# On a theorem of Shapiro

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**Abstract**

We show that a theorem of Leonid B. Shapiro which was proved under MA, is actually independent from ZFC. We also give a direct proof of the Boolean algebra version of the theorem under MA(*Cohen*).

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## 1 Introduction

L.B. Shapiro [8] recently proved the following theorem:

**Theorem 1.1** (L.B. Shapiro) (MA(*Cohen*)) *For any compact Hausdorff space $X$ of weight $< 2^{\aleph_0}$ and $\aleph_0 \leq \tau < 2^{\aleph_0}$ the following assertions are equivalent:*

- *i)* *There exists a continuous surjection from $X$ onto ${}^{\tau}\mathbb{I}$;*
- *ii)* *There exists a continuous injection from ${}^{\tau}2$ into $X$;*
- *iii)* *There exists a closed subset $Y \subseteq X$ such that $\chi(y, Y) \geq \tau$ for every $y \in Y$.*

The original proof of Theorem 1.1 by L.B. Shapiro in [8] was formulated under MA. However practically the same proof still works when merely MA(Cohen) is assumed where MA(Cohen) stands for Martin's Axiom restricted to the partial orderings of the form $\mathrm{Fn}(\kappa, 2)$.

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A part of the theorem above can be translated into the language of Boolean algebras:

**Corollary 1.2** (Boolean algebra version of Shapiro's theorem) (MA(*Cohen*)) *For any infinite Boolean algebra $B$ of cardinality $< 2^{\aleph_0}$ and any infinite $\tau$, the following are equivalent:*

*i′)* *There exists an injective Boolean mapping from* $\mathrm{Fr}\,\tau$ *into $B$;*
*ii′)* *There exists a surjective Boolean mapping from $B$ onto* $\mathrm{Fr}\,\tau$.

The implication from *ii′*) to *i′*) as well as the implication from *ii*) to *i*) can be proved already in ZFC. For the proof of *ii*) from *i*), let $g : {}^{\tau}2 \rightarrow X$ be a continuous injection. Note that $g[{}^{\tau}2]$ is a closed subset of $X$. For any fixed $y_0 \in {}^{\tau}2$ let $f' : X \rightarrow {}^{\tau}2$ be defined by

$$f'(x) = \begin{cases} g^{-1}(x) & ;\text{ if } x \in g[{}^{\tau}2], \\ y_0 & ;\text{ otherwise.} \end{cases}$$

Then $f'$ is a continuous surjection from $X$ onto ${}^{\tau}2$. Let $f''$ be a continuous surjection from ${}^{\tau}2$ to ${}^{\tau}\mathbb{I}$. E.g. let $h : {}^{\omega}2 \rightarrow \mathbb{I}$ be the continuous surjection defined by $u \mapsto$ the real represented by the binary expression $0.u(0)u(1)u(2)\cdots$. $h^{\kappa} : {}^{\kappa}({}^{\omega}2) \rightarrow {}^{\kappa}\mathbb{I}$ is then a continuous surjection. Since ${}^{\kappa}({}^{\omega}2)$ is homeomorphic to ${}^{\kappa}2$ we can find a continuous surjection $f''$ from ${}^{\tau}2$ onto ${}^{\tau}\mathbb{I}$ corresponding to $h^{\kappa}$. The mapping $g = f'' \circ f'$ is then as desired. In the next section we shall give a direct proof of *i′*) $\Rightarrow$ *ii′*). For *iii*) $\Rightarrow$ *i*) we need some deep results by Shapiro on dyadic compactum (see [8]).

The equivalence of the assertions *i′*) and *ii′*) above is not true in general for Boolean algebras of cardinality $\geq 2^{\aleph_0}$: For any $\sigma$-complete Boolean algebra $B$ and any infinite $\kappa$, there exits no surjective Boolean mapping $f : B \rightarrow \mathrm{Fr}\,\kappa$ (see Lemma 1.3 below). Hence e.g. for Boolean algebra $B = \overline{\mathrm{Fr}\,\omega}$ we have that $\mid B \mid = 2^{\aleph_0}$; $\mathrm{Fr}\,2^{\aleph_0}$ is embeddable into $B$ (by Balcar-Fraňek-Theorem, see [1]) but there exists no surjective Boolean mapping from $B$ onto $\mathrm{Fr}\,2^{\aleph_0}$. The non-existence of surjective Boolean mapping from a $\sigma$-complete Boolean algebra in the ground model onto $\mathrm{Fr}\,\tau$ is preserved in a generic extension by a partial ordering of cardinality $< \tau$ though $B$ may be no more $\sigma$-complete in such a generic extension:

**Lemma 1.3** *Let $B$ be a $\sigma$-complete Boolean algebra and $P$ a partial ordering. For any $\kappa > \mid P \mid$ we have that*

$$\Vdash_P \text{“}\textit{there exists no surjective Boolean mapping from } B \textit{ onto } \mathrm{Fr}\,\kappa\text{”}.$$

**Proof** Suppose that there would be a $P$-name $\dot{f}$ such that

$$\Vdash_P \text{“}\dot{f} : B \rightarrow \mathrm{Fr}\,\kappa \text{ is a surjective Boolean mapping”}.$$

For each $p \in P$ let

$$B_p = \{\, b \in B \,:\, p \Vdash_P \text{“}\dot{f}(b) = c \text{ for some } c \in \mathrm{Fr}\,\kappa\text{”}\,\}$$

and

$$C_p = \{\, c \in \mathrm{Fr}\,\kappa \,:\, p \Vdash_P \text{“}\dot{f}(b) = c \text{ for some } b \in B\text{”}\,\}.$$

Then $B_p$ and $C_p$ are subalgebras of $B$ and $\mathrm{Fr}\,\kappa$ respectively. Since $\bigcup_{p\in P} C_p = \mathrm{Fr}\,\kappa$ and $\kappa > |\, P \,|$ there exists some $p \in P$ such that $C_p$ is infinite. Let $c_n$, $n < \omega$ be pairwise disjoint positive elements of $C_p$. By the definition of $B_p$ and $C_p$, there exits pairwise disjoint positive elements $b_n$, $n < \omega$ of $B_p$ such that $p \Vdash_P$ “$\dot{f}(b_n) = c_n$” holds for every $n < \omega$. Let $X \subseteq \omega$ be such that there exists no $c \in \mathrm{Fr}\,\kappa$ such that $c \cdot c_n = c_n$ holds for all $n \in X$ and $c \cdot c_n = 0$ for all $n < \omega \setminus X$. Let $d = \Sigma^B_{n\in X} b_n$. Then for any $q \leq p$ there can be no $c \in \mathrm{Fr}\,\kappa$ such that $q \Vdash_P$ “$\dot{f}(d) = c$”. This is a contradiction. ❏ (Lemma 1.3)

The lemma above together with Corollary 1.2 yields the following:

**Proposition 1.4** *Let $B$ be a complete Boolean algebra with $|\, B \,| = \tau \geq \aleph_0$. Then*

$$\Vdash_{\mathrm{Fn}(\kappa,2)} \text{“\textit{there exists no surjective Boolean mapping from} } B \text{ \textit{onto} } \mathrm{Fr}\,\tau\text{”}$$

*holds if and only if $\kappa < \tau$.*

**Proof** If $\kappa < \tau$ then $|\, \mathrm{Fn}(\kappa, 2) \,| = \kappa < \tau$. Hence by Lemma 1.3,

$$\Vdash_{\mathrm{Fn}(\kappa,2)} \text{“there exists no surjective Boolean mapping from } B \text{ onto } \mathrm{Fr}\,\tau\text{”}$$

holds.

Suppose now that $\kappa \geq \tau$. Then as in the proof of Proposition 2.1, we can show that

$$\Vdash_{\mathrm{Fn}(\kappa,2)} \text{“there exists a surjective Boolean mapping from } B \text{ onto } \mathrm{Fr}\,\tau\text{”}$$

holds. ❏ (Proposition 1.4)

Now, (⫯) (read “stick”, see [2]) is the following principle:

(⫯): There exists a sequence $(x_\alpha)_{\alpha<\omega_1}$ of countable subsets of $\omega_1$ such that for any $y \in [\omega_1]^{\aleph_1}$ there exists $\alpha < \omega_1$ such that $x_\alpha \subseteq y$.

Clearly (⫯) follows from CH. Another combinatorial principle (♣), a strengthening of (⫯), is introduced in Ostaszewski [7]. Let $Lim(\omega_1) = \{\, \gamma < \omega_1 \,:\, \gamma \text{ is a limit}\,\}$.

