Weakly extendible cardinals and compactness of extended logics

渕野 昌　　　　酒井拓史

Sakaé Fuchino[*,†] and Hiroshi Sakai[*,‡]

Abstract

We introduce the notion of weakly extendible cardinals and show that these cardinals are characterized in terms of weak compactness of second order logic. The consistency strength and largeness of weakly extendible cardinals are located strictly between that of strongly unfoldable (i.e. shrewd) cardinals, and strongly uplifting cardinals.

Weak compactness of many other logics can be connected to certain variants of the notion of weakly extendible cardinals.

We also show that, under $\mathsf{V} = \mathsf{L}$, a cardinal κ is the weak compactness number of $\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat,\kappa,\omega}$ if and only if it is the weak compactness number of $\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}$. The latter condition is equivalent to the condition that κ is weakly extendible by the characterization mentioned above (this equivalence holds without the assumption of $\mathsf{V} = \mathsf{L}$).

Contents

* Graduate School of System Informatics, Kobe University
Rokko-dai 1-1, Nada, Kobe 657-8501 Japan
† fuchino@diamond.kobe-u.ac.jp, ‡ hsakai@people.kobe-u.ac.jp

1 Compactness and Löwenheim-Skolem theorems of extended logics

Suppose that $\mathcal{L}$ is a logic with its model relation $\models_{\mathcal{L}}$ (with all expected properties for a model relation, like those assumed in Lindström's theorem: see e.g. Chapter XII of [6]). In the following, we will not treat these logics and model relations associated to them in a strict axiomatic setting of an abstract model theory. Instead we just say that such logics are "*proper*"[1)] and, if necessary, mention only the specific properties of the logics explicitly which are assumed in some of the assertions. intro

For such *proper logic* $\mathcal{L}$, a cardinal κ is said to be *$\mathcal{L}$-compact*, if for any collection T of $\mathcal{L}$-sentences, T is *satisfiable* (i.e. there is a structure $\mathfrak{A}$ with $\mathfrak{A} \models_{\mathcal{L}} T$) if (and only if) T is *$<\kappa$-satisfiable* (i.e. all $T_0 \in [T]^{<\kappa}$ are satisfiable).

If κ is $\mathcal{L}$-compact then any cardinal $\lambda > \kappa$ is $\mathcal{L}$-compact as well. Thus by naming the minimal $\mathcal{L}$-compact cardinal (if it exists), we completely describe the situation with $\mathcal{L}$-compactness. We shall call this minimal cardinal the *compactness number* of $\mathcal{L}$ and denote it by $\mathfrak{cn}(\mathcal{L})$ (if it exists, otherwise we write $\mathfrak{cn}(\mathcal{L}) = \infty$). Thus

$$\mathfrak{cn}(\mathcal{L}) := \min(\{\kappa \in \mathsf{Card} : \text{for any } \mathcal{L}\text{-theory } T,\ T \text{ is satisfiable if and only if all } T_0 \in [T]^{<\kappa} \text{ are satisfiable}\} \cup \{\infty\}).$$

We denote with $\mathcal{L}^{\mathrm{II}}$ the (monadic, full) second-order logic whose formulas are defined similarly to the first-order logic but with additional second-order variables X, Y, Z etc. and built-in predicate symbol ε for which it is allowed to build

(1.1) atomic formulas of the form $x \ \varepsilon \ X$ for a first-order variable x and a second-order variable X: x-intro-a-a

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This is an extended version of a draft of the paper with the same title. All additional details not to be contained in the submitted version of the paper are either typeset in dark electric blue (the color in which this paragraph is typeset) or put in separate appendices. The numbering of the assertions is kept identical with the submitted version.

The most up-to-date file of the extended version is downloadable as:
[0] https://fuchino.ddo.jp/papers/weakly-extendible-x.pdf

[1)] In [1], logics with corresponding "natural" properties are called *regular*.

the second-order variables are interpreted as such running over all subsets of the underlying set of the structure, and the atomic formula $x \varepsilon X$ is interpreted as the (true) element relation between valuations of the variables (as element and subset of the underlying set of the structure in consideration).

For a cardinal κ, $\mathcal{L}^{\rm II}_{\kappa,\omega}$ is the $\mathcal{L}_{\kappa,\omega}$ extension of the second-order logic $\mathcal{L}^{\rm II}$. That is, the formulas of $\mathcal{L}^{\rm II}_{\kappa,\omega}$ are constructed inductively as in $\mathcal{L}^{\rm II}$ with the additional clause saying that conjunction $\bigwedge\!\!\!\bigwedge\Phi$ and disjunction $\bigvee\!\!\!\bigvee\Phi$ of set Φ of formulas of size $<\kappa$ are allowed as far as the set of free variables appearing in Φ is finite. The model relation for infinitary conjunction and disjunction is defined as expected.

For $\omega < \lambda \leq \kappa$, $\mathcal{L}^{\rm II}_{\kappa,\lambda}$ is the $\mathcal{L}_{\kappa,\lambda}$ extension of $\mathcal{L}^{\rm II}_{\kappa,\omega}$ where we ease the restriction for $\bigwedge\!\!\!\bigwedge\Phi$ and $\bigvee\!\!\!\bigvee\Phi$ such that the formulas $\bigwedge\!\!\!\bigwedge\Phi$ and $\bigvee\!\!\!\bigvee\Phi$ are now permitted if the set of the free variables in Φ has cardinality $<\lambda$, and, in addition, existential and universal quantification over a block of quantifiers of size $<\lambda$ is allowed.

For $n \geq 2$, let $\mathcal{L}^n$ be the nth-order logic and $\mathcal{L}^{\rm HO}$ be the higher-order logic defined as the union of all $\mathcal{L}^n$, $n \in \omega$. $\mathcal{L}^{\rm HO}_{\kappa,\omega}$, $\mathcal{L}^{\rm HO}_{\kappa,\lambda}$ are then defined similarly to $\mathcal{L}^{\rm II}_{\kappa,\omega}$, $\mathcal{L}^{\rm II}_{\kappa,\lambda}$.

Using this terminology, the classical characterization of extendible cardinals (for the definition of extendible cardinals see the beginning of the Section 2 below) by M. Magidor can be reformulated as follows:

Theorem 1.1 (M. Magidor [19], see also Theorem 23.4 in [15]) (1) *A cardinal κ is extendible if and only if we have $\kappa = \mathfrak{cn}(\mathcal{L}^{\rm II}_{\kappa,\omega})$; if this equality holds then we also have $\mathfrak{cn}(\mathcal{L}^{\rm II}_{\kappa,\omega}) = \mathfrak{cn}(\mathcal{L}^{\rm HO}_{\kappa,\kappa})$.* P-intro-0

(2) *For a cardinal κ, $\kappa = \mathfrak{cn}(\mathcal{L}^{\rm II})$ holds if and only if κ is the least extendible cardinal. If κ is the least extendible cardinal then we also have $\mathfrak{cn}(\mathcal{L}^{\rm HO}_{\kappa,\kappa}) = \mathfrak{cn}(\mathcal{L}^{\rm II}_{\kappa,\omega}) = \mathfrak{cn}(\mathcal{L}^{\rm II}_{\kappa,\kappa}) = \mathfrak{cn}(\mathcal{L}^{\rm II}) = \kappa$.* □

Lemma 1.2 *For any uncountable cardinal κ, we have $\kappa \leq \mathfrak{cn}(\mathcal{L}_{\kappa,\omega})$. (Actually we have $\kappa \leq \mathfrak{wcn}(\mathcal{L}_{\kappa,\omega})$ where $\mathfrak{wcn}(\cdot)$ is defined below).* P-intro-4

Proof. For all infinite $\mu < \kappa$, and infinite $\mu' \leq \mu$ the theory

$$T_{\mu'} := \{\forall_x(\bigvee\!\!\!\bigvee_{\alpha<\mu'} x \equiv c_\alpha)\} \cup \{d \not\equiv c_\alpha \,:\, \alpha < \mu'\}.$$

is $<\mu'$-satisfiable but not satisfiable. This shows $\mu < (\ \mathfrak{wcn}(\mathcal{L}_{\kappa,\omega})\ ) \leq \mathfrak{cn}(\mathcal{L}_{\kappa,\omega})$.

□ (Lemma 1.2)

It is classical that strongly compact cardinals are also characterized in a similar vein:

Lemma 1.3 *An uncountable cardinal κ is strongly compact if and only if $\kappa = \mathfrak{cn}(\mathcal{L}_{\kappa,\kappa}) = \mathfrak{cn}(\mathcal{L}_{\kappa,\omega})$.* P-intro-1

Proof. The first equality is simply the definition of strong compactness e.g. adopted in Kanamori [15]. The second equation can be obtained by a modification of the proof of Proposition 4.1 in Kanamori [15]. ❑ (Lemma 1.3)

We define the *weak compactness spectrum* $\mathsf{WCS}(\mathcal{L})$ of a logic $\mathcal{L}$ by:

$$\mathsf{WCS}(\mathcal{L}) := \{\kappa \in \mathsf{Card} : \kappa > \aleph_0, \text{ for any } \mathcal{L}\text{-theory } T \text{ of signature with at most } \leq\kappa\text{-many non-logical symbols, if } T \text{ is } <\kappa\text{-satisfiable, then } T \text{ is satisfiable}\}.$$

We shall also say that an uncountable cardinal κ is *weakly $\mathcal{L}$-compact* if $\kappa \in \mathsf{WCS}(\mathcal{L})$.

In analogy to $\mathfrak{cn}(\mathcal{L})$, we can also define the *weak compactness number* $\mathfrak{wcn}(\mathcal{L})$ as the minimum of $\mathsf{WCS}(\mathcal{L})$:

$$\mathfrak{wcn}(\mathcal{L}) := \min(\{\kappa \in \mathsf{Card} : \kappa > \aleph_0, \text{for any } \mathcal{L}\text{-theory } T \text{ of signature with at most } \leq\kappa\text{-many non-logical symbols, if } T \text{ is } <\kappa\text{-satisfiable, then } T \text{ is satisfiable}\} \cup \{\infty\}).$$

In contrast to $\mathcal{L}$-compactness, there is no guarantee that $\mathsf{WCS}(\mathcal{L})$ is an end-segment of Card and hence $\mathfrak{wcn}(\mathcal{L})$ does not necessarily decide $\mathsf{WCS}(\mathcal{L})$.

The following is a direct consequence of the definitions of the notions we introduced above:

Lemma 1.4 (1) *For any logic $\mathcal{L}$, we have $\{\kappa \in \mathsf{Card} : \kappa \geq \mathfrak{cn}(\mathcal{L})\} \subseteq \mathsf{WCS}(\mathcal{L})$.* P-intro-2

(2) *$\mathfrak{wcn}(\mathcal{L}) \leq \mathfrak{cn}(\mathcal{L})$.*

(3) *If $\mathcal{L}$ and $\mathcal{L}'$ are logics such that each formulas of $\mathcal{L}$ can be translated to formulas of $\mathcal{L}'$ then we have*

$$\mathfrak{wcn}(\mathcal{L}) \leq \mathfrak{wcn}(\mathcal{L}'),\ \mathfrak{cn}(\mathcal{L}) \leq \mathfrak{cn}(\mathcal{L}') \quad \textit{and} \quad \mathsf{WCS}(\mathcal{L}) \supseteq \mathsf{WCS}(\mathcal{L}').$$ ❑

Corollary 1.5 *For cardinals κ and κ' with $\kappa \leq \kappa'$, if κ' is the least extendible cardinal $\geq \kappa$, then* P-intro-5

$$\{\lambda \in \mathsf{Card} : \kappa' \leq \lambda\} \subseteq \mathsf{WCS}(\mathcal{L}^{\mathrm{HO}}_{\kappa,\kappa}) \subseteq \mathsf{WCS}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}) \subseteq \mathsf{WCS}(\mathcal{L}^{\mathrm{II}}).$$

Proof. By Theorem 1.1 and Lemma 1.4. ❑ (Corollary 1.5)

Lemma 1.6 (1) *An uncountable cardinal* κ *is weakly compact if and only if* $\kappa = \mathfrak{wcn}(\mathcal{L}_{\kappa,\kappa}) = \mathfrak{wcn}(\mathcal{L}_{\kappa,\omega})$. P-intro-3

(2) $\{\mu \in \mathsf{Card} : \mu \geq \kappa, \mu$ *is either weakly compact or* $\mu \geq$ *the first strongly compact cardinal above* $\kappa\} \subseteq \mathsf{WCS}(\mathcal{L}_{\kappa,\kappa})$.

Proof. (1): The first equality is just the definition of weakly compactness while the second equality can be obtained as a byproduct of the characterization of weak compactness by tree property (see e.g. the proof of Theorem 7.8 in Kanamori [15] or Lemma 32.1 in Jech [11]). Note that by the definition of $\mathfrak{wcn}(\mathcal{L}_{\kappa,\omega})$ (adopted from [15]), it is easy to see that $\kappa = \mathfrak{wcn}(\mathcal{L}_{\kappa,\omega})$ implies that κ is strongly inaccessible.

(2): By (1), Lemma 1.4, and Lemma 1.3. ❑ (Lemma 1.6)

The question about a possible solution of the following "equation" seems to be a natural one:

$$\frac{\text{weakly compact cardinals}}{\text{strongly compact cardinals}} = \frac{x}{\text{extendible cardinals}}. \tag{1.2}$$ x-intro-a

In terms of compactness and weak compactness, x above must be a large cardinal property which should be characterized by

$$\kappa = \mathfrak{wcn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}). \tag{1.3}$$ x-intro-a-0

In Section 2, we introduce a new notion of large cardinals which we named "weak extendibility" and show that these cardinals are exactly those characterized by (1.3) (Theorem 2.2).

We prove that the consistency strength of weak extendibility is strictly between that of subtleness and strong unfoldability (Theorem 2.5). Note that, by Lücke [18], strong unfoldability is equivalent to shrewdness of Michael Rathjen [21].

Remembering Lindström's Theorem which gives the characterization of the first-order logic as the maximal logic satisfying the (countable) compactness and Downward Löwenheim-Skolem Theorem down to countable (see e.g. [6]), it seems to be natural to consider also the spectrum of Löwenheim-Skolem number of logics in our context (though a straight-forward generalization of Lindström's Theorem itself seems to be impossible: see [24]).

In the following, we denote with $|\mathfrak{A}|$ the underlying set of the (first-order) structure $\mathfrak{A}$ and with $\|\mathfrak{A}\|$ the cardinality of (the underlying set of) $\mathfrak{A}$. Nevertheless, if we are talking about a set A, we continue to denote the cardinality of A with $|A|$.

Suppose that $\mathcal{L}$ is a logic with the associated notion of elementary submodel $\prec_{\mathcal{L}}$ (which should satisfy all the expected properties of an elementary submodel relation, like the properties assumed in the proof of the following Lemma 1.7). The *Löwenheim-Skolem-Tarski spectrum* of $\mathcal{L}$ is defined by:

$$\mathsf{LSTS}(\mathcal{L}) := \{\mu \in \mathsf{Card} \,:\, \text{for any structure } \mathfrak{A} \text{ of a countable signature and } S \subseteq |\mathfrak{A}| \text{ with } |S| < \mu, \text{ there is } \mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A} \text{ such that } S \subseteq |\mathfrak{B}| \text{ and } \|\mathfrak{B}\| < \mu\}.$$

The terminology with "Löwenheim-Skolem-Tarski spectrum" is chosen in accordance with that of [20]. The definition here is however slightly different since we consider not the cardinality of sentences but rather the cardinality of the signature. In [7], and [8], corresponding notion is refered to as "strong Löwenheim-Skolem property".

