 4.2)[2] .

The last assertion follows from a characterization of openly generated Boolean algebras under $\mathrm{MA}^+(\sigma\text{-closed})$ (Theorem 3.1)[2] . Using this characterization we also prove the independence of Problem 7 in Ščepin [16] (Proposition 4.3 and Theorem 4.4).

1 Introduction

Let A be a subalgebra of a Boolean algebra B ($A \leq B$). A is a *relatively complete subalgebra of* B ($A \leq_{\mathrm{rc}} B$) if, for every $b \in B$, the set $\{ a \in A : a \leq b \}$ has the greatest element c. This c is called *the projection of* b *in* A and is denoted by $\mathrm{pr}^B_A(b)$. If $A \leq B$ but A is not relative complete in B , this is denoted by $A \leq_{\neg\mathrm{rc}} B$. If $A \leq_{\mathrm{rc}} B$ and $b \in B$ then $\overline{\mathrm{pr}}^B_A(b) = -\mathrm{pr}^B_A(-b)$ is the least element of $\{ a \in A : a \geq b \}$.

For any set X, $[X]^{\aleph_0}$ denotes the set $\{ Y \subseteq X : | Y | = \aleph_0 \}$. A subset $\mathcal{C}$ of $[X]^{\aleph_0}$ is said to be *closed unbounded* (club) if $\mathcal{C}$ is closed with respect to union of an increasing chain of countable length and cofinal in $[X]^{\aleph_0}$ (with respect to $\subseteq$). An $\mathcal{S} \subseteq [X]^{\aleph_0}$ is said to be *stationary* if $\mathcal{S} \cap \mathcal{C} \neq \emptyset$ holds for any club $\mathcal{C} \subseteq [X]^{\aleph_0}$. For basic facts about closed unbounded and stationary sets of $[X]^{\aleph_0}$ the reader may consult e.g. [3] or [12].

[0] This note is to appear in the Journal of Symbolic Logic.

[1] Partially supported by the Deutsche Forschungsgemeinschaft grant Ko 490/5-1.
The author would like to thank L. Heindorf who made him acquainted with Ščepin's results on openly generated topological spaces. Also he would like to thank S. Koppelberg and L. Heindorf for reading an earlier version of the paper very carefully and making a lot of useful comments.

[2] These theorems were originally proved under MM. The author would like to thank T. Jech for suggesting him that this assumption might be still weakened. (Actually this assumption can be still weakened: Qi Feng pointed out that under almost the same idea these theorems can be proved under Freissner's Axiom R which is a consequence of $\mathrm{MA}^+(\sigma\text{-closed})$, see [4].)

For a Boolean algebra B let $\mathrm{Sub}_{\mathrm{rc}}^{\aleph_0}(B) = \{ A \in [B]^{\aleph_0} \ :\ A \leq_{\mathrm{rc}} B \}$. A Boolean algebra B is said to be *openly generated* if the set $\mathrm{Sub}_{\mathrm{rc}}^{\aleph_0}(B)$ includes a club subset of $[B]^{\aleph_0}$.

The class of openly generated Boolean algebras is closed under the union of a continuously increasing chain with respect to "$\leq_{\mathrm{rc}}$":

Theorem 1.1 (Ščepin [16]) *For a limit ordinal δ let $(B_\alpha)_{\alpha<\delta}$ be a continuous chain of openly generated Boolean algebras such that $B_\alpha \leq_{\mathrm{rc}} B_{\alpha+1}$ for every $\alpha < \delta$. Then $\bigcup_{\alpha<\delta} B_\alpha$ is also openly generated.* ▮

In the next section we shall give a proof of Theorem 1.1 for the convenience of the reader.

A Boolean algebra B is said to be *projective over a subalgebra A* of B if $B \oplus \mathrm{Fr}\,\kappa \cong_A A \oplus \mathrm{Fr}\,\kappa$ holds for $\kappa = |\, B \,| + \aleph_0$. B is said to be *projective* if B is projective over 2. Note, that this definition differs from the original one (for the original definition see e.g. [14]). A Boolean algebra B is *countably generated* over a subalgebra A of B if there exists a countable set $X \subseteq B$ such that $B = A[X]$ holds.

Projective Boolean algebras have the following nice characterization:

Theorem 1.2 (Haydon, see e.g. [14] for the proof) *For a Boolean algebra B and $A \leq B$ the following are equivalent:*

(*a*) *B is projective over A;*

(*b*) *There exist an ordinal ρ and a continuously increasing sequence $(B_\alpha)_{\alpha<\rho}$ of subalgebras of B such that $B_0 = A$, $B_\alpha \leq_{\mathrm{rc}} B_{\alpha+1}$ and $B_{\alpha+1}$ is countably generated over B_α for every $\alpha < \rho$ and $\bigcup_{\alpha<\rho} B_\alpha = B$.* ▮

From Theorem 1.2 it follows that, for every Boolean algebra B of cardinality $\leq \aleph_1$, B is openly generated if and only if B is projective.

