On weak compactness of extended logics

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Outline

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- Weak compactness of stationary logic

- Proof of Theorem 4.
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(Not so) abstract Logic

- We consider a logic $\mathcal{L}$ here as a triple $\langle \mathcal{L}, \models_{\mathcal{L}}, \prec_{\mathcal{L}} \rangle$ where $\mathcal{L}$ is a mapping which gives, to each set of signature, the corresponding set of $\mathcal{L}$-formulas in the signature, $\models_{\mathcal{L}}$ is the model relation for the logic, and $\prec_{\mathcal{L}}$ the elementary substructure relation associated to the logic $\mathcal{L}$.
- We assume that $\langle \mathcal{L}, \models_{\mathcal{L}}, \prec_{\mathcal{L}} \rangle$ satisfies all the natural properties which are expected for such a logic.
- The compactness number of the logic $\mathcal{L}$ is defined by:

$$\mathfrak{cn}(\mathcal{L}) := \min(\{\kappa \in \text{Card} : \text{for any } \mathcal{L}\text{-theory } T,\ T \text{ is satisfiable} \Leftrightarrow \text{all } S_0 \in [T]^{<\kappa} \text{ are satisfiable}\} \cup \{\infty\}).$$

- The weak compactness spectrum of $\mathcal{L}$ is the class:

$$\text{WCS}(\mathcal{L}) := \{\kappa \in \text{Card} : \text{for any } \mathcal{L}\text{-theory } T \text{ of signature of size } \leq \kappa,\ T \text{ is satisfiable} \Leftrightarrow S_0 \in [T]^{<\kappa} \text{ are satisfiable}\} \cup \{\infty\}.$$

- The weak compactness number of $\mathcal{L}$ is:

$$\mathfrak{wcn}(\mathcal{L}) := \min(\text{WCS}(\mathcal{L})).$$

Weak Compactness Spectrum

- The compactness number of the logic $\mathcal{L}$ is defined by:

$$\mathfrak{cn}(\mathcal{L}) := \min(\{\kappa \in \mathrm{Card} : \text{for any } \mathcal{L}\text{-theory } T,\ T \text{ is satisfiable} \Leftrightarrow \text{all } S_0 \in [T]^{<\kappa} \text{ are satisfiable}\} \cup \{\infty\}).$$

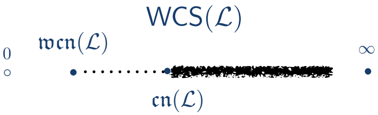
- The weak compactness spectrum of $\mathcal{L}$ is the class:

$$\mathrm{WCS}(\mathcal{L}) := \{\kappa \in \mathrm{Card} : \text{for any } \mathcal{L}\text{-theory } T \text{ of signature of size } \leq \kappa,\ T \text{ is satisfiable } \Leftrightarrow\ S_0 \in [T]^{<\kappa} \text{ are satisfiable}\} \cup \{\infty\}.$$

- The weak compactness number of $\mathcal{L}$ is:

$$\mathfrak{wcn}(\mathcal{L}) := \min(\mathrm{WCS}(\mathcal{L})).$$

Lemma 1. $\{\mathfrak{wcn}(\mathcal{L})\} \cup \{\kappa : \kappa \geq \mathfrak{cn}(\mathcal{L})\} \subseteq \mathrm{WCS}(\mathcal{L})$.



Theorem 2. (classical) (1) $\kappa = \mathfrak{wcn}(\mathcal{L}_{\kappa,\omega}) \Leftrightarrow \kappa$ is weakly compact.
(2) $\kappa = \mathfrak{cn}(\mathcal{L}_{\kappa,\omega}) \Leftrightarrow \kappa$ is strongly compact.

- $\mathcal{L}^{\mathrm{II}}$ denotes the (full) second-order logic and $\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}$ its infinitary logic extension.

