Rado's Conjecture and Hamburger's Hypothesis

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Rado Conjecture (RC)

(RC): For any tree T, if T is not special, then there is a non-special subtree T_0 of T of size $< \aleph_2$.

- A tree T is special if T is the union $T = \bigcup_{n\in\omega} T_n$ where each T_n is pairwise incomparable (or antichain in tree terminology).
- Note that T_0 as above must be of size $= \aleph_1$ since any countable tree is special.
- Note also that the assertion of RC holds, if T has height $> \omega_1$.

Proposition 1. If ω_2 is generically supercompact by σ-closed p.o.s, then RC holds. In particular, the consistency of RC follows from the existence of a supercompact cardinal (actually a strongly compact cardinal is enough to prove the consistency of RC).

- We shall see a proof of a more general assertion later.

Rado Conjecture (RC) (2/3)

- The relation of RC to other principles:

Theorem 1. (S. Todorčević, 1993) RC implies Chang's Conjecture (CC) and Singular Cardinal Hypothesis (SCH).

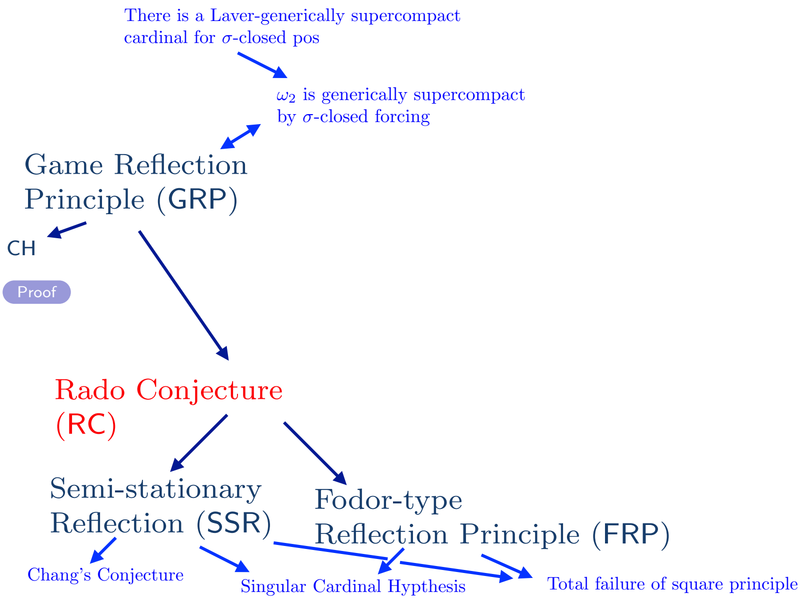
Theorem 2. (Ph. Doebler, 2013) RC implies Semi-Stationary Reflection Principle (SSR).

Theorem 3. (S.F, H. Sakai, V.Torres-Perez, T.Usuba) RC implies Fodor-type Reflection Principle (FRP).

Theorem 4. (B.König, 2004) The condition "ω_2 is a generically supercompact cardinal by σ-closed p.o.s" can be characterized as a reflection statement on non-existence of winning strategy for the second player of certain game (Game Reflection Principle (GRP)).

- By the Theorem on the previous slide, GRP implies RC.

Rado Conjecture (RC) (3/3)



Hamburger's Hypothesis (HH)

- The consistency of the following statement is still open:

(HH): For any topological space X with $\chi(x, X) \leq \aleph_0$ for all $x \in X$, if X is non-metrizable, then there is a non-metrizable subspace Y of X of cardinality $< \aleph_2$.