Openly generated Boolean algebras can be characterized in terms of forcing:

Proposition 1.3 *A Boolean algebra B is openly generated if and only if for some resp. any σ-closed p.o.-set P collapsing $|\, B \,|$ to $\leq \aleph_1$, $\Vdash_P$ " B* is projective Boolean algebra *" holds.*

Proof Let B be an openly generated Boolean algebra. Let $\mathcal{C} \subseteq \mathrm{Sub}_{\mathrm{rc}}^{\aleph_0}(B)$ be a club subset of $[B]^{\aleph_0}$. Let P be a σ-closed p.o.-set collapsing $|\, B \,|$ to $\leq \aleph_1$. Then

$$\Vdash_P \text{" } \mathcal{C} \subseteq \mathrm{Sub}_{\mathrm{rc}}^{\aleph_0}(B) \text{ and } \mathcal{C} \text{ is closed in } [B]^{\aleph_0} \text{ ".}$$

Hence we have $\Vdash_P$ " B is openly generated ". Since $\Vdash_P$ " $|\, B \,| \leq \aleph_1$ " it follows from the remark above that $\Vdash_P$ " B is projective ".

For the converse recall that every σ-closed p.o.-set is proper (see e.g. Theorem 2.3 in [3]). Now, if B is not openly generated then the set $\mathcal{B} = [B]^{\aleph_0} \setminus \mathrm{Sub}_{\mathrm{rc}}^{\aleph_0}(B)$ is stationary in $[B]^{\aleph_0}$. For any σ-closed p.o.-set P collapsing $|\, B \,|$ to $\leq \aleph_1$ we have that $\Vdash_P$ " $\mathcal{B}$ is stationary " by properness of P. Hence it follows from Theorem 1.2 that $\Vdash_P$ " B is not projective ". ▮ (Proposition 1.3)

Corollary 1.4 *Every openly generated Boolean algebra satisfies the* ccc.

Proof Let B be an openly generated Boolean algebra and P a σ-closed p.o.-set collapsing $|B|$ to $\aleph_1$. Then it holds that $\Vdash_P$ " B satisfies the ccc ". Hence B satisfies the ccc. ▮ (Corollary 1.4)

Corollary 1.5 (Ščepin [16]) *Every relatively complete subalgebra of an openly generated Boolean algebra is openly generated. In particular every relatively complete subalgebra of a free Boolean algebra is openly generated.*

Proof Let $B \leq_{\mathrm{rc}} C$ for an openly generated Boolean algebra C. Without loss of generality $\kappa = |C| \geq \aleph_1$. Let P be a σ-closed p.o.-set which collapses κ to $\aleph_1$. By Proposition 1.3 have

$\Vdash_P$ " B is a relatively complete subalgebra of a projective Boolean algebra. "

Since $\Vdash_P$ " $|B| \leq \aleph_1$ ", it follows from Proposition 2.12 in [14] that $\Vdash_P$ " B is projective ". ▮ (Corollary 1.5)

A Boolean algebra B has the *Bockstein separation property* (BSP) if every regular ideal over B is countably generated (see e.g. [14]).

Corollary 1.6 *Every openly generated Boolean algebra has the BSP.*

Proof Let B be an openly generated Boolean algebra and I a regular ideal over B. Let P be as in Proposition 1.3. Then we have

$\Vdash_P$ " B is projective Boolean algebra and I is a regular ideal over B ".

Since projective Boolean algebras have the BSP (see e.g. Theorem 1.12 in [14]) it follows that

$\Vdash_P$ " I is countably generated ".

Since P is σ-closed it follows that I is really countably generated. ▮ (Corollary 1.6)

For a cardinal κ, $\mathcal{L}_{\infty\kappa}$ is the logic whose formulas are constructed recursively just like in first order logic with the difference that conjunction and disjunction of a set of formulas as well as quantification over a block of free variables of cardinality $< \kappa$ is allowed. A Boolean algebra B is called $\mathcal{L}_{\infty\kappa}$*-free* if B is $\mathcal{L}_{\infty\kappa}$-elementarily equivalent to the free Boolean algebra $\mathrm{Fr}\,\kappa$ (see [10]). By Kueker's characterization of $\mathcal{L}_{\infty\kappa}$-freeness [15] (see also [5]) and Theorem 1.2 we obtain the following:

Proposition 1.7 *If B is an openly generated Boolean algebra then $B \oplus \mathrm{Fr}\,\aleph_1$ is $\mathcal{L}_{\infty\aleph_1}$-free.* ∎

The converse of the proposition above is not true: there exist an $\mathcal{L}_{\infty\aleph_1}$-free nonfree Boolean algebra B of cardinality $\aleph_1$ (see [10]). Such B is not projective hence not openly generated. On the other hand it is easily seen that $B \oplus \mathrm{Fr}\,\aleph_1$ is still $\mathcal{L}_{\infty\aleph_1}$-free.

The following characterization of openly generated Boolean algebras is also often quite useful.

