

DID among large cardinals and Laver-generic large cardinal axioms

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- [II] S.F., A. Ottenbreit Maschio Rodrigues, and H. Sakai, **Strong downward Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum**, Archive for Mathematical Logic, Vol.60, 3-4, (2021), 495–523. <https://fuchino.ddo.jp/papers/SDLS-II-x.pdf>
- [S.F & Sakai] S.F.. and H. Sakai, **The first order definability of generic large cardinals**, submitted. <https://fuchino.ddo.jp/papers/definability-of-glc-x.pdf>
- [S.F.1] S.F., **Maximality Principles and Resurrection Axioms under a Laver generic large cardinal**, (a pre-preprint version for the paper “Maximality Principles and Resurrection Axioms in light of a Laver generic large cardinal”, in preparation) <https://fuchino.ddo.jp/papers/RIMS2022-RA-MP-x.pdf>
- [S.F. & Usuba] S.F., and T. Usuba, **On Recurrence Axioms**, preprint. <https://fuchino.ddo.jp/papers/recurrence-axioms-x.pdf>
- [S.F.2] S.F., **Reflection and Recurrence**, to appear in the Festschrift on the occasion of the 75. birthday of Professor Janos Makowsky, Birkhäuser, (2025). https://fuchino.ddo.jp/papers/reflection_and_recurrence-Janos-Festschrift-x.pdf
- [S.F. & Gappo & Parente] S.F., T. Gappo, and F. Parente, **Generic Absoluteness revisited**, preprint. <https://fuchino.ddo.jp/papers/generic-absoluteness-revisited-x.pdf>

- ▷ References
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- ▷ Large cardinals characterized by elementary embeddings
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- ▶ One of M. Magidor's classical theorems says that it is consistent that **the first strongly compact cardinal is equal to the first measurable cardinal** while it is also consistent (modulo a supercompact cardinal) that **the first strong compact cardinal is the first supercompact cardinal**. Magidor called this kind of phenomena **identity crises**.

[Magidor] M. Magidor, [How large is the first strongly compact cardinal? or a study of identity crises](#), AML, 10 (1976), 33-57.

- ▷ Similar kind of identity crisis is also studied in the recent paper:

[Hayut-Magidor-Poveda] Y. Hayut, M. Magidor, and A. Poveda, [Identity crisis between supercompactness and Vopěnka principle](#), JSL, Vol.87 (2), 2022, 626-648.

- ▶ In this talk, we study some cases of dissociative identity disorder (**DID**, previously known as MPD (多重人格)) among large cardinals (LCs) and Laver-generic large cardinal axioms (LgLCAs) where the apparent consistency strengths of certain LCs and LgLCAs are shown to be totally different from the actual consistency strengths.

- ▶ Most of the notions of large cardinals, in particular the notions of large cardinals stronger than measurable cardinals, are characterized as critical points of certain elementary embeddings. For example:
 - ▷ A cardinal κ is said to be **supercompact** if and only if, for any $\lambda > \kappa$, there are classes j , $M \subseteq V$ s.t. (1) $j : V \xrightarrow{\kappa} M$,^[1] (2) $j(\kappa) > \lambda$, and M is sufficiently closed, or more specifically:
 - (3) ${}^\lambda M \subseteq M$.
 - ▷ The existence of j with the target model M can be considered as a strong reflection property.
 - ▶ Many notions of large cardinals are obtained by modifying the definition of supercompact cardinals around the closedness condition (3), and/or by changing “for all $\lambda > \kappa$ ” to “for some $\lambda > \kappa$ ”.

^[1] With “ $j : V \xrightarrow{\kappa} M$ ” we denote the circumstance “ M is a transitive class, j is an elementary embedding of the class structure (V, \in) into the class structure (M, \in) , and κ is the critical point of j (i.e. $\kappa = \min\{\mu \in \text{Card} : j(\mu) \neq \mu\}$)”

Large cardinals characterized by elementary embeddings (2/3) DID (6/35)

- ▷ A cardinal κ is said to be **supercompact** if and only if, for any $\lambda > \kappa$, there are classes $j, M \subseteq V$ s.t. (1) $j : V \xrightarrow{\kappa} M$, (2) $j(\kappa) > \lambda$, and M is sufficiently closed, or more specifically:
(3) ${}^\lambda M \subseteq M$.
- ▶ Many notions of large cardinals are obtained by modifying the definition of supercompact cardinals around the closedness condition (3), and/or by changing “for all $\lambda > \kappa$ ” to “for some $\lambda > \kappa$ ”.
- ▷ Thus, we obtain the definition of **super-almost-huge cardinal** by replacing (3) with (3)' $j(\kappa) > M \subseteq M$ in the definition of supercompactness.
- ▷ The definition of **superhuge cardinal** is obtained by replacing (3) with (3)'' $j(\kappa) M \subseteq M$ in the definition of supercompactness.