- The following is a theorem in ZFC:

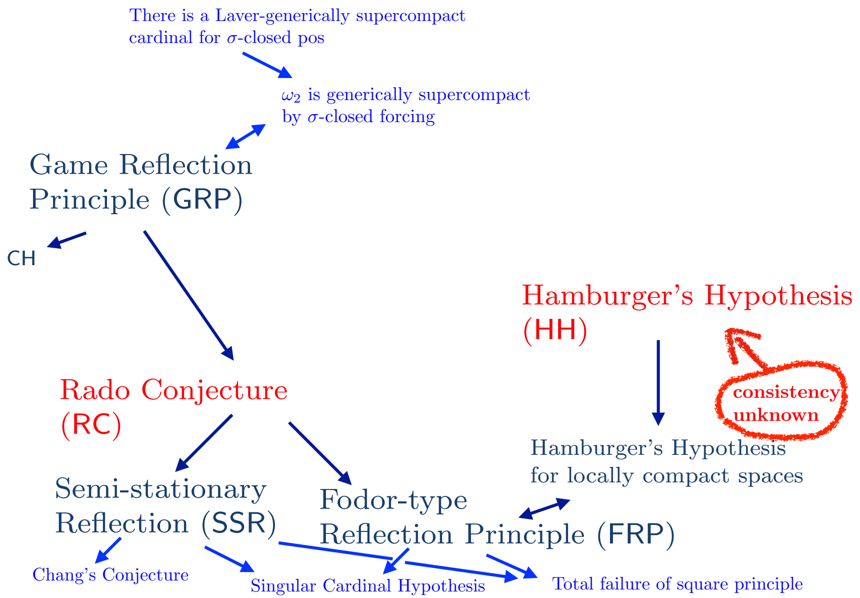
Theorem 5. (A. Dow) If X is a non-metrizable compact space then there is a non-metrizable subspace Y of X of cardinality $< \aleph_2$.

- The following statement is shown to be equivalent (over ZFC) to the Fodor-type Reflection Principle (FRP):

▷ If X is a non-metrizable locally compact space then there is a subspace Y of X of cardinality $< \aleph_2$ s.t. Y is also non-metrizable.

▷ Z. Balogh proved that the statement follows form Axiom R.
[S.F., Juhász, Soukup, Szentmiklóssy, Usuba] and [S.F., Sakai, Torres-Perez, Usuba] show the equivalence.

Hamburger's Hypothesis (HH) (2/2)



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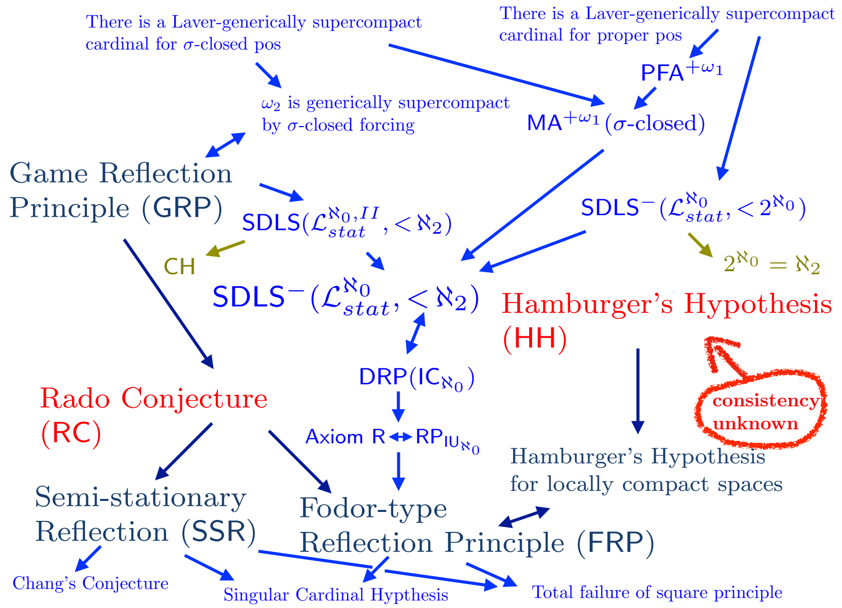

Reflection principles with higher reflection points

- We also want to consider reflection principles with the reflection point $< 2^{\aleph_0}$ or $\leq 2^{\aleph_0}$ (i.e. $< (2^{\aleph_0})^+$). More generally, for a cardinal κ, let

(RC($< \kappa$)): For any tree T, if T is not special, then there is a non-special subtree T_0 of T of size $< \kappa$.

(HH($< \kappa$)): For any topological space X with $\chi(x, X) < \kappa$ for all $x \in X$, if X is non-metrizable, then there is a non-metrizable subspace Y of X of cardinality $< \kappa$.

