Mathematical reflection principles in light of Laver generic Large Cardinal Axioms

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(September 17, 2026 (15:11 JST) printer version)

- **2026 年 9 月 10 日 (15:40～16:00 JST) : PPICTA2026**

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Shelah's Reflection Theorem (1978)

Theorem 1. (S. Shelah, presented in S. Ben-David [1978])
Assuming the consistency of a supercompact cardinal, the assertion:

> Every non-free algebra has a stationarily many subalgebras of cardinality $< 2^{\aleph_0}$ which are all non-free

is consistent.

- A modern view of Shelah's Reflection Theorem:

Theorem 2. (S.F. [2025])
The super-$C^{(\infty)}$-c.c.c.-LgLCA (Laver-generic Large Cardinal Axiom) for hyperhuge implies the assertion:

> Every non-free algebra has a stationarily many subalgebras of cardinality $< 2^{\aleph_0}$ which are all non-free.

- The super-$C^{(\infty)}$-c.c.c.-LgLCA for hyperhuge is an overkill. However, this axiom scheme is a member of a family of axiom schemes that decide most of known mathematical and set theoretical questions and that can be considered to be candidates of "true" extensions of ZFC.

Laver-generic Large Cardinals

Abbrv.: super- $C^{(\infty)}$-$\mathcal{P}$-Lg. hyperhuge

- For a class $\mathcal{P}$ of p.o.s, a cardinal κ is super-$C^{(\infty)}$-$\mathcal{P}$-Laver generic hyperhuge if

  For all $n \in \mathbb{N}$, for any $\lambda_0 > \kappa$ and p.o. $\mathbb{P} \in \mathcal{P}$, there are $\lambda > \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$ (i.e. $\lambda \in (C^{(n)})^{\mathsf{V}}$), and a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of a p.o. s.t. $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", and for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t.

  (a) $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$,
  (b) $j(\kappa) > \lambda$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$,
  (c) (tightness) $|\mathsf{RO}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})| \leq j(\kappa)$, and
  (d) $j''j(\lambda) \in M$ (the closure property corresponding to hyperhugeness),
  (e) ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$ (i.e. $j(\lambda) \in (C^{(n)})^{\mathsf{V}[\mathbb{H}]}$). ("super-$C^{(\infty)}$" feature)

- In general, the definition of super-$C^{(\infty)}$- $\mathcal{P}$-Lg. hyperhuge cardinal is not formalizable in the language of ZFC. However for most of $\mathcal{P}$, super-$C^{(\infty)}$-$\mathcal{P}$-Lg. hyperhuge cardinal κ without "super-$C^{(\infty)}$" feature (formalizable in ZFC!) is a definable cardinal (e.g. as $\kappa_{\mathfrak{refl}}$ on the next slide). So the existence of a super-$C^{(\infty)}$-$\mathcal{P}$-Lg. hyperhuge cardinal in such a case is formalizable in infinitely many formulas.

Laver generic Maximums

- The following are axioms (axiom schemes more exactly) claiming the existence of a Laver-generic large cardinal (Laver-generic Large Cardinal Axioms, Abbrv.: LgLCAs) maximal in terms of the amount of Recurrence Axiom phenomena they imply[1] .
  - $\kappa_{\mathfrak{refl}} := \max\{\aleph_2, 2^{\aleph_0}\}$.

(A) (super-$C^{(\infty)}$-subcomplete-LgLCA for hyperhuge) : $\kappa_{\mathfrak{refl}}$ is super-$C^{(\infty)}$-$\mathcal{P}$-Lg. hyperhuge for $\mathcal{P}$ = subcomplete p.o.s (generalization of σ-closed p.o.s) (maximal w.r.t. p.o.s adding no reals. This implies CH)

(B) (super-$C^{(\infty)}$-semiproper-LgLCA for hyperhuge) : $\kappa_{\mathfrak{refl}}$ is super-$C^{(\infty)}$-$\mathcal{P}$-Lg. hyperhuge for $\mathcal{P}$ = all semiproper p.o.s (maximal w.r.t. p.o.s preserving stationary subsets of ω_1. This implies $2^{\aleph_0} = \aleph_2$)

(Γ) (super-$C^{(\infty)}$-c.c.c.-LgLCA for hyperhuge) : $\kappa_{\mathfrak{refl}}$ is super-$C^{(\infty)}$-$\mathcal{P}$-Lg. hyperhuge for $\mathcal{P}$ = c.c.c. p.o.s (maximal w.r.t. p.o.s preserving all cardinals and cofinalities. This implies that the continuum is extremely large (e.g. weakly hyper Mahlo, and more) — This is the assumption of Theorem 2)

[1] S.F., and Usuba, On Recurrence Axioms, Annals of Pure and Applied Logic, Vol.176, (10), (2025).