Theorem 3. ([Magidor 1971]) (1) $\kappa = \mathfrak{cn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}) \Leftrightarrow \kappa$ is extendible.
(2) If $\mathfrak{cn}(\mathcal{L}^{\mathrm{II}}) < \infty$ then $\mathfrak{cn}(\mathcal{L}^{\mathrm{II}})$ is the least extendible cardinal.

- κ is extendible if, for any $\eta > 0$, there is a ζ and j s.t. $j : V_{\kappa+\eta} \overset{\prec}{\rightarrow}_{\kappa} V_{\zeta}$.

- What is the x in terms of the characterizations above ?

$$\frac{\text{weakly compact cardinals}}{\text{strongly compact cardinals}} = \frac{x}{\text{extendible cardinals}}.$$

- (Does $\kappa = \mathfrak{wcn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega})$characterize a large cardinal ? If so, what is the large cardinal ?)

Weakly extendible cardinals

- The following notion of weak extendibility is the solution to the equation. We can also nicely place the weak extendibility in the hierarchy of small large cardinals.

- A cardinal κ is weakly extendible $:\Leftrightarrow$ ① $\kappa = 2^{<\kappa}$ and, ② for any $\theta > \kappa$, and $M \prec V_\theta$ with $\kappa + 1 \subseteq M$, $|M| = \kappa$, there are $\overline{\theta}$ and j with $j : M \preccurlyeq_\kappa V_{\overline{\theta}}$.

Notation: If we write $j : M \overset{\prec}{\rightarrow}_\kappa N$ then M and N are transitive $\in$- (set or class)-structures and $crit(j) = \kappa$. If we write $j : M \preccurlyeq_\kappa N$ then M and N are not necessarily transitive $\in$-structures but $\kappa + 1 \subseteq M, N$.

Theorem 4. ([∞]) (1) $\kappa = \mathfrak{wcn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}) \Leftrightarrow \kappa$ is weakly extendible.
(2) If $\mathfrak{wcn}(\mathcal{L}^{\mathrm{II}}) < \infty$, then $\mathfrak{wcn}(\mathcal{L}^{\mathrm{II}})$ is the least weakly extendible cardinal.

Proof.

Weakly extendible cardinals in the hierarchy of small large cardinals

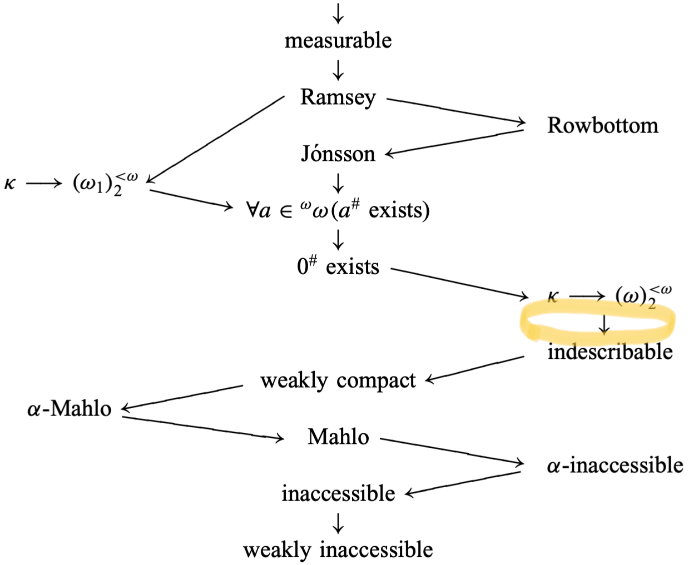
- A cardinal κ is said to be strongly unfoldable (Villaveces) if $\kappa = 2^{<\kappa}$ and, for any ordinal $\lambda > \kappa$ and any transitive model M of ZFC^- s.t. $\kappa \in M$, ${}^{\kappa>}M \subseteq M$ and $|M| = \kappa$, there is a transitive $N \supseteq V_\lambda$ with $j : M \overset{\prec}{\rightarrow}_\kappa N$ and $j(\kappa) > \lambda$. Here, ZFC^- denotes the axiom system ZFC without the Power Set Axiom.

