Set-Theoretic Multiverse

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Teoria Mnogości to ...

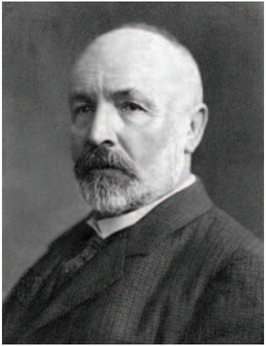
- Set Theory is a study of the (mathematical) infinity.

- It is also a study of the foundation of mathematics since (almost?) all mathematical theories we know and their proofs can be (re)formulated in the framework of the standard axioms of set theory:

  The Zermelo-Fraenkel set theory with Axiom of Choice
  abbreviated as ZFC

- Set Theory can also be a/the foundation of mathematics just because of the fact that all mathematical theories (that is, formulation of their theorems and reasoning in these theories) can be carried out in ZFC.

The beginning of the Set Theory



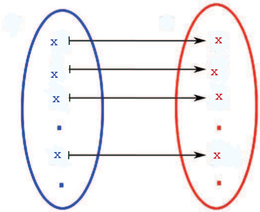
... das Wesen der Mathematik liegt gerade in ihrer Freiheit
[Cantor, 1883].

(... the essence of mathematics just lies in its freedom
[Cantor, 1883])

Georg Cantor (Saint Petersburg 1845 — 1918 Halle)

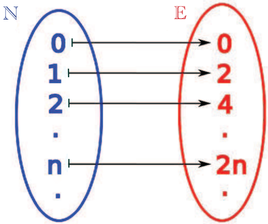
- Georg Cantor created the Set Theory around 1870.
- On December 7, 1873, Cantor found out that there are several (actually infinitely many) different "sizes" of infinitude.

Size (cardinality) of infinite sets



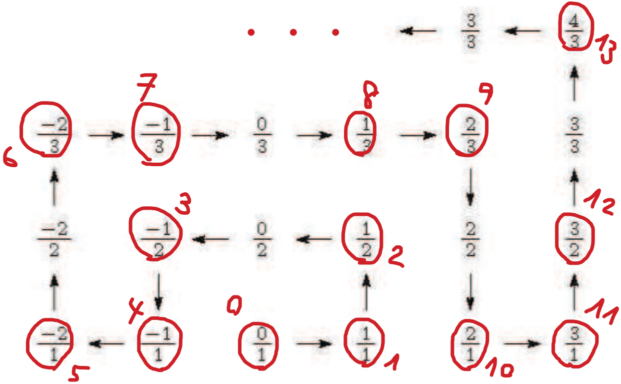
Two sets (collections of mathematical objects) are considered to be **of the same size** (**cardinality**) if there is a bijection (1-1 onto mapping) of all elements of one set to all elements of the other set.

The set $\mathbb{N}$ of all natural numbers
($\mathbb{N} = \{0, 1, 2, 3, 4, , ...\}$)
and the set $\mathbb{E}$ of all even numbers
($\mathbb{E} = \{0, 2, 4, 6, 8, ...\}$)
have the same cardinality although
$\mathbb{E}$ is a proper subset of $\mathbb{N}$ ($\mathbb{E} \subsetneq \mathbb{N}$) !!!



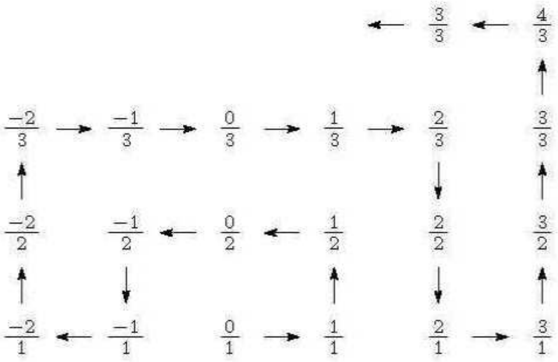
- We call a set **countable** if it is of the same cardinality with the set of all natural numbers. So the set of all even numbers is countable and a similar argument shows that the set of all odd numbers is countable as well.