A more comprehensive list:

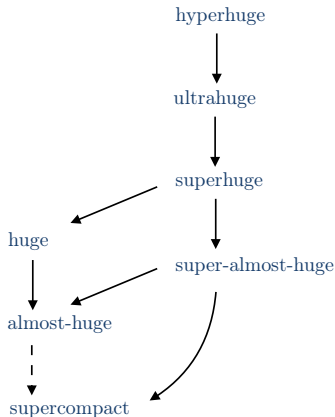
Large cardinals characterized by elementary embeddings (3/3) DID (7/35)

- ▷ A cardinal κ is said to be **supercompact** if and only if, for any $\lambda > \kappa$, there are classes $j, M \subseteq V$ s.t. (1) $j : V \xrightarrow{\lambda}_{\kappa} M$, (2) $j(\kappa) > \lambda$, and M is sufficiently closed, or more specifically:
 (3) ${}^\lambda M \subseteq M$.

- ▶ Many notions of large cardinals are obtained by modifying the definition of supercompact cardinals around the closedness condition (3), and/or by changing “for all $\lambda > \kappa$ ” to “for some $\lambda > \kappa$ ”.

	The condition (3): ${}^\lambda M \subseteq M$ replaced by	“for all $\lambda > \kappa$ ” replaced by “for some $\lambda > \kappa$ ”
hyperhuge	$j(\lambda)M \subseteq M$	-
ultrahuge	$j(\kappa)M \subseteq M$ and $V_{j(\lambda)} \in M$	-
superhuge	$j(\kappa)M \subseteq M$	-
super-almost-huge	$j(\kappa) > M \subseteq M$	-
huge	$j(\kappa)M \subseteq M$	✓
almost-huge	$j(\kappa) > M \subseteq M$	✓

- By definition:



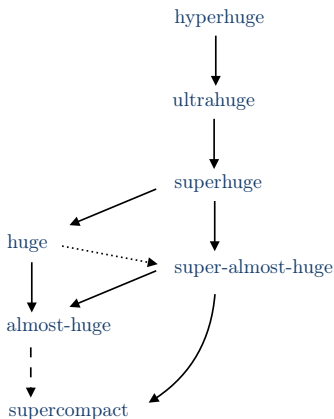
$B \leftarrow A$: “if a cardinal κ is A then κ is B .”

$B \leftarrow - A$: “if a cardinal κ is A then there are cofinally many $0 < \mu < \kappa$
s.t. μ is B in V_κ ”

Large large cardinals ordered by implication (2/2)

DID (9/35)

- ▶ Hugeness is a local property.
- ▶ Super-almost-hugeness with “for all $\lambda > \kappa \dots$ ” is rather of global character.
- ▷ This gives the impression that super-almost-hugeness might be much stronger than the hugeness in terms of consistency strength.
- ▶ However, we can show:



$B \leftarrow - A$: “if a cardinal κ is A then there are cofinally many $0 < \mu < \kappa$ μ is B in V_κ ”

$B \leftarrow \cdots A$: “if a cardinal κ is with the large cardinal property A , then there are normal measure one many λ with B in V_κ ”.

Proposition 1. Suppose that κ is huge. Then,

$$\{\alpha < \kappa : V_\kappa \models \text{“}\alpha \text{ is super almost-huge”}\}$$

is a normal measure 1 subset of κ .

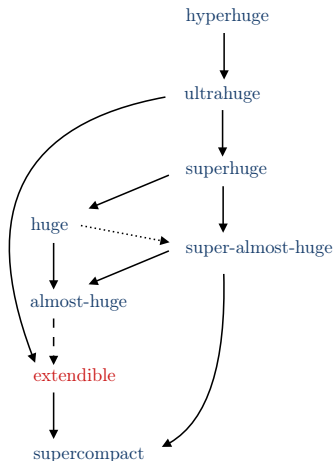
Idea of Proof: Modify Theorem 24.11 in

[kanamori] Akihiro Kanamori, The Higher Infinite, Springer Verlag (2004)

to characterize super-almost hugeness. Then solve the corresponding modification of Exercise 24.12 (see [these slides](#) for more details).

 (Proposition 1.)

	The condition (3): ${}^\lambda M \subseteq M$ replaced by	"for all $\lambda > \kappa$ " replaced by "for some $\lambda > \kappa$ "
hyperhuge	$j^{(\lambda)}M \subseteq M$	-
ultrahuge	$j^{(\kappa)}M \subseteq M$ and $V_{j^{(\lambda)}} \in M$	-
superhuge	$j^{(\kappa)}M \subseteq M$	-
super-almost-huge	$j^{(\kappa)>}M \subseteq M$	-
huge	$j^{(\kappa)}M \subseteq M$	✓
almost-huge	$j^{(\kappa)>}M \subseteq M$	✓
extendible	$V_{j^{(\lambda)}} \in M$	-



$B \leftarrow - A$: “if a cardinal κ is A then there are cofinally many $0 < \mu < \kappa$ μ is B in V_κ ”

$B \leftarrow \cdots A$: “if a cardinal κ is with the large cardinal property A , then there are normal measure one many λ with B in V_κ ”.