Proposition 1.8 (Ščepin [16], see also Heindorf [11]) *A Boolean algebra B is openly generated if and only if there exists a set $\mathcal{S}$ of subalgebras of B such that*

(0) *$A \leq_{\mathrm{rc}} B$ for all $A \in \mathcal{S}$;*

(1) *for all $A \leq B$ there exists $A' \in \mathcal{S}$ such that $A \leq A'$ and $|\,A\,| = |\,A'\,|$;*

(2) *$\mathcal{S}$ is closed with respect to union of an increasing chain.*

Proof If there exists a set $\mathcal{S}$ of subalgebras of a Boolean algebra B satisfying (0), (1) and (2) above then $\mathcal{S} \cap \mathrm{Sub}^{\aleph_0}_{\mathrm{rc}}(B)$ is a club subset of $[B]^{\aleph_0}$ and hence B is openly generated. Conversely if B is openly generated let $\mathcal{C} \subseteq \mathrm{Sub}^{\aleph_0}_{\mathrm{rc}}(B)$ be a club subset of $[B]^{\aleph_0}$. Then

$$\mathcal{S} = \{\, A \leq B \,:\, \mathcal{C} \cap [A]^{\aleph_0} \text{ is club subset of } [A]^{\aleph_0} \,\}$$

satisfies the properties (0), (1) and (2). (To prove (1), first prove (2) and use it in a proof by induction on $|\,A\,|$.) ∎ (Proposition 1.8)

For a cardinal κ let us call a Boolean algebra B *$<\kappa$-absorbing* if, for any subalgebra A of B of cardinality $< \kappa$, there exists a relatively complete subalgebra A' of B such that $A \leq A'$ and $|\,A'\,| = |\,A\,|$. B is *absorbing* if B is $<|\,B\,|$-absorbing. By Proposition 1.8, every openly generated Boolean algebra is absorbing.

Problem 3 in [16] asks if every absorbing Boolean algebra is openly generated. An $\mathcal{L}_{\infty\aleph_1}$-free nonfree Boolean algebra B of cardinality $\aleph_1$ gives a negative answer to the problem. Also an absorbing Boolean algebra of higher cardinality need not be openly generated: Let B be an $\mathcal{L}_{\infty\aleph_1}$-free nonfree Boolean algebra of cardinality $\aleph_1$ and let $C = B \oplus \mathrm{Fr}\,\kappa$ for any cardinal κ. Clearly C is absorbing but by Corollary 1.5, and since B is embeddable into C as a regular subalgebra, C is not openly generated. Since there exist nonfree $\mathcal{L}_{\infty\aleph_1}$-free Boolean algebras of cardinality $\aleph_1$ satisfying the ccc the example above show also that an uncountable absorbing Boolean algebra need not to be an openly generated Boolean algebra. In Section 3 we shall show that under $\mathrm{MA}^+(\sigma\text{-closed})$ (for the definition see below) the open generatedness follows from a slight strengthening of the notion of ccc $<\aleph_2$-absorbing Boolean algebras (Theorem 3.1).

Here $\mathrm{MA}^+(\sigma\text{-closed})$ is the axiom which asserts that, for every σ-closed p.o.-set P, set $\mathcal{D}$ of dense subsets of P with $|\,\mathcal{D}\,| = \aleph_1$ and a P-name $\dot{S}$ of a stationary subset of

ω_1, there exists a $\mathcal{D}$-generic filter G over P such that $\dot{S} = \{\, \alpha : p \Vdash_P$ " $\alpha \in \dot{S}$ " for some $p \in G \,\}$ is stationary subset of ω_1. Shelah [17] proved that $\mathrm{MA}^+(\sigma$-closed) follows from MM. Under the existence of a super-compact cardinal we can construct a model which satisfies $\mathrm{MA}^+(\sigma$-closed) and CH.

In section 3 we shall give characterizations of openly generated Boolean algebras under $\mathrm{MA}^+(\sigma$-closed) (Theorem 3.1).

Using this characterization we show in section 4 that the assertion "every $\mathcal{L}_{\infty\aleph_2}$-free Boolean algebra is openly generated" is independent (Proposition 4.1 and Theorem 4.2). Also the same characterization is used to show the independence of Problem 7 in Ščepin [16].

2 A proof of Theorem 1.1

In this section we shall give a purely algebraic proof of Theorem 1.1. A similar proof is stated also in L. Heindorf [11]. Yet another proof of Theorem 1.1, due to I. Bandlow, using elementary submodels of H_κ for large enough κ is also to be found there. Let us begin with the following lemmas:

Lemma 2.1 *Let B, B' be openly generated Boolean algebras such that $B \leq_{\mathrm{rc}} B'$. Let $\mathcal{C} \subseteq \mathrm{Sub}_{\mathrm{rc}}^{\aleph_0}(B)$ and $\mathcal{C}'' \subseteq \mathrm{Sub}_{\mathrm{rc}}^{\aleph_0}(B')$ be club subsets of $[B]^{\aleph_0}$ and $[B']^{\aleph_0}$ respectively. Then there exists $\mathcal{C}' \subseteq \mathrm{Sub}_{\mathrm{rc}}^{\aleph_0}(B')$ such that*

(0) *$\mathcal{C}'$ is a club subset of $[B']^{\aleph_0}$;*

(1) *$\mathcal{C} \subseteq \mathcal{C}' \subseteq \mathcal{C} \cup \mathcal{C}''$ and $\mathcal{C} = \{\, C \cap B : C \in \mathcal{C}' \,\}$;*