Lemma 6. (Hajnal-Juhász, 1976) HH($< \kappa$) is equivalent to the following seemingly weaker reflection principle. In particular, HH($< \aleph_2$) is equivalent to HH:

- For any topological space X with $\chi(x, X) \leq \aleph_0$ for all $x \in X$, if X is non-metrizable, then there is a non-metrizable subspace Y of X of cardinality $< \kappa$.

Reflection principles with higher reflection points (2/2)

- A p.o. $\mathbb{P}$ preserves the non-metrizability, if, for any non-metrizable topological space $X = \langle X, \tau \rangle$, we have $\Vdash_{\mathbb{P}}$ "X is non-metrizable". A property $\mathcal{P}$ of p.o.s preserves the non-metrizability, if $\mathbb{P}$ preserves the non-metrizability for all $\mathbb{P} \models \mathcal{P}$.

- A p.o. $\mathbb{P}$ preserves non-specialty, if, for any non-special tree T, we have $\Vdash_{\mathbb{P}}$ "T is non-special". A property $\mathcal{P}$ of p.o.s preserves the non-specialty, if $\mathbb{P}$ preserves the non-specialty for all $\mathbb{P} \models \mathcal{P}$.

Proposition 7. Suppose that κ is generically supercompact by $\mathcal{P}$.

(1) If $\mathcal{P}$ preserves non-metrizability, then $\mathsf{HH}(<\kappa)$ holds.

(2) If $\mathcal{P}$ preserves non-specialty, then $\mathsf{RC}(<\kappa)$ holds.

Proof.

Preservation and non-preservation

Proposition 8. (1) (Todorčević) σ-closed p.o.s preserve non-specialty.

(2) σ-centered p.o.s preserve non-specialty.

(3) FS-iterations of σ-centered p.o.s preserve non-specialty.

(4) (Todorčević) There is a ccc p.o. which does not preserve non-specialty. □

- The example ccc p.o. for the proof (4) above can be used to show the following:

Proposition 9. (Todorčević) $\mathrm{RC}(<\kappa)$ implies $\mathfrak{ma} < \kappa$.

Preservation and non-preservation (2/2)

Theorem 10. (1) (Dow, Tall, and Weiss) Generalized Cohen forcing (for adding multiple Cohen reals) preserves non-metrizability.

(2) (van Douwen) Hechler forcing does not preserve non-metrizability. □

- The topological space constructed in the proof of Theorem 2, (2) also shows the following:

Lemma 11. (van Douwen) HH($<\kappa$) implies $\mathfrak{b} < \kappa$. □

Consistency results

Theorem 12. $\mathsf{RC}(<2^{\aleph_0}) + \mathsf{HH}(<2^{\aleph_0})$ is consistent (modulo a supercompact cardinal).

Proof. Let κ be a supercompact cardinal, $\mathbb{P} = \mathrm{Fn}(\kappa, 2)$, and $\mathbb{G}$ a $(\mathsf{V}, \mathbb{P})$-generic set. Then, in $\mathsf{V}[\mathbb{G}]$, we have $\kappa = 2^{\aleph_0}$ and κ is generically supercompact for $\{\mathrm{Fn}(\lambda, 2) : \lambda \in \mathsf{On}\}$.

- By Proposition 8, (3) and Theorem 10, (1), it follows that

$$\mathsf{V}[\mathbb{G}] \models \mathsf{RC}(<2^{\aleph_0}) \ + \ \mathsf{HH}(<2^{\aleph_0}).$$ □

Theorem 13. $\mathsf{RC}(<2^{\aleph_0}) + \neg\mathsf{HH}(<2^{\aleph_0})$ is consistent (modulo a supercompact cardinal).

Proof. Superompact long FS-iteration of Heckler forcing will do (see Proposition 8, (3) and Lemma 11). □

Consistency results (2/2)

Theorem 14. $\neg\mathrm{RC}(<2^{\aleph_0}) + \neg\mathrm{HH}(<2^{\aleph_0})$ is consistent with the continuum being Laver-generically supercompact for ccc p.o.s.