Rational numbers are countable



- The examples above rather suggest that all infinite sets might be countable. But Cantor proved that this is not at all the case:

Rational numbers are countable



- The examples above rather suggest that all infinite sets might be countable. But Cantor proved that this is not at all the case:

Real numbers are uncountable

- Real numbers are the numbers which corresponds to the points on the real line. We denote with $\mathbb{R}$ the set of all real numbers.
- Cantor proved in 1873 that there can be no (1-1 onto) mapping from $\mathbb{N}$ to $\mathbb{R}$ which exhaustively enumerate real numbers.
- Suppose, toward a contradiction, that there were an enumeration of all real numbers $r_0, r_1, r_2, \ldots, r_n, \ldots \quad n \in \mathbb{N}$.

$$
\begin{array}{rl}
r_0: & 2.4161073825503356\cdots \\
r_1: & -562.4328358208955225\cdots \\
r_2: & 1.9462686567164178\cdots \\
r_3: & 0.001178224 29 \\
r_4: & -1.5490001 \\
\vdots & \quad \vdots
\end{array}
$$

- Choosing the smallest out of 1 or 2 which is different from the each of 4, 3, 6, 1, 0,..., we obtain the sequence 1, 1, 1, 2, 1 ,....
- The number $0.11121\cdots$ is different from all of $r_0, r_1, r_2, r_3, r_4, \ldots$. This is a contradiction.

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$$\begin{array}{rr}
r_0: & 2.416107382550 3356\cdots \\
r_1: & -562.4328358208955225\cdots \\
r_2: & 1.9462686567164178\cdots \\
r_3: & 0.0011782242 9 \\
r_4: & -1.5490001 \\
\vdots & \vdots
\end{array}$$

- ▷ Choosing the smallest out of 1 or 2 which is different from the each of 4, 3, 6, 1, 0,..., we obtain the sequence 1, 1, 1, 2, 1 ,....
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Continuum Hypothesis

- Cardinality of infinite sets can be also enumerated transfinitely. The smallest infinite cardinality, the cardinality of countable sets or countability, is denoted by $\aleph_0$. The next cardinality is then called $\aleph_1$, and so on. In this way we obtain a sequence of cardinalities

$$\aleph_0, \aleph_1, \aleph_2, \aleph_3, , \dots \quad \aleph_\omega, \aleph_{\omega+1}, \aleph_{\omega+2}, \dots$$

- The cardinality of the real numbers is often denoted by $2^{\aleph_0}$.
- The consideration on the last slide shows that $2^{\aleph_0} \geq \aleph_1$.
- Cantor conjectured that there is no cardinality between $\aleph_0$ and $2^{\aleph_0}$ and so the equation $2^{\aleph_0} = \aleph_1$ holds.
  - This equation is called the **Continuum Hypothesis** (Cantor himself mentioned about „Kontinuumproblem“ since he firmly believed in the validity of the equation).
- Cantor could not solve this problem and it remained unsolved until a (partial) solution was found in 1960s by Paul Cohen.

Axiomatization of Set Theory

Ernst Zermelo
(Berlin, 1871 — 1953, Freiburg)

In his 1907 paper, Zermelo proposed an axiomatization of Cantorean set theory. This system is modified and extended by some other axioms including the ones Abraham Fraenkel proposed. The final form of the axiom system based on the first order logic was established in 1930s and called now **Zermelo-Fraenkel set theory with Axiom of Choice** (ZFC).

- It was Nicolas Bourbaki who popularized the idea of set theory as the foundation of mathematics in 1950s and 1960s. Most of the members of the Bourbaki group were rather anti-logic and anti-set theory and, as a result, the roll they prescribed to set theory was not the "the study of the foundation of mathematics" but rather "the foundation of mathematics" in the sense of an introductory course of mathematics.

Consistency of the set theory

- The axiomatization of the set theory had the historical background that it is discovered at the turn of the 20th century that a careless argument in set theory leads easily to a contradiction. The set-theorists of the generation next to Cantor felt need to specify what is the correct reasoning in set theory.

- Form this point of view the consistency proof of the axiom system of set theory should be a very urgent problem. Zermelo wrote:

  Even for the very important consistency of my axioms, I cannot yet give a strict proof. [Zermelo, 1907]

However ...

Theorem 1 (The 1st Incompleteness Theorem (Gödel, Rosser 1931/1936))
For any (concretely given) formal axiom system T (over any logic) in which a large enough fragment of elementary number theory can be interpreted, if the system is consistent then it is not complete. That is, there is an assertion in the language of T which is independent from T i.e. which cannot be proved or negated from T.

Theorem 2 (The 2nd Incompleteness Theorem (Gödel 1931))
For any (concretely given) formal axiom system T (over any logic) in which a large enough fragment of elementary number theory can be interpreted, if the system T is consistent then the assertion $consis(\ulcorner\ulcorner T\urcorner\urcorner)$ in the language of the system which expresses the consistency of the system is not provable in the system itself.