(♣): There exists a sequence $(x_\gamma)_{\gamma\in Lim(\omega_1)}$ of countable subsets of $\omega_1$ such that for every $\gamma \in Lim(\omega_1)$, $x_\gamma$ is a cofinal subset of $\gamma$, $otp(x_\gamma) = \omega$ and for every $X \in [\omega_1]^{\aleph_1}$ there is $\gamma \in Lim(\omega_1)$ such that $x_\gamma \subseteq X$.

Clearly (⫯) follows from (♣). Unlike (⫯), (♣) does not follow from CH, since (♣) + CH is equivalent with $\diamondsuit$ (K. Devlin, see [7]). For more about the combinatorial principles (⫯) and (♣), and independence results connected with them, see [4].

MA(*countable*) — Martin's axiom restricted to countable partial orderings — and MA(*Cohen*) both add a lot of Cohen reals over any small model of (a sufficiently large finite subset of) ZFC and in many cases where this property is needed, MA(*countable*) is just enough. Hence it seems to be quite natural to ask if these axioms are perhaps equivalent. However they are not. I. Juhász proved in an unpublished note that $\neg$CH + MA(*countable*) + (♣) is consistent (two other constructions of models of $\neg$CH + MA(*countable*) + (♣) are to be found in [5] and [4].). On the other hand, it is easy to see that the negation of MA(Fn($\aleph_1$, 2)) follows from $\neg$CH + (♣): using (⫯) we can obtain a Boolean algebra $B$ of cardinality $\aleph_1$ such that $\mathrm{Fr}\,\omega_1$ is embeddable into $B$ but there is no surjection from $B$ onto $\mathrm{Fr}\,\omega_1$ (see Theorem 4.4). By Proposition 2.1, this shows that $m_{\mathrm{Fn}(\aleph_1,2)} = \aleph_1 < 2^{\aleph_0}$. It follows also that the assertions of Theorem 1.1 and Corollary 1.2 are independent from ZFC and MA(*countable*) is not enough to prove them.

Corollary 1.2 for other variety than Boolean algebras can be simply false. E.g., this is the case in the variety of abelian groups: in [3], an $\aleph_1$-free abelian group $G$ in $\aleph_1$ is constructed (in ZFC) which contains uncountable free subgroup but $Hom(G, Z) = 0$.

## 2 A proof of the Boolean algebra version of the theorem

In this section we shall prove Corollary 1.2. More precisely we prove the following Proposition 2.1. For any class $\mathcal{C}$ of partial orderings Let

$$\begin{aligned} m_{\mathcal{C}} = \min\{\, |\,\mathcal{D}\,| \ : & \ \mathcal{D} \text{ is a family of dense subsets of } P \text{ for some } P \in \mathcal{C} \\ & \text{such that there exists no } \mathcal{D}\text{-generic filter over } P\,\} \end{aligned}$$

If $\mathcal{C}$ is a singleton $\{\,P\,\}$, we shall write simply $m_P$ in place of $m_{\{\,P\,\}}$. Let us say that two partial orderings $P$, $Q$ are coabsolute when their completions are isomorphic. It is easy to see that for any class $\mathcal{C}$ of partial orderings $m_{\mathcal{C}} = m_{\tilde{\mathcal{C}}}$ where $\tilde{\mathcal{C}} = \{\,Q\ :\ Q \text{ is coabsolute with some } P \in \mathcal{C}\,\}$. If the class $\mathcal{C}$ is introduced by a property $\mathcal{P}$ of Boolean algebras, we also write $m_{\mathcal{P}}$ in place of $m_{\mathcal{D}}$. We also write $m_{countable} = m_{\{\,P\,:\,P \text{ is countable}\,\}}$ and $m_{Cohen} = m_{\{\,P\,:\,P=\mathrm{Fn}(\kappa,2) \text{ for some } \kappa\,\}}$. It is known that $m_{countable}$ is equal to the covering number of meager sets in $\mathbb{R}$. Clearly MA(*Cohen*) (MA(*countable*), MA etc. respectively) holds if and only if $m_{Cohen} = 2^{\aleph_0}$ ($m_{countable} = 2^{\aleph_0}$, $m_{ccc} = 2^{\aleph_0}$ etc. respectively) and we have $m_{ccc} \leq m_{Cohen} \leq m_{countable}$.

