実数値可測基数の存在公理と類似の連続体濃度が巨大であることを導く公理について

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1 A classical characterization of real-valued measurability $\leq 2^{\aleph_0}$

For a set X, a family $\mathcal{B} \subseteq \mathcal{P}(X)$ is said to be a *set algebra* if $\emptyset$, $X \in \mathcal{B}$ and $\mathcal{B}$ is closed with respect to intersection and union of finitely may sets as well as complement. A set algebra $\mathcal{B} \subseteq \mathcal{P}(X)$ is *σ-closed* if it is also closed with respect to union and intersection of countable family of elements. For a cardinal κ, a set algebra $\mathcal{B}$ is *κ-closed* if it is closed with respect to union and intersection of every $\mathcal{F} \subseteq \mathcal{B}$ with $|\,\mathcal{F}\,| < \kappa$. Thus, $\mathcal{B}$ is σ-closed if and only if it is $\aleph_1$-closed.

A probablility measure (p.m.) ν on a σ-closed set algebra $\mathcal{B} \subseteq \mathcal{P}(X)$ for a set X is a mapping $\nu : \mathcal{B} \to [0,1]$ such that

(1.1) $\nu(\emptyset) = 0$, $\nu(X) = 1$; pm-0

(1.2) $\nu(\dot{\bigcup}_{n\in\omega} u_n) = \sum_{n\in\omega} \nu(u_n) \ \left(= \sup\{\sum_{n\in s} \nu(u_n) \ : \ s \in [\omega]^{<\aleph_0}\}\right)$ for every pairwise pm-1 disjoint[1] $u_n \in \mathcal{B}$, $n \in \omega$.

The property (1.2) is refered to as the σ-*additivity* of $\mathcal{B}$.

P.M.-0

Lemma 1.1. *For a p.m. on a σ-closed set algebra $\mathcal{B}$, we have:*

(0) $\nu(u_0 \dot\cup \cdots \dot\cup u_{n-1}) = \nu(u_0) + \cdots + \nu(u_{n-1})$ *for every pairwise disjoint* $u_0,\ldots, u_{n-1} \in \mathcal{B}$.

(1) $\nu(u) \leq \nu(v)$ *for any u, $v \in \mathcal{B}$ with $u \subseteq v$.*

Proof. (0): This can be seen as a special case of (1.2), letting $u_k = \emptyset$ for all $k > n$.

(1): $\nu(u) \leq \nu(u) + \nu(v \setminus u) = \nu(u \dot\cup (v \setminus u)) = \nu(v)$ by (0). ❑ (Lemma 1.1)

For a cardinal κ, a p.m. ν on a set algebra $\mathcal{B}$ is κ-*additive* if $\mathcal{B}$ is κ-*closed* and

(1.3) $\nu(\bigcup_{\alpha\in\lambda} u_\alpha) = \sum_{\alpha\in\lambda} \nu(u_\alpha) \left(= \sup\{\sum_{n\in s} \nu(u_n) \,:\, s \in [\lambda]^{<\aleph_0}\}\right)$ for all $\lambda < \kappa$ and pairwise disjoint $u_\alpha \in \mathcal{B}$, $\alpha \in \lambda$. pm-2

Thus, σ-additivity (1.2) of ν is just ω_1-additivity in this terminology.

The additivity of a p.m. is decided by the additivity of the ideal of null sets with respect to the p.m. (Lemma 1.3).

RVM-0-a

Lemma 1.2. *Suppose that ν is a p.m. on some $\mathcal{B} \subseteq \mathcal{P}(X)$. If $\mathcal{F} \subseteq \mathcal{B}$ is an uncountable family of pairwise disjoint subsets of $\mathcal{B}$ then*

$$\mathcal{F}' = \{u \in \mathcal{F} \,:\, \nu(u) \neq 0\}$$

is countable.