From large cardinals to generic large cardinals

DID (14/35)

- ▶ Small cardinals like \aleph_1 , \aleph_2 , 2^{\aleph_0} cannot be large cardinals! But they can have many features of large cardinals by being generic large cardinals.
- ▷ An important ingredient for the composition of the notion of generic large cardinal is Proposition 22.4 (b) in [kanamori].
- ▶ For a class \mathcal{P} of p.o.s, κ is said to be \mathcal{P} -generic supercompact if, for all $\lambda > \kappa$ there is $\mathbb{P} \in \mathcal{P}$ s.t. for a (V, \mathbb{P}) -generic \mathbb{G} there are $j : M \subseteq V[\mathbb{G}]$ s.t. $j : V \xrightarrow{\check{}}_{\kappa} M$, $j(\kappa) > \lambda$, and $(3)^* j''\lambda \in M$.
- ▷ The equivalence in Proposition 22.4 (b) in [kanamori] is no more valid in the generic elementary embedding context but $(3)^*$ is still a closedness property of the target model M . This fact is summarized in Lemma 3.5 in S.F., A. Ottenbreit Maschio Rodrigues, and H. Sakai [II].
- ▶ A small cardinal can be \mathcal{P} -generic large cardinal. For example, in the standard model of Proper Forcing Axiom (PFA), 2^{\aleph_0} is \mathcal{P} -generic supercompact (for $\mathcal{P} =$ proper p.o.s).

- ▶ A small cardinal can be \mathcal{P} -generic large cardinal. For example, in the standard model of Proper Forcing Axiom (PFA), 2^{\aleph_0} is \mathcal{P} -generic supercompact (for $\mathcal{P} = \text{proper p.o.s.}$).
- ▷ Similarly, in the standard model of Martin's Maximum (MM), 2^{\aleph_0} is \mathcal{P} -generic supercompact (for $\mathcal{P} = \text{semi-proper p.o.s.}$).
- ▶ Analyzing the standard models of PFA and Martin's Maximum (MM), we obtain the notion of Laver-generic large cardinal:
 - ▷ A cardinal κ is **tightly \mathcal{P} -Laver-generic supercompact** if, for any $\lambda > \kappa$, and for any $\mathbb{P} \in \mathcal{P}$, there is a \mathbb{P} -name \mathbb{Q} s.t. $\Vdash_{\mathbb{P}} \text{"}\mathbb{Q} \in \mathcal{P}\text{"}$ and for any $(\mathbb{V}, \mathbb{P} * \mathbb{Q})$ -generic \mathbb{H} , there are $j, M \subseteq \mathbb{V}[\mathbb{H}]$ s.t. $j : \mathbb{V} \xrightarrow{\sim}_{\kappa} M, j(\kappa) > \lambda, \mathbb{P}, \mathbb{P} * \mathbb{Q}, \mathbb{H} \in M, (4) j''\lambda \in M,$ and (5) $|RO(\mathbb{P} * \mathbb{Q})| \leq j(\kappa)$.

The word “tightly” refers to the condition (5).

- ▷ The \mathcal{P} -Laver-generic large cardinal axiom for the notion of supercompactness (**\mathcal{P} -LgLCA for supercompact**, for short) is the assertion that $\kappa_{\text{refl}} := \max\{2^{\aleph_0}, \aleph_2\}$ is tightly \mathcal{P} -Laver-generic supercompact cardinal.

- ▶ The condition “ $\kappa = \kappa_{\text{refl}}$ ” is (almost) a consequence of Laver-gen. supercompactness.

Proposition 2. (Theorem 5.9 in [II]) For $\mathcal{P} = \sigma$ -closed p.o.s, proper p.o.s, semi-proper p.o.s, ccc p.o.s, etc., if κ is tightly \mathcal{P} -Laver gen. supercompact then $\kappa = \kappa_{\text{refl}}$.

- ▶ Along with the hierarchy of large cardinals, we can introduce corresponding LgLCA by modifying the condition (4) in the definition of \mathcal{P} -LgLCA for supercompact.

\mathcal{P} -LgLCA for	The condition (4): $j''\lambda \in M$ is replaced by
hyperhuge	$j''j(\lambda) \in M$
ultrahuge	$j''j(\kappa) \in M$ and $V_{j(\lambda)}^{V^{\mathbb{H}}} \in M$
superhuge	$j''j(\kappa) \in M$
super-almost-huge	$j''j(\mu) \in M$ for all $\mu < j(\kappa)$

- By definition:

\mathcal{P} -LgLC for hyperhuge



\mathcal{P} -LgLC for ultrahuge



\mathcal{P} -LgLC for superhuge



\mathcal{P} -LgLC for super-almost-huge



\mathcal{P} -LgLC for supercompact

$B \longleftarrow A$: “the axiom A implies the axiom B ”

- By Theorem 5.3 in S.F., and T. Usuba [S.F. & Usuba], it follows that \mathcal{P} -LgLCA for hyperhuge, and for transfinitely iterable \mathcal{P} is equiconsistent with the existence of an hyperhuge cardinal.

