DID among large cardinals and Laver-generic large cardinal axioms

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References

[II] S.F., A. Ottenbreit Maschio Rodrigues, and H. Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum, Archive for Mathematical Logic, Vol.60, 3-4, (2021), 495–523. https://fuchino.ddo.jp/papers/SDLS-II-x.pdf

[S.F & Sakai] S.F.. and H. Sakai, The first order definability of generic large cardinals, submitted. https://fuchino.ddo.jp/papers/definability-of-glc-x.pdf

[S.F.1] S.F., Maximality Principles and Resurrection Axioms under a Laver generic large cardinal, (a pre-preprint version for the paper "Maximality Principles and Resurrection Axioms in light of a Laver generic large cardinal", in preparation) https://fuchino.ddo.jp/papers/RIMS2022-RA-MP-x.pdf

[S.F. & Usuba] S.F., and T. Usuba, On Recurrence Axioms, preprint. https://fuchino.ddo.jp/papers/recurrence-axioms-x.pdf

[S.F.2] S.F., Reflection and Recurrence, to appear in the Festschrift on the occasion of the 75. birthday of Professor Janos Makowsky, Birkhäuser, (2025). https://fuchino.ddo.jp/papers/reflection_and_recurrence-Janos-Festschrift-x.pdf

[S.F. & Gappo & Parente] S.F., T. Gappo, and F. Parente, Generic Absoluteness revisited, preprint.

https://fuchino.ddo.jp/papers/generic-absoluteness-revisited-x.pdf

Outline

Identity crises

- One of M. Magidor's classical theorems says that it is consistent that the first strongly compact cardinal is equal to the first measurable cardinal while it is also consistent (modulo a supercompact cardinal) that the first strong compact cardinal is the first supercompact cardinal. Magidor called this kind of phenomena identity crises.

[Magidor] M. Magidor, How large is the first strongly compact cadinal? or a study of identity cirses, AML, 10 (1976), 33-57.

- Similar kind of identity crisis is also studied in the recent paper:

[Hayut-Magidor-Poveda] Y. Hayut, M. Magidor, and A. Poveda, Identity crisis between supercompactness and Vopěnka principle, JSL, Vol.87 (2), 2022, 626-648.

- In this talk, we study some cases of dissociative identity disorder (DID, previously known as MPD (多重人格)) among large cardinals (LCs) and Laver-generic large cardinal axioms (LgLCs) where the apparent consistency strengths of certain LCs and LgLCs are shown to be totally different from the actual consistency strengths.

Large cardinals characterized by elementary embeddings

- Most of the notions of large cardinals, in particular the notions of large cardinals stronger than measurable cardinals, are characterized as critical points of certain elementary embeddings. For example:

  - A cardinal κ is said to be supercompact if and only if, for any $\lambda > \kappa$, there are classes j, $M \subseteq \mathsf{V}$ s.t. (1) $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$,[1] (2) $j(\kappa) > \lambda$, and M is sufficiently closed, or more specifically: (3) ${}^{\lambda}M \subseteq M$.

  - The existence of j with the target model M can be considered as a strong reflection property.

- Many notions of large cardinals are obtained by modifying the definition of supercompact cardinals around the closedness condition (3), and/or by changing "for all $\lambda > \kappa$" to "for some $\lambda > \kappa$".

[1] With "$j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$" we denote the circumstance "M is a transitive class, j is an elementary embedding of the class structure $(\mathsf{V}, \in)$ into the class structure $(M, \in)$, and κ is the critical point of j (i.e. $\kappa = \min\{\mu \in \mathsf{Card} : j(\mu) \neq \mu\}$)"

Large cardinals characterized by elementary embeddings (2/3)

▷ A cardinal κ is said to be supercompact if and only if, for any $\lambda > \kappa$, there are classes j, $M \subseteq V$ s.t. (1) $j : V \xrightarrow{\prec}_{\kappa} M$,(2) $j(\kappa) > \lambda$, and M is sufficiently closed, or more specifically: (3) ${}^{\lambda}M \subseteq M$.

- Many notions of large cardinals are obtained by modifying the definition of supercompact cardinals around the closedness condition (3), and/or by changing "for all $\lambda > \kappa$" to "for some $\lambda > \kappa$".

▷ Thus, we obtain the definition of super-almost-huge cardinal by replacing (3) with (3)' ${}^{j(\kappa)>}M \subseteq M$ in the definition of supercompactness.

▷ The definition of superhuge cardinal is obtained by replacing (3) with (3)" ${}^{j(\kappa)}M \subseteq M$ in the definition of supercompactness.