Proof. Suppose not. Then, by Pegion Hole Principle, there are $m \in \omega \setminus \{0\}$ and uncountable $\mathcal{F}'' \subseteq \mathcal{F}'$ such that $\nu(u) > \frac{1}{m}$ for all $u \in \mathcal{F}''$. For any distinct $u_0,\ldots, u_{mn-1} \in \mathcal{F}''$, we have

(1.4) $$\mu(u_0 \dot\cup \cdots \dot\cup u_{mn-1}) = \mu(u_0) + \cdots + \mu(u_{mn-1}) > \frac{mn}{m} = n$$

by Lemma 1.1, (a). By Lemma 1.1, (b), this is a contradiction to (1.1). ❑ (Lemma 1.2)

RVM-0-0

Lemma 1.3. *Suppose that ν is a p.m. on a κ-closed $\mathcal{B}$ then the following are equivalent:*

(a) *ν is κ-additive.*

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[1] As usual "$\dot\bigcup$" and "$\dot\cup$" denote disjoint union of sets.

(b) $\nu(\bigcup_{\alpha\in\lambda} u_\alpha) = 0$ *holds for all* $\lambda < \kappa$, *and* $u_\alpha \in \mathcal{B}$ *with* $\nu(u_\alpha) = 0$ *for all* $\alpha < \lambda$.

(c) $\nu(\bigcup_{\alpha\in\lambda} u_\alpha) = 0$ *holds for all* $\lambda < \kappa$, *and pairwise disjoint* $u_\alpha \in \mathcal{B}$ *with* $\nu(u_\alpha) = 0$ *for all* $\alpha < \lambda$.

Proof. (a) $\Rightarrow$ (c) $\Leftrightarrow$ (b) is clear.

To prove (c) $\Rightarrow$ (a), suppose that ν is not κ-additive. Then there are $\lambda < \kappa$ and pairwise disjoint $u_\alpha \in \mathcal{B}$ such that

$$(1.5) \quad \sum_{\alpha<\gamma} \nu(u_\alpha) < \nu(\dot{\bigcup_{\alpha<\gamma}} u_\alpha)$$ pm-4

Note that it is easy to see that the inequality "$\leq$" always holds.

By Lemma 1.2, $I = \{\alpha < \lambda : \nu(u_\alpha) > 0\}$ is countable. By σ-additivity of ν,

$$(1.6) \quad \nu(\dot{\bigcup_{\alpha\in I}} u_\alpha) = \sum_{\alpha\in I} \nu(u_\alpha) = \sum_{\alpha\in I} \nu(u_\alpha) + \sum_{\alpha\in\lambda\setminus I} \nu(u_\alpha) = \sum_{\alpha\in\lambda} \nu(u_\alpha).$$ pm-5

Since $\dot{\bigcup_{\alpha<\lambda}} u_\alpha = (\dot{\bigcup_{\alpha\in I}} u_\alpha) \mathbin{\dot\cup} (\dot{\bigcup_{\alpha\in\lambda\setminus I}} u_\alpha)$,

$$(1.7) \quad \nu(\dot{\bigcup_{\alpha<\lambda}} u_\alpha) = \nu(\dot{\bigcup_{\alpha\in I}} u_\alpha) + \nu(\dot{\bigcup_{\alpha\in\lambda\setminus I}} u_\alpha)$$ pm-6

again by σ-additivity of ν. Putting together (1.5), (1.6) and (1.7), we obtain

$$\nu(\dot{\bigcup_{\alpha\in I}} u_\alpha) = \sum_{\alpha<\gamma} \nu(u_\alpha) < \nu(\dot{\bigcup_{\alpha<\gamma}} u_\alpha) = \nu(\dot{\bigcup_{\alpha\in I}} u_\alpha) + \nu(\dot{\bigcup_{\alpha\in\lambda\setminus I}} u_\alpha).$$

Hence $\nu(\dot{\bigcup_{\alpha\in\lambda\setminus I}} u_\alpha) > 0$.