Our present definition of the Löwenheim-Skolem-Tarski spectrum corresponds to the Löwenheim-Skolem property in [19]:

Lemma 1.7 *For a logic $\mathcal{L}$, we have* P-intro-6

$$\mathsf{LSTS}(\mathcal{L}) = \{\mu \in \mathsf{Card} \,:\, \textit{for any structure } \mathfrak{A} \textit{ with signature of size} < \mu, \textit{ there is } \mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A} \textit{ such that } \|\mathfrak{B}\| < \mu\}.$$

Proof. "$\subseteq$": Suppose that $\mu \in \mathsf{LSTS}(\mathcal{L})$ and let $\mathfrak{A}$ be a structure of signature of size $\nu < \mu$. Without loss of generality, we may assume that $\mathfrak{A}$ is a relational structure and $\mathfrak{A} = \langle |\mathfrak{A}|, R_{n,\alpha}\rangle_{n\in\omega,\alpha<\nu}$ where $R_{n,\alpha}$ is an n-ary relation on $|\mathfrak{A}|$ for $n \in \omega$ and $\alpha < \nu$. We may also assume, without loss of generality, that $\|\mathfrak{A}\| \geq \mu$ and $\nu \subseteq |\mathfrak{A}|$.

Let $R_n := \bigcup_{\alpha<\nu}\{\alpha\} \times R_{n,\alpha}$ for each $n \in \omega$. Let $\mathfrak{A}^- := \langle |\mathfrak{A}|, R_n\rangle_{n\in\omega}$.

Applying our assumption on μ, we find $\mathfrak{B}^- \prec_{\mathcal{L}} \mathfrak{A}^-$ with $\|\mathfrak{B}^-\| < \mu$ and $\nu \subseteq |\mathfrak{B}^-|$. By the last condition, we can reconstruct a submodel $\mathfrak{B}$ of $\mathfrak{A}$ from $\mathfrak{B}^-$ with $|\mathfrak{B}| = |\mathfrak{B}^-|$ and $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$.

"$\supseteq$": Suppose now that μ is in the set on the right side of the equality. Let $\mathfrak{A}$ be a structure of size $\geq \mu$ with a countable signature, and $S \in [\,|\mathfrak{A}|\,]^{<\mu}$.

Let $\mathfrak{A}^+ = \langle \mathfrak{A}, a\rangle_{a\in S}$. Applying the assumption on μ, we obtain $\mathfrak{B}^+ \prec_{\mathcal{L}} \mathfrak{A}^+$ of size $<\mu$. Denoting by $\mathfrak{B}$ the structure $\mathfrak{B}^+$ reduced to the original signature, we have $\|\mathfrak{B}\| < \mu$, $S \subseteq |\mathfrak{B}|$ and $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$. ❑ (Lemma 1.7)

Lemma 1.8 *For any logic $\mathcal{L}$, $\mathsf{LSTS}(\mathcal{L})$ is a closed class of cardinals.* P-intro-7

Proof. Suppose that $\langle \kappa_\alpha \,:\, \alpha < \delta\rangle$ is a strictly increasing sequence in $\mathsf{LSTS}(\mathcal{L})$ and $\kappa = \sup_{\alpha<\delta}\kappa_\alpha$. We want to show that $\kappa \in \mathsf{LSTS}(\mathcal{L})$.

Suppose that $\mathfrak{A}$ is a structure of countable signature and $S \subseteq [\,|\mathfrak{A}|\,]^{<\kappa}$. Let $\alpha < \delta$ be such that $|S| < \kappa_\alpha$. Since $\kappa_\alpha \in \mathsf{LSTS}(\mathcal{L})$, there is a $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$ such that $S \subseteq |\mathfrak{B}|$ and $\|\mathfrak{B}\| < \kappa_\alpha < \kappa$. This shows that $\kappa \in \mathsf{LSTS}(\mathcal{L})$. ❑ (Lemma 1.8)

For $\mathcal{L}^{\rm II}$, one detail of the definition of the $\mathcal{L}^{\rm II}$-elementary submodel relation must be emphasized: for structures $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{A} \prec_{\mathcal{L}^{\rm II}} \mathfrak{B}$ if and only if for all $\mathcal{L}^{\rm II}$-formula $\varphi(x_0, ...)$ (in the signature of the structures) without free second-order variables and $a_0, ... \in |\mathfrak{A}|$,

$$\mathfrak{A} \models_{\mathcal{L}^{\rm II}} \varphi(a_0, ...) \;\Leftrightarrow\; \mathfrak{B} \models_{\mathcal{L}^{\rm II}} \varphi(a_0, ...). \tag{1.4}$$ x-intro-0

Exclusion of the second-order parameters from the definition is justified by the fact that we would have only trivial cases of elementary substructure relation $\mathfrak{A} \prec_{\mathcal{L}^{\rm II}} \mathfrak{B}$ (namely when $\mathfrak{A} = \mathfrak{B}$) if we would have included the second-order parameters in the definition (1.4).

By Lemma 1.8, the proof of Theorem 1 and Theorem 2 in [19] can be recast to show the following:

Theorem 1.9 (M. Magidor [19]) P-intro-7-0

$\mathsf{LSTS}(\mathcal{L}^{\rm II}) = \mathsf{LSTS}(\mathcal{L}^{\rm HO}) = \{\kappa \in \mathsf{Card} : \kappa$ *is a supercompact cardinal, or a limit of supercompact cardinals*$\}$ □

Similarly to $\mathfrak{cn}(\mathcal{L})$ and $\mathfrak{wcn}(\mathcal{L})$, we define the Löwenheim-Skolem-Tarski number of a logic $\mathcal{L}$ be the least element of $\mathsf{LSTS}(\mathcal{L})$. More precisely, for a logic $\mathcal{L}$, we let

$$\mathfrak{lstn}(\mathcal{L}) := \min(\mathsf{LSTS}(\mathcal{L}) \cup \{\infty\}). \tag{1.5}$$ x-intro-1-0

Theorem 1.9 implies

$$\mathfrak{lstn}(\mathcal{L}^{\rm II}) = \mathfrak{lstn}(\mathcal{L}^{\rm HO}) = \text{the least supercompact cardinal (if there is one).} \tag{1.6}$$ x-intro-1-1

$\mathcal{L}^{\aleph_0,\rm II}$ denotes the weak (monadic) second-order logic with second-order variables X, Y, Z etc. whose intended interpretation is that they run over countable subsets of the underlying set of the structure in consideration. We shall call this type of second-order variables weak second-order variables (in $\aleph_0$ -interpretation).

The formulas of $\mathcal{L}^{\aleph_0,\rm II}$, $\mathcal{L}^{\aleph_0,\rm II}_{\kappa,\omega}$, $\mathcal{L}^{\aleph_0,\rm II}_{\kappa,\lambda}$ are defined in exactly the same way as the formulas of $\mathcal{L}^{\rm II}$, $\mathcal{L}^{\rm II}_{\kappa,\omega}$, $\mathcal{L}^{\rm II}_{\kappa,\lambda}$ but the inductive definition of the semantics uses the $\aleph_0$-interpretation:

$$\begin{aligned} &\mathfrak{A} \models_{\mathcal{L}^{\aleph_0,\rm II}} \exists X \varphi(a_0, ..., X, A_0, ...) \\ &\quad :\Leftrightarrow \;\text{there is } B \in [\,|\mathfrak{A}|\,]^{\aleph_0} \text{ such that } \mathfrak{A} \models_{\mathcal{L}^{\aleph_0,\rm II}} \varphi(a_0, ..., B, A_0, ...). \end{aligned}$$

In case of this weak second-order logic, a definition of the elementary submodel relation with second-order parameters also makes sense. Here, for simplicity, we assume below that the elementarity $\prec_{\mathcal{L}^{\aleph_0,\rm II}}$, $\prec_{\mathcal{L}^{\aleph_0,\rm II}_{\kappa,\omega}}$ etc. are always defined similarly to (1.4) without second-order parameters.

The Löwenheim-Skolem theorems of stationary quantifier in the context of the weak second-order logic and its interplay with various reflection principles were studied in [7].

The logic $\mathcal{L}^{\aleph_0}_{stat}$ is defined as the logic with monadic second-order variables with the second-order quantifier *stat* (and its dual *a.a.*) such that the recursive definition of $\mathcal{L}^{\aleph_0}_{stat}$-formulas includes (1.1) together with the following in addition to the clauses in the usual definition of the first-order logic:

> If φ is a $\mathcal{L}^{\aleph_0}_{stat}$-formula and X a second-order variable, then $stat X\,\varphi$ is also a $\mathcal{L}^{\aleph_0}_{stat}$-formula.

The model relation of this logic is then defined as usual with the following additional clause:

> For any $\mathcal{L}^{\aleph_0}_{stat}$-formula $\varphi(x_0, ..., X, X_0, ...)$, $a_0, ... \in |\mathfrak{A}|$ and $A_0, ... \in [\,|\mathfrak{A}|\,]^{\aleph_0}$,
>
> $$\begin{aligned}&\mathfrak{A} \models_{\mathcal{L}^{\aleph_0}_{stat}} stat\, X\ \varphi(a_0, ..., X, A_0, ...)\\ :\Leftrightarrow\ &\{A \in [\,|\mathfrak{A}|\,]^{\aleph_0} : \mathfrak{A} \models_{\mathcal{L}^{\aleph_0}_{stat}} \varphi(a_0, ..., A, A_0, ...)\} \text{ is a stationary subset of } [\,|\mathfrak{A}|\,]^{\aleph_0}.\end{aligned}$$

$\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}$ is the extension of $\mathcal{L}^{\aleph_0}_{stat}$ with the weak second-order existential (and universal) quantification. $\mathcal{L}^{\aleph_0}_{stat,\kappa,\lambda}$ and $\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat,\kappa,\lambda}$ are then the infinitary versions of $\mathcal{L}^{\aleph_0}_{stat}$ and $\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}$ defined as expected.

Lemma 1.10 (1) *The expressive power of $\mathcal{L}^{\aleph_0,\mathrm{II}}_{\kappa,\omega}$ exceeds that of $\mathcal{L}_{\kappa,\omega}$.* P-intro-8

(2) *For $\omega < \lambda \leq \kappa$, $\mathcal{L}^{\aleph_0,\mathrm{II}}_{\kappa,\lambda}$ has the same expressive power as $\mathcal{L}_{\kappa,\lambda}$.*

(3) *For any $\omega < \lambda \leq \kappa$, $\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat,\kappa,\lambda}$ is interpretable in $\mathcal{L}^{\mathrm{III}}_{\kappa,\lambda}$ where $\mathcal{L}^{\mathrm{III}}$ denotes the third-order logic, and $\mathcal{L}^{\mathrm{III}}_{\kappa,\lambda}$ its infinitary extension.*

Proof. (1): For a binary relation symbol R, "R is well-founded" can be expressed by the $\mathcal{L}^{\aleph_0,\mathrm{II}}$-sentence:

$$\forall X \exists_y (y\ \varepsilon\ X\ \wedge\ \forall_x (x \not\!R\ y)). \tag{1.7}$$

On the other hand, $\mathcal{L}_{\kappa,\omega}$ cannot express the well-foundedness of R by a theorem of Lopez-Escobar [17].

(2): We define the translation of $\mathcal{L}^{\aleph_0,\mathrm{II}}_{\kappa,\lambda}$-formula φ into $\mathcal{L}_{\kappa,\lambda}$-formula φ_* by assigning each second-order variable X to countably many new first order variables x^X_i, $i \in \omega$; assigning each atomic formula of the form $x\ \varepsilon\ X$ to the formula

$$\bigvee\{x \equiv x^X_i\ :\ i \in \omega\};$$

and assigning each $\mathcal{L}^{\aleph_0,\mathrm{II}}_{\kappa,\lambda}$-formula φ of the form $\exists X\,\psi(...,,X,...)$ to $\mathcal{L}_{\kappa,\lambda}$-formula φ_* of the form $\exists_{x_0^X}\exists_{x_1^X}...(\bigwedge\!\!\!\bigwedge_{i<j<\omega} x_i^X \not\equiv x_j^X \;\wedge\; \psi_*(...,x_0^X,x_1^X,...,...))$.

(3): A translation of $\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat,\kappa,\lambda}$-formula φ into $\mathcal{L}^{\mathrm{III}}_{\kappa,\lambda}$-formula φ_{**} will do which starts similarly to $\varphi\mapsto\varphi_*$ in (2) and continues with the following details of the recursive definition:

If the $\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat,\kappa,\lambda}$-formula φ is of the form $\exists X\psi$, the translation φ_{**} is defined to be the $\mathcal{L}^{\mathrm{III}}_{\kappa,\lambda}$-formula:

$$\exists X\exists_{x_0^X}\exists_{x_1^X}\cdots(\bigwedge\!\!\!\bigwedge_{i<j<\omega} x_i^X \not\equiv x_j^X \;\wedge\; \forall_y(y\,\varepsilon\,X \;\leftrightarrow\; \bigvee\!\!\!\bigvee_{k<\omega} y\equiv x_k^X) \;\wedge\; \psi_{**}).$$

If the $\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat,\kappa,\lambda}$-formula φ is of the form $stat\,X\psi$, the translation φ_{**} is defined to be the $\mathcal{L}^{\mathrm{III}}_{\kappa,\lambda}$-formula:

$$\begin{aligned}&\exists\mathcal{X}\,(\,\forall X\,(X\,\varepsilon\,\mathcal{X}\;\rightarrow\\&\qquad\exists_{x_0^X}\exists_{x_1^X}...(\bigwedge\!\!\!\bigwedge_{i<j<\omega} x_i^X \not\equiv x_j^X \;\wedge\; \forall_y(y\,\varepsilon\,X \;\leftrightarrow\; \bigvee\!\!\!\bigvee_{k<\omega} y\equiv x_k^X) \;\wedge\\&\psi_{**}))\\&\quad\wedge\;\forall\mathcal{Y}\,(\text{“}\mathcal{Y}\text{ is a club of countable sets”}\;\rightarrow\exists X\,(X\,\varepsilon\,\mathcal{X}\;\wedge\;X\,\varepsilon\,\mathcal{Y})))\end{aligned}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are third order variables. ❑ (Lemma 1.10)

Proposition 1.11 (1) *An uncountable cardinal κ is weakly compact if and only if $\kappa=\mathfrak{wcn}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{\kappa,\omega})$.* P-intro-9

(2) $\mathfrak{wcn}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat,\kappa,\omega})\le\mathfrak{wcn}(\mathcal{L}^{\mathrm{III}}_{\kappa,\omega})\le$ *the first weakly extendible cardinal above κ.*

Proof. By Lemma 1.6, Lemma 1.4, (3) and Lemma 1.10. The rightmost inequality in (2) follows from Theorem 2.2 (for the definition of weakly extendibility, see around (2.1)). ❑ (Proposition 1.11)

In Section 4, we discuss about further results related to Proposition 1.11 above. In particular, we shall show that $\mathfrak{wcn}(\mathcal{L}^{\aleph_0,\mathrm{II}})$ is above certain large cardinals (see the remark after Proposition 4.1) while, in L, the condition $\kappa=\mathfrak{wcn}((\mathcal{L}^{\aleph_0}_{stat})_{\kappa,\omega})$ is equivalent to $\kappa=\mathfrak{wcn}((\mathcal{L}^{\mathrm{II}})_{\kappa,\omega})$ (Theorem 4.6).