A more comprehensive list:

Large cardinals characterized by elementary embeddings (3/3)

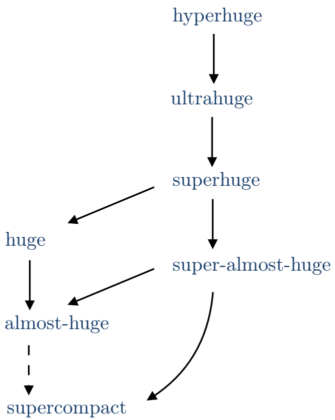
▷ A cardinal κ is said to be **supercompact** if and only if, for any $\lambda > \kappa$, there are classes j, $M \subseteq \mathsf{V}$ s.t. (1) $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$,(2) $j(\kappa) > \lambda$, and M is sufficiently closed, or more specifically: (3) ${}^{\lambda}M \subseteq M$.

- Many notions of large cardinals are obtained by modifying the definition of supercompact cardinals around the closedness condition (3), and/or by changing "for all $\lambda > \kappa$" to "for some $\lambda > \kappa$".

	The condition (3): ${}^{\lambda}M \subseteq M$ replaced by	"for all $\lambda > \kappa$" replaced by "for some $\lambda > \kappa$"
hyperhuge	${}^{j(\lambda)}M \subseteq M$	-
ultrahuge	${}^{j(\kappa)}M \subseteq M$ and $V_{j(\lambda)} \in M$	-
superhuge	${}^{j(\kappa)}M \subseteq M$	-
super-almost-huge	${}^{j(\kappa)>}M \subseteq M$	-
huge	${}^{j(\kappa)}M \subseteq M$	√
almost-huge	${}^{j(\kappa)>}M \subseteq M$	√

Large large caridnals ordered by implication

- By definition:



B ⟵ A : "if a cardinal κ is A then κ is B."

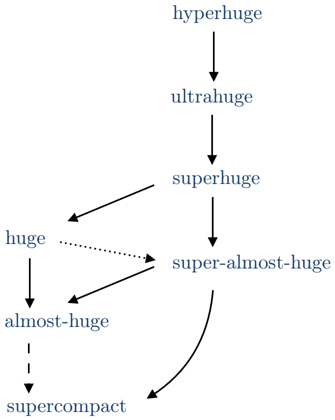
B ⇠ A : "if a cardinal κ is A then there are cofinally many $0 < \mu < \kappa$ s.t. μ is B in V_κ"

Large large cardinals ordered by implication (2/2)

- The global character "for all $\lambda > \kappa$ ..." of super-almost-huge cardinal gives the impression that it might be much stronger than hugeness in terms of consistency strength.

- However, we can show in the following:

Ordered by implication, and by "normal measure one occurrence below"



B ← − A : "if a cardinal κ is A then there are cofinally many $0 < \mu < \kappa$ μ is B in V_κ"

B ←⋯ A: "if a cardinal κ is with the large cardinal property A, then there are normal measure one many λ with B in V_κ".

There are many super-almost-huge cardinals in V_κ for a huge κ

Proposition 1. Suppose that κ is huge. Then,

$$\{\alpha < \kappa : V_\kappa \models \text{“}\alpha \text{ is super almost-huge”}\}$$

is a normal measure 1 subset of κ.

Idea of Proof: Modify Theorem 24.11 in

[kanamori] Akihiro Kanamori, The Higher Infinite, Springer Verlag (2004)

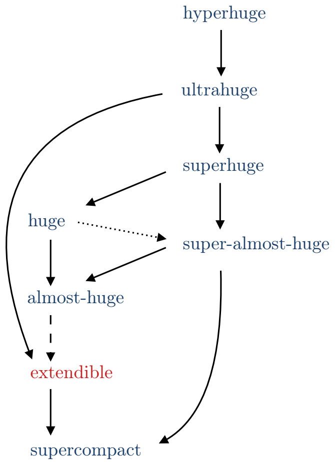
to characterize super-almost hugeness. Then solve the corresponding modification of Exercise 24.12 (see these slides for more details).

□ (Proposition 1.)

Another (classical) DID with extendible cardinal

	The condition (3): ${}^{\lambda}M \subseteq M$ replaced by	"for all $\lambda > \kappa$" replaced by "for some $\lambda > \kappa$"
hyperhuge	${}^{j(\lambda)}M \subseteq M$	-
ultrahuge	${}^{j(\kappa)}M \subseteq M$ and $V_{j(\lambda)} \in M$	-
superhuge	${}^{j(\kappa)}M \subseteq M$	-
super-almost-huge	${}^{j(\kappa)>}M \subseteq M$	-
huge	${}^{j(\kappa)}M \subseteq M$	$\surd$
almost-huge	${}^{j(\kappa)>}M \subseteq M$	$\surd$
extendible	$V_{j(\lambda)} \in M$	-

Another (classical) DID with extendible cardinal (2/2)



B ← – A : "if a cardinal κ is A then there are cofinally many $0 < \mu < \kappa$ μ is B in V_κ"

B ←······ A: "if a cardinal κ is with the large cardinal property A, then there are normal measure one many λ with B in V_κ".

