Laver generic Large Cardinal Axioms and Laver Generic Maximum

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Laver-generic Maximum

- Does there exist a family of reasonable extensions of ZFC s.t. each of such extensions proves (almost) all known consistent mathematical statements either as its consequence (and this holds "very often" for the "preferable" statements) or as a theorem holding in "many" grounds ?

- We show that strengthenings ("super-$C^{(\infty)}$-" versions) of

  Laver-generic Large Cardinal Axioms (LgLCAs, for short)

  provide such extensions.

References

[0] S.F., André Ottenbreit-M.-R., and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics II, Archive for Mathematical Logic, Vol.60, 3-4, (2021). post-print.

[1] S.F., and Toshimichi Usuba, On Recurrence Axioms, APAL, Vol.176, (10), (2025). post-print.

[2] S.F., Extendible cardinals, and Laver-generic large cardinal axioms for extendibility, preprint, arXiv.

[3] S.F., Takehiko Gappo, and Francesco Parente, Generic Absoluteness Revisited, preprint, arXiv.

Outline

Recurrence Axioms

- For a class $\mathcal{P}$ of p.o.s, a set A, and a set Γ of $\mathcal{L}_\varepsilon$-formulas, the Recurrence Axiom for $\mathcal{P}$, A, and Γ (notation: $\mathrm{RcA}_\Gamma(\mathcal{P}, A)$) is the axiom scheme:

$\mathrm{RcA}_\Gamma(\mathcal{P}, A)$: for each $\varphi = \varphi(\overline{x})$ in Γ, for any $\mathbb{P} \in \mathcal{P}$, and $\overline{a} \in A$, if $\Vdash_{\mathbb{P}}$ " $\varphi(\overline{a})$ " then there is a ground M of V s.t. $\overline{a} \in M$ and $M \models \varphi(\overline{a})$.

- $M \subseteq \mathsf{V}$ is a ground of V if it is an inner model (of ZFC) in V s.t. there is a set forcing $\mathbb{P} \in M$ and $(M, \mathbb{P})$-generic $\mathbb{G} \in \mathsf{V}$ s.t. $\mathsf{V} = M[\mathbb{G}]$.
- We drop Γ and write $\mathrm{RcA}(\mathcal{P}, A)$ if Γ is the set of all $\mathcal{L}_\varepsilon$-formulas.
- $\mathrm{RcA}(\mathcal{P}, A)$ is a weak version of Maximality Principle $\mathrm{MP}(\mathcal{P}, A)$:

$\mathrm{MP}(\mathcal{P}, A)$: For any formula φ and $\overline{a} \in A$, if there is $\mathbb{P} \in \mathcal{P}$ s.t. for any $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ " $\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$ ", we have $\Vdash_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}}$ " $\varphi(\overline{a})$ ", then $\varphi(\overline{a})$ holds in V.

Recurrence Axioms (2/2)

Lemma 0. For (two-step) iterable $\mathcal{P}$, $\mathsf{MP}(\mathcal{P}, A)$ is equivalent to

$\mathsf{RcA}^+(\mathcal{P}, A)$: for any $\mathbb{P} \in \mathcal{P}$, and $\bar{a} \in A$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\bar{a})$" then there is a $\mathcal{P}$-ground M of V s.t. $\bar{a} \in M$ and $M \models \varphi(\bar{a})$.

- ▷ $M \subseteq \mathsf{V}$ is a $\mathcal{P}$-ground of V, if it is an inner model (of ZFC) in V s.t. there is a set forcing $\mathbb{P} \in M$ with $M \models \mathbb{P} \in \mathcal{P}$, and $(M, \mathbb{P})$-generic $\mathbb{G} \in \mathsf{V}$ s.t. $\mathsf{V} = M[\mathbb{G}]$.

- ► With $\mathsf{RcA}(\mathcal{P}, A)$, or even with $\mathsf{MP}(\mathcal{P}, A)$, we realize the second-part of the objective of the talk mentioned in a previous slide.

- ▷ Thus, what we have to find are strong enough "natural" axioms which imply Recurrence Axiom or even Maximality Principle.