2 Weakly extendible cardinals

Let us begin with recalling the definition of extendible cardinals: a cardinal κ is *extendible* if, for any $\eta>0$, there is a ζ and j such that $j:V_{\kappa+\eta}\overset{\prec}{\rightarrow}_\kappa V_\zeta$ (see Kanamori [15])[2]. weakl-ext

[2] In the following, write $j:M\overset{\prec}{\rightarrow}_\kappa N$ to denote the situation that j is an elementary embedding (in the sense of elementarity in the first-order logic) of $M=\langle M,\in\rangle$ into $N=\langle N,\in\rangle$, both M and fn-0

As it is well known, weakly and strongly compact cardinals are characterized in terms of elementary embeddings:

Lemma 2.1 (1) (see e.g. [9]) *A cardinal κ is weakly compact if and only if $\kappa = 2^{<\kappa}$ and satisfies the following Embedding Property: for any transitive M with $\kappa \in M$ and $|\, M\,| = \kappa$, there is a transitive N with j such that $j : M \overset{\prec}{\rightarrow}_\kappa N$.* P-w-ext-a

(2) (Theorem 22.17 in [15]) *A cardinal κ is strongly compact if and only if, for any cardinal $\lambda > \kappa$, there are classes M, j such that $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$ and, for any $a \in [M]^{\leq\lambda}$, there is $b \in [M]^{<j(\kappa)} \cap M$ with $a \subseteq b$.* ❑

Comparing the definition of extendibility with the characterizations of weak and strong compact cardinals in Lemma 2.1, The following notion seems to be a good candidate of the large cardinal property which should be characterized by (1.3):

We shall say that a cardinal κ is *weakly extendible* if

(2.1) $\kappa = 2^{<\kappa}$ and, x-w-ext-0

(2.2) for any $\theta > \kappa$, and $M \prec V_\theta$ with $\kappa + 1 \subseteq M$, and $|\, M\,| = \kappa$, there are $\overline{\theta}$ and j with $j : M \preccurlyeq_\kappa V_{\overline{\theta}}$. x-w-ext-1

Here, when we write $j : M \preccurlyeq_\kappa N$, we assume that $\kappa + 1 \subseteq M, N$ (but $M = \langle M, \in\rangle$ and $N = \langle N, \in\rangle$ are not necessarily transitive), and $j : M \rightarrow N$ is an elementary embedding with $j \restriction \kappa = id_\kappa$ and $j(\kappa) > \kappa$. Similarly to the notation $j : M \overset{\prec}{\rightarrow}_\kappa N$, the "elementary embedding" here is meant in terms of first-order logic. Note that, if $m : M \overset{\cong}{\rightarrow} M_0$, $n : N \overset{\cong}{\rightarrow} N_0$ are the Mostowski collapses of M and N, we have $n \circ j \circ m^{-1} : M_0 \overset{\prec}{\rightarrow}_\kappa N_0$.

A weakly extendible cardinal is (strongly) inaccessible. This can be shown by an easy direct argument (see Lemma A 2.1 below). However, by the definition of the weak extendibility and the characterization of weak compactness (Lemma 2.1, (1)), we see immediately that a weakly extendible cardinal is weakly compact and we know that a weakly compact cardinal is inaccessible (e.g. Proposition 4.4 in [15]).

Lemma A 2.1 *If κ is weakly extendible then κ is (strongly) inaccessible.* P-w-ext-a-0

Proof. κ is not a successor cardinal: Suppose otherwise, say $\kappa = \mu^+$. Let $M \prec V_{\kappa+1}$ be such that $\kappa + 1 \subseteq M$ and $|\, M\,| = \kappa$. Let $j : M \preccurlyeq_\kappa V_{\overline{\theta}}$. Then $j(\mu) = \mu$ and

---

N are transitive, and κ is the critical point of j. M, N, j are sets in the definition of extendibility but later we shall also treat the cases where these are proper classes possibly in some generic extension of V.

hence $V_{\overline{\theta}} \models$ "$j(\kappa)$ is the successor of μ". It follows that $j(\kappa) = (\mu^+)^{V_{\overline{\theta}}} \leq \mu^+ = \kappa$. This is a contradiction.

κ is not a singular cardinal: Suppose otherwise, say $\kappa = \lim_{\alpha<\mu} \kappa_\alpha$ for $\mu < \kappa$ and $\kappa_\alpha < \kappa$ for $\alpha < \mu$. For an $M \prec V_{\kappa+1}$ with $\langle \kappa_\alpha : \alpha < \mu \rangle \in M$ and $|M| = \kappa$ let $j : M \preccurlyeq_\kappa V_{\overline{\theta}}$. By elementarity, we have

$$V_{\overline{\theta}} \models \text{“}j(\kappa) \text{ is the limit of } j(\langle \kappa_\alpha : \alpha < \mu \rangle)\text{”}.$$

Since $j(\langle \kappa_\alpha : \alpha < \mu \rangle) = \langle \kappa_\alpha : \alpha < \mu \rangle$, it follows that $j(\kappa) = \kappa$. This is again a contradiction.

κ is a strong limit: Otherwise there is a $\mu < \kappa$ such that $2^\mu = \kappa$ (Note that the assumption $2^{<\kappa} = \kappa$ implies $2^\mu \leq \kappa$).

Let $f : {}^\mu 2 \to \kappa$ be a bijection and let $M \prec V_{\kappa+1}$ be such that $f \in M$, $\kappa+1 \subseteq M$ and $|M| = \kappa$. Note that ${}^\mu 2 \subseteq M$ and hence $({}^\mu 2)^M = {}^\mu 2$.

Let $j : M \preccurlyeq_\kappa V_{\overline{\theta}}$. Then $j({}^\mu 2) = {}^\mu 2$ and $j(f) = j(f) \restriction {}^\mu 2 = f$. By elementarity, it follows that $V_{\overline{\theta}} \models$ "$j(\kappa)$ is the least upper bound of $f''\,{}^\mu 2$" which implies $j(\kappa) = \kappa$. This is a contradiction. ❑ (Lemma 2.1)

The next theorem shows that our notion of weak extendibility is exactly what we are looking for.

After we had written the first version of this paper, Will Boney told us that he also obtained a slight variation of the following theorem which also can be seen as a characterization of the weak compactness of infinitary second-order logics (cf. Theorem 4.5 in [4]). We keep our proof of the theorem here since it will be modified to obtain further results in this and next sections.

Theorem 2.2 *For a cardinal κ, the following are equivalent:* P-w-ext-0

(a) $\kappa = \mathfrak{wcn}(\mathcal{L}^{\rm II}_{\kappa,\omega})$.

(b) $\kappa = \mathfrak{wcn}(\mathcal{L}^{\rm HO}_{\kappa,\kappa})$.

(c) *κ is weakly extendible.*

Proof. "(c) ⇒ (b)": Assume that κ is weakly extendible. κ is then inaccessible (see the remark after the definition of weakly extendibility). Suppose that T is a $<\kappa$-satisfiable $\mathcal{L}^{\rm HO}_{\kappa,\kappa}$-theory of signature of size $\leq \kappa$. We want to show that T is satisfiable.

Since T has cardinality $\leq \kappa$, we may assume that T is a subset of κ by some reasonable coding.

Let θ be large enough. In particular, such that

(2.3) $\quad \theta \geq \kappa^{+\omega}$ and x-w-ext-2

(2.4) $\quad V_\theta \models$ " T is $<\kappa$-satisfiable". x-w-ext-3

Let $M \prec V_\theta$ be such that $\kappa+1 \subseteq M$, $T \in M$, $|\,M\,| = \kappa$, and let $\overline{\theta}$, j be such that $j : M \preccurlyeq_\kappa V_{\overline{\theta}}$. Then we have $V_{\overline{\theta}} \models$ " $j(T)$ is $<j(\kappa)$-satisfiable" by (2.4) and by elementarity of j.

Since $V_{\overline{\theta}} \models$ " $|\,T\,| < j(\kappa)$ and $\overbrace{T}^{=\,j(T)\,\cap\,\kappa} \subseteq j(T)$ ", it follows that there is $\mathfrak{A} \in V_{\overline{\theta}}$ such that $V_{\overline{\theta}} \models$ " $\mathfrak{A} \models_{\mathcal{L}^{\mathrm{HO}}_{\kappa,\kappa}} T$ ". Now $\overline{\theta} \geq j(\kappa)^{+\omega} \geq \kappa^{+\omega}$ by (2.3) and by elementarity of j. Thus, it follows that $\mathfrak{A} \models_{\mathcal{L}^{\mathrm{HO}}_{\kappa,\kappa}} T$. Thus, T is realizable.

"(b) $\Rightarrow$ (a)": By Lemma 1.4, (3) and Lemma 1.2.

"(a) $\Rightarrow$ (c)": Assume that $\kappa = \mathfrak{wcn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega})$ holds. Then we have $\kappa = \mathfrak{wcn}(\mathcal{L}_{\kappa,\omega})$ by Lemma 1.2. Hence κ is weakly compact by Lemma 1.6. In particular, κ is inaccessible. Thus it is enough to show that κ satisfies (2.2).

Suppose that $\theta > \kappa$ and $M \prec V_\theta$ is such that $\kappa+1 \subseteq M$ and $|\,M\,| = \kappa$.

Let φ^* be an $\mathcal{L}^{\mathrm{II}}$-sentence in the signature $\{\underset{\sqcup}{\in}\}$ such that

(2.5) $\quad \langle\,|\mathfrak{A}|\,, \underset{\sqcup}{\in}^{\mathfrak{A}}\rangle \models \varphi^* \;\Leftrightarrow\; \underset{\sqcup}{\in}^{\mathfrak{A}}$ is well-founded and extensional binary relation, and the Mostowski collapse of $(\langle\,|\mathfrak{A}|\,, \underset{\sqcup}{\in}^{\mathfrak{A}}\rangle)$ is $\langle V_\gamma, \in\rangle$ for some γ x-w-ext-3-0

Let

(2.6) $\quad T := \{\varphi^*\} \cup \{\varphi(\underset{\sqcup}{c}_{a_0}, ...) \,:\, \varphi$ is a first-order formula in the signature $\{\underset{\sqcup}{\in}\}$, $a_0, ... \in M$ and $M \models \varphi(a_0, ...)\}$ x-w-ext-4

$$\left.\begin{array}{l} \cup\ \{\forall_x(x \underset{\sqcup}{\in} \underset{\sqcup}{c}_\alpha \;\leftrightarrow\; \bigvee\!\!\!\bigvee_{\beta<\alpha} x \equiv \underset{\sqcup}{c}_\beta) \,:\, \alpha < \kappa\} \\ \cup\ \{\underset{\sqcup}{c}_\alpha \underset{\sqcup}{\in} \underset{\sqcup}{d} \,:\, \alpha < \kappa\} \\ \cup\ \{\underset{\sqcup}{d} \underset{\sqcup}{\in} \underset{\sqcup}{c}_\kappa\}. \end{array}\right\} \quad (*)$$

The signature of the $\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}$-theory T is $\{\underset{\sqcup}{\in}, \underset{\sqcup}{d}\} \cup \{\underset{\sqcup}{c}_a \,:\, a \in M\}$ and it is of cardinality κ.

Claim 2.2.1 *T is $<\kappa$-satisfiable.*

$\vdash$ Suppose that $T_0 \in [T]^{<\kappa}$. We have to show that T_0 has a model.

Let $C_0 := \{a \in M \,:\, \underset{\sqcup}{c}_a$ appears in $T_0\}$ and

$$\alpha^* := \sup\{\alpha+1 \,:\, \alpha < \kappa,\ \underset{\sqcup}{c}_\alpha \text{ appears in } T_0\}.$$

Let $\mathfrak{M} = \langle V_\theta, \underset{\sqcup}{d}^{\mathfrak{M}}, (\underset{\sqcup}{c}_a)^{\mathfrak{M}}, \underset{\sqcup}{\in}^{\mathfrak{M}}\rangle_{a\in C_0}$ where $\underset{\sqcup}{d}^{\mathfrak{M}} := \alpha^*$ and $(\underset{\sqcup}{c}_a)^{\mathfrak{M}} := a$ for all $a \in C_0$. Then $\mathfrak{M} \models_{\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}} T_0$. $\dashv$ (Claim 2.2.1)

By the assumption on κ, it follows that T is satisfiable. Let $\mathfrak{B}$ be a model of T. By $\mathfrak{B} \models_{\mathcal{L}^{\mathrm{II}}} \varphi^*$, we can take the Mostowski collapse $\mathfrak{B}^*$ of $\mathfrak{B}$, and $|\mathfrak{B}^*| = V_{\overline{\theta}}$

for some ordinal $\overline{\theta}$. Note that we have $\underset{\smile}{\in}^{\mathfrak{B}^*} = \in$. By the definition $j : M \to V_{\overline{\theta}}$; $a \mapsto [\underset{\smile}{c}_a]^{\mathfrak{B}^*}$, we obtain $j : M \preccurlyeq_\kappa V_{\overline{\theta}}$. Note that κ is the critical point of j by (2.6) (*). □ (Theorem 2.2)

We also obtain a characterization of elements of $\mathsf{WCS}(\mathcal{L}^{\mathrm{II}})$ by modifying the proof of Theorem 2.2.

Let us call a cardinal κ *weakly sub-extendible* if, for any $\theta > \kappa$ and $M \prec V_\theta$ with $\kappa + 1 \subseteq M$ and $|\,M\,| = \kappa$, there are $\overline{\theta}$ and j with $j : M \preccurlyeq_\mu V_{\overline{\theta}}$ for some $\mu \leq \kappa$ such that (2.7): $j(\kappa) > \sup(j''\kappa)$. x-w-ext-4-0

Theorem 2.3 *For a cardinal κ, $\kappa \in \mathsf{WCS}(\mathcal{L}^{\mathrm{II}}) \Leftrightarrow \kappa$ is weakly sub-extendible.* P-w-ext-0-0

Proof. ($\Leftarrow$): Suppose that κ is weakly sub-extendible and T is a $<\kappa$-satisfiable $\mathcal{L}^{\mathrm{II}}$-theory of cardinality κ (note that, since $\mathcal{L}^{\mathrm{II}}$ is finitary, the cardinality of T ($+\aleph_0$) is equal to the cardinality of the signature ($+\aleph_0$). We may assume that T is is nicely coded as a subset of κ.

Let $\theta > \kappa$ be large enough regular cardinal such that $V_\theta \models$ "T is $<\kappa$-satisfiable". Let $M \prec V_\theta$ be such that $\kappa + 1 \subseteq M$, $T \in M$ and $|\,M\,| = \kappa$. By assumption there are j and $\overline{\theta}$ such that $j : M \preccurlyeq_\mu V_{\overline{\theta}}$ for some $\mu \leq \kappa$ and $j(\kappa) > \kappa$.