From large cardinals to generic large cardinals

- Small cardinals like $\aleph_1$, $\aleph_2$, $2^{\aleph_0}$ cannot be a large cardinal! But they can have many features of large cardinals by being generic large cardinals.

- An important ingredient for the composition of the notion of generic large cardinal is Proposition 22.4 (b) in [kanamori].

- For a class $\mathcal{P}$ of p.o.s, κ is said to be $\mathcal{P}$-generic supercompact if, for all $\lambda > \kappa$ there is $\mathbb{P} \in \mathcal{P}$ s.t. for a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ there are j $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M$, $j(\kappa) > \lambda$, and $(3)^*$ $j''\lambda \in M$.

- The equivalence in Proposition 22.4 (b) in [kanamori] is no more valid in the generic elementary embedding context but $(3)^*$ is still a closedness property of the target model M. This fact is summarized in Lemma 3.5 in S.F., A. Ottenbreit Maschio Rodrigues, and H. Sakai [II].

- A small cardinal can be $\mathcal{P}$-generic large cardinal. For example, in the standard model of Proper Forcing Axiom (PFA), $2^{\aleph_0}$ is $\mathcal{P}$-generic supercompact (for $\mathcal{P}$ = proper p.o.s).

- ▶ A small cardinal can be $\mathcal{P}$-generic large cardinal. For example, in the standard model of Proper Forcing Axiom (PFA), $2^{\aleph_0}$ is $\mathcal{P}$-generic supercompact (for $\mathcal{P}$ = proper p.o.s).
- ▷ Similarly, in the standard model of Martin's Maximum (MM), $2^{\aleph_0}$ is $\mathcal{P}$-generic supercompact (for $\mathcal{P}$ = semi-proper p.o.s).
- ▶ Analyzing the standard model of PFA and Martin's Maximum MM, we obtain the notion of Laver-generic large cardinal:
- ▷ A cardinal κ is tightly $\mathcal{P}$-Laver-generic supercompact if, for any $\lambda > \kappa$, and for any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ s.t. $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$, (4) $j''\lambda \in M$, and (5) $|RO(\mathbb{P} * \underset{\sim}{\mathbb{Q}})| \leq j(\kappa)$.

  The word "tightly" refers to the condition (5).
- ▷ The $\mathcal{P}$-Laver-generic large cardinal axiom for the notion of supercompactness ($\mathcal{P}$-LgLC for supercompact, for short) is the assertion that $\kappa_{\mathfrak{refl}} := \max\{2^{\aleph_0}, \aleph_2\}$ is tightly $\mathcal{P}$-Laver-generic supercompact cardinal.

Laver-generic large cardinals

- The condition "$\kappa = \kappa_{\mathfrak{refl}}$" is (almost) a consequence of Laver-gen. supercompactness.

Proposition 2. (Theorem 5.9 in [II]) For $\mathcal{P}$ = σ-closed p.o.s, proper p.o.s, semi-proper p.o.s, ccc p.o.s, etc., if κ is tightly $\mathcal{P}$-Laver gen. supercompact then $\kappa = \kappa_{\mathfrak{refl}}$.

- Along with the hierarchy of large cardinals, we can introduce corresponding LgLC by modifying the condition (4) in the definition of $\mathcal{P}$-LgLC for supercompact.

$\mathcal{P}$-LgLC for	The condition (4): $j''\lambda \in M$ is replaced by
hyperhuge	$j''j(\lambda) \in M$
ultrahuge	$j''j(\kappa) \in M$ and ${V_{j(\lambda)}}^{V[\mathbb{H}]} \in M$
superhuge	$j''j(\kappa) \in M$
super-almost-huge	$j''j(\mu) \in M$ for all $\mu < j(\kappa)$

Laver-generic large cardinals (2/3)

- By definition:

$\mathcal{P}$-LgLC for hyperhuge

$\downarrow$

$\mathcal{P}$-LgLC for ultrahuge

$\downarrow$

$\mathcal{P}$-LgLC for superhuge

$\downarrow$

$\mathcal{P}$-LgLC for super-almost-huge

$\downarrow$

$\mathcal{P}$-LgLC for supercompact

B $\longleftarrow$ A : "the axiom A implies the axiom B"

- By Theorem 5.3 in S.F., and T. Usuba [S.F. & Usuba], it follows that $\mathcal{P}$-LgLC for hyperhuge, and for transfinitely iterable $\mathcal{P}$ is equiconsistent with the existence of an hyperhuge cardinal.

Laver-generic large cardinals (3/3)

$\exists$ a hyperhuge cardinal ←- - - - → $\mathcal{P}$-LgLC for hyperhuge

Theorem 5.3 in [S.F. & Usuba]

$\downarrow$

$\mathcal{P}$-LgLC for ultrahuge

$\downarrow$

$\mathcal{P}$-LgLC for superhuge

$\downarrow$

$\mathcal{P}$-LgLC for super-almost-huge

$\downarrow$

$\mathcal{P}$-LgLC for supercompact

B ←—— A : "the axiom A implies the axiom B"

B ←- - -→ A : "the axioms A and B are equi-consistent."