Laver-generic large cardinals (3/3)

DID (18/35)

\exists a hyperhuge cardinal \longleftrightarrow \mathcal{P} -LgLC for hyperhuge

Theorem 5.3 in [S.F. & Usuba]

\mathcal{P} -LgLC for ultrahuge

\mathcal{P} -LgLC for superhuge

\mathcal{P} -LgLC for super-almost-huge

\mathcal{P} -LgLC for supercompact

$B \longleftarrow A$: “the axiom A implies the axiom B”

$B \longleftrightarrow A$: “the axioms A and B are equi-consistent.”

- \mathcal{P} -LgLCA for hyperhuge, for transitively iterable \mathcal{P} is one of only few families of strong axioms of set-theory whose **exact consistency strength is known.**

Proposition 3. (ccc-LgLCA for supercompact) For any non-free algebra A (in a universal algebraic class of structures) there is non-free subalgebra B of A of size $< 2^{\aleph_0}$.

Proof. Note that ccc-LgLCA for supercompact implies that the continuum is extremely large and hence $\kappa_{\text{refl}} = 2^{\aleph_0}$.

- ▶ Suppose toward a contradiction, that A is a non-free algebra s.t. all subalgebras of A of size $< 2^{\aleph_0}$ are free.
- ▶ Let $\lambda := 2^{|A|}$. W.l.o.g., the underlying set of A is $\mu < \lambda$. Let \mathbb{P} be a ccc p.o. adding $\lambda' \geq \lambda$ many reals and let \mathbb{Q} be a \mathbb{P} -name of a ccc p.o. s.t. for a $(V, \mathbb{P} * \mathbb{Q})$ -generic \mathbb{H} , there are j , $M \subseteq V[\mathbb{H}]$ as in the definition of ccc-LgLCA with the critical point $\kappa = 2^{\aleph_0}$.
- ▶ Then $A \in M$. Since $M \models A \leq j(A)$ and $M \models |A| < j(\kappa) = 2^{\aleph_0}$, by elementarity, it follows that $M \models A$ is free.
- ▶ On the other hand, since $\mathbb{P} * \mathbb{Q}$ is ccc, $V[\mathbb{H}] \models A$ is not free. Hence $M \models A$ is not free (see [fuchino 1992], Theorem 2.1). This is a contradiction.

□ (Proposition 3)

Proposition 4. (Cohen-LgLCA for supercompact) Any non-metrizable topological space X with character $< 2^{\aleph_0}$ has a non-metrizable subspace Y of size $< 2^{\aleph_0}$.

Proof. Similarly to Proposition 3. Using a result of Dow, Tall, and Weiss, Cohen forcing preserve non-metrizability of a topological space. \square (Proposition 3.)

Proposition 5. (1) For any σ -closed generically supercompact cardinal κ , if T is non-special tree then there is $T' \in [T]^{<\kappa}$ which is also non-special.
 (2) If σ -closed-LgLCA for supercompact holds, then Rado Conjecture (RC) holds.
 (3) If \mathcal{P} contains all ccc p.o.s, then \mathcal{P} -LgLCA for supercompact implies \neg RC.

Proof. (1),(2): Similarly to Proposition 3. Using the fact that σ -closed p.o.s preserve non-specialty of trees (Todorćević).
 (3): Since MA implies \neg RC and by Theorem 6 below. \square (Proposition 5)

Theorem 6. (Theorem 5.7 in S.F., A. Ottenbreit Maschio Rodrigues, and H. Sakai [II])
 (\mathcal{P} -LgLCA for supercompact for a stationary preserving \mathcal{P})
 $MA^{+ < \kappa_{\text{refl}}}(\mathcal{P})$ holds. □


Corollary 7. Suppose that \mathcal{P} is stationary preserving and contains all σ -closed p.o.s. Then \mathcal{P} -LgLCA for supercompact implies the Fodor-type Reflection Principle (FRP).


Proof. By Theorem 6, it follows that \mathcal{P} -LgLCA implies


$MA^+(\sigma\text{-closed})$. It is known that this principle implies FRP (See Section 2 of S.F., I.Juhász, L.Soukup, Z.Szentmiklóssy and T.Usuba, Fodor-type Reflection Principle and reflection of metrizable and meta-Lindelöfness). □ (Corollary 7)


► In contrast:

Proposition 8. FRP is independent over \mathcal{P} -LgLCA for supercompact (actually for any large cardinal property) for any class \mathcal{P} of ccc p.o.s as far as the axiom " \mathcal{P} -LgLCA for supercompact" is consistent. □

Theorem 6. (Theorem 5.7 in S.F., A. Ottenbreit Maschio Rodrigues, and H. Sakai [II])
(\mathcal{P} -LgLCA for supercompact for stationary preserving \mathcal{P})
 $MA^{+ < \kappa_{\text{refl}}}(\mathcal{P})$ holds. 