By elementarity, we have $V_{\overline{\theta}} \models$ "$j(T)$ is $<j(\kappa)$-satisfiable". Since $V_{\overline{\theta}} \models$ "$j''T \in [j(T)]^{<j(\kappa)}$", it follows that $V_{\overline{\theta}} \models$ "$j''T$ is satisfiable". Let $\mathfrak{A} \in V_{\overline{\theta}}$ be such that $V_{\overline{\theta}} \models$ "$j''T \models_{\mathcal{L}^{\mathrm{II}}} \mathfrak{A}$". Since $V_{\overline{\theta}}$ interpret $\models_{\mathcal{L}^{\mathrm{II}}}$ correctly (later we introduce the terminology with which we would say V_θ is $\mathcal{L}^{\mathrm{II}}$-truthful), we have $\mathfrak{A} \models_{\mathcal{L}^{\mathrm{II}}} j''T$. Thus, by renaming the structure of $\mathfrak{A}$ we obtain a model $\mathfrak{A}'$ of T.

($\Rightarrow$): Suppose that $\kappa \in \mathsf{WCS}(\mathcal{L}^{\mathrm{II}})$. Suppose that θ and M are as in the definition of the weakly sub-extendibility. Let

$$
\begin{aligned}
(2.8)\qquad T := \{\varphi^*\} \cup \{\varphi(\underset{\smile}{c}_{a_0}, \ldots) \,:\, & \varphi \text{ is a first-order formula in the signature } \{\underset{\smile}{\in}\}, \\
& a_0, \ldots \in M \text{ and } M \models \varphi(a_0, \ldots)\} \\
\left.\begin{aligned} &\cup \{\underset{\smile}{c}_\alpha \,\underset{\smile}{\in}\, \underset{\smile}{d} \,:\, \alpha < \kappa\} \\ &\cup \{\underset{\smile}{d} \,\underset{\smile}{\in}\, \underset{\smile}{c}_\kappa\} \end{aligned}\right\} \quad (**)
\end{aligned}
$$
x-w-ext-5

where φ^* is the $\mathcal{L}^{\mathrm{II}}$-sentence in (2.5).

T is a $\mathcal{L}^{\mathrm{II}}$-theory of size κ. Similarly to the proof of Theorem 2.2 we can show that T is $<\kappa$-satisfiable. Thus, by definition of κ, T has a model $\mathfrak{A}$. The Mostowski collapse $\mathfrak{A}^*$ of the model $\mathfrak{A}$ with respect to $\underset{\smile}{\in}^{\mathfrak{A}}$ is $V_{\overline{\theta}}$ for some $\overline{\theta} > \kappa$ and $j : M \to V_{\overline{\theta}}$; $a \mapsto (\underset{\smile}{c}_a)^{\mathfrak{A}^*}$ is an elementary embedding of M into $V_{\overline{\theta}}$. Because of missing first line of (2.6) (*) in T of (2.6), our T in (2.8) cannot guarantee that κ is the critical point of the elementary embedding but its critical point must be some cardinal $\leq \kappa$ by (2.8) (**), and (2.7) holds.

This shows that κ is weakly sub-extendible. ❑ (Theorem 2.3)

To place the weakly extendible cardinal in the hierarchy (or a zoo?) of (small) large cardinals, let us recall some notions of large cardinals we are going to mention in Theorem 2.5:

A cardinal κ is said to be *strongly unfoldable* if $\kappa = 2^{<\kappa}$ and, for any ordinal $\lambda > \kappa$ and any transitive model M of ZFC^- such that $\kappa \in M$, ${}^{\kappa>}M \subseteq M$ and $|\,M\,| = \kappa$, there is a transitive $N \supseteq V_\lambda$ with $j : M \xrightarrow{\prec}_\kappa N$ and $j(\kappa) > \lambda$. Here, ZFC^- denotes the axiom system ZFC without the Power Set Axiom.

The notion of strongly unfoldable cardinal was introduced by A. Villaveces in [25]. Recently, P. Lücke [18] proved that the strong unfoldability is equivalent to the shrewdness, a natural strengthening of the total indescribability which was introduced by M. Rathjen in [21].

Similarly to the argument after the definition (2.1), (2.2) of weakly extendible cardinals, we can also easily prove that a strongly unfoldable cardinal is weakly compact and hence inaccessible in particular.

An uncountable cardinal κ is said to be *subtle* if, for any club $C \subseteq \kappa$, and any sequence $\langle A_\alpha \,:\, \alpha \in C\rangle$ such that $A_\alpha \subseteq \alpha$ for all $\alpha \in C$, there are α, $\beta \in C$ with $\alpha < \beta$ such that $A_\alpha = A_\beta \cap \alpha$. The notion of subtle cardinal was first considered by Jensen and Kunen in [14]. Baumgartner [2], [3] studied further its combinatorial properties and associated ideals.

Note that subtle cardinals are compatible with $\mathsf{V} = \mathsf{L}$: if κ is subtle, then so it is in L.

Strongly uplifting cardinals are introduced in Hamkins and Johnstone [10]: an inaccessible cardinal κ is *strongly uplifting* if, for every $A \subseteq \kappa$ there are arbitrarily large regular $\theta > \kappa$ such that $\langle V_\kappa, \in, A\rangle \prec \langle V_\theta, \in, \overline{A}\rangle$ for some $\overline{A} \subseteq V_\theta$.

The following Lemma A 2.2 and Proposition A 2.3 were originally used in a direct proof of Theorem 2.5, (4).

Lemma A 2.2 *Suppose that κ is a subtle cardinal. Then:* (1) *κ is inaccessible.* P-w-ext-0-1
(2) *For any club $C \subseteq \kappa$ and any sequence $\langle A_\alpha \,:\, \alpha \in C\rangle$ such that $A_\alpha \subseteq \alpha$ for all $\alpha \in C$, there are inaccessible α, $\beta \in C$ with $\alpha < \beta$ and $A_\alpha = A_\beta \cap \alpha$.*

Proof. (1): κ is regular: Suppose that κ is a singular cardinal. Say, $\kappa = \sup_{\xi<\mu} \kappa_\xi$ for some $\mu < \kappa$ and a continuously and strictly increasing sequence of ordinals such that $\kappa_0 > \mu$. Let $C = \{\kappa_\xi \,:\, \xi < \mu\}$ and, for each $\kappa_\xi \in C$, let $A_{\kappa_\xi} = \{\xi\}$. Then $\langle A_\eta \,:\, \eta \in C\rangle$ is a counter example for the subtleness of κ.

κ is a strong limit: Suppose that κ is not a strong limit. Then there is $\mu < \kappa$ such that $2^\mu \geq \kappa$. Let $C = \kappa \setminus \mu$ and let $f : C \to \mathcal{P}(\mu)$ be an injection. Then

$\langle f(\xi) : \xi \in C \rangle$ is a counterexample for the subtleness of κ.

(2): Let $C \subseteq \kappa$ be a club and let $\langle A_\alpha : \alpha \in C \rangle$ be such that $A_\alpha \subseteq \alpha$ for all $\alpha \in C$. We want to show that there are inaccessible α, $\beta \in C$ with $\alpha < \beta$ and $A_\alpha = A_\beta \cap \alpha$.

Since κ is inaccessible by (1), we may assume that

(ℵ2.1) all $\alpha \in C$ are strong limit cardinals. x-w-ext-8-0

For each α,

$$E_\alpha := \text{even ordinals} \subseteq \alpha, \quad \text{and} \quad O_\alpha := \text{odd ordinals} \subseteq \alpha.$$

Let $f_\alpha : \alpha \to O_\alpha$ be the unique order preserving bijection for each $\alpha \in C$. Note that

(ℵ2.2) for all α, $\beta \in C$ with $\alpha < \beta$, we have $f_\alpha = f_\beta \restriction \alpha$. x-w-ext-8-1

Let

(ℵ2.3) $B'_\alpha := f_\alpha{}'' A_\alpha$, x-w-ext-8-2

(ℵ2.4) x-w-ext-8-3

$$B''_\alpha = \begin{cases} \text{a cofinal subset } D \text{ of } E_\alpha \text{ of order-type } cf(\alpha)\text{, such that 0 is the} \\ \text{least element of } D \text{ and } cf(\alpha) \text{ is the one but least element of } D, \\ \qquad\qquad \text{if } \alpha \text{ is singular;} \\ \emptyset, \qquad\qquad \text{otherwise} \end{cases}$$

and

(ℵ2.5) $B_\alpha = B'_\alpha \mathbin{\dot\cup} B''_\alpha$ x-w-ext-8-4

for all $\alpha \in C$.

Applying the subtleness of κ to the sequence $\langle B_\alpha : \alpha \in C \rangle$, let α, $\beta \in C$ be such that $B_\alpha = B_\beta \cap \alpha$. By (ℵ2.3) (and (ℵ2.2)), it follows that $A_\alpha = A_\beta \cap \alpha$.

Both of α and β must be regular by (ℵ2.4) and hence inaccessible by (ℵ2.1).

□ (Lemma 2.3)

Proposition A 2.3 *A cardinal κ is subtle if and only if* P-w-ext-0-2

(ℵ2.6) *for any club $C \subseteq \kappa$ and any sequence $\langle B_\alpha : \alpha < \kappa \rangle$ such that $B_\alpha \subseteq V_\alpha$ for all $\alpha \in C$, there are α, $\beta \in C$ such that $\alpha < \beta$, α and β are inaccessible, and $B_\alpha = B_\beta \cap V_\alpha$.* x-w-ext-9

Proof. It is clear that (ℵ2.6) implies that κ is subtle. To prove the converse, assume that κ is subtle. Let $f : \kappa \to V_\kappa$ be a bijection such that $f''\alpha \subseteq V_\alpha$ for all $\alpha < \kappa$ and $f''\alpha = V_\alpha$ if and only if $|\,\alpha\,| = |\,V_\alpha\,|$ (this is possible by Lemma 2.2, (1)).

Let $A_\alpha := f^{-1}{}''B_\alpha$ for $\alpha \in C$. By Lemma 2.2, (2), there are inaccessible α, $\beta < \kappa$ with $\alpha < \beta$ and $A_\alpha = A_\beta \cap \alpha$. Noticing that the mappings $f \restriction \alpha : \alpha \to V_\alpha$ and $f \restriction \beta : \beta \to V_\beta$ are bijective, we obtain

$$B_\alpha = f''A_\alpha = f''(A_\beta \cap \alpha) = (f''A_\beta) \cap (f''\alpha) = B_\beta \cap V_\alpha. \qquad \square \text{ (Proposition 2.3)}$$

Theorem 2.4 (Hamkins and Johnstone [10] Theorem 7) *Any subtle cardinal is a stationary limit of strongly uplifting cardinals.* □ P-w-ext-0-1-0

Theorem 2.5 (1) *If κ is weakly extendible, then there is a weakly compact $\lambda > \kappa$. On the other hand, strong unfoldability of κ does not imply the existence of inaccessible $\lambda > \kappa$.* P-w-ext-1

(2) *If κ is weakly extendible and ν is the first inaccessible cardinal above κ (which exists by (1)), then $V_\nu \models$ "κ is strongly unfoldable but not weakly extendible". Also $V_\nu \models$ "there is no inaccessible cardinal above κ". It follows that* ZFC + *"there is a weakly extendible cardinal" proves consis*(⌜⌜ZFC⌝⌝+⌜there is a strongly unfoldable cardinal⌝).

(3) (Boney, Dimopoulos, Gitman, and Magidor [5], Proposition 4.8) *If κ is a strongly uplifting cardinal then κ is weakly extendible and κ is a stationary limit of weakly extendible cardinals.*

(4) *If κ is a subtle cardinal, then κ is a stationary limit of weakly extendible cardinals.*

Proof. (1): Suppose that κ is a weakly extendible cardinal. By Lemma 2.1, (1), κ is weakly compact.

Let $M \prec V_\theta$ for a $\theta > \kappa$ be such that $V_\theta \models$ "κ is weakly compact", $\kappa + 1 \subseteq M$ and $|\,M\,| = \kappa$. Let $\overline{\theta} > \kappa$ and j be such that $j : M \preccurlyeq_\kappa V_{\overline{\theta}}$.

By elementarity, $V_{\overline{\theta}} \models$ "$j(\kappa)$ is weakly compact". Hence $j(\kappa) > \kappa$ is really weakly compact.

The second assertion follows from (2).

(2): Suppose that κ is weakly extendible and $\nu > \kappa$ is the first inaccessible cardinal above κ.

Since $\kappa = 2^{<\kappa}$, we also have $V_\nu \models \kappa = 2^{<\kappa}$.

Suppose that $M \in V_\nu$ is a transitive model of ZFC^- such that $\kappa \in M$, ${}^{\kappa>}M \subseteq M$ and $|\,M\,| = \kappa$. Let $\kappa < \lambda < \nu$ $(= \mathrm{On}^{V_\nu})$. We have to show that, in V_ν, there is an elementary embedding of M into a target model satisfying the conditions in the definition of strong unfoldability.

Let $\kappa < \theta < \nu$ be such that $M \in V_\theta$ and let $M^* \prec V_\theta$ be such that $M \in M^*$, $M \subseteq M^*$, $\kappa + 1 \subseteq M^*$, and $|\,M^*\,| = \kappa$. By weak extendibility of κ, there is $\overline{\theta} > \kappa$ and j such that $j : M^* \preccurlyeq_\kappa V_{\overline{\theta}}$. Note that $j(\kappa) \geq \nu$ since $j(\kappa)$ must be inaccessible.

Let $N \prec V_{\overline{\theta}}$ be such that V_λ, $j''M^* \subseteq N$ and $|\,N\,| < \nu$. Let $m : N \to N^*$ be the Mostowski collapse. Then we have $N^* \in V_\nu$, and, letting $j^* := m \circ j$, $j^* : M^* \preccurlyeq_\kappa N^*$. Note that we also have $j^* \in V_\nu$.

Since $V_\lambda \subseteq N$, we have $m \restriction V_\lambda = id_{V_\lambda}$ and hence $V_\lambda \subseteq N^*$. Also, it follows that $j^*(\kappa) \geq \lambda$.

By elementarity, $N^* \models$ "$j^*(M)$ is transitive". Since N^* is transitive it follows that $j^*(M)$ is really transitive. Also, by elementarity $N^* \models [j^*(M)]^{<j^*(\kappa)} \subseteq j^*(M)$. It follows that $V_\lambda \subseteq j^*(M)$.

It is easy to check that $j^* \restriction M : M \overset{\preceq}{\to}_\kappa j^*(M)$ and all of these objects are elements of V_ν.

This shows that κ is strongly unfoldable in V_ν.

Clearly, $V_\nu \models$ "there is no inaccessible cardinal $> \kappa$" and this shows the second half of (1).

Since V_ν as above is also a model of ZFC, we obtain $consis(\ulcorner\ulcorner$ZFC$\urcorner\urcorner + \ulcorner$there is a strongly unfoldable cardinal$\urcorner)$.

(3): For the proof of the first claim see [5] (noticing Theorem 2.2). The second claim is obtained by a slight modification of the proof given in [5]: Suppose that κ is a strongly uplifting cardinal and C is a club subset of κ. By the first claim, $\kappa = \mathfrak{wcn}(\mathcal{L}^{\rm II}_{\kappa,\omega})$. Let θ be sufficiently large such that $V_\theta \models$ "$\kappa = \mathfrak{wcn}(\mathcal{L}^{\rm II}_{\kappa,\omega})$" and there is $\overline{C} \subseteq \theta$ such that $\langle V_\kappa, \in, C\rangle \prec \langle V_\theta, \in, \overline{C}\rangle$.