- P. Lücke [Lücke 2022] proved that the strong unfoldability is equivalent to the shrewdness, a natural strengthening of the total indescribability which was introduced by M. Rathjen.

- An inaccessible cardinal κ is strongly uplifting (Hamkins and Johnstone) if, for every $A \subseteq \kappa$ there are arbitrarily large regular $\theta > \kappa$ such that $\langle V_\kappa, \in, A\rangle \prec \langle V_\theta, \in \overline{A}\rangle$ for some $\overline{A} \subseteq V_\theta$.

- Any subtle cardinal is a stationary limit of strongly uplifting cardinals ([Hamkins-Johnstone 2017]).

Weakly extendible cardinals in the hierarchy of small large cardinals (2/4)



The lower half of the chart of large cardinals in [Kanamori 2003]

Weakly extendible cardinals in the hierarchy of small large cardinals (3/4)

Theorem 5.([∞]) (1) If κ is weakly extendible, then there is a weakly compact $\lambda > \kappa$. On the other hand, strong unfoldability of κ does not imply the existence of inaccessible $\lambda > \kappa$.

(2) Assume κ is weakly extendible and ν is the first inaccessible cardinal above κ (which exists by (1)).
Then $V_\nu \models$ "κ is strongly unfoldable but not weakly extendible".
Also $V_\nu \models$ "there is no inaccessible cardinal above κ". In particular, ZFC + "there is a weakly extendible cardinal" proves
consis(⌜⌜ZFC⌝⌝ + ⌜there is a strongly unfoldable cardinal⌝).

(3) ([Boney et al.] Proposition 4.8 + Theorem 4.) If κ is a strongly uplifting cardinal then κ is weakly extendible and κ is a stationary limit of weakly extendible cardinals.

(4) If κ is a subtle cardinal, then κ is a stationary limit of weakly extendible cardinals. □

Weakly extendible cardinals in the hierarchy of small large cardinals (4/4)

subtle

↓ [Hamkins-Johnstone]

strongly uplifting

↓ [Boney etal.] + [∞]

weakly extendible

↓ [∞]

strongly unfoldable = shrewd

↓ [Lücke] [Dzamonja-Hamkins]

totally indescribable

↓

weakly compact

Large cardinal characterizations of the weak compactness of many more logics

▷ (The proof of) Theorem 4 has the following generalizations:

- Let us call a logic $\mathcal{L}$ finitary if the set of free variables in any $\mathcal{L}$-formula is finite, the set of all $\mathcal{L}$-formulas of given signature S of cardinality $\leq \kappa$, for an infinite κ has size $\leq \kappa$, and, for any infinite ordinal θ and $\varphi \in V_\theta$, $V_\theta \models$"φ is an $\mathcal{L}$-formula" if and only if φ is (really) an $\mathcal{L}$-formula.

- For $N \models \mathrm{ZFC}^-$ and a structure $\mathfrak{A} \in N$ s.t. the index sets of the components of $\mathfrak{A}$ are all included in N also as subsets in N, let $\mathfrak{A}^N := \mathfrak{A} \restriction N$.

- For a logic $\mathcal{L}$, if N is s.t. $N \models \mathrm{ZC}^-$ and N contains all parameters needed to define $\mathcal{L}$, we shall say that N is $\mathcal{L}$-truthful if, for all structures $\mathfrak{A}$ as above (in connection with this N), $N \models$"$\mathfrak{A} \models_{\mathcal{L}} \varphi$" is equivalent to $\mathfrak{A}^N \models_{\mathcal{L}} \varphi$.

Theorem 6. ([∞]) (1) Suppose that $\mathcal{L}$ is a finitary logic s.t.

① V_θ for all regular uncountable θ is $\mathcal{L}$-truthful; and

② "$\underline{\in}$ is well-founded" is expressible by a formula $\varphi^*_{\mathcal{L}}$ in $\mathcal{L}$.