Theorem 9. ([S.F.& Usuba], see Theorem 6.1 in [S.F.&Gappo&Parente])
(\mathcal{P} -LgLCA for ultrahuge) The restricted version of Recurrence Axiom
 $(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))_{\Gamma}\text{-RcA}^+$ holds for $\Gamma =$ conjunctions of Σ_2 and Π_2 formulas. 

Theorem 10. (Theorem 7.2 in [S.F.1]) (\mathcal{P} -LgLCA for ultrahuge)
The Unbounded Resurrection Axiom for \mathcal{P} of Tsaprounis [Tsaprounis]
holds. 

Theorem 11. (Theorem 4.10 in [S.F.& Usuba]) (Super- $C^{(\infty)}$ - \mathcal{P} -LgLCA
for ultrahuge) The Maximality Principle (Hamkins [Hamkins1])
 $MP(\mathcal{P}, \mathcal{H}(\kappa_{\text{refl}}))$ holds. 

LgLCAs provide an integrated picture of axioms and principles of set theory (2/2)DID (23/35)

Theorem 12. (Theorems 5.2 and 5.3 in [S.F.& Usuba])

(\mathcal{P} -LgLCA for hyperhuge (for any \mathcal{P})) The bedrock exists and κ_{refl} is hyperhuge in the bedrock. Note that this implies $\neg\text{GA}$. \square

Theorem 13. (1) (Proposition 2.8 in [II]) Suppose that κ is \mathcal{P} -generically supercompact and all elements of \mathcal{P} are μ -cc for a cardinal μ . Then Singular Cardinal Hypothesis (SCH) above $\max\{2^{<\kappa}, \mu\}$ holds.

(2) (Corollary 5.2 in [S.F.& Usuba]) (\mathcal{P} -LgLCA for hyperhuge (for an arbitrary \mathcal{P})) There are class many huge cardinals, and SCH holds above some cardinal.

Proof. (1): A modification of the proof of Solovay's theorem on SCH above a supercompact cardinal will do.

(2): By Theorem 12.

\square (Theorem 13)

- ▶ A cardinal κ is **tightly \mathcal{P} -Laver generically extendible** if if, for any $\lambda > \kappa$, and for any $\mathbb{P} \in \mathcal{P}$, there is a \mathbb{P} -name \mathbb{Q} s.t. $\Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{P}\text{”}$ and for any $(V, \mathbb{P} * \mathbb{Q})$ -generic \mathbb{H} , there are $j, M \subseteq V[\mathbb{H}]$ s.t. $j : V \xrightarrow{\sim}_{\kappa} M, j(\kappa) > \lambda$, (1): $V_{j(\lambda)}^{V[\mathbb{H}]} \in M$, and (2): $|RO(\mathbb{P} * \mathbb{Q})| \leq j(\kappa)$.
- ▷ The \mathcal{P} -Laver-generic large cardinal axiom for the notion of extendibility (**\mathcal{P} -LgLCA for extendible**, for short) is the assertion that κ_{refl} is tightly \mathcal{P} -Laver-generic extendible cardinal.
- ▶ A cardinal κ is **tightly super- $C^{(\infty)}$ - \mathcal{P} -Laver generically extendible** if, for any $n \in \mathbb{N}$, $\lambda_0 > \kappa$, and $\mathbb{P} \in \mathcal{P}$, there are $\lambda \geq \lambda_0$ and a \mathbb{P} -name \mathbb{Q} s.t. $V_{\lambda} \prec_{\Sigma_n} V, \Vdash_{\mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{P}\text{”}$ and for any $(V, \mathbb{P} * \mathbb{Q})$ -generic \mathbb{H} , there are $j, M \subseteq V[\mathbb{H}]$ s.t. (3): $V_{j(\lambda)}^{V[\mathbb{H}]} \prec_{\Sigma_n} V[\mathbb{H}], j : V \xrightarrow{\sim}_{\kappa} M, j(\kappa) > \lambda$, (1): $V_{j(\lambda)}^{V[\mathbb{H}]} \in M$, and (2): $|RO(\mathbb{P} * \mathbb{Q})| \leq j(\kappa)$.
- ▷ The super- $C^{(\infty)}$ - \mathcal{P} -Laver-generic large cardinal axiom for the notion of extendibility (**super- $C^{(\infty)}$ - \mathcal{P} -LgLCA for extendible**, for short) is the assertion that κ_{refl} is tightly super $C^{(\infty)}$ - \mathcal{P} -Laver-generic extendible cardinal.