Laver generic Maximums (2/3)

- Axioms (A), (B), (Γ) can be seen as generalizations of the following Reflection Statement (in ZFC) :

For any uncountable set X there is an uncountable subset Y of cardinality $< \kappa_{\mathfrak{refl}}$.

Theorem 3. (1) (Lemma 2.5, (1) in S.F. [2]) If $\kappa_{\mathfrak{refl}}$ is tightly $\mathcal{P}$-Lg. extendible (this is much weaker than our assumption), then $\mathcal{P}$ is stationary preserving.

(2) (Theorem 5.9, (1),(2) in S.F., et al. [3]) If (A) or (B) holds then the Strong Downward Löwenheim-Skolem Theorem down to $< \kappa_{\mathfrak{refl}}$ holds for the stationary logic $\mathcal{L}^{\aleph_0}_{stat}$ (i.e. $\mathsf{SDLS}(\mathcal{L}^{\aleph_0}_{stat}, < \kappa_{\mathfrak{refl}})$) holds.

(3) (Theorem 5.9, (3) in S.F., et al. [3]) If (Γ) holds, then the Strong Downward Löwenheim-Skolem Theorem down to $< \kappa_{\mathfrak{refl}}$ holds for the internal interpretation of stationary logic $\mathcal{L}^{\aleph_0}_{stat}$ (i.e. $\mathsf{SDLS}^{int}(\mathcal{L}^{\aleph_0}_{stat}, < \kappa_{\mathfrak{refl}})$) holds.

[2] S.F., Maximality and resurrection in light of Laver-generic large cardinal axioms, preprint.

[3] S.F., A.Ottenbreit Maschio Rodrigues, and H.Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, II, Arch. for Math. Logic, Vol.60, (2021), 495–532.

Laver generic Maximums (3/3)

- The following is also considered as an instance of Laver genericity maximal in terms of Recurrence Axiom.

(Δ) (super-$C^{(\infty)}$-LgLCAA for hyperhuge) : $2^{\aleph_0}$ is super-$C^{(\infty)}$-$\mathcal{P}$-Lg. hyperhuge for $\mathcal{P} =$ all p.o.s (unconditional maximality but the parameter set allowed for the Recurrence Axiom is $\mathcal{H}(\omega_1)$; This implies CH).

- Models of each of (A), (B), (Γ), and (Δ) can be constructed starting from "ZF + there is a 2-huge cardinal" (Lemma 4.5 and Lemma 4.7 in S.F., and Usuba [1]). Chart of Large Cardinals

- In the following slides, we see how (A), (B), (Γ), and (Δ) decide various mathematical reflection statements.

Singular Cardinal Hypothesis (SCH)

- SCH is a (topological) Reflection Property (see e.g. Corollary 2.5 in [4]).

Theorem 4. Each of (A), (B), (Γ), (Δ) implies SCH.

- ▷ More generally, we have the following. Note that "super-$C^{(\infty)}$-$\mathcal{P}$-Lg. hyperhuge" is a stronger condition than "tightly $\mathcal{P}$-Lg. hyperhuge".

Theorem 5. (Corollary 5.4 in S.F., and Usuba [1]) If there is a cardinal κ which is tightly $\mathcal{P}$-Lg. hyperhuge for some class $\mathcal{P}$ of p.o.s, then SCH above κ holds.

Proof. If κ is tightly $\mathcal{P}$-Lg. hyperhuge, then the bedrock $\overline{\mathsf{W}}$ exists, $\overline{\mathsf{W}}$ is a $\leq\kappa$-ground of V, and κ is hyperhuge in $\overline{\mathsf{W}}$ (Theorem 5.2 and Theorem 5.3 in S.F., and Usuba [1]). Hence, in $\overline{\mathsf{W}}$, SCH holds above κ. Since $\overline{\mathsf{W}}$ is a $\leq\kappa$-ground of V, SCH above κ is preserved in V. □ (Theorem 5)

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[4] S.F., and A.Rinot, Openly generated Boolean algebras and the Fodor-type Reflection Principle, Fundamenta Mathematicae 212, (2011), 261–283.

Fodor-type Reflection Principle (FRP)

- FRP is a (purely combinatorial) Reflection Principle which can be reformulated as:

  For any locally countably compact topological space X if X is not metrizable, then there is a subspace Y of X of cardinality $< \aleph_2$ which is neither metrizable.

Theorem 6. (Proposition 6.7 in [5]) $(\,1\,)$ Each of (A) and (B) implies FRP.
$(\,2\,)$ FRP is independent over $(\,\Gamma\,)$.