(2) *For any $C \in \mathcal{C}'$ and $b \in B$ it holds that $\mathrm{pr}^{B'}_{C\cap B}(b) = \mathrm{pr}^{B'}_{C}(b)$.*

Proof Let

$$\mathcal{C}^* = \{\, C \in \mathcal{C}'' : C \cap B \in \mathcal{C},\ \mathrm{pr}^{B'}_{B}(c) \in C \text{ holds for all } c \in C \,\}.$$

Claim 2.1.1 *For every $C \in \mathcal{C}^*$ and $b \in B$, $\mathrm{pr}^{B'}_{C\cap B}(b) = \mathrm{pr}^{B'}_{C}(b)$ holds.*

Proof of Claim 2.1.1 Let $b \in B$. It is enough to show: $\mathrm{pr}^{B'}_{C\cap B}(b) \geq \mathrm{pr}^{B'}_{C}(b)$. Since $\mathrm{pr}^{B'}_{C}(b) \in C$ we have $\mathrm{pr}^{B'}_{C}(b) \leq \overline{\mathrm{pr}}^{B'}_{B}(\mathrm{pr}^{B'}_{C}(b)) \leq b$. By the definition of $\mathcal{C}^*$ we have $\overline{\mathrm{pr}}^{B'}_{B}(\mathrm{pr}^{B'}_{C}(b)) \in C$. Hence $\overline{\mathrm{pr}}^{B'}_{B}(\mathrm{pr}^{B'}_{C}(b)) \leq \mathrm{pr}^{B'}_{C\cap B}(b)$. ▮ (Claim 2.1.1)

Clearly $\mathcal{C}^*$ is club in $[B']^{\aleph_0}$. Hence $\mathcal{C}' = \mathcal{C} \cup \mathcal{C}^*$ is as desired. ▮ (Lemma 2.1)

Lemma 2.2 *For a limit ordinal δ let $(B_\alpha)_{\alpha\in\delta}$ be an increasing chain of Boolean algebras such that $B_\alpha \leq_{\mathrm{rc}} B_\beta$ for all $\alpha \leq \beta < \delta$. Assume that $(\mathcal{C}_\alpha)_{\alpha<\delta}$ is an increasing chain such that*

(0) *$\mathcal{C}_\alpha \subseteq \mathrm{Sub}_{\mathrm{rc}}^{\aleph_0}(B_\alpha)$ and $\mathcal{C}_\alpha$ is a club subset of $[B_\alpha]^{\aleph_0}$;*

(1) *For all* $\alpha \leq \beta < \delta$, $\mathcal{C}_\alpha = \{\, C \cap B_\alpha \,:\, C \in \mathcal{C}_\beta \,\}$ *holds;*

(2) *For all* $\alpha \leq \beta < \delta$, $C \in \mathcal{C}_\beta$ *and* $b \in B_\alpha$, $\mathrm{pr}^{B_\beta}_{C \cap B_\alpha}(b) = \mathrm{pr}^{B_\beta}_{C}(b)$ *holds;*

(3) *For all limit* $\gamma < \delta$,

$$\mathcal{C}_\gamma = \{\, C \in [B]^{\aleph_0} \,:\, C \cap B_\alpha \in \mathcal{C}_\alpha \text{ for all } \alpha < \gamma \,\}$$

holds.

Let $B_\delta = \bigcup_{\alpha<\delta} B_\alpha$ *and*

$$\mathcal{C}_\delta = \{\, C \in [B_\delta]^{\aleph_0} \,:\, C \cap B_\alpha \in \mathcal{C}_\alpha \text{ for all } \alpha < \delta \,\}.$$

Then $\mathcal{C}_\delta \supseteq \bigcup_{\alpha<\delta} \mathcal{C}_\alpha$, $\mathcal{C}_\delta \subseteq \mathrm{Sub}^{\aleph_0}_{\mathrm{rc}}(B_\delta)$, $\mathcal{C}_\delta$ *is club in* $[B_\delta]^{\aleph_0}$ *and* $(\mathcal{C}_\alpha)_{\alpha<\delta+1}$ *satisfies the conditions* (1), (2) *and* (3) *for* $\delta+1$ *in place of* δ.

Proof Let $C \in \mathcal{C}_\delta$ and $b \in B_\delta$, say $b \in B_\alpha$ for some $\alpha < \delta$. Then by (2) we have

$$\mathrm{pr}^{B_\alpha}_{C \cap B_\alpha}(b) = \mathrm{pr}^{B_\beta}_{C \cap B_\alpha}(b) = \mathrm{pr}^{B_\beta}_{C \cap B_\beta}(b)$$

for every $\beta < \delta$ with $\alpha \leq \beta$. Thus $\mathrm{pr}^{B_\delta}_{C}(b)$ exists and is equal to $\mathrm{pr}^{B_\delta}_{C \cap B_\alpha}(b)$. Hence $\mathcal{C}_\delta \subseteq \mathrm{Sub}^{\aleph_0}_{\mathrm{rc}}(B)$. This also shows that $(\mathcal{C}_\alpha)_{\alpha<\delta+1}$ satisfies the condition (2). By the definition of $\mathcal{C}_\delta$ it is clear that $(\mathcal{C}_\alpha)_{\alpha<\delta+1}$ also satisfies the conditions (1), (3). Also from the definition of $\mathcal{C}_\delta$ it follows that $\mathcal{C}_\delta \supseteq \bigcup_{\alpha<\delta} \mathcal{C}_\alpha$ and $\mathcal{C}_\delta$ is closed.