- $\mathcal{P}$-LgLC for hyperhuge, for transfinitely iterable $\mathcal{P}$ is one of only few families of strong axioms of set-theory whose exact consistency strength is known.

Proposition 3. (ccc-LgLC for supercompact) For any non-free algebra A (in universal algebra) there is non-free subalgebra B of A of size $< 2^{\aleph_0}$.

Proof. Note that ccc-LgLC for supercompact implies that the continuum is extremely large and hence $\kappa_{\mathfrak{refl}} = 2^{\aleph_0}$.

- Suppose toward a contradiction, that A is a non-free algebra s.t. all subalgebras of A of size $< 2^{\aleph_0}$ are free.
- Let $\lambda := 2^{|A|}$. W.l.o.g., the underlying set of A is $\mu < \lambda$. Let $\mathbb{P}$ be a ccc p.o. adding $\lambda' \geq \lambda$ many reals and let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name of a ccc p.o. s.t. for a $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ as in the definition of ccc-LgLC with the critical point $\kappa = 2^{\aleph_0}$.
- Then $A \in M$. Since $M \models A \leq j(A)$ and $M \models |A| < j(\kappa) = 2^{\aleph_0}$., by elementarity, it follows that $M \models A$ is free.
- On the other hand, since $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is ccc, $\mathsf{V}[\mathbb{H}] \models A$ is not free. Hence $M \models A$ is not free. This is a contradiction,. □ (Proposition 3)

Proposition 4. (Cohen-LgLC for supercompact) Any non-metrizable topological space X with character $< 2^{\aleph_0}$ has a non-metrizable subspace Y of size $< 2^{\aleph_0}$.

Proof. Similarly to Proposition 3. Using a result of Dow, Tall, and Weiss, Cohen forcing preserve non-metrizability of a topological space.

(Proposition 3.)

Proposition 5. (1) For any σ-closed generically supercompact cardinal κ, if T is non-special tree then there is $T' \in [T]^{<\kappa}$ which is also non-special.

(2) If σ-closed-LgLC for supercompact holds, then Rado Conjecture (RC) holds.

(3) If $\mathcal{P}$ contains all ccc p.o.s, then $\mathcal{P}$-LgLC for supercompact implies $\neg$RC.

Proof. (1),(2): Similarly to Proposition 3. Using the fact that σ-closed p.o.s preserve non-specialty of trees (Todorcévic).

(3): Since MA implies $\neg$RC and by Theorem 6 below.

(Proposition 5)

LgLCs imply strong mathematical reflection theorems (3/3)

Theorem 6. (Theorem 5.7 in S.F., A. Ottenbreit Maschio Rodrigues, and H. Sakai [II])
($\mathcal{P}$-LgLC for supercompact for a stationary preserving $\mathcal{P}$)
$\mathsf{MA}^{+<\kappa_{\mathfrak{refl}}}(\mathcal{P})$ holds. □

Corollary 7. Suppose that $\mathcal{P}$ is stationary preserving and contains all σ-closed p.o.s. Then $\mathcal{P}$-LgLC for supercompact implies the Fodor-type Reflection Principle (FRP).

Proof. By Theorem 6, it follows that $\mathcal{P}$-LgLC implies $\mathsf{MA}^{+}(\sigma\text{-closed})$.
It is known that this principle implies FRP (See Section 2 of S.F., I.Juhász, L.Soukup, Z.Szentmiklóssy and T.Usuba, Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness). □ (Corollary 7)

- In contrast:

Proposition 8. FRP is independent over $\mathcal{P}$-LgLC for supercompact (actually for any large cardinal property) for any class $\mathcal{P}$ of ccc p.o.s as far as the axiom "$\mathcal{P}$-LgLC for supercompact" is consistent. □

Theorem 6. (Theorem 5.7 in S.F., A. Ottenbreit Maschio Rodrigues, and H. Sakai [II]) ($\mathcal{P}$-LgLC for supercompact for stationary preserving $\mathcal{P}$)
$\mathsf{MA}^{+<\kappa_{\mathfrak{refl}}}(\mathcal{P})$ holds.

Theorem 9. ([S.F.& Usuba], see Theorem 6.1 in [S.F.&Gappo&Parente]) ($\mathcal{P}$-LgLC for ultrahuge) The restricted version of Recurrence Axiom
$(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_\Gamma$-$\mathsf{RcA}^+$ holds for $\Gamma =$ conjunctions of Σ_2 and Π_2 formulas.

Theorem 10. (Theorem 7.2 in [S.F.1]) ($\mathcal{P}$-LgLC for ultrahuge)
The Unbounded Resurrection Axiom for $\mathcal{P}$ of Tsaprounis [Tsaprounis] holds.