LgLCAs for extendible (2/2)

DID (25/35)

- Note that, in general, “ κ is tightly super- $\mathcal{C}^{(\infty)}$ - \mathcal{P} -Laver generically extendible” is not formalizable in the language of ZF. In contrast, the axiom “super- $\mathcal{C}^{(\infty)}$ - \mathcal{P} -LgLCA for extendible” is formalizable in the language of ZF in infinitely many formulas. This is because the axiom refers to the definable cardinal κ_{refl} .

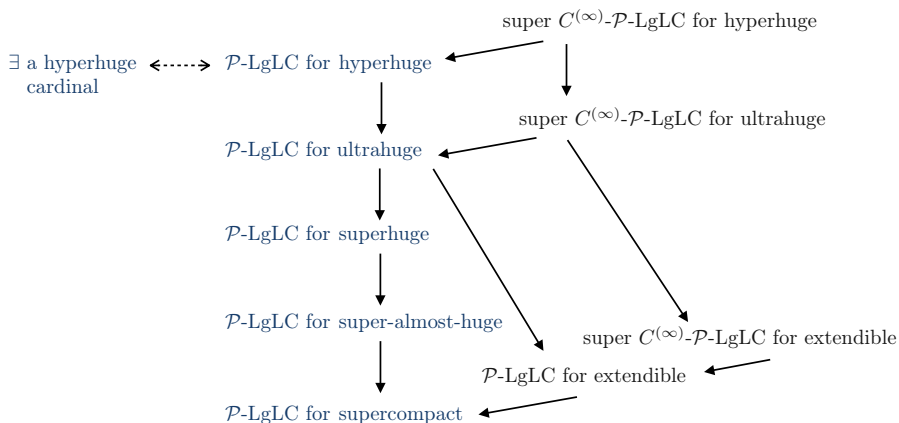
\mathcal{P} -LgLCA for	The condition (4): $j''\lambda \in M$ is replaced by
hyperhuge	$j''j(\lambda) \in M$
ultrahuge	$j''j(\kappa) \in M$ and $V_{j(\lambda)}^{V[\mathbb{H}]} \in M$
superhuge	$j''j(\kappa) \in M$
super-almost-huge	$j''j(\mu) \in M$ for all $\mu < j(\kappa)$
extendible	$V_{j(\lambda)}^{V[\mathbb{H}]} \in M$

- ▶ In Theorems 9 and 11, \mathcal{P} -LgLCA for ultrahuge, and super- $\mathcal{C}^{(\infty)}$ - \mathcal{P} -LgLCA for ultrahuge can be replaced by \mathcal{P} -LgLCA for extendible, and super- $\mathcal{C}^{(\infty)}$ - \mathcal{P} -LgLCA for extendible, respectively.
- ▶ In the proof of Theorem 10, it seems that \mathcal{P} -LgLCA for ultrahuge is used in its full strength. However we have

Theorem 14. (Theorem 7.1 in [S.F.1]) (\mathcal{P} -LgLCA for extendible)
The Boldface Resurrection Axiom for \mathcal{P} of Hamkins [Hamkins2] holds.



- ▶ \mathcal{P} -LgLCA for extendible has consistency strength below that of an extendible cardinal (see Theorem 15 below).
- ▶ Super- $\mathcal{C}^{(\infty)}$ - \mathcal{P} -LgLCA for extendible have consistency strength strictly less than that of an almost-huge cardinal (see Theorem 16).



$B \longleftarrow A$: “the axiom A implies the axiom B”

$B \longleftrightarrow A$: “the axioms A and B are equi-consistent.”

- The following theorem was suggested by Gabe Goldberg:

Theorem 15. Suppose that κ is extendible. Then for many natural classes \mathcal{P} of p.o.s consisting of stationary preserving p.o.s (including the class of all ccc p.o.s, all σ -closed p.o.s, all proper p.o.s, all semi-proper p.o.s, etc.), there is a p.o. \mathbb{P}_κ s.t.

$\Vdash_{\mathbb{P}_\kappa}$ “ $\kappa = \kappa_{\text{refl}}$ and κ is tightly \mathcal{P} -Laver generic extendible”.[2]

Lemma 15.0. If κ is extendible then there are class many measurable cardinals.

Proof. If κ is extendible then it is supercompact (Proposition 23.6 in [kanamori]). Hence, in particular κ is measurable. If $j_0 : V_\gamma \xrightarrow{\kappa} V_\delta$ with $j_0(\kappa) > \gamma$ then $V_\delta \models$ “there is a normal ultrafilter over $j_0(\kappa)$ ” by elementarity. Since the normal ultrafilter over $j_0(\kappa)$ in V_δ is really a normal ultrafilter, $j_0(\kappa)$ is measurable. \square (Lemma 15.0)

[2] The corresponding theorem for the super $\mathcal{C}^{(\infty)}$ - \mathcal{P} -Laver generic ultrahugeness can be formulated for all transfinitely iterable classes \mathcal{P} .