Proof. (1): Each of (A) and (B) implies $\mathsf{MA}^+(\sigma\text{-closed})$ (Theorem 5.7, (1) in S.F., A.Ottenbreit Maschio Rodrigues, and Sakai [3]).

▷ $\mathsf{MA}^+(\sigma\text{-closed})$ implies FRP (Corollary 2.6, in [6]).

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[5] S.F., Extendible cardinals, and Laver-generic large cardinal axioms for extendibility, preprint.

[6] S.F., István Juhász, Lajos Soukup, Zoltán Szentmiklóssy and Toshimichi Usuba, Fodor-type Reflection Principle and reflection of metrizability and meta-Lindelöfness, Topology and its Applications, Vol.157, 8 (June 2010), 1415–1429.

Fodor-type Reflection Principle (FRP) (2/2)

(2): In a (set-)model of ZFC $+$ there is a super-$C^{(\infty)}$-hyperhuge cardinal κ, force FRP using the first supercompact in the model.

- Then force in such a way that κ becomes super-$C^{(\infty)}$-c.c.c.-Lg. hyperhuge by a c.c.c. p.o..
- ▷ FRP survives the second extension. (Theorem 3.4 in [6])
- To prove the consistency of the negation of FRP, start from the same model as above, and force $\square_\mu$ for some $\mu < \kappa$ by a small forcing.
- ▷ If we force now s.t. κ becomes super-$C^{(\infty)}$-Lg. hyperhuge by a c.c.c. p.o., the $\square_\mu$ sequence created in the first generic extension survives.

(Theorem 6)

Suslin Hypothesis (SH)

- Suslin Hypothesis can be also regarded as a reflection statement (about the height of c.c.c. trees).

Theorem 7. (Proposition 6.11 in [5]) (1) Each of (B) and (Γ) implies SH. (2) Each of (A) and (Δ) implies $\neg$SH.

Proof. (1): Each of (B) and (Γ) implies $\mathsf{MA}(\omega_1)$ (Theorem 5.7, (1),(2) in S.F., A.Ottenbreit Maschio Rodrigues, and Sakai [3]).

(2): (A) implies $\diamondsuit_{\omega_1}$: Assume (A). $\diamondsuit_{\omega_1}$ can be forced by a σ-closed (and hence subcomplete) forcing. $\diamondsuit_{\omega_1}$ is preserved by subcomplete forcing. Since (A) implies the Maximality Principle $\mathsf{MP}(\text{subcomplete}, \mathcal{H}(\aleph_2))$ (Theorem 4.10 in S.F., and Usuba [1]). It follows that $\mathsf{V} \models \diamondsuit_{\omega_1}$.

- (Γ) implies $\diamondsuit_{\omega_1}$: (Γ) implies $\diamondsuit^{\mathcal{P},\mathit{extendible}}_{\mathit{Laver},\omega_1}$ (Proposition 4.6 in [7])
- $\diamondsuit^{\mathcal{P},\mathit{extendible}}_{\mathit{Laver},\omega_1}$ implies $\diamondsuit_{\omega_1}$ (Proposition 4.5 in [7]). $\square$ (Theorem 7)

[7] S.F., Two further aspects of Laver-generic Large Cardinal Axioms, RIMS Kôkûroku, No.2349, Recent Developments in Axiomatic Set Theory,(2026), 1–31.

Suslin Hypothesis (SH) (2/2)

Proposition 8. (Proposition 6.11, (2) in [5]) Suppose that $\mathcal{P}$ is the class of all productively ccc posets containing Cohen posets. Then the super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for hyperhuge implies that continuum is very large and $\neg$SH holds.

Problem 9. (H.Sakai) Is there a class $\mathcal{P}$ of subcomplete p.o.s s.t. the super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for hyperhuge implies $\neg$SH ?

- Note that the super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for hyperhuge for $\mathcal{P}$ as in Problem 9 implies CH.

Kurepa Hypothesis (KH)

- Recall that an ω_1-tree T is a Kurepa tree if it has $\geq \aleph_2$ (uncountable) branches.
  - We say that Kurepa Hypothesis (KH) holds if there is a Kurepa tree.

Theorem 10. $(\,1\,)$ (Proposition 6.8 (1) in [5]) $(\,\Delta\,)$ implies KH.
$(\,2\,)$ (Proposition 6.8 (3) (4) in [5]) (B) and $(\,\Gamma\,)$ imply $\neg$KH.

Proposition 11. (Proposition 6.8, (2) in [5]) If $\mathcal{P} =$ all σ-closed p.o.s, then the super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for hyperhuge implies KH.