$\mathcal{C}_\delta$ is unbounded: For $X \in [B_\delta]^{\aleph_0}$ let $(\alpha_n)_{n\in\omega}$ be an increasing sequence of ordinals $< \delta$ such that $X \subseteq \bigcup_{n\in\omega} B_{\alpha_n}$ holds. Let $(C_n)_{n\in\omega}$ be an increasing chain of countable subalgebras of B such that $X \cap B_{\alpha_n} \subseteq C_n$ and $C_n \in \mathcal{C}_{\alpha_n}$ for all $n \in \omega$. Let $C = \bigcup_{n\in\omega} C_n$. Clearly $X \subseteq C$.

Claim 2.2.1 $C \in \mathcal{C}_\delta$.

Proof of Claim 2.2.1 Let $\alpha^* = \sup_{n\in\omega} \alpha_n$. For any $\alpha < \alpha^*$ let $n_0 < \omega$ be such that $\alpha \leq \alpha_{n_0}$. We have $C \cap B_\alpha = \bigcup_{n \geq n_0} C_n \cap B_\alpha$. By (1), $(C_n \cap B_\alpha)_{n_0 \leq n < \omega}$ is an increasing sequence of elements of $\mathcal{C}_\alpha$. It follows that $C \cap B_\alpha \in \mathcal{C}_\alpha$.
$C \cap B_{\alpha^*} = C$ and, by (3), $C \in \mathcal{C}_{\alpha^*}$ holds.
For $\alpha > \alpha^*$ we have $\mathcal{C}_\alpha \supseteq \mathcal{C}_{\alpha^*}$. Hence again $C \cap B_\alpha \in \mathcal{C}_\alpha$ holds. ▮ (Claim 2.2.1)

▮ (Lemma 2.2)

Proof of Theorem 1.1 Using Lemma 2.1 and Lemma 2.2 inductively we can construct an increasing chain $(\mathcal{C}_\alpha)_{\alpha<\delta}$ satisfying the conditions (0), (1), (2) and (3) in Lemma 2.2. Then $\mathcal{C}_\delta$ defined as in Lemma 2.2 witnesses that $\bigcup_{\alpha<\delta} B_\alpha$ is openly generated. ▮

3 Openly generated Boolean algebras under $\mathrm{MA}^+(\sigma\text{-closed})$

Under $\mathrm{MA}^+(\sigma\text{-closed})$ we have the following characterization of openly generated Boolean algebras:

Theorem 3.1 ($\mathrm{MA}^+(\sigma\text{-closed})$) *For a Boolean algebra B the following are equivalent:*

(*a*) *B is openly generated;*

(*b*) *B satisfies the ccc and has the BSP, and every relatively complete subalgebra A of B of cardinality $\leq 2^{\aleph_0}$ is openly generated;*

(*c*) *B is $<\aleph_1$-absorbing and every relatively complete subalgebra A of B of cardinality $\leq \aleph_1$ is projective;*

(*d*) *B is $<\aleph_1$-absorbing and*

(*) $\{ A \leq B : |A| = \aleph_1$ and A is projective$\}$ *is cofinal in* $[B]^{\aleph_1}$ *with respect to* $\subseteq$.

Proof $(a) \Rightarrow (b)$: By Corollary 1.6 and Corollary 1.5.
$(b) \Rightarrow (c)$: Let B be as in (b). Let C be a countable subalgebra of B. Since B satisfies the ccc and has the BSP, by Theorem 2 in [1] (see also [11]) there exists an $A \leq_{\mathrm{rc}} B$ such that $C \leq A$ and $|A| \leq 2^{\aleph_0}$. By the assumption A is openly generated. Hence by Proposition 1.8 there exists a countable $A' \leq_{\mathrm{rc}} A$ such that $C \leq A'$. This proves that B is $<\aleph_1$-absorbing. If $A \leq_{\mathrm{rc}} B$ is of cardinality $\leq \aleph_1$ A is openly generated by the assumption. Hence A is projective by the remark after Theorem 1.2.

$(c) \Rightarrow (d)$: Trivial.

$(d) \Rightarrow (c)$: Let B be as in (d). If $A \leq_{\mathrm{rc}} B$ is of cardinality $\leq \aleph_1$ then $A \leq C$ for some projective $C \leq B$. Since we have $A \leq_{\mathrm{rc}} B$ it follows from Proposition 2.12 in [14] that A is projective.