Theorem 11. (Theorem 4.10 in [S.F.& Usuba]) (Super-$C^{(\infty)}$-$\mathcal{P}$-LgLC for ultrahuge) The Maximality Principle (Hamkins [Hamkins1]) $\mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))$ holds.

Theorem 12. (Theorems 5.2 and 5.3 in [S.F.& Usuba])
($\mathcal{P}$-LgLC for hyperhuge (for any $\mathcal{P}$)) The bedrock exists and $\kappa_{\mathfrak{refl}}$ is hyperhuge in the bedrock. Note that this implies ¬GA. □

Theorem 13. (1) (Proposition 2.8 in [II]) Suppose that κ is $\mathcal{P}$-generically supercompact and all elements of $\mathcal{P}$ are μ-cc for a cardinal μ. Then Singular Cardinal Hypothesis (SCH) above $\max\{2^{<\kappa}, \mu\}$ holds.

(2) (Corollary 5.2 in [S.F.& Usuba]) ($\mathcal{P}$-LgLC for hyperhuge (for an arbitrary $\mathcal{P}$)) There are class many huge cardinals, and SCH holds above some cardinal.

Proof. (1): A modification of the proof of Solovay's theorem on SCH above a supercompact cardinal will do.

(2): By Theorem 12. □ (Theorem 13)

LgLCs for extendible

- A cardinal κ is tightly $\mathcal{P}$-Laver generically extendible if if, for any $\lambda > \kappa$, and for any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ s.t. $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \overset{\preceq}{\rightarrow}_{\kappa} M$, $j(\kappa) > \lambda$, (1): $V_{j(\lambda)}^{\mathsf{V}[\mathbb{H}]} \in M$, and (2): $|\,RO(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| \leq j(\kappa)$.

- The $\mathcal{P}$-Laver-generic large cardinal axiom for the notion of extendibility ($\mathcal{P}$-LgLC for extendible, for short) is the assertion that $\kappa_{\mathfrak{refl}}$ is tightly $\mathcal{P}$-Laver-generic extendible cardinal.

- A cardinal κ is tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver generically extendible if, for any $n \in \mathbb{N}$, $\lambda_0 > \kappa$, and $\mathbb{P} \in \mathcal{P}$, there are $\lambda \geq \lambda_0$ and a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ s.t. $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. (3): ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$, $j : \mathsf{V} \overset{\preceq}{\rightarrow}_{\kappa} M$, $j(\kappa) > \lambda$, (1): $V_{j(\lambda)}^{\mathsf{V}[\mathbb{H}]} \in M$, and (2): $|\,RO(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| \leq j(\kappa)$.

- The super-$C^{(\infty)}$-$\mathcal{P}$-Laver-generic large cardinal axiom for the notion of extendibility (super-$C^{(\infty)}$-$\mathcal{P}$-LgLC for extendible, for short) is the assertion that $\kappa_{\mathfrak{refl}}$ is tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver-generic extendible cardinal.

LgLCs for extendible (2/2)

- Note that, in general, "κ is tightly super-$C^{(\infty)}$-$\mathcal{P}$-Laver generically extendible" is not formalizable in the language of ZF. In contrast, the axiom "super-$C^{(\infty)}$-$\mathcal{P}$-LgLC for extendible" is formalizable in the language of ZF in infinitely many formulas. This is because the axiom refers to the definable cardinal $\kappa_{\mathfrak{refl}}$.

$\mathcal{P}$-LgLC for	The condition (4): $j''\lambda \in M$ is replaced by
hyperhuge	$j''j(\lambda) \in M$
ultrahuge	$j''j(\kappa) \in M$ and $V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \in M$
superhuge	$j''j(\kappa) \in M$
super-almost-huge	$j''j(\mu) \in M$ for all $\mu < j(\kappa)$
extendible	$V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \in M$

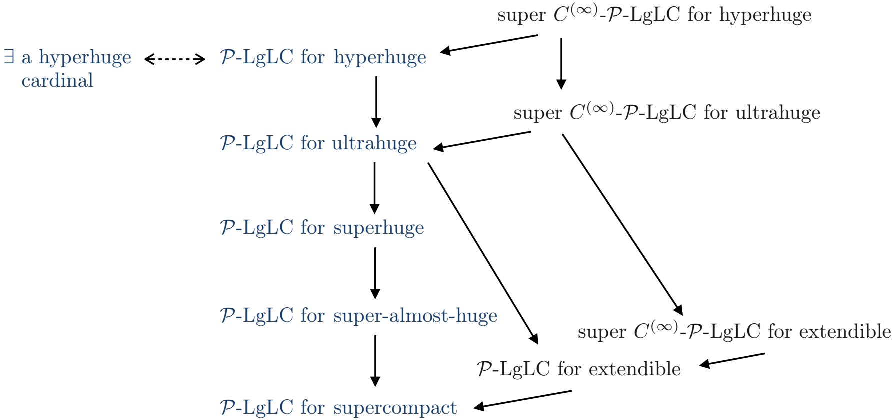
DID around LgLCs for extendible

- In Theorems 9 and 11, $\mathcal{P}$-LgLC for ultrahuge, and super-$C^{(\infty)}$-$\mathcal{P}$-LgLC for ultrahuge can be replaced by $\mathcal{P}$-LgLC for extendible, and super-$C^{(\infty)}$-$\mathcal{P}$-LgLC for extendible, respectively.
- In the proof of Theorem 10, it seems that $\mathcal{P}$-LgLC for ultrahuge is used in its full strength. However we have