Thus $\langle u_\alpha : \alpha \in \lambda \setminus I\rangle$ (reenumerated with the index set $\lambda = |\lambda \setminus I|$) is a counter-example to (c). ❏ (Lemma 1.3)

A p.m. ν on $\mathcal{B} \subseteq \mathcal{P}(X)$ is *continuous* if $\nu(\{x\}) = 0$ for all $x \in X$.

A cardinal κ is *real-valued measurable* if there is a continuous κ-additive p.m. on $\mathcal{P}(\kappa)$.

The following is the main theorem of the present section:

RVM-0

Theorem 1.4. *The following are equivalent:*

(a) *There is a σ-additive measure ν on $\mathcal{P}(\mathbb{R})$ extending the Lebesgue measure.*

(b) *There is a real-valued measurable cardinal $\leq 2^{\aleph_0}$.*

Theorem 1.4 is proved in the course of the following lemmas.

"(a) $\Rightarrow$ (b)" of Theorem 1.4 follows from Lemma 1.5.

RVM-1

Lemma 1.5. *Suppose that there is a continuous p.m. on $\mathcal{P}(\lambda)$ for some cardinal λ. Then there is $\kappa \leq \lambda$ such that κ is real-valued measurable.*

Proof. Let

$$\kappa = \min\{\lambda' \leq \lambda \,:\, \mathcal{P}(\lambda') \text{ carries a continuous p.m.}\}.$$

By the assumption, $\{\lambda' \leq \lambda \,:\, \mathcal{P}(\lambda') \text{ carries a continuous p.m.}\} \neq \emptyset$. Hence thiere is a continuous p.m. ν on κ.

It is enough to show that this ν is κ-additive. Suppose not. Then, by Lemma 1.3, there is some $\delta < \kappa$ and pairwise disjoint $u_\alpha \in \mathcal{P}(\kappa)$, $\alpha < \delta$ such that $\nu(u_\alpha) = 0$ for all $\alpha < \delta$ but $\nu(\dot{\bigcup}_{\alpha<\delta} u_\alpha) > 0$. Let $\mu : \mathcal{P}(\delta) \rightarrow [0,1]$ be defined by

$$\mu(X) = \nu(\bigcup_{\alpha \in X} u_\alpha)$$

for all $X \in \mathcal{P}(\delta)$.

Claim 1.5.1. *μ is a continuous p.m. on $\mathcal{P}(\delta)$.*

$\vdash$ Exercise. $\dashv$ (Claim 1.5.1)

This is a contradiction to the minimality of κ. ❑ (Lemma 1.5)

For a p.m. ν on $\mathcal{P}(X)$, $x \subseteq X$ is said to be an *atom with respect to* ν if $\nu(x) > 0$ but there is no partition $x = x_0 \dot{\cup} x_1$ of x such that $\nu(x_0)$, $\nu(x_1) > 0$. Thus, $x \subseteq X$ is an atom with respect to ν if and only if $[x]_I$ is an atom in the boolean algebra $\mathcal{P}(X)/I$ for the ideal I of null sets with respect to ν.

A p.m. ν on $\mathcal{P}(X)$ is said to be *atomless* if there is no atom with respect to ν.

RVM-2

Lemma 1.6. *If ν is a κ-additive p.m. on $\mathcal{P}(\kappa)$ for some $\kappa \leq 2^{\aleph_0}$, then ν is atomless.*

Thus, if $\kappa \leq 2^{\aleph_0}$ is real-valued measurable then any witness of this is an atomless p.m.

Proof. Suppose that $x \in \mathcal{P}(\kappa)$ is an atom with respect to ν. Then

$$U = \{u \in \mathcal{P}(x) \,:\, \nu(u) = 1\}$$

is a κ-closed non-principal ultra-filter over x. Thus $\kappa = |\,x\,| > 2^{\aleph_0}$ (and much more, see Section 3). ❑ (Lemma 1.6)

Inverse of the lemma above also holds (see Lemma 1.8 below).