Laver-generic large cardinals

- For a class $\mathcal{P}$ of p.o.s, and a notion LC of large cardinal, a cardinal κ is said to be tight $\mathcal{P}$-Laver-generic LC (tight $\mathcal{P}$-Lg LC, for short), if

  for any $\lambda > \kappa$ *and* $\mathbb{P} \in \mathcal{P}$, *there is a* $\mathbb{P}$-*name* $\underset{\sim}{\mathbb{Q}}$ *with* $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" *s.t. for* $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-*generic* $\mathbb{H}$, *there are* j, $M \subseteq \mathsf{V}[\mathbb{H}]$ *with*

  (a) $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$,
  (b) $j(\kappa) > \lambda$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$,
  (c) *(tightness)* $|\,\mathsf{RO}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| \leq j(\kappa)$, *and*
  (d) M *satisfies the closedness property corresponding to* LC.

LC	Closedness property
hyperhuge	$j''j(\lambda) \in M$
ultrahuge	$j''j(\kappa) \in M$ and ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$
superhuge	$j''j(\kappa) \in M$
super-almost-huge	$j''\mu \in M$ for all $\mu < j(\kappa)$
extendible	${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$
supercompact	$j''\lambda \in M$

see [kanamori], Proposition 22.4,(b)

Laver-generic Large Cardinal Axioms (LgLCAs)

- For many ω_1 preserving "natural" classes $\mathcal{P}$ of p.o.s, the condition "κ is tight $\mathcal{P}$-Lg LC" implies $\kappa = \kappa_{\mathfrak{refl}} := \max\{\aleph_2, 2^{\aleph_0}\}$. (This is the case with σ-closed p.o.s, proper p.o.s, semi-proper p.o.s, ccc p.o.s, etc.)

- The $\mathcal{P}$-Laver-generic Large Cardinal Axiom for LC (the $\mathcal{P}$-LgLCA for LC, for short) is the axiom asserting:

  - $\kappa_{\mathfrak{refl}}$ is a tight $\mathcal{P}$-Lg LC.

Theorem 1. ([2], Theorem 4.5) If the $\mathcal{P}$-LgLCA for LC holds then elements of $\mathcal{P}$ are stationary preserving. If $\mathcal{P}$ contains a p.o. collapsing $\aleph_2$ or a p.o. adding a new real, then the $\mathcal{P}$-LgLCA for LC implies that the continuum is $\aleph_1$ or $\aleph_2$ or very large (weakly mahlo, and more). □

LgLCA for all p.o.s

▷ The Laver-generic Large Cardinal Axiom for all p.o.s for LC (the LgLCAA for LC, for short) is the axiom asserting:

- $2^{\aleph_0}$ is a tight $\mathcal{P}$-Lg LC for $\mathcal{P}^\infty :=$ all p.o.s.

Theorem 2. For transfinitely iterable stationary preserving, Σ_2-definable $\mathcal{P}$, a model of the $\mathcal{P}$-LgLCA for LC can be obtained by starting from a model with a (genuine) LC κ and iterating κ-times (with the support appropriate for $\mathcal{P}$) along with a Laver function for LC.
(see e.g. [2], Theorem 5.2)

- For $\mathcal{P}^\infty :=$ all p.o.s, LgLCAA p.o.s for LC can be obtained by starting from a model with a LC κ and iterating κ-times with FS along with a Laver function for LC. (see e.g. [2], Theorem 8.2)

Laver-generic Large Cardinal Axioms (LgLCAs) (2/3)

[0] S.F., André Ottenbreit-M.-R., and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics II, Archive for Mathematical Logic, Vol.60, 3-4, (2021). post-print.
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[2] S.F., Extendible cardinals, and Laver-generic large cardinal axioms for extendibility, preprint, arXiv.
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▷ The $\mathcal{P}$-LgLCA for supercompact implies: ▶ Chart in [2]

- $\mathrm{MA}^{**}(\mathcal{P})$ ([2], Theorem 6.4).
- Various reflection principles (see e.g. [2], Theorem 6.1, Theorem 6.8, Theorem 6.9).
- SCH (if elements of $\mathcal{P}$ satisfy μ-c.c. for some fixed small enough μ or $\mathcal{P} \supseteq$ proper p.o.s, see [0] Prop.2.8).
- $\neg$CCA ([2], p.36).

▷ The $\mathcal{P}$-LgLCA for extendible implies:

- $\mathbb{RA}^{\mathcal{P}}_{\mathcal{H}(\kappa_{\mathfrak{refl}})}$ (Resurrection Axiom, see [2], Theorem 4.2).
- $(\mathcal{P}, \mathcal{H}(\aleph_1))_\Gamma$-$\mathrm{RcA}^+$ (A fragment of Maximality Principle, [3] Theorem 6.1).
- Viale's Absoluteness Theorem for $\mathcal{P}$ ([3], Theorem 5.7).