**Proposition 2.1** *Let $B$ be a Boolean algebra containing $\mathrm{Fr}\,\kappa$ as a subalgebra. If $\mid B \mid < m_{\mathrm{Fn}(\kappa,2)}$, then there exists a surjective Boolean mapping from $B$ onto $\mathrm{Fr}\,\kappa$.*

**Proof** By Sikorski's theorem, there is a Boolean mapping from $B$ to $\overline{\mathrm{Fr}\,\kappa}$ — the completion of $\mathrm{Fr}\,\kappa$, extending the inverse of the canonical embedding of $\mathrm{Fr}\,\kappa$ into $B$. Hence without loss of generality we may assume that $B$ is a subalgebra of $\overline{\mathrm{Fr}\,\kappa}$. Now let $P = \mathrm{Fn}(\kappa, 3)$. Note that $P$ is coabsolute with $\mathrm{Fn}(\kappa, 2)$. We shall define a family $\mathcal{D}$ of dense subsets of $P$ such that $\mid D \mid < m_{\mathrm{Fn}(\kappa,2)}$ so that among other things (see below), for $\mathcal{D}$-generic set $G$, $g = \bigcup G$ will be a function from $\kappa$ to 3 and $X = \{\, \alpha < \kappa \,:\, g(\alpha) = 2 \,\}$ will be of cardinality $\kappa$. Then we let $f$ be the function on $\kappa$ defined by:

$$f(\alpha) = \begin{cases} 0_B & ;\text{ if } g(\alpha) = 0, \\ 1_B & ;\text{ if } g(\alpha) = 1, \\ \alpha & ;\text{ otherwise.} \end{cases}$$

Let $\bar{f}$ be the Boolean mapping from $\mathrm{Fr}\,\alpha$ to $\mathrm{Fr}\,X$ generated by $f$.

Now we are done, if we can show that $\bar{f}$ extends to a Boolean mapping $\tilde{f}$ from $B$ onto $\mathrm{Fr}\,X$. But by the following Lemma 2.2, we can choose $\mathcal{D}$ appropriate for this purpose.

For $p \in P$, let $B_p = \mathrm{Fr}\,\mathrm{dom}(p)$ (hence $B_p \leq B$) and $f_p : B_p \to \mathrm{Fr}\,(p^{-1}[\{2\}])$ be the Boolean mapping generated by the mapping $f_p^0$ on $\mathrm{dom}(p)$ defined by:

$$f_p^0(\alpha) = \begin{cases} 0_B & ;\text{ if } p(\alpha) = 0, \\ 1_B & ;\text{ if } p(\alpha) = 1, \\ \alpha & ;\text{ otherwise.} \end{cases}$$

**Lemma 2.2** *For any $b \in B$ and $p \in P$ there exists $q \leq p$ and $b_1$, $b_2 \in B_q$ such that $b_1 \leq b$, $b_2 \leq -b$ and $f_q(b_1) + f_q(b_2) = 1$ ( i.e, $q$ "forces" $\tilde{f}(b) = f_q(b_1)$ ).*

For the proof of the Lemma 2.2 we use the following Lemma whose proof is left to the reader:

**Lemma 2.3** *Let $b \in \overline{\mathrm{Fr}\,\kappa}$ and let $Y \subseteq \kappa$ be a countable set such that $b \in \overline{\mathrm{Fr}\,Y}$ holds. Let $Y = \{\, \alpha_n \,:\, n < \omega \,\}$. Then there exist an increasing sequence $(l_n)_{n<\omega}$ with $l_n < \omega$ for $n < \omega$ and a sequence $(i_n)_{n<\omega}$ with $i_n \in {}^{l_n}\{\, -1, 1 \,\}$ for $n < \omega$ such that, letting $p_n = \Sigma_{k<l_n} i_n(k) \cdot \alpha_k$ for $n < \omega$,* *i)* *either* $p_n \leq b$ *or* $p_n \leq -b$ *and*

*ii)* $\Sigma_{n<\omega} p_n = 1$.

*In particular we have* $b = \Sigma\{\, p_n \,:\, n<\omega,\ p_n \leq b \,\}$. ❑

**Proof of Lemma 2.2** Let $Y = \{\, \alpha_n \,:\, n<\omega \,\}$, $(l_n)_{n<\omega}$, $(i_n)_{n<\omega}$ and $p_n$, $n<\omega$ be as in Lemma 2.3 for our $b \in B$. Without loss of generality we may assume that $\mathrm{dom}(p) \cap Y = \{\, \alpha_n \,:\, n<k \,\}$ for some $k<\omega$. Let ${}^k\{\, -1,1 \,\} = \{\, \tau_m \,:\, m<2^k \,\}$. By induction we can take $n_m < \omega$ for $m < 2^k$ such that

*a)* $i_{n_m}$ is compatible (as an element of $\mathrm{Fn}(Y, \{\, -1,1 \,\})$) with $\tau_m$ and

*b)* $\{\, i_{n_m} \restriction (\mathrm{dom}(i_{n_m}) \setminus k) \,:\, m<2^k \,\}$ is pairwise compatible.