Proof. If $2^{\aleph_0}$ is Laver-generically supercompact for ccc p.o.s, then MA holds. □

- Existence of a Laver-generically supercompact for ccc p.o.s implies that the continuum is fairly large.

Mixed support iteration

- The consistency results in the previous slides can be still strengthened by adding the consistency of the following principles. The proof is done by Mixed support iteration of supercompact lenth along with a Laver function with a preparatory iteration.
  - $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{\aleph_0}_{stat}, < 2^{\aleph_0})$, $\mathsf{GRP}^{<2^{\aleph_0}}(\leq 2^{\aleph_0})$,
  - $\mathsf{SDLS}^{int}_{+}(\mathcal{L}^{PKL}_{stat}, < 2^{\aleph_0})$;
  - A certain fragment of $\mathsf{MA}^{++}(ccc)$.

Some Open Prblems:

- Hamburger's Problem.
- Consistency of $\mathsf{HH}(< 2^{\aleph_0}) + \neg\mathsf{RC}(< 2^{\aleph_0})$.
- Cardinal invariants under $\mathsf{HH}(< 2^{\aleph_0})$ or $\mathsf{RC}(< 2^{\aleph_0})$.

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Thank you for your attention!
ご清聴ありがとうございました.
Muchas gracias por su atención.


asante sana kwa umakini wako
끝까지 들어 주셔서 감사합니다.
谢谢您的倾听。
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Laver generically large cardinals

- A cardinal κ is said to be a Laver-generically supercompact for $\mathcal{P}$ for a property $\mathcal{P}$ of p.o.s, if, for any $\lambda \geq \kappa$ and any $\mathbb{P} \models \mathcal{P}$, there are a p.o. $\mathbb{Q} \models \mathcal{P}$ with $\mathbb{P} \lessdot \mathbb{Q}$ and $(\mathsf{V}, \mathbb{Q})$-generic filter $\mathbb{H}$ s.t. there are classes $M, j \subseteq \mathsf{V}[\mathbb{H}]$ with
  - M is an inner model of $\mathsf{V}[\mathbb{G}]$ and $j : \mathsf{V} \overset{\preccurlyeq}{\rightarrow} M$,
  - $crit(j) = \kappa$,
  - $j(\kappa) > \lambda$,
  - $\mathbb{P}$, $\mathbb{H} \in M$ and
  - $j''\lambda \in M$.

Back to the 1.diag.

Back to the 2.diag.

Back to the main diag.

Proof of Proposition 7.

- We prove (1). (2) is shown similarly. We have to show the following:

  • Suppose that κ is generically supercompact by $\mathcal{P}$ where $\mathcal{P}$ preserves non-metrizability. Then $\mathsf{HH}(<\kappa)$ holds.

  Proof. Let $\langle X, \tau \rangle$ be a non-metrizable space with (†) $\chi(x, X) < \kappa$ for all $x \in X$. W.l.o.g., we may assume that $X = \lambda$ for a cardinal λ and (‡) $|\,\tau\,| \leq |\,X\,|$.

- Let $\mathbb{P} \models \mathcal{P}$ and $(V, \mathbb{P})$-generic $\mathbb{G}$ be s.t. there are classes j, $M \in \mathsf{V}[\mathbb{G}]$ s.t. (1) M is an inner model of $\mathsf{V}[\mathbb{G}]$ and $j : \mathsf{V} \overset{\preccurlyeq}{\to} M$, (2) $crit(j) = \kappa$, (3) $j(\kappa) > \lambda$, and (4) $j''\lambda \in M$.

- Let $X^* = j''X$ and $\tau^* = \{j''O \,:\, O \in \tau\}$. By (4) and (‡), $\langle X^*, \tau^* \rangle \in M$. By (†), $\langle X^*, \tau^* \rangle$ is a subspace of $\langle j(X), j(\tau) \rangle$.

- Since $\langle X^*, \tau^* \rangle \cong \langle X, \tau \rangle$ in $\mathsf{V}[\mathbb{G}]$,

  $$\mathsf{V}[\mathbb{G}] \models \text{“} \langle X^*, \tau^* \rangle \text{ is non-metrizable”}$$

  by $\mathbb{P} \models \mathcal{P}$.