▷ The LgLCAA for supercompact/extendible also imply corresponding principles.

- See [2], Theorem 8.3 $\sim$ 8.6.

Laver-generic Large Cardinal Axioms (LgLCAs) (3/3)

[0] S.F., André Ottenbreit-M.-R., and Hiroshi Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics II, Archive for Mathematical Logic, Vol.60, 3-4, (2021). post-print.
[1] S.F., and Toshimichi Usuba, On Recurrence Axioms, Vol.176, (10), (2025). post-print.
[2] S.F., Extendible cardinals, and Laver-generic large cardinal axioms for extendibility, preprint, arXiv.
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► Chart in [2]

▷ The $\mathcal{P}$-LgLCA for hyperhuge implies:

- The bedrock exists, and $\kappa_{\mathfrak{refl}}$ is a hyperhuge cardinal in the bedrock ([1], Theorem 5.2, Theorem 5.3).
- There are class many huge cardinals ([1], Corollary 5.4, (1)).
- SCH holds ([1], Corollary 5.4, (2)).

Super-$C^{(\infty)}$ Laver-generic Large cardinals

- For a class $\mathcal{P}$ of p.o.s, and a notion LC of large cardinal, a cardinal κ is super-$C^{(\infty)}$-$\mathcal{P}$-Laver-generic LC (super-$C^{(\infty)}$-$\mathcal{P}$-Lg LC, for short), if

  for any $n \in \omega$, $\lambda_0 > \kappa$ *and* $\mathbb{P} \in \mathcal{P}$, *there are* $\lambda \geq \lambda_0$ *with* $V_\lambda \prec_{\Sigma_n} \mathsf{V}$ *and a* $\mathbb{P}$*-name* $\underset{\sim}{\mathbb{Q}}$ *with* $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$"
  s.t. for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$*-generic* $\mathbb{H}$, *there are* j, $M \subseteq \mathsf{V}[\mathbb{H}]$ *with*
  (a) $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$,
  (b) $j(\kappa) > \lambda$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$,
  (c) *(tightness)* $|\,\mathsf{RO}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| \leq j(\kappa)$,
  (d) M *satisfies the closedness property corresponding to* LC, *and*
  (e) $V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$.

- In general, "κ is super-$C^{(\infty)}$-$\mathcal{P}$-Lg LC" is not expressible in the language of ZF since we need infinitely many formulas to express it (and since a variable cannot be shared by infinitely many formulas). However, if κ is a definable cardinal (like $2^{\aleph_0}$ or $\kappa_{\mathfrak{refl}}$), we can formulate this in an axiom scheme.

Super-$C^{(\infty)}$-Laver-generic Large Cardinal Axioms

- The super-$C^{(\infty)}$-$\mathcal{P}$-Laver-generic Large Cardinal Axiom for LC (the super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for LC, for short) is the axiom asserting:
  - $\kappa_{\mathfrak{refl}}$ is a super-$C^{(\infty)}$-$\mathcal{P}$-Lg LC.
- The super-$C^{(\infty)}$-Laver-generic Large Cardinal Axiom for all p.o.s for LC (the super-$C^{(\infty)}$-LgLCAA for LC, for short) is the axiom asserting:
  - $2^{\aleph_0}$ is a super-$C^{(\infty)}$-$\mathcal{P}^\infty$-Lg LC for $\mathcal{P}^\infty :=$ all p.o.s.

Super-$C^{(\infty)}$-Laver-generic Large Cardinal Axioms (2/2)

▷ The super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for extendible implies:

- $\mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))$ (Maximality Principle, see [1], Theorem 4.10).

▷ The super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for hyperhuge implies:

- The bedrock exists, and $\kappa_{\mathfrak{refl}}{}^{\mathsf{V}}$ is a super-$C^{(\infty)}$-hyperhuge cardinal in the bedrock ([1], Theorem 5.8).

▷ The super-$C^{(\infty)}$-LgLCAA for extendible implies:

- $\mathsf{MP}(\mathcal{P}^\infty, \mathcal{H}(\aleph_1))$ for $\mathcal{P}^\infty :=$ all p.o.s ([1], Theorem 4.10).

▷ The super-$C^{(\infty)}$-LgLCAA for hyperhuge implies:

- The bedrock exists, and $\aleph_1{}^{\mathsf{V}}$ is a super-$C^{(\infty)}$-hyperhuge cardinal in the bedrock ([1], Theorem 5.8).