Let $\tilde{n} = \max\{\, n_m \,:\, m<2^k \,\}$, $\tilde{l} = l_{\tilde{n}}$ and $\tilde{i} = \bigcup\{\, i_{n_m} \restriction (\mathrm{dom}(i_{n_m}) \setminus k) \,:\, m<2^k \,\}$. Let $q \leq p$ be such that $\mathrm{dom}(q) = \mathrm{dom}(p) \cup \{\, \alpha_k, \ldots, \alpha_{\tilde{l}-1} \,\}$, $q \restriction \mathrm{dom}(p) = p$ and

$$q(\alpha_m) = \begin{cases} 1 & ;\ \text{if } \tilde{i}(\alpha_m) = 1, \\ 0 & ;\ \text{if } \tilde{i}(\alpha_m) = -1. \end{cases}$$

Then $q$ as above together with $b_1 = \Sigma\{\, p_n \,:\, n<\tilde{n},\ p_n \leq b \,\}$ and $b_2 = \Sigma\{\, p_n \,:\, n<\tilde{n},\ p_n \leq -b \,\}$ is as desired. ❑ (Lemma 2.2)

Now by the lemma above

$$\begin{aligned} \mathcal{D} = & \{\, \{\, p \in P \,:\, \alpha \in \mathrm{dom}(p) \,\} \,:\, \alpha < \kappa \,\} \\ & \cup \{\, \{\, p \in P \,:\, \exists \beta > \alpha\ p(\beta) = 2 \,\} \,:\, \alpha < \kappa \,\} \\ & \cup \{\, \{\, q \in P \,:\, f_q(b_1) + f_q(b_2) = 1 \text{ for some } b_1 \leq b,\ b_2 \leq -b \,\} \,:\, b \in B \,\} \end{aligned}$$

is a family of dense subsets of $P$. Clearly the mapping $\bar{f}$ defined as above with respect to this $\mathcal{D}$ can be extended to a Boolean mapping $\tilde{f}$ from $B$ onto $\mathrm{Fr}\,X$.

❑ (Proposition 2.1)

# 3 Pcf and the theorem of Shapiro

**Proposition 3.1** *Assume that*

$\oplus_{\mu,\kappa,\lambda}$ *for any* $\mathcal{F} \subseteq [\lambda]^{\aleph_0}$ *with* $|\,\mathcal{F}\,| < \mu$, *there is* $Y \in [\lambda]^{\kappa}$ *such that* $a \cap Y$ *is finite for all* $a \in \mathcal{F}$.

*Then, for any Boolean algebra* $B$ *of cardinality* $< \mu$, *if* $\mathrm{Fr}\,\lambda$ *is embeddable into* $B$ *then there is a surjective Boolean mapping from* $B$ *onto* $\mathrm{Fr}\,\kappa$.

**Proof** As in the proof of Proposition 2.1, we may assume without loss of generality that $\mathrm{Fr}\,\lambda \leq B \leq \overline{\mathrm{Fr}\,\lambda}$ holds. Let $|\,B\,| = i^* \ (< \mu)$ and let $(y_i)_{i<i^*}$ be an enumeration of $B$. Let $y_i = \sum_{n<\omega} \tau_i^n(\alpha(i,n,0),\ldots,\alpha(i,n,m_{i,n}))$ where $\tau_i^n$ is a Boolean term with $m_{i,m}+1$ variables and $\alpha(i,n,0),\ldots,\alpha(i,n,m_{i,n}) < \lambda$ for $i < i^*$ and $n < \omega$. For $i < i^*$, let $w_i = \{\,\alpha(i,n,l) \,:\, n<\omega,\ l \leq m_{i,n}\,\}$. By the assumption, there exists a $Y \in [\lambda]^\kappa$ such that $w_i \cap Y$ is finite for every $i < i^*$. Let $g : B \to \mathrm{Fr}\,Y$ be defined by

$$g(y_i) = \sum_{n<\omega} \tau_i^n(\alpha^*(i,n,0),\ldots,\alpha^*(i,n,m_{i,n}))$$

where

$$\alpha^*(i,n,l) = \begin{cases} \alpha(i,n,l) & ;\ \text{if } \alpha(i,n,l) \in Y \\ 0_B & ;\ \text{otherwise.} \end{cases}$$