$(c) \Rightarrow (a)$: Let B be as in (c). Toward a contradiction assume that B is not openly generated. Then $T = \{ C : C \leq_{\neg\mathrm{rc}} B, |C| = \aleph_0 \}$ is stationary in $[B]^{\aleph_0}$. Let

$$P = \{ p : \text{for some } \alpha < \omega_1,\ p : \alpha \to B \text{ and } p \text{ is 1-1} \}$$

be the σ-closed p.o.-set with the partial ordering $p \leq q \Leftrightarrow q \subseteq p$. For each $\alpha < \omega_1$ let

$$D_\alpha = \{ p \in P : p[\beta] \leq_{\mathrm{rc}} B \text{ for some } \beta \subseteq \mathrm{dom}(p) \text{ such that } \alpha \leq \beta \},$$

$$E_\alpha = \{ p \in P : \begin{array}{l} \alpha \in \mathrm{dom}(p),\ \text{if } p[\alpha] \leq_{\neg\mathrm{rc}} B \text{ then there is some} \\ \beta \in \mathrm{dom}(p) \text{ such that } p(\beta) \text{ has no projection to } p[\alpha] \end{array} \}.$$

Then D_α and E_α are dense in P. Let $\mathcal{D}_1 = \{ D_\alpha : \alpha < \omega_1 \}$, $\mathcal{D}_2 = \{ E_\alpha : \alpha < \omega_1 \}$ and $\mathcal{D} = D_1 \cup D_2$ Let $\dot{S}$ be a P-name such that

$$\Vdash_P \text{“ } \dot{S} = \{ \alpha < \omega_1 : (\bigcup \dot{G})[\alpha] \in T \} \text{ ”}$$

where $\dot{G}$ is the standard P-name of a generic filter. Then we have $\Vdash_P$ " $\dot{S}$ is stationary subset of ω_1 ". Hence by $\mathrm{MA}^+(\sigma\text{-closed})$ there exists a $\mathcal{D}$-generic filter G such that $\dot{S}^G$ is stationary. Let $A = (\bigcup G)[\omega_1]$. By $\mathcal{D}_1$-genericity of G we have $A \leq_{\mathrm{rc}} B$. But since $\{ (\bigcup G)[\alpha] \, : \, \alpha \in \dot{S}^G \}$ is stationary in $[A]^{\aleph_0}$, and since G is $\mathcal{D}_2$-generic, A cannot be projective by Theorem 1.1. This is a contradiction. ▮ (Theorem 3.1)

Note that in teh proof of Theorem 3.1 we used $\mathrm{MA}^+(\sigma\text{-closed})$ only for the proof of $(c) \Rightarrow (a)$. Note also that, by the Boolean algebra constructed in the proof of Proposition 4.1 below, none of (b) – (d) characterizes the openly generated Boolean algebras under $V = L$.

4 Some independence results

In ZFC we can construct $\mathcal{L}_{\infty\aleph_2}$-free Boolean algebras of cardinality $\aleph_2$ which are nonfree but openly generated ([9]). We show that the existence of $\mathcal{L}_{\infty\aleph_2}$-free non openly generated Boolean algebras is independent from set theory.

Proposition 4.1 ($V = L$) *For every non-weakly-compact regular κ there exists an $\mathcal{L}_{\infty\kappa}$-free Boolean algebra B which is not openly generated.*

Proof Under $V = L$ there exists a non-reflecting stationary $S \subseteq \{ \alpha < \kappa \, : \, \mathrm{cof}(\alpha) = \omega \}$ for every non weakly compact regular κ. In [9] it is shown that we can construct an $\mathcal{L}_{\infty\kappa}$-free Boolean algebra B which is a union of a continuously increasing chain of free subalgebras $(B_\alpha)_{\alpha<\kappa}$ of cardinality $< \kappa$ such that $S = \{ \alpha < \kappa \, : \, B_\alpha \leq_{\neg\mathrm{rc}} B \}$. By Proposition 1.8, B is not openly generated. ▮ (Proposition 4.1)

On the other hand it is easily seen that every $\mathcal{L}_{\infty\aleph_2}$-free Boolean algebra is $< \aleph_1$-absorbing and satisfies the condition (*) in Theorem 3.1. Hence we obtain the following theorem:

Theorem 4.2 ($\mathrm{MA}^+(\sigma\text{-closed})$) *Every $\mathcal{L}_{\infty\aleph_2}$-free Boolean algebra is openly generated.* ▮

For a cardinal κ let us say that a Boolean algebra B satisfies $\mathrm{C}(\kappa)$ if the following conditions hold: There exist an upward directed p.o.-set I and an indexed set $(B_i)_{i\in I}$ of subalgebras of B such that $B_i \leq B_j$ holds for every i, $j \in I$ with $i \leq j$; every increasing chain C in I of length $< \kappa$ has its supremum i_C in I and $\bigcup_{i\in C} B_i = B_{i_C}$; B_i is projective for every $i \in I$ and $B = \bigcup_{i\in I} B_i$.

E. V. Ščepin asked, translated into the language of Boolean algebras, if every Boolean algebra B satisfying $\mathrm{C}(\aleph_2)$ is openly generated (Problem 7 in [16]). For some other problems about openly generated Boolean algebras and solutions of some of them see Heindorf [11]. In the following we show that this problem is independent from set theory (modulo some large cardinal). Note that, by Proposition 1.8, every openly generated Boolean algebra satisfies the condition $\mathrm{C}(\aleph_2)$.