Then a cardinal κ is weakly $\mathcal{L}$-compact (i.e. $\kappa \in \mathrm{WCS}(\mathcal{L})$) $\Leftrightarrow$

for any regular $\theta \geq \kappa$ and $M \prec V_\theta$ s.t. $\kappa + 1 \subseteq M$, $|M| = \kappa$, there are j, N s.t. $\kappa + 1 \subseteq N$, $j : M \preccurlyeq N$, $j(\kappa) > \min(\mathrm{On}^N \setminus \sup(j''\kappa))$, and N is $\mathcal{L}$-truthful.

(2) Suppose that $\mathcal{L}^*$ is a logic obtained from a finitary logic $\mathcal{L}$ which satisfies ① and ② above, by extending $\mathcal{L}$ by taking the closure of the set of $\mathcal{L}$ formulas w.r.t. infinitary conjunction and disjunction of set of formulas of size $< \kappa$ and first order logical operations. Then κ is weakly $\mathcal{L}^*$-compact $\Leftrightarrow$

③ for any regular $\theta \geq \kappa$ and $M \prec V_\theta$ s.t. $\kappa + 1 \subseteq M$, $|M| = \kappa$, there are j, N s.t. $j : M \preccurlyeq_\kappa N$, and N is $\mathcal{L}$-truthful.

Theorem 6. ([∞]) (1) Suppose that $\mathcal{L}$ is a finitary logic s.t.

① V_θ for all regular uncountable θ is $\mathcal{L}$-truthful; and

② "$\underset{\sqcup}{\in}$ is well-founded" is expressible by a formula $\varphi^*_{\mathcal{L}}$ in $\mathcal{L}$.

Then ...

(2) Suppose that $\mathcal{L}^*$ is a logic obtained from a finitary logic $\mathcal{L}$ which satisfies ① and ② above, by extending $\mathcal{L}$ by taking the closure of the set of $\mathcal{L}$ formulas w.r.t. infinitary conjunction and disjunction of set of formulas of size $<\kappa$ and first order logical operations. Then κ is weakly $\mathcal{L}^*$-compact $\Leftrightarrow$

③ for any regular $\theta \geq \kappa$ and $M \prec V_\theta$ s.t. $\kappa + 1 \subseteq M$, $|M| = \kappa$, there are j, N s.t. $j : M \preccurlyeq_\kappa N$, and N is $\mathcal{L}$-truthful.

(3) Suppose that $\mathcal{L}$ is a finitary logic satisfying ① and ② in (1). Then $\mathfrak{wcn}(\mathcal{L})$ is the least cardinal κ satisfying ③. □

Weak compactness of stationary logic

- $\mathcal{L}^{\aleph_0,\mathrm{II}}$ denotes the weak second-order logic in which second order variables are interpreted as countable subsets of the underlying set of the structure in consideration.
- $\mathcal{L}^{\aleph_0,II}_{stat}$ is the weak second-order logic with stationarity quantifier:
  $\mathfrak{A} \models stat\, x\, \varphi(x, ...) \;\Leftrightarrow$
  $\{a \in [\,|\mathfrak{A}|\,]^{\aleph_0} : \mathfrak{A} \models \varphi(a, ...)\}$ is stationary in $[\,|\mathfrak{A}|\,]^{\aleph_0}$.
- $\mathcal{L}^{\aleph_0}_{stat}$ is as above but without second-order existential (and universal) quantification.

- $\mathcal{L}^{\aleph_0,\mathrm{II}}_s tat$ satisfies ①, ② of Theorem 6. Hence $\mathfrak{wcn}(\mathcal{L}^{\aleph_0,\mathrm{II}}_s tat)$ and cardinals κ with $\kappa = \mathfrak{wcn}((\mathcal{L}^{\aleph_0,\mathrm{II}}_s tat)_{\kappa,\omega})$ are large cardinals (at least weakly compact).
- We can say slightly more than this:

Weak compactness of stationary logic (2/2)

Theorem 7. ([∞]) Assume that $\kappa = \mathfrak{wcn}((\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat})_{\kappa,\omega})$. Then

$L \models$ "κ is a weakly compact cardinal and it is a stationary limit of weakly compact cardinals".