Theorem 14. (Theorem 7.1 in [S.F.1]) ($\mathcal{P}$-LgLC for extendible) The Boldface Resurrection Axiom for $\mathcal{P}$ of Hamkins [Hamkins2] holds.

- $\mathcal{P}$-LgLC for extendible has consistency strength below that of an extendible cardinal (see Theorem 15 below).
- Super-$C^{(\infty)}$-$\mathcal{P}$-LgLC for extendible have consistency strength strictly less than that of an almost-huge cardinal (see Conjecture 16).

DID around LgLCs for extendible (2/2)



B $\longleftarrow$ A : "the axiom A implies the axiom B"

B $\dashleftarrow\dashrightarrow$ A : "the axioms A and B are equi-consistent."

Consistency proof of LgLCs for extendible

- The following theorem was suggested by Gabe Goldberg:

Theorem 15. Suppose that κ is extendible. Then for many natural classes $\mathcal{P}$ of p.o.s consisting of stationary preserving p.o.s (including the class of all ccc p.o.s, all σ-closed p.o.s, all proper p.o.s, all semi-proper p.o.s, etc.), there is a p.o. $\mathbb{P}_\kappa$ s.t. $\Vdash_{\mathbb{P}_\kappa}$ "$\kappa = \kappa_{\mathfrak{refl}}$ and κ is tightly $\mathcal{P}$-Laver generic extendible".[2]

Lemma 15.0. If κ is extendible then there are class many measurable cardinals.

Proof. If κ is extendible then it is supercompact (Proposition 23.6 in [kanamori]). Hence, in particular κ is measurable. If $j_0 : V_\gamma \xrightarrow{\prec}_\kappa V_\delta$ with $j_0(\kappa) > \gamma$ then $V_\delta \models$ " there is a normal ultrafilter over $j_0(\kappa)$" by elementarity. Since the normal ultrafilter over $j_0(\kappa)$ in V_δ is really a normal ultrafilter, $j_0(\kappa)$ is measurable. $\square$ (Lemma 15.0)

[2] The corresponding theorem for the super $C^{(\infty)}$-$\mathcal{P}$-Laver generic ultrahugeness can be formulated for all transfinitely iterable classes $\mathcal{P}$.

Consistency proof of LgLCs for extendible (2/6)

- We call a mapping $f : M \to N$ cofinal (in N) if for all $b \in N$ there is $a \in M$ s.t. $b \in f(a)$.

Lemma 15.1. (A special case of Lemma 6 in [S.F. & Sakai]) For any cardinal θ and $j_0 : \mathcal{H}(\theta) \overset{\prec}{\to} N$, letting $N_0 = \bigcup j_0{}''\mathcal{H}(\theta)$, we have $j_0 : \mathcal{H}(\theta) \overset{\prec}{\to} N_0$ and j_0 is cofinal in N_0. □

Lemma 15.2. (A special case of Lemma 7 in [S.F. & Sakai]) For any regular cardinal θ and cofinal $j_0 : \mathcal{H}(\theta) \overset{\prec}{\to} N$, there are j, $M \subseteq \mathsf{V}$ s.t. $j : \mathsf{V} \overset{\prec}{\to} M$, $N \subseteq M$, and $j_0 \subseteq j$. □

Lemma 15.3. For a cardinal κ, the following are equivalent:
(a) κ is extendible. (b) For all $\lambda > \kappa$, there are j, M s.t. $j : \mathsf{V} \overset{\prec}{\to}_{\kappa} M$ s.t. $j(\kappa) > \lambda$ and $V_{j(\lambda)} \in M$.

Proof. (b) $\Rightarrow$ (a) is trivial. The other direction follows from Lemma 15.0, Lemma 15.1, and Lemma 15.2. □ (Lemma 15.3)

Lemma 15.4. An extendible cardinal κ admits a Laver-function. I.e., there is a mapping $f : \kappa \to V_\kappa$ s.t. for any x, and $\lambda > \kappa$ there are j, M s.t. $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$ s.t. $j(\kappa) > \lambda$, $V_{j(\lambda)} \in M$ and $j(f)(\kappa) = x$.[3]

Proof. A modification of the proof of Theorem 20.21 in [Millennium book] (Th. Jech, Set Theory, The Third Millennium Edition) will do.