Lemma 15.4. An extendible cardinal κ admits a Laver-function. I.e., there is a mapping $f : \kappa \rightarrow V_\kappa$ s.t. for any x , and $\lambda > \kappa$ there are j, M s.t. $j : V \overset{\sim}{\rightarrow}_\kappa M$ s.t. $j(\kappa) > \lambda$, $V_{j(\lambda)} \in M$ and $j(f)(\kappa) = x$.^[3]

Proof. A modification of the proof of Theorem 20.21 in [Millennium book] (Th. Jech, Set Theory, The Third Millennium Edition) will do.

► Assume, toward a contradiction, that there is no Laver function $f : \kappa \rightarrow V_\kappa$.

▷ Let $\varphi(f)$ be the formula


$$\begin{aligned} \exists \underline{\alpha} \exists \underline{\delta} \exists \underline{x} (& f : \underline{\alpha} \rightarrow V_{\underline{\alpha}} \wedge \underline{\alpha} < \underline{\delta} \wedge \underline{\delta} \text{ is inaccessible} \wedge \underline{x} \in V_{\underline{\delta}} \\ & \wedge \forall \underline{\delta}' \forall \underline{j} ((\underline{j} : V_{\underline{\delta}} \overset{\sim}{\rightarrow} V_{\underline{\delta}'} \wedge \underline{j} \text{ is cofinal in } V_{\underline{\delta}'}) \rightarrow j(f)(\underline{\alpha}) \neq \underline{x}) \end{aligned}$$

▷ If $\varphi(f)$ holds then the witness of $\underline{\alpha}$ in $\varphi(f)$ is uniquely determined. In this case, let δ_f and x_f be witnesses for $\underline{\delta}$ and \underline{x} in $\varphi(x)$. Let $\mu_f := \text{rank}(x_f)$. We choose δ_f, x_f and μ_f so that δ_f minimal among the possible witnesses of $\underline{\delta}$ and x_f is chosen so that μ_f is minimal. ▷ If $\varphi(f)$ does not hold, we let $\delta_f := 0$ and $\mu_f := 0$.

Consistency proof of LgLCAs for extendible (4/6)

DID (31/35)

- ⊕ By assumption, we have $\varphi(f)$ for all $f : \kappa \rightarrow V_\kappa$.
- ▶ Let ν be an inaccessible cardinal
 $\geq \max\{\delta_f, \mu_f : f : \alpha \rightarrow V_\alpha \text{ for inaccessible } \alpha \leq \kappa\}$.
- ▷ Let $j^* : V \xrightarrow{\sim}_{\nu, \kappa} M$ be s.t. (1[†]): $j^*(\kappa) > \nu$ and (2[†]): $V_\nu \in M$.
- ▷ Let $A := \{\alpha < \kappa : \forall f (f : \alpha \rightarrow V_\alpha \rightarrow \varphi(f))\}$.
- ▶ By assumption, $V \models \text{“}\forall f ((f : \kappa \rightarrow V_\kappa) \rightarrow \varphi(f))\text{”}$. By (2[†]), it follows
- ▷ $M \models \text{“}\forall f ((f : \kappa \rightarrow V_\kappa) \rightarrow \varphi(f))\text{”}$. Thus we have
 $M \models j^*(A) \ni \kappa$.
- ▶ Let $f^* : \kappa \rightarrow V_\kappa$ be defined by
$$f^*(\alpha) := \begin{cases} x_{f^* \upharpoonright \alpha}, & \text{if } \alpha \in A; \\ \emptyset, & \text{otherwise.} \end{cases}$$
- ▶ Let $x^* := j^*(f^*)(\kappa)$. ▷ By definition of f^* , by \oplus , and since $j^*(f^*) \upharpoonright \kappa = f^*$, x^* together with δ_{f^*} and μ_{f^*} witnesses $\varphi(f^*)$.
(x^* may be different from x_{f^*} but this does not matter.)
- ▶ In particular, $x^* \neq ((j^* \upharpoonright V_{\delta_{f^*}})(f^*)(\kappa) = j^*(f^*)(\kappa))$. This is a contradiction. □ (Lemma 15.4)

^[3] Lemma 15.4 is well-known. See e.g. [corraza]. I go through the details of the proof in the present setting so that I can reuse them in the future proof of 

A (sketch of a) proof of Theorem 15: ► We show the Theorem for the case that \mathcal{P} is the class of all proper p.o.s. ► Let f be a Laver function for extendible cardinal κ (f exists by Lemma 15.2).

► Let $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ be an CS-iteration of elements of \mathcal{P} s.t.

$$\mathbb{Q}_\beta := \begin{cases} f(\beta), & \text{if } f(\beta) \text{ is a } \mathbb{P}_\beta\text{-name} \\ & \text{and } \Vdash_{\mathbb{P}_\beta} "f(\beta) \in \mathcal{P}"; \\ \mathbb{P}_\beta\text{-name of the trivial forcing,} & \text{otherwise.} \end{cases}$$

► We show that $\Vdash_{\mathbb{P}_\kappa}$ “ \mathcal{P} -LgLCA for extendible”.