- These theorems also apply to the axiom system ZFC.

There are even mathematical assertions independent from ZFC

- The independent assertion constructed in the proof of Theorem 1 is rather artificial. However we know today that there are "mathematical" natural assertions which are independent from ZFC.

Theorem 3 (Gödel, 1940)

If ZF (ZFC without Axiom of Choice) is consistent then ZFC is also consistent.

Theorem 4 (Cohen, 1963, 1964)

(1) Axiom of Choice is independent over ZF (if ZF is consistent).

(2) Continuum Hypothesis is independent over ZFC (if ZFC is consistent).

Farther examples of independence from ZF and ZFC

- The following assertions are known to be independent from ZF:
  - All vector spaces have linear basis.
  - All subsets of real numbers $\mathbb{R}$ are Lebesgue measurable.
- The following assertions are known to be independent from ZFC:
  - All sets of real numbers $\mathbb{R}$ of cardinality strictly less than continuum are null-sets.
  - There are uncountable co-analytic sets which do not contain any perfect set.
  - There are projective sets which are non-Lebesgue measurable.
  - There is a measure extending the Lebesgue measure defined for all subsets of the real numbers $\mathbb{R}$.

Consistency proofs as constructions of models of ZFC

- The proof of [Gödel, 1940] is obtained by constructing an inner model (a special kind of submodel) of a model of ZF (the universe of constructible sets denoted by L (Gödel's L)). In the consistency the Axiom of Choice is then proved by showing that L satisfies the Axiom of Choice.
- The proof in [Cohen, 1953, 1954] is done by starting from a model M of set-theory to construct so-called generic extensions $M[G_0]$, $M[G_1]$ of M which are models of the Continuum Hypothesis and the negation of the Continuum Hypothesis respectively.

Set-theoretic multiverse

- Working with the constructions of different models of set theory for independence proofs, set theorists obtain more and more the feeling that what they study in set theory are not phenomena in a single universe of set theory but rather relationships of many different universes of set theory constructed by by Gödel's and Cohen's construction methods and others.
- The standpoint that we are dealing with the class of universes of set theory is called set-theoretic multiverse and is getting attention in recent years.
- The terminology of "set-theoretic multiverse" was introduced by Hugh Woodin who is the champion of the research in Gödel's Program. Actually we can discuss about the universe among many universes of the set-theoretic multiverse which should be the model of the "correct" axioms extending ZFC.

- There are many new type of problems in set theory which become first apparent seen from the viewpoint of the set-theoretic multiverse. Two examples:

- ▷ A set theoretic assertion φ is called a button if it has the property that when, it is made true in a generic extension of a universe, then it remains true in all further generic extensions. Is it possible that all buttons are pushed in a universe (i.e. all such properties are already true in a universe without making it true in a generic extension) $\Rightarrow$ Maximality Principles of Joel Hamkins (e.g. [Hamkins, 2003])

- ▷ We call an inner model M of a universe U a ground if U is a generic extension of M. Is the intersection of all grounds (this is called the mantle by Hamkins) also a ground? $\Rightarrow$ Yes if there is a very large large cardinal (Toshinichi Usuba [Usuba, ∞]).

Summary

- The set-theoretic multiverse provides a pluralistic viewpoint to the Continuum Problem and many other independence results in set theory.

  - It also provides us a possibility to discuss about the significance of some models (and corresponding axioms of set theory) in the multiverse.

- There are many interesting set-theoretic problems which became apparent seen from the viewpoint of set-theoretic multiverse. We are possibly standing right at the beginning of an exciting new development of set theory.

Some Further readings

- "The downward directed grounds hypothesis and large large cardinals", by Toshimichi Usuba, to appear in Journal of Symbolic Logic.
- "集合論的多元宇宙" by S.F. and Toshimichi Usuba, a monograph in preparation.
- "On the set-generic multiverse", by Sy-David Friedman, S.F. and Hiroshi Sakai, National University of Singapore, Vol.33, Sets and Computations, eds.: Sy-David Friedman, Dilip Raghavan and Yue Yang, World Scientific Publishing (March, 2017), 25–44.
- "The Set-theoretic multiverse as a mathematical plenitudinous Platonism viewpoint", by S.F., Annals of the Japan Association for the Philosophy of Science, Vol.20 (2012), 49–54.

Dziękuję za uwagę.