Proposition 4.3 ($V = L$) *For every cardinal κ there exists a non openly generated B satisfying* $\mathrm{C}(\kappa)$.

Proof It is enough to show the existence of B as above for every regular non-weakly-compact κ. Clearly the Boolean algebra B in the proof of Proposition 4.1 is such a Boolean algebra. ▮ (Proposition 4.3)

It follows from Proposition 4.3 that under $V = L$ Problem 7 (and also Problem 8) in [16] has the negative solution.

In contrast to this we have the following theorem which gives yet another characterization of openly generated Boolean algebras under $\mathrm{MA}^+(\sigma\text{-closed})$:

Theorem 4.4 ($\mathrm{MA}^+(\sigma$-closed)) *For every Boolean algebra B, B is openly generated if and only if B satisfies* $\mathrm{C}(\aleph_2)$.

Proof Assume that B is a Boolean algebra satisfying the condition $\mathrm{C}(\aleph_2)$. Since (*) in Theorem 3.1 (d) follows immediately from $\mathrm{C}(\aleph_2)$, it is enough to show that B is $< \aleph_1$-absorbing.

Claim 4.4.1 *B has the BSP.*

Proof Let $(B_i)_{i\in I}$ be as in the definition of $\mathrm{C}(\aleph_2)$ for B. Let

$$I^* = \{ (i, A) : i \in I,\ A \leq_{\mathrm{rc}} B_i,\ |A| \leq \aleph_1 \}$$

be the p.o.-set with the partial ordering

$$(i, A) \leq (i', A') \quad \Leftrightarrow \quad i \leq i' \text{ and } A \leq A'.$$

For $i^* \in I^*$ with $i^* = (i, A)$, let $A_{i^*} = A$. We show that every increasing chain in I^* of length ω_1 has its supremum: Let $(i^*_\alpha)_{\alpha<\omega_1}$ be an increasing sequence in I^*. Say $i^*_\alpha = (i_\alpha, A_\alpha)$ for $\alpha < \omega_1$. Let i be the supremum of $(i_\alpha)_{\alpha<\omega_1}$ and $A = \bigcup_{\alpha<\omega_1} A_\alpha$. Then $A \leq B_i$. We show that $A \leq_{\mathrm{rc}} B_i$ holds. For contradiction let us assume that there is some $b \in B_i$ without its projection on A. Without loss of generality we may assume that $b \in B_{i_0}$. Then $(\mathrm{pr}^{B_{i_\alpha}}_{A_\alpha}(b))_{\alpha<\omega_1}$ is non-eventually constant increasing sequence of elements of B_i. This is a contradiction since B_i satisfies the ccc.

Now suppose that there would be a non-countably generated regular ideal J over B. Let $(i^*_\alpha)_{\alpha<\omega_1}$ and $(C_\alpha)_{\alpha<\omega_1}$ be such that

$(i^*_\alpha)_{\alpha<\omega_1}$ is an increasing chain in I^* and $A_{i^*_\alpha}$ is countable for every $\alpha < \omega_1$;

$(C_\alpha)_{\alpha<\omega_1}$ is a continuously increasing chain of countable subalgebras of B such that $(C_\alpha, J \cap C_\alpha) \prec (B, J)$ for every $\alpha < \omega_1$;

$C_\alpha \leq A_{i^*_\alpha} \leq C_{\alpha+1}$ for every $\alpha < \omega_1$;

$J \cap C_\alpha$ does not generate $J \cap C_{\alpha+1}$ for every $\alpha < \omega_1$.

Let $i^* = (i, A)$ be the supremum of $(i^*_\alpha)_{\alpha<\omega_1}$. Since we have $(A, J\cap A) \prec (B, J)$, $J\cap A$ is a regular ideal over A. Since $A \leq_{\rm rc} B_i$, $|\,A\,| \leq \aleph_1$ and B_i is projective, A is also projective (see e.g. Proposition 2.12 in [14]). Hence A has the BSP (see e.g. Theorem 1.12 in [14]). But this is a contradiction since $J \cap A$ is not countably generated by the last condition of the construction. ▮ (Claim 4.4.1)

For a countable set X and $Y \supseteq X$ it is easily seen that there are only countably many (model theoretic) types over $\mathrm{Fr}\,X$ realized in $\mathrm{Fr}\,Y$. Using this fact we can prove the following:

Claim 4.4.2
If

$$\{\,A \,:\, A \leq B,\ |\,A\,| = \aleph_1 \textit{ and } A \textit{ is a subalgebra of a free Boolean algebra }\,\}$$

is cofinal in $[B]^{\aleph_1}$ *then for every countable* $C \leq B$ *the set*

$$S(C) = \{\,\{\,c \in C \,:\, c \leq b\,\} \,:\, b \in B\,\}$$

is countable. ▮

Let A be an arbitrary countable subalgebra of B. Let $(A_\alpha)_{\alpha<\omega_1}$ be a continuously increasing chain of countable subalgebras of B such that $A_0 = A$ and for every $t \in S(A_\alpha)$ a countable generator of the ideal

$$I_t = \{\,b \in B \,:\, a \cdot b = 0 \text{ for every } a \in t\,\}$$

is included in $A_{\alpha+1}$. The last condition is possible by Claim 4.4.2 and since B satisfies the BSP by Claim 4.4.1. Note that I_t as above is a regular ideal.