RVM-3

Lemma 1.7. *Suppose that ν is an atomless p.m. on $\mathcal{P}(X)$. Then for any $x \in \mathcal{P}(X)$ and $0 < a < 1$, there is $y \subseteq x$ with $\nu(y) = a\nu(x)$.*

Proof. First, we prove a consequence of the lemma:

RVM-3-0

Claim 1.7.1. *For any $x \in \mathcal{P}(X)$ with $\nu(x) = r > 0$ and $\varepsilon > 0$, there is $y \subseteq x$ such that $0 < \nu(y) < \varepsilon$.*

$\vdash$ Suppose not. Then there is an $\varepsilon > 0$ such that, for any $y \subseteq x$ with $\nu(y) > 0$, we have $\nu(y) \geq \varepsilon$. Let $n \in \omega$ be such that $n\varepsilon > r$. Since ν is atomless it is easy to find a partition $x = x_0 \dot\cup x_1 \dot\cup \cdots \dot\cup x_{n-1}$ of x such that $\nu(x_i) > 0$ for all $i < n$. Thus $\nu(x_i) > \varepsilon$ for all $i < n$ by the assumption. It follows that

$$r = \nu(x) = \nu(x_0) + \nu(x_1) + \cdots + \nu(x_{n-1}) \geq n\varepsilon > r.$$

This is a contradiction. $\dashv$ (Claim 1.7.1)

Now, suppose that $x \in \mathcal{P}(X)$. If $\nu(x) = 0$ then the claim of the lemma clearly holds with $y = x$. So assume $\nu(x) > 0$.

For $\alpha < \omega_1$, let $x_\alpha \in \mathcal{P}(x)$ and $a_\alpha \in [0, 1]$ be defined such that

(1.8) $a_\alpha = a\nu(x) - \nu(\bigcup_{\beta<\alpha} x_\beta)$; rvm-8

(1.9) $x_\alpha \subseteq x \setminus \sum_{\beta<\alpha} x_\beta$ for all $\alpha < \omega_1$; rvm-8-0

(1.10) if $a_\alpha > 0$ then $0 < \nu(x_\alpha) < a_\alpha$ for all $\alpha < \omega_1$; rvm-9

(1.11) if $a_\alpha = 0$ then $x_\alpha = \emptyset$. rvm-10

Note that x_α, $\alpha < \omega_1$ are pairwise disjoint by (1.9). Note also that $a_\alpha \geq 0$ for all $\alpha < \omega_1$ by (1.10) and (1.11). (1.10) is possible by Claim 1.7.1. By Lemma 1.2, there is $\alpha < \omega_1$ such that $x_\alpha = \emptyset$. Let $\alpha^* < \omega_1$ be the smallest $\alpha < \omega_1$ with $x_\alpha = \emptyset$. By (1.8) and (1.10), $y = \bigcup_{\beta<\alpha^*} x_\alpha$ is as desired. ❑ (Lemma 1.7)

Proof of Theorem 1.4: "(a) ⇒ (b)" follows from Lemma 1.5.

For "(b) ⇒ (a)", suppose that $\kappa \leq 2^{\aleph_0}$ is real-valued measurable and ν be a witness of this. ν is atomless by Lemma 1.6. Hence, by Lemma 1.7, we can find $u_t \in \mathcal{P}(\kappa)$, $t \in {}^{\omega>}2$ such that[2)]

(1.12) $u_\emptyset = \kappa$; rvm-11

(1.13) $u_t = u_{t^\frown 0} \dot\cup u_{t^\frown 1}$; rvm-12

(1.14) $\nu(u_t) = \dfrac{1}{2^{\ell(t)}}$. rvm-13

To finish the proof, it is enough to find a p.m. on $\mathcal{P}({}^{\omega}2)$ extending the Borel measure over the Cantor space ${}^{\omega}2$. For $x \in \mathcal{P}({}^{\omega}2)$, let

$$\mu(x) = \nu(\bigcap_{n\in\omega} \bigcup \{u_{f\restriction n} \,:\, f \in x\}).$$