Since $\langle V_\theta, \in, \overline{C}\rangle \models$ "$\underset{\smile}{C}(\kappa)$ and $\kappa = \mathfrak{wcn}(\mathcal{L}^{\rm II}_{\kappa,\omega})$", we have

$$\langle V_\theta, \in, \overline{C}\rangle \models \text{"there is a cardinal } \delta \text{ in } \underset{\smile}{C} \text{ with } \delta = \mathfrak{wcn}(\mathcal{L}^{\rm II}_{\delta,\omega})\text{"}$$

(we denote with $\underset{\smile}{C}$ the unary predicate symbol corresponding to C and $\overline{C}$), it follows that

$$\langle V_\kappa, \in, C\rangle \models \text{"there is a cardinal } \delta \text{ in } \underset{\smile}{C} \text{ with } \delta = \mathfrak{wcn}(\mathcal{L}^{\rm II}_{\delta,\omega})\text{"}$$

by elementarity. Since κ is strongly uplifting, it follows that "$\delta = \mathfrak{wcn}(\mathcal{L}^{\rm II}_{\delta,\omega})$" holds in V for δ as above.

(4): follows from Theorem 2.4 and (3). Suppose that κ is a subtle cardinal and $C \subseteq \kappa$ is a club. By Theorem 2.4, there is $\lambda \in Lim(C)$ such that λ is strongly uplifting. Now since $C \cap \lambda$ is a club in λ, there is a weakly extendible $\mu \in C \cap \lambda$ by (3). ❑ (Theorem 2.5)

Theorem 2.3 implies that $\mathfrak{wcn}(\mathcal{L}^{\rm II})$ cannot be a small cardinal. This follows from the next observation which should be well-known:

Lemma 2.6 *Suppose that* ① $M \prec V_\theta$, $\kappa+1 \subseteq M$, N *and* ② $j : M \preccurlyeq_\mu N$ *for* $\mu \leq \kappa$. *Then,* μ *is* (1) *a regular cardinal,* (2) *weakly inaccessible, and* (3) *weakly Mahlo, weakly hyper Mahlo, etc.* P-w-ext-3

Proof. Without loss of generality, we may assume that N is transitive.

(1): Suppose toward a contradiction that μ is singular. Then, by the elementarity ①, there are $\nu < \mu$ and $f \in M$ such that $M \models$ "$f : \nu \to \mu$ is cofinal". Since $j(f) = j(f) \restriction j(\nu) = j(f) \restriction \nu = f$ it follows, by the elementarity ②, that $N \models$ "$f : \nu \to j(\mu)$ is cofinal". Thus $\mu = j(\mu)$. This is a contradiction.

(2): By (1) (and since μ cannot be ω), it is enough to show that μ is not a successor cardinal. Suppose it were, say $\mu = \nu^+$. $j(\mu) = \mu$ by definition of $\mu > \nu$. By the elementarity ①, we have

$$M \models \text{"}\forall x < \mu\, (x \neq 0 \ \rightarrow\ \exists y (y : \nu \to x \text{ is a surjection}))\text{"}.$$

By the elementarity ②, it follows that

$$N \models \text{"}\forall x < j(\mu)\, (x \neq 0 \ \rightarrow\ \exists y (y : \nu \to x \text{ is a surjection}))\text{"}.$$

Thus we have $\mu \leq j(\mu) \leq \nu^+ = \mu$ and hence $j(\mu) = \mu$. This is a contradiction to the choice of μ.

(3): Suppose that $C \in M$ is a club subset of μ. Then $j(C) \cap \mu = C$ and $N \models$ "$j(C)$ is a club subset of $j(\mu)$". This and (2) together with the elementarity ② imply $N \models$ "$\mu \in j(C)$ and μ is a weakly inaccessible cardinal". Again by the elementarity ② and since C was arbitrary, we obtain

$$M \models \text{"}\forall_x\, (x \text{ is a club subset of } \mu \ \rightarrow \\ x \text{ contains an element which is weakly inaccessible})\text{"}.$$

By the elementarity ②, the same statement holds in V_θ and hence also in V. This shows that μ is weakly Mahlo. The same argument can be repeated to show that μ is weakly hyper Mahlo, weakly hyper hyper Mahlo, etc. ❑ (Lemma 2.6)

Proposition 2.7 (1) *If* $\mathfrak{wcn}(\mathcal{L}^{\mathrm{II}}) < \infty$, $\mathfrak{wcn}(\mathcal{L}^{\mathrm{II}})$ *is greater than the first weakly Mahlo cardinal, first weakly hyper Mahlo cardinal, etc.* P-w-ext-4

(2) $2^{\aleph_0} \notin \mathsf{WCS}(\mathcal{L}^{\mathrm{II}})$, $(2^{\aleph_0})^+ \notin \mathsf{WCS}(\mathcal{L}^{\mathrm{II}})$, *etc.*

Proof. (1): By Theorem 2.3 and Lemma 2.6.

(2): $T :=$ "the $\mathcal{L}^{\mathrm{II}}$-theory of $\langle \mathcal{P}(\omega), a, \in \rangle_{a \in \mathcal{P}(\omega)}$" $\cup$ $\{\underset{\sim}{d} \neq \underset{\sim}{c}_a : a \in \mathcal{P}(\omega)\}$ is a counter-example to $2^{\aleph_0} \in \mathsf{WCS}(\mathcal{L}^{\mathrm{II}})$.

For $n \in \omega \setminus 1$, let $f : \mathcal{P}(\omega) \to 2^{\aleph_0}$ be a bijection, and let

$$\mathcal{F}_n := \{g \,:\, g : (2^{\aleph_0})^{+k} \to \alpha \text{ is a surjection for } k \in n,\ 0 < \alpha < (2^{\aleph_0})^{+(k+1)}\}.$$

Then

$$\begin{aligned} T_n := \text{"the } \mathcal{L}^{\mathrm{II}}\text{-theory of} \\ \langle \mathcal{P}(\omega) \cup (2^{\aleph_0})^{+n}, \mathcal{P}(\omega), 2^{\aleph_0}, (2^{\aleph_0})^{+1}, ..., (2^{\aleph_0})^{+(n-1)}, f, g, \in \rangle_{g \in \mathcal{F}_n}\text{"} \\ \cup \{ \underset{\sqcup}{d} \neq \underset{\sqcup}{c}_a \,:\, a \in \mathcal{P}(\omega) \cup (2^{\aleph_0})^{+n} \} \end{aligned}$$

is a counter example to $(2^{\aleph_0})^{+n} \in \mathsf{WCS}(\mathcal{L}^{\mathrm{II}})$. ❑ (Proposition 2.7)

3 General characterization of weakly $\mathcal{L}$-compact cardinals

We show in this section that Theorem 2.2 can be generalized to a wider class of logics. comp-card

In the following, ZC^- denotes the Zermelo set-theory with the Axiom of Choice minus the Power-set Axiom.

Suppose that $N \models \mathsf{ZC}^-$. For a signature $\mathcal{S} \in N$ with $\mathcal{S} = \langle \underset{\sqcup}{c}_i, \underset{\sqcup}{f}_j, \underset{\sqcup}{r}_k \rangle_{i \in \kappa_0, j \in \kappa_1, k \in \kappa_2}$ where $\underset{\sqcup}{c}_i$, is a constant symbol, $\underset{\sqcup}{f}_j$ an m_j-place function symbol, and $\underset{\sqcup}{r}_k$ an n_k-place relation symbol for each $i \in \kappa_0$, $j \in \kappa_1$, $k \in \kappa_2$; $\kappa_0 + 1$, $\kappa_1 + 1$, $\kappa_2 + 1 \subseteq N$, and

$$N \models \text{"}\mathfrak{A} \text{ is a structure in the signature } \{\underset{\sqcup}{c}_i, \underset{\sqcup}{f}_j, \underset{\sqcup}{r}_k\}_{i \in \kappa_0, j \in \kappa_1, k \in \kappa_2}\text{"} \tag{3.1}$$ x-LS-24-a

we denote with $\mathfrak{A}^N$ the structure

$$\mathfrak{A}^N := \langle A \cap N, c_i, f_j \restriction (A \cap N)^{m_j}, r_k \cap (A \cap N)^{n_k} \rangle_{i \in \kappa_0, j \in \kappa_1, k \in \kappa_2} \tag{3.2}$$ x-LS-24-0

where $c_i = (\underset{\sqcup}{c}_i)^{\mathfrak{A}}$, $f_j = (\underset{\sqcup}{f}_j)^{\mathfrak{A}}$, etc.

Note that,

$$\text{if } m : N \overset{\cong}{\to} N_0 \text{ is the Mostowski collapse, then } \mathfrak{A}^N \cong m(\mathfrak{A}) = (m(\mathfrak{A}))^{N_0}. \tag{3.3}$$ x-LS-24

For a logic $\mathcal{L}$, if N is such that $N \models \mathsf{ZC}^-$ and N contains all parameters needed to define $\mathcal{L}$, we shall say that N is *$\mathcal{L}$-truthful* if, for all structures $\mathfrak{A}$ as above (in connection with this N), $N \models$ "$\mathfrak{A} \models_{\mathcal{L}} \varphi$" is equivalent to $\mathfrak{A}^N \models_{\mathcal{L}} \varphi$. By (3.3), if N is $\mathcal{L}$-truthful, then its Mostowski collapse N_0 is also $\mathcal{L}$-truthful.

Note that, for a sentence φ in a proper logic $\mathcal{L}$, there is a first-order formula φ_* such that

$$N \models \text{"}\mathfrak{A} \models_{\mathcal{L}} \varphi\text{"} \ \Leftrightarrow\ N \models \varphi_*(\mathfrak{A}), \tag{3.4}$$ x-LS-24-1

and an $\mathcal{L}$-formula φ_{**} such that

$$\mathfrak{A}^N \models_{\mathcal{L}} \varphi \;\Leftrightarrow\; N \models_{\mathcal{L}} \varphi_{**}(\mathfrak{A}). \tag{3.5}$$

x-LS-24-2

Thus, letting

$$\Phi^*_{\mathcal{L}} := \{\forall_x(\text{“}x \text{ is a structure in the signature of } \varphi\text{”} \rightarrow (\varphi_*(x) \leftrightarrow \varphi_{**}(x))) \;:\; \varphi \text{ is an } \mathcal{L}\text{-formula}\}, \tag{3.6}$$

x-LS-24-3

we have

$$N \models \Phi^*_{\mathcal{L}} \;\Leftrightarrow\; N \text{ is } \mathcal{L}\text{-truthful} \tag{3.7}$$

x-LS-24-4

for any N with $N \models \mathsf{ZC}^-$

Let us call here a logic $\mathcal{L}$ *finitary* if the set of free variables in any $\mathcal{L}$-formula is finite, the set of all $\mathcal{L}$-formulas of given signature S of cardinality $\leq \kappa$, for an infinite κ has size $\leq \kappa$, and, for any infinite ordinal θ and $\varphi \in V_\theta$, $V_\theta \models$ “φ is an $\mathcal{L}$-formula” if and only if φ is (really) an $\mathcal{L}$-formula.

$\mathcal{L}_{\kappa,\omega}$ for an uncountable cardinal κ is an example of non finitary logic (since the size of the set of formulas in a signature can exceed the size of the signature).

Theorem 3.1 (1) *Suppose that $\mathcal{L}$ is a finitary proper logic such that*

P-comp-2

$$V_\theta \textit{ for all regular uncountable } \theta \textit{ is } \mathcal{L}\textit{-truthful; and} \tag{3.8}$$

x-comp-6

$$\text{“}\underline{\in} \textit{ is well-founded” is expressible by a formula } \varphi^*_{\mathcal{L}} \textit{ in } \mathcal{L}. \tag{3.9}$$

x-comp-7

Then a cardinal κ is weakly $\mathcal{L}$-compact (i.e. $\kappa \in \mathsf{WCS}(\mathcal{L})$) $\Leftrightarrow$

(3.10) *for any regular $\theta \geq \kappa$ and $M \prec V_\theta$ such that $\kappa+1 \subseteq M$, $|M| = \kappa$, there are j, N such that $\kappa+1 \subseteq N$, $j : M \preccurlyeq N$, $j(\kappa) > \min(\mathrm{On}^N \setminus \sup(j''\kappa))$, and N is $\mathcal{L}$-truthful.* *x-comp-8*

(2) *Suppose that $\mathcal{L}^*$ is a logic obtained from a finitary proper logic $\mathcal{L}$ which satisfies (3.8) and (3.9), by extending $\mathcal{L}$ by taking the closure of the set of $\mathcal{L}$ formulas with respect to infinitary conjunction and disjunction of set of formulas of size $< \kappa$ and first order logical operations.*

Then κ is weakly $\mathcal{L}^*$-compact $\Leftrightarrow$ $2^{<\kappa} = \kappa$ and

(3.11) *for any regular $\theta \geq \kappa$ and $M \prec V_\theta$ such that $\kappa+1 \subseteq M$, $|M| = \kappa$, there are j, N such that $j : M \preccurlyeq_\kappa N$,[3] and N is $\mathcal{L}$-truthful.* *x-comp-8-0*

Proof. (1): “$\Leftarrow$”: Assume that (3.10) holds for κ. Let T be a $<\kappa$-satisfiable $\mathcal{L}$-theory of signature of size $\leq \kappa$. Since $\mathcal{L}$ is finitary, we may assume that $|T| = \kappa$.

---

[3] For the notation $j : M \preccurlyeq_\kappa N$ see directly after (2.2).

Without loss of generality, we may further assume that T is coded as a subset of κ. Let θ be sufficiently large regular cardinal such that $V_\theta \models$ "T is $<\kappa$-satisfiable".

Let $M \prec V_\theta$ be such that $T \in M$, $\kappa + 1 \subseteq M$ (note that this implies $T \subseteq M$), and $|\,M\,| = \kappa$. By (3.10), there are j and N such that

(3.12) $\quad \kappa + 1 \subseteq N,$ x-comp-9

(3.13) $\quad j : M \preccurlyeq N,$ x-comp-10

(3.14) $\quad j(\kappa) > \min(\mathrm{On}^N \setminus \sup(j''\kappa))$, and x-comp-11

(3.15) $\quad N$ is $\mathcal{L}$-truthful. x-comp-12

By elementarity (3.13), we have

(3.16) $\quad N \models$ "$j(T)$ is a $<j(\kappa)$-satisfiable $\mathcal{L}$-theory". x-comp-13

Let $\alpha^* := \min(\mathrm{On}^N \setminus \sup(j''\kappa))$ and $T^* := j(T) \cap \alpha^*$. Then $T^* \in N$ and $j''T \subseteq T^*$. By (3.14) and (3.16), it follows that $N \models$ "$\mathfrak{A} \models T^*$" for some structure $\mathfrak{A} \in N$. By (3.15), it follows that $\mathfrak{A}^N \models T^*$ and hence $\mathfrak{A}^N \models j''T$. Thus, by renaming the components of the structure $\mathfrak{A}^N$, we obtain a model of T.

"$\Rightarrow$": Assume that κ is weakly $\mathcal{L}$-compact.