The function $g$ is well-defined since, for each $i < \omega$, $\tau_i^n(\alpha^*(i,n,0),\ldots,\alpha^*(i,n,m_{i,n}))$ is an element of $\mathrm{Fr}\,(w_i \cap Y)$ and $\mathrm{Fr}\,(w_i \cap Y)$ is finite. Clearly this $g$ is as desired.

❏ (Proposition 3.1)

**Theorem 3.2** *Assume that*

$(*)_{\mu,\lambda,\kappa}$ *there are* $\underset{\circ}{a}_i \in [Reg \cap (\lambda^+ \setminus \kappa^+)]^{<\aleph_0}$ *for* $i < \kappa$ *such that for every* $a \in [\kappa]^{\aleph_0}$, $\max \mathrm{pcf}(\bigcup_{i\in a} \underset{\circ}{a}_i) \geq \mu$ *holds.*

*Then for any Boolean algebra* $B$ *of cardinality* $< \mu$, *if* $\mathrm{Fr}\,\kappa$ *is embeddable into* $B$ *then there is a surjective Boolean mapping* $g$ *from* $B$ *onto* $\mathrm{Fr}\,\kappa$.

( For more about $(*)_{\mu,\lambda,\kappa}$ see [10]. For pcf theory in general, the reader may consult [11].) The theorem follows from Proposition 3.1 and the following:

**Lemma 3.3** *Assume that* $(*)_{\mu,\lambda,\kappa}$ *(as in Theorem 3.2) holds. Then* $\oplus_{\mu,\kappa,\kappa}$ *holds.*

**Proof** Since max pcf is always regular, we may assume that $\mu$ is regular. Let $\underset{\circ}{a} = \bigcup_{i<\kappa} \underset{\circ}{a}_i$. In place of $[\kappa]^{\aleph_0}$, we consider $[Z]^{\aleph_0}$ for $Z = \dot{\bigcup}_{i<\kappa} Z_i$ where $Z_i = \{\,i\,\} \times \prod \underset{\circ}{a}_i$. Hence we assume that $\mathcal{F} \subseteq [Z]^{\aleph_0}$ and $|\,\mathcal{F}\,| < \mu$.

For each $a \in \mathcal{F}$, let $g_a \in \prod \underset{\circ}{a}$ be defined by

$$g_a(\theta) = \sup\{\,\eta(\theta) \,:\, \eta \in a,\ \theta \in \mathrm{dom}(\eta)\,\}$$

for each $\theta \in \underset{\circ}{a}$, where we put $\sup \emptyset = 0$. Since $\prod \underset{\circ}{a}/J_{<\mu}[\underset{\circ}{a}]$ is $\mu$-directed and $|\,\mathcal{F}\,| < \mu$, there is $f^* \in \prod \underset{\circ}{a}$ such that $g_a <_{J_{<\mu}[\underset{\circ}{a}]} f^*$ holds for all $a \in \mathcal{F}$. For $i < \kappa$, let $z_i = \{\,(0,i)\,\} \cup (f^* \restriction \underset{\circ}{a}_i)$. Then $z_i \in Z_i$ for $i < \kappa$. We show that $Y = \{\,z_i \,:\, i < \kappa\,\}$ is as required. Suppose not. Then $Y \cap a$ would be infinite for some $a \in \mathcal{F}$. By the assumption, it follows that $\bigcup_{z_i\in Y\cap a}\underset{\circ}{a}_i \notin J_{<\mu}[\underset{\circ}{a}]$. But for $z_i \in Y\cap a$ we have $\{\,(0,i)\,\}\cup(f^*\restriction \underset{\circ}{a}_i)\in a$. It follows that for $\theta\in\underset{\circ}{a}_i$ we have $f^*(\theta)\le g_a(\theta)$. This is a contradiction to $g_a <_{J_{<\mu}[\underset{\circ}{a}]} f^*$. ❑ (Lemma 3.3)