Open Problem: Does the super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA or the super-$C^{(\infty)}$-LgLCAA for extendible also imply the existence of bedrock?

▷ For genuine large cardinals the assumption of a hyperhuge cardinal can be reduced to the existence of an extendible cardinal in connection with the existence of the bedrock (see [usuba1] and [usuba2]).

Super $C^{(n)}$-large cardinals and their characterizations

- For $n \in \mathbb{N}$, and a notion LC of large cardinals, we say that a cardinal κ is super $C^{(n)}$-LC, if the following holds:

> *for any $\lambda_0 > \kappa$ there is (equivalently, for all) $\lambda \geq \lambda_0$ with $V_\lambda \prec_{\Sigma_n}$ V there are j, $M \subseteq$ V s.t.*
> (a) $j : \mathsf{V} \xrightarrow{\prec}_\kappa M$,
> (b) $j(\kappa) > \lambda$, (d) M *satisfies the closedness property of* LC, *and*
> (e) $V_{j(\lambda)} \prec_{\Sigma_n} \mathsf{V}$.

- ▷ A cardinal κ is super-$C^{(\infty)}$-LC if κ is super-$C^{(n)}$-LC for all $n \in \mathbb{N}$ (or for all $n \in \omega$ if we are talking about this property in a set model).

- Similarly to the definition of the super-LgLC, "κ is super-$C^{(\infty)}$-LC" is unformalizable in general.

From a super $C^{(\infty)}$-large cardinal to the super-$C^{(\infty)}$-LgLCA

- For stationary preserving transfinitely iterable $\mathcal{P}$, inaccessible μ and $\kappa < \mu$, if
  - $V_\mu \models$ "κ is super-$C^{(\infty)}$LC with a Laver function for LC",

  then the iteration in $\mathcal{P}$ of length κ with an appropriate support along with the Laver function in V_μ creates a generic extension $V_\mu[\mathbb{H}]$ which satisfies the super-$C^{(\infty)}$- $\mathcal{P}$-LgLCA for super-$C^{(\infty)}$-LC. (see e.g. [2], Theorem 5.2)

- If μ is almost huge, there are cofinally many $\kappa < \mu$ s.t.
  - $V_\mu \models$ "κ is super-$C^{(\infty)}$extendible with a Laver function".

  (see [2], Lemma 3.1, Proposition 3.2, and Lemma 5.1)

- If μ is 2-huge, there are stationarily many $\kappa < \mu$ s.t.
  - $V_\mu \models$ "κ is super-$C^{(\infty)}$hyperhuge with a Laver function".

Super $C^{(n)}$-large cardinals and their characterizations (2/2)

- Andreas Lietz proved that super $C^{(n)}$-extendible cardinals are equivalent to $C^{(n)}$-extendible cardinals (of Joan Bagaria).

Theorem 2.6 in [2]

Open Problem: Dose a similar equivalence theorem hold for “hyperhuge” ?

The Laver-generic Maximum (2/2)

- The following axiomatic setting of set theory is one of the combinations of the axioms we considered so far:
  - ▷ ZFC + the $\mathcal{P}$-LgLCA for hyperhuge where $\mathcal{P}$ = all semi-proper p.o.s + There is a semi-proper ground W of V (in particular $\omega_1{}^{\mathsf{W}} = \omega_1{}^{\mathsf{V}}$) s.t. W $\models$ "LgLCAA for hyperhuge".
  - ▷ As it was shown in the previous slides Laver-generic Large Cardinal Axioms (LgLCAs) (2/3), (3/3), Super-$C^{(\infty)}$-Laver-generic Large Cardinal Axioms (2/2), this axiom systems various natural principles and axioms including MP($\mathcal{P}, \kappa_{\mathfrak{refl}}$) and MP($\mathcal{P}^\infty, \mathcal{H}(\aleph_1)^{\overline{W}}$) where the $\overline{W}$ is the bedrock to V.
  - ▷ In this axiom system, Martin's Maximum^{++} is a theorem while Cicho/'n's Maximum is a phenomena in many $\mathcal{P}$-grounds.
- Another alternative axiom system would be:
  - ▷ ZFC + LgLCAA for hyperhuge.

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Thank you for your attention!
ご清聴ありがとうございました.
Vielen Dank für die Aufmerksamkeit.