Problem 12. Does $(\,\mathrm{A}\,)$ imply KH ?

Katowice Problem (KP)

(KP) $\mathcal{P}(\omega)/fin \cong \mathcal{P}(\omega_1)/fin \quad (\beta\omega \setminus \omega \cong \beta\omega_1 \setminus \omega_1)$.

- Katowice Problem asks the consistency of KP.

Lemma 13. CH implies $\neg$KP.

Theorem 14. (Balcar and Frankiewicz) (1) For cardinals any $\kappa < \lambda$ with $(\kappa, \lambda) \neq (\omega, \omega_1)$, we have $\mathcal{P}(\kappa)/fin \not\cong \mathcal{P}(\lambda)/fin$ (in ZFC).
(2) KP implies $\mathfrak{b} = \mathfrak{d} = \aleph_1$.

Proposition 15. Each of (A),(B),(Γ),(Δ) implies $\neg$KP.

Proof. Both of (A) and (Δ) imply CH (see e.g. Lemma 2.5, (2) and $(2')$ in [2]). Thus $\neg$KP follows by Lemma 13.

- Both of (B) and (Γ) imply $\mathfrak{b} > \aleph_1$ (e.g. by MA($\aleph_1$)). Thus $\neg$KP follows by Theorem 14, (2). □ (Proposition 15)

- Does Proposition 15 point to the possibility that $\neg$KP is a theorem in ZFC ?

Total failure of square principles

- Square principles create counter-examples to various structural reflection statements. Thus the total failure of square principes is a consequence of such structural reflection statements, and we may view the total failure of the square as a (very weak) reflection principle.

Theorem 16. (1) Each of (A) and (B) implies $\neg\square_\kappa$ for all uncountable cardinals κ.

(2) (Γ) implies that there is $\kappa_0 < 2^{\aleph_0}$ s.t. $\neg\square_\kappa$ holds for all $\kappa > \kappa_0$.

(3) (Δ) implies that there are only set many κ s.t. $\square_\kappa$ holds.

Proof. (1): This holds since both of (A) and (B) imply $\mathsf{MA}^+(\sigma\text{-closed})$.

Total failure of square principles (2/2)

(2): The super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for hyperhuge for any $\mathcal{P}$, and hence $(\,\Gamma\,)$ in particular, implies that there are extendible cardinals (Theorem 5.8 and Theorem 4.2 in S.F., and Usuba [1]).

▷ Since extendible cardinals are supercompact, it follows that we have the total failure of the square principles above some cardinal κ^* by a Theorem by Solovay. Since square principles survive c.c.c. generic extension, we have

$$\Vdash_{\mathbb{P}} \text{“ there is } \kappa_0 < 2^{\aleph_0} \text{ s.t. we have the total failure of the square above } \kappa_0\text{”.}$$

for a c.c.c. p.o. $\mathbb{P}$ forcing the continuun larger than κ^*.

▷ By the Maximality Principle available under the strong version of LgLCA, it follows that there is a c.c.c.-ground of V satisfying the statement of the tota failure. Thus V, as a c.c.c. generic extension of the ground, also satisfies the statement.

(3): Similarly to (2), the super-$C^{(\infty)}$-LgLCAA for hyperhuge implies that there is a supecompact cardinal. Hence all the cardinal κ with $\square_\kappa$ are below this supercompact cardinal by the Theorem by Solovay.

□ (Theorem 16)

Thank you for your attention!

ご清聴ありがとうございました.

Vielen Dank für die Aufmerksamkeit.

Grazie mille per l’attenzione.

Muchas gracias por la atención.

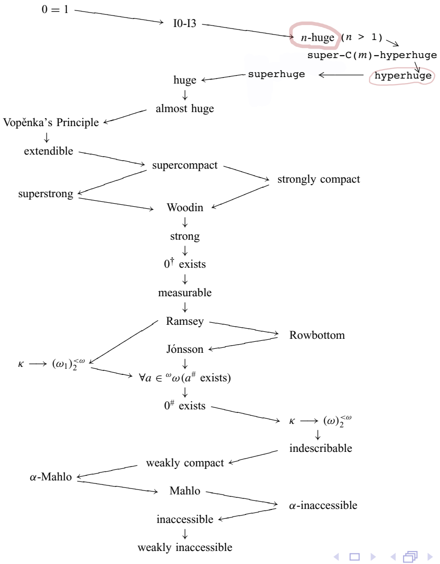
경청해 주셔서 감사합니다

Bardzo dziękuję za uwagę.

謝謝你的傾聽

Chart of Cardinals in Akihiro Kanamori: "The Higher Infinite" (extended)

The arrows indicate direct implications or relative consistency implications, often both.



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