# 4 Independence of the theorem of Shapiro

The principle ($\text{⫯}$) suggests the following cardinal invariant $\text{⫯}$:

$$\text{⫯} = \min\{\,|\,X\,|\;:\;X\subseteq[\omega_1]^{\aleph_0},\ \forall y\in[\omega_1]^{\aleph_1}\ \exists x\in X\ x\subseteq y\,\}.$$

Clearly $\aleph_1\le\text{⫯}\le 2^{\aleph_0}$ and ($\text{⫯}$) holds if and only if $\text{⫯}=\aleph_1$. We can also consider the following variants of $\text{⫯}$:

$$\begin{aligned}\text{⫯}' = \min\{\,\kappa\;:\kappa\ge\aleph_1,&\text{ there is an } X\subseteq[\kappa]^{\aleph_0}\\ &\text{such that } |\,X\,|=\kappa \text{ and } \forall y\in[\kappa]^{\aleph_1}\ \exists x\in X\ x\subseteq y\,\},\end{aligned}$$

$$\begin{aligned}\text{⫯}'' = \min\{\,\kappa\;:\kappa\ge\aleph_1,&\text{ there is an } X\subseteq[\kappa]^{\aleph_0}\\ &\text{such that } |\,X\,|=\kappa \text{ and } \forall y\in[\kappa]^{\kappa}\ \exists x\in X\ x\subseteq y\,\}.\end{aligned}$$

We have $\aleph_1\le\text{⫯}''\le\text{⫯}'\le 2^{\aleph_0}$ and ($\text{⫯}$) holds if and only if $\text{⫯}=\text{⫯}'=\text{⫯}''=\aleph_1$ holds.

It can be easily shown that $\text{⫯}\le\text{⫯}'$ holds. Moreover if $\text{⫯}<\aleph_{\omega_1}$, then $\text{⫯}=\text{⫯}'$ holds. The question, if $\text{⫯}<\text{⫯}'$ is consistent, is connected with some very fundamental unsolved problems on cardinal arithmetic while we can show that $\text{⫯}''<\text{⫯}$ is consistent. For more, see [4] and [10].

**Proposition 4.1** *There exists a Boolean algebra $B$ such that $|\,B\,|=\text{⫯}'$, $\mathrm{Fr}\,\text{⫯}'$ is embeddable into $B$ but there is no surjective Boolean mapping from $B$ onto $\mathrm{Fr}\,\omega_1$.*

**Proof** Let $\Phi:\kappa\to\kappa;\ \alpha\mapsto\xi_\alpha$ be the continuously increasing function defined inductively by $\xi_0=\omega$ and $\xi_{\alpha+1}=\xi_\alpha+|\,\xi_\alpha\,|$. Let $\kappa=\text{⫯}'$ and let $X\subseteq[\kappa\times\mathrm{Fr}\,\omega_1]^{\aleph_0}$ be such that $|\,X\,|=\kappa$, $\omega\times\mathrm{Fr}\,\omega\in X$ and $\forall y\in[\kappa\times\mathrm{Fr}\,\omega_1]^{\aleph_1}\ \exists x\in X\ x\subseteq y$ holds. Let $(x_\alpha)_{\alpha<\kappa}$ be an enumeration of $X$ such that $x_\alpha\subseteq\xi_\alpha\times\mathrm{Fr}\,\omega_1$ for all $\alpha<\kappa$.

Now let $(B_\alpha)_{\alpha<\kappa}$ be a continuously increasing sequence of Boolean algebras such that for all $\alpha<\kappa$

*1)* the underlying set of $B_\alpha$ is $\xi_\alpha$;

*2)* there exits a $b_\alpha\in B_{\alpha+1}$ such that $b_\alpha$ is free over $B_\alpha$;

*3)* if $x_\alpha$ generates a Boolean mapping $f_\alpha$ from a subalgebra of $B_\alpha$ onto an infinite subalgebra of $\mathrm{Fr}\,\omega_1$ then $B_{\alpha+1}$ contains an element $c_\alpha$ of the form $\Sigma^{B_{\alpha+1}}_{n\in Z_\alpha} b^\alpha_n$ where $Z_\alpha \subseteq \omega$, $b^\alpha_n$, $n<\omega$ are pairwise disjoint elements in $\mathrm{dom}(f_\alpha)$, $f_\alpha(b^\alpha_n) \neq 0$ for all $n<\omega$ and there is no $d \in \mathrm{Fr}\,\omega_1$ such that $d \cdot f_\alpha(b^\alpha_n) = f(b^\alpha_n)$ for all $n \in Z_\alpha$ and $d \cdot f_\alpha(b^\alpha_n) = 0$ for all $n < \omega \setminus Z_\alpha$ holds.