Suppose that $\theta \geq \kappa$ is a regular cardinal, and M is such that $M \prec V_\theta$, $\kappa{+}1 \subseteq M$, and $|\,M\,| = \kappa$. Let T be the $\mathcal{L}$-theory defined by:

(3.17) x-comp-14
$$
\begin{aligned}
T := {} & \{\varphi^*_{\mathcal{L}}\} \\
& \cup \{\varphi(\underline{c}_{a_0}, \ldots) \,:\, \varphi \text{ is a first-order formula in the signature } \{\underline{\in}\}, \\
& \qquad\qquad a_0, \ldots \in M \text{ and } M \models \varphi(a_0, \ldots)\} \\
& \cup \{\underline{c}_\alpha \mathrel{\underline{\in}} \underline{d} \,:\, \alpha < \kappa\} \\
& \cup \{\underline{d} \mathrel{\underline{\in}} \underline{c}_\kappa\} \\
& \cup \Phi^*_{\mathcal{L}}
\end{aligned}
$$

where $\varphi^*_{\mathcal{L}}$ is an $\mathcal{L}$-formula as in (3.9) and $\Phi^*_{\mathcal{L}}$ a set of $\mathcal{L}$-formulas in (3.6).

T is $<\kappa$-satisfiable: by (3.8) and (3.7), V_θ can be expanded to a model $\mathfrak{A}_0$ of any subset T_0 of T of size $<\kappa$ by letting $(\underline{c}_a)^{\mathfrak{A}} := a$ for all constant symbols of this form appearing T_0, and $(\underline{d})^{\mathfrak{A}} := \sup(\{\alpha < \kappa \,:\, \underline{c}_\alpha \text{ appears in } T_0\})$. Note that $\sup(\{\alpha < \kappa \,:\, \underline{c}_\alpha \text{ appears in } T_0\}) < \kappa$ since κ is regular.

By assumption, there is a model $\mathfrak{B}$ of T. By $\varphi^*_{\mathcal{L}} \in T$, we can take the Mostowski collapse $\mathfrak{B}^*$ of $\mathfrak{B}$ (with respect to $\underline{\in}^{\mathfrak{B}}$). We then have $\underline{\in}^{\mathfrak{B}^*} = \in$ and $|\mathfrak{B}^*|$ is a transitive set.

By (3.7) and since $\Phi^*_{\mathcal{L}} \subseteq T$, $N := |\mathfrak{B}^*|$ is $\mathcal{L}$-truthful.

Let $j : M \to N$ be defined by $j(a) := (\underline{c}_a)^{\mathfrak{B}^*}$ for $a \in M$. Then these N, j are as desired in (3.10).

(2): This can be proved similarly to (1). For the direction "⇒", note the following Lemma 3.2. Also, note that $2^{<\kappa} = \kappa$ follows from the assumption of the weakly $\mathcal{L}^*$-compactness of κ since κ is then at least weakly compact.

For the proof of the direction "⇐", consider

$$
\begin{aligned}
T' := \{\varphi^*_{\mathcal{L}}\} \\
&\cup \{\varphi(\underset{\sqcup}{c}_{a_0}, \ldots) : \varphi \text{ is a first-order formula in the signature } \{\underset{\sqcup}{\in}\}, \\
&\qquad a_0, \ldots \in M \text{ and } M \models \varphi(a_0, \ldots)\} \\
&\cup \{\forall_x (x \underset{\sqcup}{\in} \underset{\sqcup}{c}_\alpha \leftrightarrow \textstyle\bigvee\!\!\!\bigvee_{\beta<\alpha} x \equiv \underset{\sqcup}{c}_\beta) : \alpha < \kappa\} \\
&\cup \{\underset{\sqcup}{c}_\alpha \underset{\sqcup}{\in} \underset{\sqcup}{d} : \alpha < \kappa\} \\
&\cup \{\underset{\sqcup}{d} \underset{\sqcup}{\in} \underset{\sqcup}{c}_\kappa\} \\
&\cup \Phi^*_{\mathcal{L}^*}
\end{aligned}
$$

in place of T in the proof of (1). ❑ (Theorem 3.1)

Lemma 3.2 *Suppose that $\mathcal{L}$ and $\mathcal{L}^*$ are as in Theorem 3.1. If M is a model of a sufficiently large fragment of set theory, $M \supseteq \kappa + 1$, and M is $\mathcal{L}$-truthful, then M is also $\mathcal{L}^*$-truthful.* ❑ P-comp-2-0

Lemma A 3.1 *Suppose that $\mathcal{L}$ is a finitary proper logic and M is a transitive model of* ZC⁻ *such that* (3.18): *M is $\mathcal{L}$-truthful, $N \in M$, and* (3.19): *$M \models$ "N is an $\mathcal{L}$-truthful model of ZC⁻".*[3)a] *Then N is $\mathcal{L}$-truthful.* P-comp-3-0 x-comp-16 x-comp-17

Proof. Let $\mathfrak{A} \in N$ and φ is a $\mathcal{L}$-formula in N in the signature of $\mathfrak{A}$. Then we have

$$
N \models \text{“}\mathfrak{A} \models_{\mathcal{L}} \varphi\text{”} \overbrace{\Leftrightarrow}^{\text{by (3.18) and since } \mathcal{L} \text{ contains the first-order logic}} M \models \text{“}N \models \text{“}\mathfrak{A} \models_{\mathcal{L}} \varphi\text{”}\text{”} \underbrace{\Leftrightarrow}_{\text{by (3.19)}} M \models \text{“}\mathfrak{A}^N \models_{\mathcal{L}} \varphi\text{”} \underbrace{\Leftrightarrow}_{\text{by (3.18)}} \mathfrak{A}^N \models_{\mathcal{L}} \varphi.
$$

❑ (Lemma 3.2)

Theorem 3.1 is a generalization of Theorem 2.2. This can be seen in the following:

Lemma 3.3 *If a transitive set N (seen as an $\in$-structure) is elementary equivalent (in first-order logic) to V_θ for some limit ordinal θ and N is $\mathcal{L}^{\mathrm{II}}$-truthful then $N = V_{\theta'}$ for some limit ordinal θ'.* P-comp-4

Proof. Note that, by elementarity,

[3)a] In general, this can be formulated by infinitely many conditions: one for each $\mathcal{L}$-formula.

$$N \models \text{“}\forall_x \exists_\alpha \exists_f\, (\text{“}\alpha \text{ is an ordinal”} \ \wedge\ \text{“}f \text{ is a mapping on } \alpha\text{”}$$
$$\wedge\ \text{“}f \text{ is an initial segment of the cumulative hierarchy”}$$
$$\wedge\ \text{“}x \in f(\beta) \text{ for some } \beta < \alpha)\text{”}.$$

Thus, it is enough to show that, if $A \in N$, then $\mathcal{P}(A) \subseteq N$. Let $\mathfrak{A} \in N$ be such that $N \models \text{“}\mathfrak{A} = \langle \mathcal{P}(A) \cup A, A, \in \rangle\text{”}$. Since N is transitive, we have (3.20) : $\mathfrak{A}^N = \mathfrak{A}$. x-comp-17-0

$$N \models \text{“}\mathfrak{A} \models_{\mathcal{L}^{\mathrm{II}}} \forall X (\forall_y (y \,\varepsilon\, X \rightarrow y \,\underline{\in}\, \underline{A}) \ \rightarrow\ \exists_z \forall_y\, (y \,\underline{\in}\, z \leftrightarrow y \,\varepsilon\, X))\text{”},$$

since N is elementary equivalent to V_θ.

It follows that

$$\mathfrak{A} \models_{\mathcal{L}^{\mathrm{II}}} \forall X (\forall_y (y \,\varepsilon\, X \rightarrow y \,\underline{\in}\, \underline{A}) \ \rightarrow\ \exists_z \forall_y\, (y \,\underline{\in}\, z \leftrightarrow y \,\varepsilon\, X))$$

by $\mathcal{L}^{\mathrm{II}}$-truthfulness of N (and (3.20)).

It follows that $|\mathfrak{A}^N| = \mathcal{P}(A) \cup A$. Thus, $\mathcal{P}(A) \subseteq N$. ❑ (Lemma 3.3)

$\mathfrak{wcn}(\mathcal{L})$ for a finitary logic $\mathcal{L}$ as in Theorem 3.1, (1) is always fairly large since the following analog of Proposition 2.7 holds for such a logic:

Proposition 3.4 *Suppose that $\mathcal{L}$ is a finitary proper logic satisfying (3.8) and (3.9).* *P-comp-4-0*

(1) *If $\mathfrak{wcn}(\mathcal{L}) < \infty$, $\mathfrak{wcn}(\mathcal{L})$ is greater than the first weakly Mahlo cardinal, first weakly hyper Mahlo cardinal, etc.*

(2) $2^{\aleph_0} \notin \mathsf{WCS}(\mathcal{L})$, $(2^{\aleph_0})^+ \notin \mathsf{WCS}(\mathcal{L})$, *etc.*

Proof. (1): By Theorem 3.1, (1) and Lemma 2.6.

(2): The theories similarly to T and T_n in the proof of Proposition 2.7 with $\mathcal{L}^{\mathrm{II}}$-theories of the structures replaced by corresponding $\mathcal{L}$-theories show the inequality.

❑ (Proposition 3.4)

4 Weak compactness of stationary logic

As is already noted, $\mathfrak{lstn}(\mathcal{L}^{\mathrm{II}})$ is the least supercompact cardinal by one of Magidor's Theorems (see Theorem 1.9). In contrast, $\mathfrak{lstn}(\mathcal{L}^{\aleph_0,\mathrm{II}})$ is $(2^{\aleph_0})^+$ while $\mathfrak{lstn}(\mathcal{L}^{\aleph_0}_{stat})$ can be (consistently) $\leq 2^{\aleph_0}$ though the consistency strength of $\mathsf{LSTS}(\mathcal{L}^{\aleph_0}_{stat}) \neq \emptyset$ is still rather high (see the following Lemma 4.1, (4) and (5)). The size of $\mathfrak{lstn}(\mathcal{L}^{\aleph_0}_{stat})$ is wildly undecided under ZFC (see Lemma 4.1, (10)). stat

We shall call a cardinal κ $\cdot^{\aleph_0}$*-closed* if $\mu^{\aleph_0} < \kappa$ holds for all $\mu < \kappa$.

For a class $\mathcal{P}$ of posets, a cardinal κ is said to be $\mathcal{P}$-generically supercompact ($\mathcal{P}$-gen. supercompact, for short) if for, any $\lambda \geq \kappa$, there is a poset $\mathbb{P} \in \mathcal{P}$ such that, for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, there are classes j and M (in $\mathsf{V}[\mathbb{G}]$) such that

(4.1) $\quad \mathsf{V}[\mathbb{G}] \models$ "$j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$", $j(\kappa) > \lambda$, and $j''\lambda \in M$ x-stat-a

where we are using here the convention in the footnote 2).

Proposition 4.1 (1) $\mathsf{LSTS}(\mathcal{L}^{\aleph_0,\mathrm{II}}) = \{\kappa : \kappa$ *is* $\cdot^{\aleph_0}$ *-closed*$\}$. P-stat-0

(2) $\mathfrak{lstn}(\mathcal{L}^{\aleph_0,\mathrm{II}}) = (2^{\aleph_0})^+$.

(3) *If* $\square_\kappa$ *holds then* $\mathfrak{lstn}(\mathcal{L}^{\aleph_0}_{stat}) > \kappa^+$.

(4) *If* $\square_\kappa$ *holds for class many* κ, *then* $\mathsf{LSTS}(\mathcal{L}^{\aleph_0}_{stat}) = \emptyset$.

(5) $\mathsf{MA}^{+\omega_1}(\sigma\text{-}closed)$ *implies* $\mathfrak{lstn}(\mathcal{L}^{\aleph_0}_{stat}) = \aleph_2$. *In particular,* $\mathfrak{lstn}(\mathcal{L}^{\aleph_0}_{stat}) = 2^{\aleph_0}$ *is consistent (modulo a large cardinal, e.g. a supercompact). Note that, by (2),* $\mathfrak{lstn}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}) \leq 2^{\aleph_0}$ *is inconsistent.*

(6) $\mathfrak{lstn}(\mathcal{L}^{\aleph_0}_{stat}) = \aleph_2$ *implies* $2^{\aleph_0} \leq \aleph_2$. *(*$\mathsf{CH}$ *is also possible under* $\mathfrak{lstn}(\mathcal{L}^{\aleph_0}_{stat}) = \aleph_2$*).*

(7) *If* $\kappa \in \mathsf{LSTS}(\mathcal{L}^{\aleph_0}_{stat})$ *and* $\kappa > \aleph_2$ *then* $\kappa > 2^{\aleph_0}$.

(8) $\mathfrak{lstn}(\mathcal{L}^{\aleph_0}_{stat}) = 2^{\aleph_0}$ *implies* $2^{\aleph_0} = \aleph_2$.

(9) *If* κ *is* σ*-closed-gen. supercompact then* $\mathfrak{lstn}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}) \leq \kappa$.

(10) *Assuming the consistency of* ZFC + *"there is a supercompact cardinal", the following assertion is consistent with* ZFC *for any natural number* $n \geq 2$*:*

(4.2) $\quad \mathfrak{lstn}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}) = \aleph_n$. x-stat-0

Proof. (1): is easy to prove. Let $\mathcal{C} := \{\kappa \in \mathsf{Card} : \kappa$ is $\cdot^{\aleph_0}$-closed $\}$.

$\mathcal{C} \subseteq \mathsf{LSTS}(\mathcal{L}^{\aleph_0,\mathrm{II}})$: Suppose that κ is $\cdot^{\aleph_0}$-closed. Let $\mathfrak{A}$ be an arbitrary structure of countable signature and $S \in [\,\mathfrak{A}\,]^{<\kappa}$. Let θ be a sufficiently large regular cardinal ($\mathfrak{A} \in \mathcal{H}(\theta)$ in particular).

Let $M \prec \mathcal{H}(\theta)$ be such that $\mathfrak{A} \in M$, $S \subseteq M$, $[M]^{\aleph_0} \subseteq M$ and $|\,M\,| < \kappa$ (this is possible since κ is $\cdot^{\aleph_0}$-closed). Then $\mathfrak{A} \restriction (\,|\mathfrak{A}| \cap M)$ contains S as a subset of its underlying set $|\mathfrak{A}| \cap M$, $|\,|\mathfrak{A}| \cap M\,| < \kappa$, and $\mathfrak{A} \restriction (\,|\mathfrak{A}| \cap M) \prec_{\mathcal{L}^{\aleph_0,\mathrm{II}}} \mathfrak{A}$.

$\mathsf{LSTS}(\mathcal{L}^{\aleph_0,\mathrm{II}}) \subseteq \mathcal{C}$: Suppose that κ is not in $\mathcal{C}$ and let $\mu < \kappa$ be such that $\mu^{\aleph_0} \geq \kappa$.

Let $\mathfrak{A} := \langle \mathcal{H}(\mu^+), \in \rangle$. Note that

$$\mathfrak{A} \models_{\mathcal{L}^{\aleph_0,\mathrm{II}}} \forall X \exists_x \forall_y \, (y \mathrel{\varepsilon} X \;\leftrightarrow\; y \in x).$$

If $\mathfrak{B} \prec_{\mathcal{L}^{\aleph_0,\mathrm{II}}} \mathfrak{A}$, then

$$\mathfrak{B} \models_{\mathcal{L}^{\aleph_0,\mathrm{II}}} \forall X \exists_x \forall_y \, (y \mathrel{\varepsilon} X \;\leftrightarrow\; y \in x)$$

by elementarity, and hence $[\,|\mathfrak{B}|\,]^{\aleph_0} \subseteq |\mathfrak{B}|$. Thus, if $\mu \subseteq |\mathfrak{B}|$ then $[\mu]^{\aleph_0} \subseteq |\mathfrak{B}|$, and hence we have $\kappa \leq \mu^{\aleph_0} \leq \|\mathfrak{B}\|$. This shows that $\kappa \notin \mathsf{LSTS}(\mathcal{L}^{\aleph_0,\mathrm{II}})$.