Let $A' = \bigcup_{\alpha<\omega_1} A_\alpha$. We have $A \leq A' \leq B$ and $|\,A'\,| = \aleph_1$. We show that

Claim 4.4.3 $A' \leq_{\rm rc} B$.

Proof Let $b \in B$ and let D be a maximal pairwise disjoint subset of $\{\,a \in A' \,:\, a\cdot b = 0\,\}$. Since B satisfies the ccc we have $|\,D\,| \leq \aleph_0$. Let $\alpha < \omega_1$ be such that $D \subseteq A_\alpha$ and let $t = \{\,c \in A_\alpha \,:\, c \leq -b\,\}$. By the construction there exists a countable generator $X \subseteq A_{\alpha+1}$ of I_t. Without loss of generality we may assume that X is closed under $+$. Since $b \in I_t$ there exists an $a \in X$ such that $b \leq a$. We show that a is minimal above b: If $a' \in A'$ is such that $b \leq a'$ then we have that $(a \cdot -a') \leq -b$. If $(a \cdot -a') \neq 0$ then by the maximality of D there would be some $a'' \in D$ such that $(a \cdot -a') \cdot a'' \neq 0$. But this is a contradiction to $a \in I_t$. ▮ (Claim 4.4.3)

Now by the assumption there exists a projective $P \leq B$ such that $A' \leq P$. Since $A' \leq_{\rm rc} P$ by Claim 4.4.3 and $|\,A'\,| = \aleph_1$, it follows that A' is also projective and hence there exists a countable $A'' \leq_{\rm rc} A'$ such that $A \leq A''$. This proves that B is $< \aleph_1$-absorbing. ▮ (Theorem 4.4)

References

[1] I. Bandlow, Factorization theorems and 'strong' sequences in bicompacta, Soviet Mathematics Doklady, 22:1 (1980), 196–200.

[2] J. Barwise and S. Feferman (Eds.), Model-Theoretic Logics, (Springer-Verlag, New York, Heidelberg, Berlin, 1985)

[3] J.E. Baumgartner, Applications of the Proper Forcing Axiom, in Handbook of Set Theoretic Topology, Eds.: K. Kunen and J.E. Vaughan (North-Holland, Amsterdam, New York, Oxford, 1984).

[4] R.E. Beaudoin, Strong analogues of Martin's axiom imply Axiom R, the Journal of Symbolic Logic, Vol. 52, No. 1, (1987) 216–218.

[5] P. C. Eklof, Applications to Algebra, in [2](1985), 423–441.

[6] Q. Feng and T. Jech, Local clubs, reflection, and preserving stationary sets, Proceeding of London Mathematical Society (3)58 (1989), 237–257.

[7] M. Foreman, M. Magidor and S. Shelah, Martin's Maximum, saturated ideals, and non-regular ultrafilters. Part I, Annals of Mathematics 127(1) (1988).

[8] S. Fuchino, Potential embedding and versions of Martin's axiom, Notre Dame Journal of Symbolic Logic 33(4) (1992), 481 – 492.

[9] ____, $\mathcal{L}_{\infty\kappa}$-Cohen algebras, preprint.

[10] S. Fuchino, S. Koppelberg and M. Takahashi, On $\mathcal{L}_{\infty\kappa}$-free Boolean algebras, Annals of Pure and Applied Logic 55 (1992), 265–284.

[11] L. Heindorf, Openly generated Boolean algebras, mimeograph dated on October 9, 1992.

[12] T. Jech, Multiple Forcing, (Cambridge University Press, 1987).

[13] S. Koppelberg, General Theory of Boolean Algebras, in: Handbook of Boolean Algebras, J. D. Monk with R. Bonnet(Eds.) Vol. 1, (North-Holland, Amsterdam, New York, Oxford, Tokyo, 1989).

[14] ____, Projective Boolean Algebras, in: Handbook of Boolean Algebras, J. D. Monk with R. Bonnet(Eds.) Vol. 3, (North-Holland, Amsterdam, New York, Oxford, Tokyo, 1989), 741–773.

[15] D. Kueker, $\mathcal{L}_{\infty\omega_1}$-elementarily equivalent models of power ω_1, in: Logic Year 1979-80, Lecture Notes in Mathematics Vol. 859, (Springer-Verlag, 1981), 120–131.

[16] E. V. Ščepin, Functors and uncountable powers of compacta, Uspechi Matematicheskikh Nauk, 36: 3 (1981), 3–62. (English translation: Russian Mathematical Surveys 36 (1981), 1–71.)

[17] S. Shelah, Semiproper forcing axiom implies Martin's Maximum but not PFA^+, Journal of Symbolic Logic Vol. 52, No. 2 (1987), 360–367.

Sakaé Fuchino

Institut für Mathematik II, Freie Universität Berlin
Arnimallee 3, W-1000 Berlin 33, Federal Republic of Germany
e-mail: fuchino@math.fu-berlin.de