Proof of Proposition 1. (2/2)

- It follows that

  $M \models$ "there is a non-metrizable subspace of $j(X)$ of size $< j(\kappa)$".

- By elementarity, it follows that

  $V \models$ "there is a non-metrizable subspace of X of size $< \kappa$".

$\square$

Back

Fodor-type Reflection Principle (FRP)

(FRP) For any regular $\kappa > \omega_1$, any stationary $E \subseteq E^\kappa_\omega$ and any mapping $g : E \to [\kappa]^{\aleph_0}$ with $g(\alpha) \subseteq \alpha$ for all $\alpha \in E$, there is $\gamma \in E^\kappa_{\omega_1}$ s.t.

(*) for any $I \in [\gamma]^{\aleph_1}$ closed w.r.t. g and club in γ, if $\langle I_\alpha : \alpha < \omega_1 \rangle$ is a filtration of I then $\sup(I_\alpha) \in E$ and $g(\sup(I_\alpha)) \subseteq I_\alpha$ hold for stationarily many $\alpha < \omega_1$.

- $\mathcal{F} = \langle I_\alpha : \alpha < \lambda \rangle$ is a filtration of I if $\mathcal{F}$ is a continuously increasing $\subseteq$-sequence of subsets of I of cardinality $< |\, I \,|$ s.t. $I = \bigcup_{\alpha < \lambda} I_\alpha$.

- FRP is also equivalent to the reflection of uncountable coloring number of graphs down to cardinality $< \aleph_2$.

Back to RC frame Back to HH frame

GRP implies CH

Proposition 1. Suppose that ω_2 is generically supercompact by σ-closed forcing. Then CH holds.

Proof. Suppose that ω_2 is generically supercompact by σ-closed forcing but $\neg$CH holds. Then there is a 1-1 $i : \omega_2 \to \mathcal{P}(\omega)$.

- Let $\lambda = 2^{\aleph_0}$. Let $\mathbb{P}$ be a σ-closed p.o., and $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic set with classes j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. (1) $j : V \overset{\preceq}{\to} M$, (2) $crit(j) = \omega_2$, (3) $j(\omega_2) > \lambda$, and (4) $j''\lambda \in M$.
- By elementarity (1), $M \models$ "$j(i) : j(\omega_2) \to \mathcal{P}(\omega)$ is 1-1".
- This is a contradiction as $\mathcal{P}(\omega)^{\mathsf{V}} = \mathcal{P}(\omega)^{\mathsf{V}[\mathbb{G}]}$. □

- Actually, the proposition above and its proof does not help us very much, since, to establish the proof of the equivalence of GRP and this generic supercompactness, we have to prove that GRP implies CH. This is done (in the proof of Bernhard König) by using a game defined from a Bernstein set.

Back

Generically supercompact κ by $\mathcal{P}$

- A cardinal κ is said to be a generically supercompact cardinal by $\mathcal{P}$ for a property (of p.o.s) $\mathcal{P}$ if, for any cardinal $\lambda \geq \kappa$, there is a p.o. $\mathbb{P}$ with $\mathbb{P} \models \mathcal{P}$ and a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ s.t. there are classes j, $M \subseteq \mathsf{V}[\mathbb{G}]$ with
  - M is an inner model of $\mathsf{V}[\mathbb{G}]$ and $j : \mathsf{V} \overset{\preccurlyeq}{\rightarrow} M$;
  - $crit(j) = \kappa$;
  - $j(\kappa) > \lambda$ and
  - $j''\lambda \in M$.

Proposition 1. If κ is a supercompact and $\mu < \kappa$ is an uncountable regular cardinal then for $\mathbb{P} = \mathrm{Col}(\mu, \kappa)$ and $(\mathsf{V}, \mathbb{P})$-generic filter $\mathbb{G}$, we have $\mathsf{V}[\mathbb{G}] \models \kappa = \mu^+$ and κ is generically supercompact by $< \mu$-closed forcing.

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