(2): follows from (1).

(3): If $\square_\kappa$ holds then there is a non-reflecting stationary set $S \subseteq E^{\kappa^+}_\omega$.

$S^* := \{s \in [\kappa^+]^{\aleph_0} : \sup(\{\alpha+1 : \alpha \in s\}) \in S\}$ is a non-reflecting stationary subset of $[\kappa^+]^{\aleph_0}$. S^* can be recast to a counter-example of $\mathfrak{lstn}(\mathcal{L}^{\aleph_0}_{stat}) \leq \kappa^+$.

(4): follows from (3).

(5): This can be proved by a modification of arguments in [7]. The second assertion holds since $\mathsf{MA}^{+\omega_1}(\sigma\text{-closed}) + 2^{\aleph_0} = \aleph_2$ is consistent (e.g. modulo a supercompact).

(6): $\mathfrak{lstn}(\mathcal{L}^{\aleph_0}_{stat}) = \aleph_2$ implies the principle RP in [12]. RP implies $2^{\aleph_0} \leq \aleph_2$ (Todorčević, see Theorem 37.17 in [12]).

Suppose V satisfies CH and there is a supercompact cardinal κ then collapsing κ to $\aleph_2$, we obtain a model of $\mathfrak{lstn}(\mathcal{L}^{\aleph_0}_{stat}) = \aleph_2$ (see (9)) and CH.

(7): If $\kappa \in \mathfrak{lstn}(\mathcal{L}^{\aleph_0}_{stat})$ then $\mathsf{SDLS}^-(\mathcal{L}^{\aleph_0}_{stat}, <\kappa)$ holds (see [7] for definition). Thus, by Proposition 2.1 in [8], if $\kappa > \aleph_2$ then $\kappa > 2^{\aleph_0}$ holds.

(8): This follows from Corollary 2.3 in [8].

(9): See [7] (for the detail of a direct proof, see [0]).

Suppose that $\mathfrak{A}$ is a structure with $\|\mathfrak{A}\| \geq \kappa$ and $S \in [\,|\mathfrak{A}|\,]^{<\kappa}$. Let $\mathbb{P}$ be a σ-closed poset with a $(\mathsf{V},\mathbb{P})$-generic $\mathbb{G}$ such that there are $j, M \subseteq \mathsf{V}[\mathbb{G}]$ with

($\aleph$4.1) $\quad j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, x-LS-25

($\aleph$4.2) $\quad j(\kappa) > \|\mathfrak{A}\|$ and x-LS-26

($\aleph$4.3) $\quad j''\mu \in M$ where $\mu := \|\mathfrak{A}\|^{\aleph_0}$. x-LS-27

Without loss of generality, we may assume that (4.3): $|\mathfrak{A}| = \|\mathfrak{A}\|$. x-LS-27-a

Let $\mathfrak{B} := j(\mathfrak{A}) \restriction j''\,|\mathfrak{A}|$. By ($\aleph$4.3) and (4.3), $\mathfrak{B}$, $j \restriction |\mathfrak{A}| \in M$. Thus, $\mathfrak{A} \in M$ and we have

($\aleph$4.4) $\quad M \models$ "$j \restriction |\mathfrak{A}| : \mathfrak{A} \overset{\cong}{\rightarrow} \mathfrak{B}$". x-LS-27-0

Since $([\,|\mathfrak{A}|\,]^{\aleph_0})^{\mathsf{V}} = ([\,|\mathfrak{A}|\,]^{\aleph_0})^{\mathsf{V}[\mathbb{G}]}$ by σ-closedness of $\mathbb{P}$ and $([\,|\mathfrak{A}|\,]^{\aleph_0})^{\mathsf{V}} \subseteq ([\,|\mathfrak{A}|\,]^{\aleph_0})^{M}$, we have

($\aleph$4.5) $\quad ([\,|\mathfrak{A}|\,]^{\aleph_0})^{\mathsf{V}} = ([\,|\mathfrak{A}|\,]^{\aleph_0})^{M}$. x-LS-28

Claim 4.1.1 *For any $\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}$-formula $\varphi(x_0,...,X_0,...)$, $a_0,... \in |\mathfrak{A}|$ and $A_0,... \in [\,|\mathfrak{A}|\,]^{\aleph_0}$,* *Cl-LS-8*

$$(\aleph 4.6)\quad \mathsf{V} \models \text{“}\mathfrak{A} \models \varphi(a_0, ..., A_0, ...)\text{”} \;\Leftrightarrow\; M \models \text{“}\mathfrak{A} \models \varphi(a_0, ..., A_0, ...)\text{”}.$$ *x-LS-29*

$\vdash$ We prove ($\aleph$4.6) by induction on φ. The most crucial step is when φ is of the form $\mathit{stat}\,X\,\psi(x_0, ..., X, X_0, ...)$:

Suppose that $a_0, ... \in |\mathfrak{A}|$ and $A_0, ... \in [\,|\mathfrak{A}|\,]^{\aleph_0}$.

First assume that $\mathsf{V} \models$ “$\mathfrak{A} \models \mathit{stat}\,X\,\psi(a_0, ..., X, A_0, ...)$”. This is equivalent to

$$\mathsf{V} \models \text{“}\{A \in [\,|\mathfrak{A}|\,]^{\aleph_0} : \mathfrak{A} \models \psi(a_0, ..., A, A_0, ...)\} \text{ is stationary”}.$$

We have

$$\begin{aligned}&(\{A \in [\,|\mathfrak{A}|\,]^{\aleph_0} : \mathfrak{A} \models \psi(a_0, ..., A, A_0, ...)\})^{\mathsf{V}}\\ &\qquad = (\{A \in [\,|\mathfrak{A}|\,]^{\aleph_0} : \mathfrak{A} \models \psi(a_0, ..., A, A_0, ...)\})^{M}\end{aligned}$$

by ($\aleph$4.5) and induction hypothesis. Since $\mathbb{P}$ is proper, it follows that $M \models$ “$\{A \in [\,|\mathfrak{A}|\,]^{\aleph_0} : \mathfrak{A} \models \psi(a_0, ..., A, A_0, ...)\}$ is stationary”. Thus $M \models$ “$\mathfrak{A} \models \mathit{stat}\,X\,\psi(a_0, ..., X, A_0, ...)$”.

Now assume that $\mathsf{V} \not\models$ “$\mathfrak{A} \models \mathit{stat}\,X\,\psi(a_0, ..., X, A_0, ...)$”. This means that there is a club $C \subseteq [\,|\mathfrak{A}|\,]^{\aleph_0}$ such that $\mathsf{V} \models$ “for all $A \in C$, $\mathfrak{A} \not\models \psi(a_0, ..., A, A_0, ...)$”. Now $C \in M$ by ($\aleph$4.3). By induction hypothesis

$$M \models \text{“for all } A \in C,\ \mathfrak{A} \not\models \psi(a_0, ..., A, A_0, ...)\text{”}.$$

By σ-closedness of $\mathbb{P}$, C is club in $\mathsf{V}[\mathbb{G}]$ and hence also in M.

Thus, it follows that $M \not\models$ “$\mathfrak{A} \models \mathit{stat}\,X\,\psi(a_0, ..., X, A_0, ...)$”. $\dashv$ (Claim 4.1.1)

For an $\mathcal{L}^{\aleph_0,\mathrm{II}}_{\mathit{stat}}$-formula $\varphi(x_0, ..., X_0, ...)$, $a_0, ... \in |\mathfrak{A}|$ and $A_0, ... \in [\,|\mathfrak{A}|\,]^{\aleph_0}$

$$\begin{aligned}&M \models \text{“}j(\mathfrak{A}) \models \varphi(j(a_0), ..., j(A_0), ...)\text{”} \overbrace{\Leftrightarrow}^{\text{by elementarity of } j} \mathsf{V} \models \text{“}\mathfrak{A} \models \varphi(a_0, ..., A_0, ...)\text{”}\\ &\underbrace{\Leftrightarrow}_{\text{by Claim 4.1.1}} M \models \text{“}\mathfrak{A} \models \varphi(a_0, ..., A_0, ...)\text{”}\\ &\underbrace{\Leftrightarrow}_{\text{by } (\aleph 4.4)} M \models \text{“}\mathfrak{B} \models \varphi(j(a_0), ..., j(A_0), ...)\text{”}.\end{aligned}$$

Since $M \models \|\mathfrak{B}\| < j(\kappa)$, $j(S)\ (= j''S) \subseteq |\mathfrak{B}|$, we obtain

$$M \models \text{“there is an } \mathcal{L}^{\aleph_0,\mathrm{II}}_{\mathit{stat}}\text{-elementary substructure } X \text{ of } j(\mathfrak{A}) \text{ of size} < j(\kappa) \text{ with } j(S) \subseteq X\text{”}.$$

By elementarity, it follows that

$$\mathsf{V} \models \text{“there is an } \mathcal{L}^{\aleph_0,\mathrm{II}}_{\mathit{stat}}\text{-elementary substructure } X \text{ of } \mathfrak{A} \text{ of size} < \kappa \text{ with } S \subseteq X\text{”}.$$

This shows that $\kappa \in \mathsf{LSTS}(\mathcal{L}^{\aleph,\mathrm{II}}_{stat})$.

(10): By (3) and (5). Note that, if κ is a supercompact cardinal, and $\mathbb{P} = \mathrm{Col}(\aleph_k, \kappa)$ for $k > 0$ (in the notation of Kanamori [15]) then, in $\mathsf{V}[\mathbb{G}]$ for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\kappa = \aleph_{k+1}$ and κ is σ-closed-gen. supercompact. ❑ (Proposition 4.1)

In contrast to the possible smallness of $\mathfrak{lstn}(\mathcal{L}^{\aleph_0,\mathrm{II}})$ shown in Proposition 4.1, (2) (e.g. it can be $\aleph_2$ under CH), $\mathfrak{wcn}(\mathcal{L}^{\aleph_0,\mathrm{II}})$ is fairly a large cardinal. This is because we can apply Proposition 3.4 to $\mathcal{L}^{\aleph_0,\mathrm{II}}$.

For a logic $\mathcal{L}$, the Hanf number $\mathfrak{hn}(\mathcal{L})$ of $\mathcal{L}$ is defined by

$$\mathfrak{hn}(\mathcal{L}) := \min(\{\kappa : \text{for any } \mathcal{L}\text{-sentence } \varphi \text{ if } \varphi \text{ has a model of size at least } \kappa \text{ then } \varphi \text{ has arbitrarily large model}\}). \tag{4.4}$$

x-stat-1

Note that all proper logics have Hanf number assuming the properness include the properties that, there is a cardinal $\kappa_{\mathcal{L}}$ such that each $\mathcal{L}$-formula is in a signature of size $<\kappa_{\mathcal{L}}$ and there are only set-many $\mathcal{L}$-formulas in each signature. The following should be well-known:

Lemma 4.2 *For a proper logic $\mathcal{L}$, we have $\mathfrak{hn}(\mathcal{L}) \leq \mathfrak{cn}(\mathcal{L})$.* P-stat-2

Proof. Suppose that $\kappa < \mathfrak{hn}(\mathcal{L})$ then there is an $\mathcal{L}$-sentence φ such that φ has a model $\mathfrak{A}$ with $\kappa \leq \|\mathfrak{A}\| < \mathfrak{hn}(\mathcal{L})$ such that any model of φ has size $<\mathfrak{hn}(\mathcal{L})$.

Let $T := \{\varphi\} \cup \{\underset{\sim}{c}_\alpha \not\equiv \underset{\sim}{c}_\beta : \alpha < \beta < \mathfrak{hn}(\mathcal{L})\}$. Then T does not have any model but, for any $T_0 \in [T]^{<\kappa^+}$, $\mathfrak{A}$ can be expanded to a model of T_0. This shows that $\mathfrak{cn}(\mathcal{L}) > \kappa^+$. (This proof actually shows that $\mathfrak{hn}(\mathcal{L}) < \mathfrak{cn}(\mathcal{L})$ if $\mathfrak{hn}(\mathcal{L})$ is a successor. But this fact is irrelevant for the application in the following Lemma). ❑ (Lemma 4.2)

In the following Lemma 4.3, Q_1 denotes the (first-order) quantifier "there exists uncountably many" and $\mathcal{L}(Q_1)$ the logic obtained by adding the quantifier Q_1 to the first-order logic.

The next lemma follows immediately from Lemma 4.2. Note the quantifier Q_1 is interpretable in $\mathcal{L}^{\aleph_0}_{stat}$.

[${Q_1}_x\varphi \ \mapsto\ stat\, X \exists_x (x \not\varepsilon\ X \wedge \varphi^*)$ where φ^* is the $\mathcal{L}^{\aleph_0}_{stat}$-interpretation of φ.]

Lemma 4.3 (1) $\mathfrak{wcn}(\mathcal{L}^{\aleph_0}_{stat}) \geq \mathfrak{wcn}(\mathcal{L}(Q_1)) \geq \aleph_\omega$. P-stat-3

(2) (Shelah [22]) *For a proper Logic $\mathcal{L}$, $\mathfrak{hn}(\mathcal{L})$ is a strong limit. If $\mathfrak{hn}(\mathcal{L}) > \aleph_0$, then $\mathfrak{cn}(\mathcal{L}) \geq \mathfrak{hn}(\mathcal{L}) \geq \beth_\omega$.*

(3) (Shelah [23]) $\mathfrak{hn}(\mathcal{L}^{\aleph_0}_{stat}) > \beth_\omega$. *In particular $\mathfrak{cn}(\mathcal{L}^{\aleph_0}_{stat}) > \beth_\omega$, while we have $\mathfrak{cn}(\mathcal{L}^{\aleph_0}_{stat}) \geq \mathfrak{cn}(\mathcal{L}(Q_1)) \geq \mathfrak{hn}(\mathcal{L}(Q_1)) = \beth_\omega$.* ❑

For the Proposition 4.5 below, we use Jensen's global square C (for the existence of the class C as below, see the proof of Theorem 5.1 in [13]):

Let

$$\mathsf{SING}_{\mathsf{L}} := \{\alpha : \mathsf{L} \models \text{“}\alpha \text{ is a singular limit ordinal”}\}.$$

A global square is a class function C (i.e. the corresponding $\mathcal{L}_\in$-formula defining the class function) such that the following are provable from ZFC^-:

(I) C is a class function with $dom(\mathsf{C}) = \mathsf{SING}_{\mathsf{L}}$.

(II) For each $\alpha \in \mathsf{SING}_{\mathsf{L}}$, $\mathsf{C}(\alpha) \in L$ is a club subset of α with $otp(\mathsf{C}(\alpha)) < \alpha$.

(III) (Coherence) If $\alpha \in \mathsf{SING}_{\mathsf{L}}$, and $\beta \in Lim(\mathsf{C}(\alpha))$, then $\beta \in \mathsf{SING}_{\mathsf{L}}$ and $\mathsf{C}(\beta) = \mathsf{C}(\alpha) \cap \beta$.