Then it is easy to show that μ is a p.m. on $\mathcal{P}({}^{\omega}2)$. For any $t \in {}^{\omega>}2$, we have

$$\mu([t]) = \nu(\bigcap_{n\in\omega} \bigcup \{u_{f\restriction n} \,:\, t \subseteq f\}) = \nu(u_t) = \frac{1}{2^{\ell(t)}}.$$

Hence μ extends the Borel measure over ${}^{\omega}2$. ❑ (Theorem 1.4)

RVM-4

Lemma 1.8. *If there is an atomless κ-additive continuous p.m. on $\mathcal{P}(\kappa)$ then $\kappa \le 2^{\aleph_0}$.*

Proof. Let ν be an atomless κ-additive continuous p.m. on $\mathcal{P}(\kappa)$ and let u_t, $t \in {}^{\omega>}2$ be as in the proof of Lemma 1.5 above. Then we have

$$\kappa = \bigcup_{f\in{}^{\omega}2} (\bigcap_{n\in\omega} u_{f\restriction n}).$$

If $\kappa > 2^{\aleph_0}$ then by κ-additivity of ν it follows that

$$\nu(\kappa) = \sum_{f\in{}^{\omega}2} \nu(\bigcap_{n\in\omega} u_{f\restriction n}) = 0.$$

This is a contradiction. ❑ (Lemma 1.8)

2 More combinatorics around real-valued measurable cardinals

combinatorics

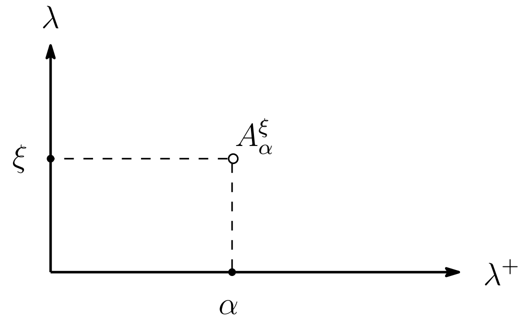
The infinite matrix of sets $\langle A^{\xi}_{\alpha} \,:\, \alpha < \lambda^+, \xi < \lambda\rangle$ in the following theorem is known as *Ulam matrix.*

Theorem 2.1. (S. Ulam, 1930) *For any cardial λ there is a sequence $A^{\xi}_{\alpha} \in \mathcal{P}(\lambda^+)$, $\alpha \in \lambda^+$, $\xi < \lambda$ such that*

(2.1) *For all $\xi < \lambda$ and for all $\alpha < \beta < \lambda^+$, $A^{\xi}_{\alpha} \cap A^{\xi}_{\beta} = \emptyset$;* rvm-14

(2.2) *For all $\alpha < \lambda^+$, $|\,\lambda^+ \setminus \bigcup_{\xi<\lambda} A^{\xi}_{\alpha}\,| < \lambda^+$.* rvm-15

[2] We denote here: ${}^{\omega>}2 = \{t \,:\, t : n \to 2 \text{ for some } n \in \omega\}$. For $t \in {}^{\omega>}2$, we denote $\ell(t) = |\,t\,| = \operatorname{dom}(t)$ and $t^\frown i = t \cup \{\langle \ell(t), i\rangle\}$ for $i \in 2$.



Proof. For each $\eta < \lambda^+$, let $f_\eta : \lambda \to \eta + 1$ be a surjection. We show that

$$A_\alpha^\xi = \{\eta < \lambda^+ \,:\, f_\eta(\xi) = \alpha\}$$

for $\alpha < \lambda^+$, $\xi < \lambda$ are as desired.