Let $B = \bigcup_{\alpha<\kappa} B_\alpha$. We show that this $B$ is as desired. By *1)* the underlying set of $B$ is $\kappa$. By *2)* $\{\, b_\alpha \,:\, \alpha < \kappa \,\}$ is an independent subset of $B$. Hence $\mathrm{Fr}\,\kappa$ is embeddable into $B$.

Suppose now that there would be a surjective Boolean mapping $f$ from $B$ onto $\mathrm{Fr}\,\omega_1$. Then there is a bijection $g \subseteq f$ from a subset of $B$ onto $\mathrm{Fr}\,\omega_1$. Since $g$ is uncountable there is an $\alpha < \kappa$ such that $x_\alpha \subseteq g$. Since $x_\alpha \subseteq f$, $x_\alpha$ satisfies the condition in *3)*. Hence there is a $c_\alpha \in B_{\alpha+1}$ such that $c_\alpha = \Sigma^{B_{\alpha+1}}_{n\in Z_\alpha} b^\alpha_n$ for $Z_\alpha$ and $b^\alpha_n$, $n<\omega$ as un *3)*. But then $f(c_\alpha) \cdot f_\alpha(b^\alpha_n) = f(b^\alpha_n)$ for all $n \in Z_\alpha$ and $f(c_\alpha) \cdot f_\alpha(b^\alpha_n) = 0$ for all $n < \omega \setminus Z_\alpha$ holds. This is a contradiction to the choice of $Z_\alpha$. ❑ (Proposition 4.1)

**Corollary 4.2** $m_{\mathrm{Fn}(\omega_1,2)} \leq \text{⫯}'$.

**Proof** By Proposition 2.1 and Proposition 4.1. ❑ (Corollary 4.2)

With almost the same proof as in Proposition 4.1 we can also prove the following:

**Proposition 4.3** *There exists a Boolean algebra $B$ such that $\mid B \mid = \text{⫯}''$, $\mathrm{Fr}\,\text{⫯}''$ is embeddable into $B$ but there is no surjective Boolean mapping from $B$ onto $\mathrm{Fr}\,\text{⫯}''$.* ❑

Since we have $\text{⫯}' = \aleph_1$ under (⫯), we obtain the following theorem:

**Theorem 4.4** *If (⫯) holds then there exists a Boolean algebra $B$ of cardinality $\aleph_1$ such that $\mathrm{Fr}\,\omega_1$ is embeddable into $B$ but there is no surjection from $B$ onto $\mathrm{Fr}\,\omega_1$.* ❑

Hence if $\neg$CH and (⫯) holds, by Theorem 4.4, there exists a counter-example to the theorem of Shapiro. This shows that we cannot just drop MA(*Cohen*) from Theorem 1.1. Since MA(*countable*) + $\neg$CH + (⫯) is consistent (see e.g. [5] or [4]), we see that MA(*countable*) is not enough for Theorem 1.1.

**Corollary 4.5** $m_{\mathit{Cohen}} \leq \text{⫯}''$.

**Proof** By Proposition 2.1 and Proposition 4.3. ❑ (Corollary 4.5)

If a Boolean algebra $B$ is atomless then $\mathrm{Fr}\,\omega$ can be embedded into $B$. By Proposition 2.1, if MA(*countable*) holds and $B$ is of cardinality $< 2^{\aleph_0}$, there exists a surjection from $B$ onto $\mathrm{Fr}\,\omega$. Here again we cannot simply drop the assumption of MA(*countable*):

**Proposition 4.6** *It is consistent that there is an atomless Boolean algebra $B$ of cardinality $\aleph_1 < 2^{\aleph_0}$ such that there is no surjective Boolean mapping from $B$ onto* $\mathrm{Fr}\,\omega$.

**Proof** By [9, Theorem 5.12], there is a model of ZFC+ $\neg$CH in which there is an endo-rigid atomless Boolean algebra $B$ of cardinality $\aleph_1$. In particular there is no surjection from $B$ onto $\mathrm{Fr}\,\omega$. ❑ (Proposition 4.6)

Note that, since (⫯) is consistent with $\neg$CH and MA(*countable*), (⫯) cannot supply such a Boolean algebra as in the proposition above.

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