- Assume, toward a contradiction, that there is no Laver function $f : \kappa \to V_\kappa$.
- ▷ Let $\varphi(f)$ be the formula

$$\exists \underline{\alpha} \exists \underline{\delta} \exists \underline{x} \, (f : \underline{\alpha} \to V_{\underline{\alpha}} \ \wedge \ \underline{\alpha} < \underline{\delta} \ \wedge \ \underline{\delta} \text{ is inaccessible } \wedge \ \underline{x} \in V_{\underline{\delta}}$$
$$\wedge \ \forall \underline{\delta}' \forall \underline{j} \, (\underline{j} : V_{\underline{\delta}} \overset{\prec}{\to} V_{\underline{\delta}'} \ \wedge \ j \text{ is cofinal in } V_{\underline{\delta}'} \ \to \ j(f)(\underline{\alpha}) \neq \underline{x}))$$

- ▷ If $\varphi(f)$ holds then the witness of $\underline{\alpha}$ in $\varphi(f)$ is uniquely determined. In this case, let δ_f and x_f be witnesses for $\underline{\delta}$ and $\underline{x}$ in $\varphi(x)$. Let $\mu_f := rank(x_f)$. We choose δ_f, x_f and μ_f so that δ_f minimal among the possible witnesses of $\underline{\delta}$ and x_f is chosen so that μ_f is minimal. ▷ If $\varphi(f)$ does not hold, we let $\delta_f := 0$ and $\mu_f := 0$.

Consistency proof of LgLCs for extendible (4/6)

- ⊕ By assumption, we have $\varphi(f)$ for all $f : \kappa \to V_\kappa$.
- ▶ Let ν be an inaccessible cardinal $\geq \max\{\delta_f, \mu_f \ : \ f : \alpha \to V_\alpha$ for inaccessible $\alpha \leq \kappa\}$.
- ▷ Let $j^* : \mathsf{V} \overset{\prec}{\to}_\kappa M$ be s.t. $(1^\dagger)$: $j^*(\kappa) > \nu$ and $(2^\dagger)$: $V_\nu \in M$.
- ▷ Let $A := \{\alpha < \kappa \ : \ \forall f\,(\,f : \alpha \to V_\alpha \ \to \ \varphi(f))\}$.
- ▶ By assumption, $\mathsf{V} \models$ "$\forall f\,(f : \kappa \to V_\kappa \ \to \ \varphi(f))$". By $(2^\dagger)$, it follows that
- ▷ $M \models$ "$\forall f\,(f : \kappa \to V_\kappa \ \to \ \varphi(f))$". Thus we have $M \models j^*(A) \ni \kappa$.
- ▶ Let $f^* : \kappa \to V_\kappa$ be defined by $$f^*(\alpha) := \begin{cases} x_{f^*\restriction\alpha}, & \text{if } \alpha \in A; \\ \emptyset, & \text{otherwise.} \end{cases}$$
- ▶ Let $x^* := j^*(f^*)(\kappa)$. ▷ By definition of f^*, by ⊕, and since $j(f^*) \restriction \kappa = f^*$, x^* together with δ_{f^*} and μ_{f^*} witnesses $\varphi(f^*)$. (x^* may be different from x_{f^*} but this does not matter.)
- ▶ In particular, $x^* \neq ((j^* \restriction V_{\delta_{f^*}})(f^*)(\kappa) = j(f^*)(\kappa)$. This is a contradiction. ⧉ (Lemma 15.4)

[3] Lemma 15.4 is well-known. See e.g. [corraza]. I go through the details of the proof in the present setting so that I can reuse them in the future proof of Conjecture 16 .

A (sketch of a) proof of Theorem 15: ▶ We show the Theorem for the case that $\mathcal{P}$ is the class of all proper p.o.s. ▶ Let f be a Laver function for extendible cardinal κ (f exists by Lemma 15.2).

▶ Let $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be an CS-iteration of elements of $\mathcal{P}$ s.t.

$$\underset{\sim}{\mathbb{Q}}_\beta := \begin{cases} f(\beta), & \text{if } f(\beta) \text{ is a } \mathbb{P}_\beta\text{-name} \\ & \text{and } \Vdash_{\mathbb{P}_\beta} \text{“} f(\beta) \in \mathcal{P} \text{”;} \\ \mathbb{P}_\beta\text{-name of the trivial forcing,} & \text{otherwise.} \end{cases}$$

▶ We show that $\Vdash_{\mathbb{P}_\kappa}$ “$\mathcal{P}$-LgLC for extendible”.

▷ First, note that $\Vdash_{\mathbb{P}_\kappa}$ “$\kappa = 2^{\aleph_0} = \kappa_{\mathfrak{refl}}$” by definition of $\mathbb{P}_\kappa$.