$\langle A_\alpha^\xi \,:\, \alpha < \lambda^+, \xi < \lambda\rangle$ satisfies (2.1): For $\xi < \lambda$ and distinct $\alpha, \beta < \lambda^+$, we have

$$\eta \in A_\alpha^\xi \;\Leftrightarrow\; f_\eta(\xi) = \alpha \;\Rightarrow\; f_\eta(\xi) \neq \beta \;\Leftrightarrow\; \eta \notin A_\beta^\xi$$

for all $\eta \in \lambda^+$. Thus $A_\alpha^\xi \cap A_\beta^\xi = \emptyset$.

$\langle A_\alpha^\xi \,:\, \alpha < \lambda^+, \xi < \lambda\rangle$ satisfies (2.2): For $\alpha < \lambda^+$

$$\begin{aligned}\lambda^+ \setminus \bigcup_{\xi<\lambda} A_\alpha^\xi &= \{\eta < \lambda^+ \,:\, f_\eta(\xi) \neq \alpha \text{ for all } \xi < \lambda\}\\ &= \{\eta < \lambda^+ \,:\, \eta < \alpha\} = \alpha.\end{aligned}$$

❑ (Theorem 2.1)

Corollary 2.2. *If κ is real-valued measurable then κ is weakly inaccessible.*

Proof. Let ν be a κ-additive continuous p.m. on $\mathcal{P}(\kappa)$.

κ is a regular cardinal: Suppose not. Then there is a sequence κ_α, $\alpha < \lambda$ of cardinals $< \kappa$ with $\lambda < \kappa$ such that $\kappa = \bigcup_{\alpha<\lambda} \kappa_\alpha$. By continuity and κ-additivity of ν, we have $\nu(\kappa_\alpha) = 0$ for all $\alpha < \lambda$. Again by κ-additivity of ν, it follows that $\nu(\kappa) = \nu(\bigcup_{\alpha<\lambda} \kappa_\alpha) = \sum_{\alpha<\lambda} \nu(\kappa_\alpha) = 0$. This is a contradiction.

Thus it is enough to show that κ is a limit cardinal. Suppose not. Then there is a cardinal λ such that $\kappa = \lambda^+$. Let $\langle A_\alpha^\xi \,:\, \alpha < \lambda^+,\, \xi < \lambda\rangle$ be a Ulam matrix. Then, for each $\alpha < \lambda^+$, we have $\nu(\bigcup_{\xi<\lambda} A_\alpha^\xi) = 1$, by (2.2) and by continuity and κ-additivity of ν. By κ-additivity of ν, it follows that there is a $\xi_\alpha < \lambda$ such that $\nu(A_\alpha^{\xi_\alpha}) > 0$. By Pigeon Hole Principle, there is a $\xi^* < \lambda$ such that $S = \{\alpha < \lambda^+ \,:\, \xi_\alpha = \xi^*\}$ is uncountable. But this is a contradiction to Lemma 1.2 as $A_\alpha^{\xi_\alpha} = A_\alpha^{\xi^*}$, $\alpha \in S$ are pairwise disjoint by (2.1). ❑ (Corollary 2.2)

A sequence $\langle f_\alpha : \alpha < \lambda \rangle$ of elements of ${}^\omega\omega$ is said to be a scale (or λ-scale) if

(2.3) $f_\alpha \leq^* f_\beta$ for all $\alpha < \beta < \lambda$ and scale-0

(2.4) $\{f_\alpha : \alpha < \kappa\}$ is cofinal in $\langle {}^\omega\omega, \leq^* \rangle$ scale-1

where $f \leq^* g$ for $f, g \in {}^\omega\omega$ denotes the almost majorizing relation, i.e., $f \leq^* g$ if and only if $\{n \in \omega : f(n) > g(n)\}$ is finite.

Theorem 2.3. *Suppose that $\kappa \leq 2^{\aleph_0}$ is a real-valued measurable cardinal. Then there is no λ-scale for any $\lambda \geq \kappa$.*

Proof.

❑ (Theorem 2.3)

3 Measurable cardinals

MEAS

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