▷ First, note that $\Vdash_{\mathbb{P}_\kappa}$ “ $\kappa = 2^{\aleph_0} = \kappa_{\text{refl}}$ ” by definition of \mathbb{P}_κ .

▷ Let \mathbb{G}_κ be a $(\mathbb{V}, \mathbb{P}_\kappa)$ -generic filter. In $\mathbb{V}[\mathbb{G}_\kappa]$, suppose that $\mathbb{P} \in \mathcal{P}$ and let \mathbb{P} be a \mathbb{P}_κ -name for \mathbb{P} .

▷ Suppose that $\lambda > \kappa$. By Lemma 15.0, there is an inaccessible $\lambda^* > \lambda$. Let $j : \mathbb{V} \xrightarrow{\lambda^*} M$ be s.t. (1*): $j(\kappa) > \lambda^*$, (2*): $V_{j(\lambda^*)} \in M$ and (3*): $j(f)(\kappa) = \mathbb{P}$. (This is possible since f is a Laver function for extendible κ .)

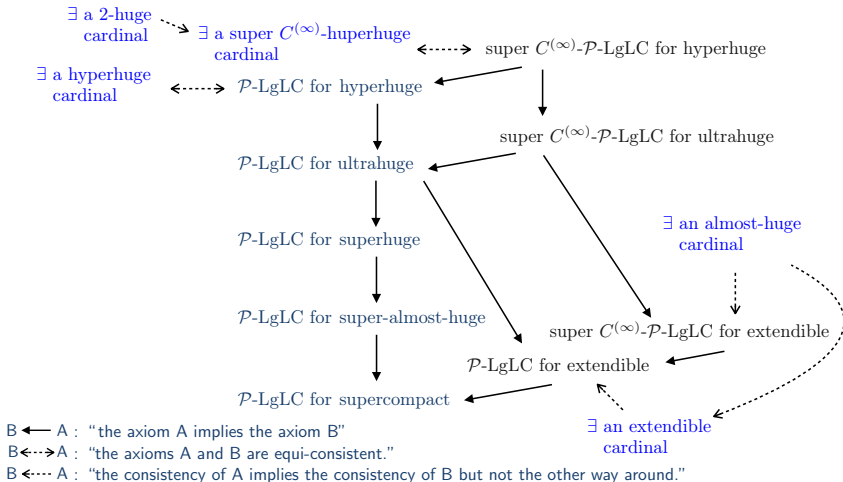
- ▶ In M , there is a $\mathbb{P}_\kappa * \mathbb{P}$ -name \mathbb{Q} s.t. $\Vdash_{\mathbb{P}_\kappa * \mathbb{P}} \text{“}\mathbb{Q} \in \mathcal{P} \text{ and } \mathbb{Q} \text{ is the direct limit of CS-iteration of small p.o.s in } \mathcal{P} \text{ of length } j(\kappa)\text{, and } \mathbb{P}_\kappa * \mathbb{P} * \mathbb{Q} \sim j(\mathbb{P}_\kappa)\text{”}$, \triangleright By (2*), the same situation holds in V .
- ▷ We have $j(\mathbb{P}_\kappa)/\mathbb{G}_\kappa \sim \mathbb{P} * \mathbb{Q}$. *Here, we are identifying \mathbb{Q} with corresponding \mathbb{P} -name.*
- ▶ Let \mathbb{H} be $(V, j(\mathbb{P}_\kappa))$ -generic filter with $\mathbb{G}_\kappa \subseteq \mathbb{H}$.
- ▶ The lifting $\tilde{j} : V[\mathbb{G}_\kappa] \xrightarrow{\sim}_\kappa M[\mathbb{H}]; \mathfrak{a}[\mathbb{G}_\kappa] \mapsto j(\mathfrak{a})[\mathbb{H}]$ witnesses that $\kappa = (\kappa_{\text{refl}})^{V[\mathbb{G}_\kappa]}$ is tightly \mathcal{P} -Laver generic extendible. For this, it suffices to show:

Claim 15.5 $V_\alpha^{V[\mathbb{H}]} \in M[\mathbb{H}]$ for all $\alpha \leq j(\lambda)$.

- ⊢ By induction on $\alpha \leq j(\lambda)$. The successor step from $\alpha < j(\lambda)$ to $\alpha + 1$ can be proved by showing that \mathbb{P}_κ -names of subsets of $V_\alpha^{V[\mathbb{H}]}$ can be chosen as elements of M . This is possible because of (2*).

⊢ (Claim 15.5.)

□ (Theorem 15)



Theorem 16. A model of super- $C^{(\infty)}$ - \mathcal{P} -LgLCA for extendible, for arbitrary tranfinitely iterable \mathcal{P} can be obtained starting from a model with a super- $C^{(\infty)}$ extendible cardinal, and this cardinal has consistency strength way below that of almost huge.



Thank you for your attention!
ご清聴ありがとうございました。