▷ Let $\mathbb{G}_\kappa$ be a $(\mathsf{V}, \mathbb{P}_\kappa)$-generic filter. In $\mathsf{V}[\mathbb{G}_\kappa]$, suppose that $\mathbb{P} \in \mathcal{P}$ and let $\underset{\sim}{\mathbb{P}}$ be a $\mathbb{P}_\kappa$-name for $\mathbb{P}$.

▷ Suppose that $\lambda > \kappa$. By Lemma 15.0, there is an inaccessible $\lambda^* > \lambda$. Let $j : \mathsf{V} \overset{\preceq}{\to}_\kappa M$ be s.t. (1^*): $j(\kappa) > \lambda^*$, (2^*): $V_{j^*(\lambda^*)} \in M$ and (3^*): $j(f)(\kappa) = \underset{\sim}{\mathbb{P}}$. (This is possible since f is a Laver function for extendible κ.)

Consistency proof of LgLCs for extendible (6/6)

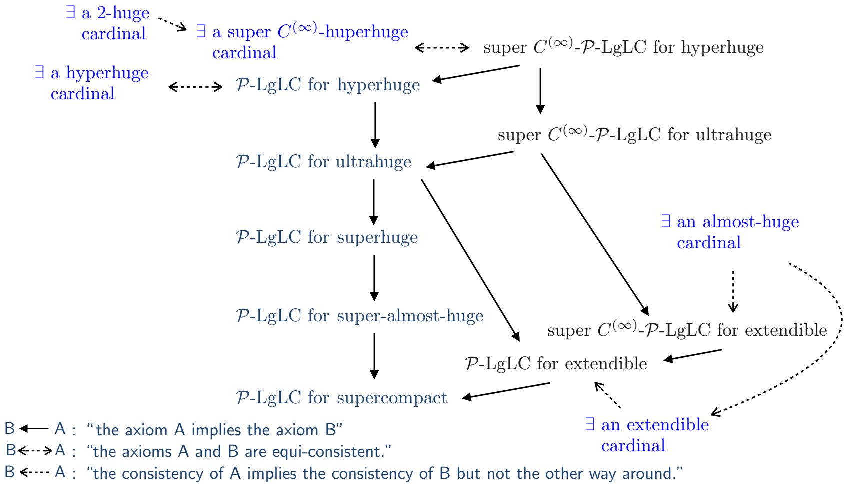
- In M, there is a $\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}}$-name $\underset{\sim}{\mathbb{Q}}$ s.t. $\Vdash_{\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$ and $\underset{\sim}{\mathbb{Q}}$ is the direct limit of CS-iteration of small p.o.s in $\mathcal{P}$ of length $j(\kappa)$, and $\mathbb{P}_\kappa * \underset{\sim}{\mathbb{P}} * \underset{\sim}{\mathbb{Q}} \sim j(\mathbb{P}_\kappa)$", ▷ By (2^*), the same situation holds in V.
- ▷ We have $j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa \ \sim\ \mathbb{P} * \underset{\sim}{\mathbb{Q}}$. *Here, we are identifying $\underset{\sim}{\mathbb{Q}}$ with corresponding $\mathbb{P}$-name.*
- Let $\mathbb{H}$ be $(\mathsf{V}, j(\mathbb{P}_\kappa))$-generic filter with $\mathbb{G}_\kappa \subseteq \mathbb{H}$.
- The lifting $\tilde{j} : \mathsf{V}[\mathbb{G}_\kappa] \overset{\prec}{\to}_\kappa M[\mathbb{H}]$; $\underset{\sim}{a}[\mathbb{G}_\kappa] \mapsto j(\underset{\sim}{a})[\mathbb{H}]$ witnesses that $\kappa = (\kappa_{\mathfrak{refl}})^{\mathsf{V}[\mathbb{G}_\kappa]}$ is tightly $\mathcal{P}$-Laver generic extendible. For this, it suffices to show:

Claim 15.5 ${V_\alpha}^{\mathsf{V}[\mathbb{H}]} \in M[\mathbb{H}]$ for all $\alpha \leq j(\lambda)$.

$\vdash$ By induction on $\alpha \leq j(\lambda)$. The successor step from $\alpha < j(\lambda)$ to $\alpha + 1$ can be proved by showing that $\mathbb{P}_\kappa$-names of subsets of ${V_\alpha}^{\mathsf{V}[\mathbb{H}]}$ can be chosen as elements of M. This is possible because of (2^*). $\dashv$ (Claim 15.5.)

□ (Theorem 15)

LgLCs



The following conjecture has been solved positively in the meantime.

Conjecture 16. A model of super-$C^{(\infty)}$-$\mathcal{P}$-LgLC for extendible, for arbitrary tranfinitely iterable $\mathcal{P}$ can be obtained starting from a model with a super-$C^{(\infty)}$ extendible cardinal, and this cardinal has consistency strength below that of almost huge.

Thank you for your attention!

ご清聴ありがとうございました.