(IV) For any transitive set model W of ZFC^-, we have $\mathsf{C}^W = \mathsf{C} \restriction (\mathsf{SING}_{\mathsf{L}})^W$.

Lemma 4.4 (see e.g. Kunen [16], Lemma 4.11) *Assume* $\mathsf{V} = \mathsf{L}$. *Then* $L_\kappa = \mathcal{H}(\kappa)$ *whenever* $\kappa > \omega$ *and* κ *is regular.* ❑ P-stat-3-0

Proposition 4.5 *Assume that* $\kappa = \mathfrak{wcn}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat,\kappa,\omega})$ *and suppose that* $\lambda > \kappa$ *is a regular cardinal and* $M \prec L_\lambda$ *is such that* $\kappa + 1 \subseteq M$ *and* $|\,M\,| = \kappa$. *Then there are* $\overline{\lambda} > \kappa$ *and* j *such that* P-stat-4

(4.5) $\overline{\lambda}$ *is a regular cardinal in* L *of uncountable cofinality (in* V*), and* x-stat-1-a

(4.6) $j : M \preccurlyeq_\kappa L_{\overline{\lambda}}$. x-stat-1-a-0

Proof. If $0^\#$ exists, then, since the regular cardinals κ, λ belong to the indiscernibles associated with $0^\#$, we can easily find j as above for any regular $\overline{\lambda} \geq \kappa^+$.

In the following, we assume that $0^\#$ does not exist. Observe first that it is enough to prove the following variation of the statement of the present proposition:

(4.7) For any $M \prec L_\lambda$ with $\kappa + 1 \subseteq M$ and $|\,M\,| = \kappa$, there is $M^* \prec L_\lambda$ such that $M \subseteq M^*$ and $|\,M^*\,| = \kappa$, with $\overline{\lambda} > \kappa$ and j^* such that x-stat-2-0

(4.5) $\overline{\lambda}$ is a regular cardinal, and

(4.6)′ $j^* : M^* \preccurlyeq_\kappa L_{\overline{\lambda}}$.

This is because, if M^*, $\overline{\lambda}$, j^* are as in (4.7) for given M, then M, $\overline{\lambda}$ and $j := j^* \restriction M$ satisfy (4.5) and (4.6).

Let $E := (E^\lambda_\omega \setminus \omega_2)^{\mathsf{V}}$. For any $\alpha \in E$, we have $(cf(\alpha))^{\mathsf{L}} < \omega_2$ by Covering Lemma. Thus, $\mathsf{C}(\alpha)$ exists for such α and $otp(\mathsf{C}(\alpha)) < \alpha$ by (II). By Fodor's Lemma, there is $\nu < \lambda$ such that

(4.8) $E_0 := \{\alpha \in E : otp(\mathsf{C}(\alpha)) = \nu\}$ is stationary. x-stat-2-1

Since $cf(\nu) = \omega$ (in V), this statement is equivalent to

(4.9) $E_1 := \{a \in [\lambda]^{\aleph_0} : a$ does not have the maximal element, $\sup(a) \in E_0\}$ is stationary subset of $[\lambda]^{\aleph_0}$. x-stat-2-2

Now suppose that $M \prec L_\lambda$ is such that $\kappa+1 \subseteq M$ and $|M| = \kappa$. Let $M^* \prec L_\lambda$ be such that $M \cup \{\nu\} \subseteq M^*$ and $|M^*| = \kappa$.

Let

(4.10) x-stat-2-3
$$\begin{aligned} T := &\overbrace{\{\varphi(\underset{\sim}{c}_{a_0}, \ldots) : a_0, \ldots \in M^*, \varphi(x_0, \ldots) \text{ is an } \mathcal{L}^{\aleph_0,\mathrm{II}}_{stat,\kappa,\omega}\text{-formula with } L_\lambda \models \varphi(a_0, \ldots)\}}^{(*)} \\ &\left.\begin{aligned} &\cup \{\underset{\sim}{c}_\alpha < \underset{\sim}{d} : \alpha \in \kappa\} \\ &\cup \{\underset{\sim}{d} < \underset{\sim}{c}_\kappa\} \\ &\cup \{\forall_x\, ((x \text{ is an ordinal} \land x < \underset{\sim}{c}_\alpha) \to \textstyle\bigvee\!\!\!\bigvee_{\beta<\alpha} x \equiv \underset{\sim}{c}_\beta) : \alpha < \kappa\}. \end{aligned}\right\}(**) \end{aligned}$$

Apparently T is of a signature of size κ. As before we can also show that T is $<\kappa$-satisfiable. Hence, by $\kappa = \mathfrak{wcn}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat,\kappa,\omega})$, T has a model $\mathfrak{A}$.

Since $\underset{\sim}{\in}^{\mathfrak{A}}$ is well-founded (which is declared in the $(*)$ part of T), we can take the Mostowski collapse $\mathfrak{A}^*$ of $\mathfrak{A}$. The underlying set of $\mathfrak{A}^*$ is of the form $L_{\overline{\lambda}}$ (also because this is formulated in the $(*)$ part of T).

Letting $j^* : M^* \to L_{\overline{\lambda}}$; $a \mapsto (\underset{\sim}{c}_a)^{\mathfrak{A}^*}$, we have

(4.11) $j^* : M^* \preccurlyeq_\kappa L_{\overline{\lambda}}$ x-stat-2-4

(the elementarity follows from $(*)$ part of T, and $crit(j) = \kappa$ from $(**)$ of T), and

(4.12) $\langle L_\lambda, a\rangle_{a\in M^*} \equiv_{\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat,\kappa,\omega}} \langle L_{\overline{\lambda}}, j^*(a)\rangle_{a\in M^*}$ x-stat-2-5

by the $(*)$ part of T.

Note that $cf(\overline{\lambda}) > \omega$ by (4.12). Thus, we are done by showing that $\overline{\lambda}$ is regular in L. Suppose not, by way of contradiction. Then $\mathsf{C}(\overline{\lambda})$ exists. $C := \{\alpha < \overline{\lambda} : \alpha$ is a limit of $\mathsf{C}(\overline{\lambda})\}$ is a club subset of $\overline{\lambda}$. $E_2 := \{\alpha < \overline{\lambda} : otp(\mathsf{C}(\alpha)) = j^*(\nu)\}$ is stationary by (4.9), the elementarity (4.12)) and (IV).

Let $\xi_0, \xi_1 \in C \cap E_2$ be with $\xi_0 < \xi_1$. Then $\mathsf{C}(\xi_0) = \mathsf{C}(\xi_1) \cap \xi_0$ by (III). This is a contradiction to $\xi_0, \xi_1 \in E_2$. ❑ (Proposition 4.5)

Under $\mathsf{V} = \mathsf{L}$, the condition $\kappa = \mathfrak{wcn}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat,\kappa,\omega})$ becomes equivalent to $\kappa = \mathfrak{wcn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega})$:

Theorem 4.6 *Assume $\mathsf{V} = \mathsf{L}$. Then $\kappa = \mathfrak{wcn}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat,\kappa,\omega})$ holds if and only if κ is weakly extendible.* P-stat-7

Theorem 4.6 follows from Proposition 4.5 and the following characterization of weak extendibility:

Lemma 4.7 *For a cardinal κ, the following are equivalent:* P-stat-8

(a) *κ is weakly extendible.*

(b) *$2^{<\kappa} = \kappa$ holds, and, for any regular $\lambda > \kappa$ and any $M \prec \mathcal{H}(\lambda)$ with $\kappa{+}1 \subseteq M$ and $|\,M\,| = \kappa$, there are regular $\overline{\lambda} > \kappa$ and $j : M \preccurlyeq_\kappa \mathcal{H}(\overline{\lambda})$.*

Proof. (a) $\Rightarrow$ (b): Assume that κ is weakly extendible. Then we have $2^{<\kappa} = \kappa$ by definition. For a regular $\lambda > \kappa$ and $M \prec \mathcal{H}(\lambda)$ with $\kappa + 1 \subseteq M$ and $|\,M\,| = \kappa$, let $\theta := \lambda + \omega$. Then M, $\mathcal{H}(\lambda) \in V_\theta$. Let M^* be such that

(4.13) $\quad M^* \prec V_\theta,$ x-stat-9

(4.14) $\quad M,\, \mathcal{H}(\lambda) \in M^*,\ M \subseteq M^*$ and x-stat-10

(4.15) $\quad |\,M^*\,| = \kappa.$ x-stat-11

Since κ is weakly extendible, there are $\overline{\theta}$ and $\overline{j}$ with (4.16) : $\overline{j} : M^* \preccurlyeq_\kappa V_{\overline{\theta}}$. Note x-stat-12 that $\overline{\theta}$ is a limit ordinal by the choice of θ and by the elementarity (4.13) and (4.16).

We have $V_{\overline{\theta}} \models$ "$\overline{j}(\lambda)$ is a regular cardinal" by elementarity (4.16), and hence $\overline{j}(\lambda)$ is really a regular cardinal. By the elementarity, we also have $V_{\overline{\theta}} \models$ "$\overline{j}(\mathcal{H}(\lambda)) = \mathcal{H}(\overline{j}(\lambda))$". Since $\mathcal{H}(\overline{j}(\lambda)) \in V_{\overline{\theta}}$, it follows that $\overline{j}(\mathcal{H}(\lambda)) = \mathcal{H}(\overline{j}(\lambda))$.

Thus, letting $j := \overline{j} \restriction M$ and $\overline{\lambda} := \overline{j}(\lambda)$, we obtain $j : M \preccurlyeq_\kappa \mathcal{H}(\overline{\lambda})$.

(b) $\Rightarrow$ (a): Assume that (b) holds. We want to show that (2.2) holds. Suppose $\theta > \kappa$ and $M \prec V_\theta$ is such that $\kappa + 1 \subseteq M$ and $|\,M\,| = \kappa$.

Let $\lambda > \theta$ be a regular cardinal such that $V_\theta \in \mathcal{H}(\lambda)$, and let $M^* \prec \mathcal{H}(\lambda)$ be such that $M,\, \theta,\, \lambda \in M^*$, $M \subseteq M^*$ and $|\,M^*\,| = \kappa$.

By (b), there are regular $\overline{\lambda} > \kappa$ and j such that $\overline{j} : M^* \preccurlyeq_\kappa \mathcal{H}(\overline{\lambda})$. Then, by letting $j := \overline{j} \restriction M$ and $\overline{\theta} := \overline{j}(\theta)$, we obtain $j : M \preccurlyeq_\kappa V_{\overline{\theta}}$. ❑ (Lemma 4.7)

Proof of Theorem 4.6: If κ is weakly extendible then $\kappa = \mathfrak{wcn}(\mathcal{L}^{\mathrm{HO}}_{\kappa,\kappa})$ holds (Theorem 2.2). By Lemma 1.10, (3), it follows that $\kappa = \mathfrak{wcn}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat,\kappa,\omega})$.

Now assume that $\kappa = \mathfrak{wcn}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat,\kappa,\omega})$. We want to show that (b) in Lemma 4.7 holds.

We have $2^{<\kappa} = \kappa$ since $\kappa = \mathfrak{wcn}(\mathcal{L}_{\kappa,\omega})$ and hence κ is weakly compact.

Suppose that $\lambda > \kappa$ is regular and $M \prec \mathcal{H}(\lambda)$ is such that $\kappa + 1 \subseteq M$ and $|\,M\,| = \kappa$. By Lemma 4.4, we have $\mathcal{H}(\lambda) = L_\lambda$. Thus, by Proposition 4.5, there are regular $\overline{\lambda} > \kappa$ and j with $j : M \preccurlyeq_\kappa L_{\overline{\lambda}}$. Again by Lemma 4.4, we have $\mathcal{H}(\overline{\lambda}) = L_{\overline{\lambda}}$. This shows that (b) in Lemma 4.7 holds. ❑ (Theorem 4.6)

For a cardinal κ with $\kappa = \mathfrak{wcn}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat,\kappa,\omega})$, we do not know if the same equation holds in L. However we can show that quite strong large cardinal properties hold around κ in L:

Theorem 4.8 *Assume that $\kappa = \mathfrak{wcn}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat,\kappa,\omega})$. Then we have:* P-stat-6

(1) $\mathsf{L} \models$ "*κ is weakly compact, and there are stationarily many weakly compact cardinals below κ*".

(2) $\mathsf{L} \models$ "*there is a weakly compact cardinal $> \kappa$ which is a limit of weakly compact cardinals*".

Proof. (1): As already noted in the proof of Theorem 4.6, κ is weakly compact. It follows that κ is also weakly compact in L (see e.g. Theorem 17.22 in [12]).

Suppose that $D \in L$ is a closed unbounded subset of κ. We have to show that there is $\mu \in D$ such that $\mathsf{L} \models$ "μ is weakly compact".

Let $\lambda > \kappa$ be a regular cardinal and let (4.17): $M \prec L_\lambda$ be such that $\kappa + 1 \subseteq M$, $D \in M$, and $|\,M\,| = \kappa$. By Proposition 4.5, there are $\overline{\lambda} > \kappa$ regular in L, and j such that (4.18): $j : M \preccurlyeq_\kappa L_{\overline{\lambda}}$. Since $\overline{\lambda}$ is regular in L, we have $L_{\overline{\lambda}} \models$ "κ is weakly compact" (this can be seen e.g. in the characterization of weak compactness in terms of the tree property). By the elementarity (4.17) and (4.18), and since $D = j(D) \cap \kappa$ is unbounded in κ, we have $L_{\overline{\lambda}} \models$ "$\kappa \in j(D)$". Thus x-stat-13 x-stat-14

$$L_{\overline{\lambda}} \models \text{"there is weakly compact } \mu \in j(D)\text{"}.$$

Hence $L_\lambda \models$ "there is weakly compact $\mu \in D$" by the elementarity (4.17) and (4.18). Since λ is regular, it follows that $\mathsf{L} \models$ "there is weakly compact $\mu \in D$".

(2): Let $\lambda > \kappa$ be regular and (4.19): $M \prec L_\lambda$ be such that $\kappa + 1 \subseteq M$ and $|\,M\,| = \kappa$. By Proposition 4.5, there are regular $\overline{\lambda} > \kappa$ and j with (4.20): $j : M \preccurlyeq_\kappa L_{\overline{\lambda}}$. By (1) and since $\lambda > \kappa$ is regular, we have x-stat-15 x-stat-16

$$L_\lambda \models \text{"}\kappa \text{ is weakly compact and is stationary limit of weakly compact cardinals."}$$

By the elementarity (4.19) and (4.20), it follows that

$$L_{\overline{\lambda}} \models \text{"}j(\kappa) \text{ is weakly compact and is stationary limit of weakly compact cardinals."}$$

Since $\overline{\lambda}$ is regular in L, it follows that

$$\mathsf{L} \models \text{"}j(\kappa) \text{ is weakly compact and is stationary limit of weakly compact cardinals."}$$

□ (Theorem 4.8)

There are still many open problems concerning the compactness of weak second-order logic and stationary logic including the one we mentioned before Theorem 4.8. We neither know the answer to the following:

Problem 4.9 *Is "$\mathfrak{wcn}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat}) < \mathfrak{wcn}(\mathcal{L}^{\mathrm{II}})$" consistent?*

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