$L \models$ " there is a weakly compact cardinal $> \kappa$ which is a stationary limit of weakly compact cardinals".

Theorem 8. ([∞]) Assume $\mathsf{V} = \mathsf{L}$. Then $\kappa = \mathfrak{wcn}(\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat})$ or $\kappa = \mathfrak{wcn}((\mathcal{L}^{\aleph_0,\mathrm{II}}_{stat})_{\kappa,\omega})$ if and only if κ is weakly extendible.

Some open Questions:

- Is $\mathfrak{wcn}(\mathcal{L}^{\aleph_0}_{stat})$ a large cardinal ?
- is $\mathfrak{wcn}(\mathcal{L}^{\aleph_0,II}_{stat}) < \mathfrak{wcn}(\mathcal{L}^{\mathrm{II}})$ consistent ?

Thank you for your attention!
ご清聴ありがとうございました.

관심을 가져 주셔서 감사합니다
Gracias por su atención.
Dziękuję za uwagę.
Grazie per l'attenzione.
Dank u voor uw aandacht.

Ich danke Ihnen für Ihre Aufmerksamkeit.

Proof of Theorem 4.

Theorem 4. ([∞]) (1) $\kappa = \mathfrak{wcn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}) \Leftrightarrow \kappa$ is weakly extendible.

Proof. "$\Leftarrow$": Assume that κ is weakly extendible. Then κ is inaccessible. Suppose that T is a $<\kappa$-satisfiable $\mathcal{L}^{\mathrm{HO}}_{\kappa,\kappa}$-theory of signature of size $\leq \kappa$. We want to show that T is satisfiable.

- Since T has cardinality $\leq \kappa$, we may assume that T is a subset of κ by some reasonable coding.
- Let θ be large enough. In particular, s.t.
  - ① $\theta \geq \kappa^{+\omega}$ and
  - ② $V_\theta \models$" T is $<\kappa$-satisfiable".
- Let $M \prec V_\theta$ be s.t. $\kappa + 1 \subseteq M$, $T \in M$, $|M| = \kappa$, and let $\overline{\theta}$, j be s.t. $j : M \preccurlyeq_\kappa V_{\overline{\theta}}$. Then we have $V_{\overline{\theta}} \models$"$j(T)$ is $<j(\kappa)$-satisfiable" by ② and by elementarity of j.
- Since $V_{\overline{\theta}} \models$"$|T| < j(\kappa)$ and $\overbrace{T}^{=j(T)\cap\kappa} \subseteq j(T)$", it follows that there is $\mathfrak{A} \in V_{\overline{\theta}}$ s.t. $V_{\overline{\theta}} \models$"$\mathfrak{A} \models_{\mathcal{L}^{\mathrm{HO}}_{\kappa,\kappa}} T$". Now $\overline{\theta} \geq j(\kappa)^{+\omega} \geq \kappa^{+\omega}$ by ① and by elementarity of j. Thus, it follows that $\mathfrak{A} \models_{\mathcal{L}^{\mathrm{HO}}_{\kappa,\kappa}} T$. Thus, T is realizable.

Proof of Theorem 4. (2/3)

Theorem 4. ([∞]) (1) $\kappa = \mathfrak{wcn}(\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}) \Leftrightarrow \kappa$ is weakly extendible.

"⇒": Assume that $\kappa = \mathfrak{wcn}(\mathcal{L}^{II}_{\kappa,\omega})$ holds. Then we have $\kappa = \mathfrak{wcn}(\mathcal{L}_{\kappa,\omega})$. Hence κ is weakly compact. In particular, κ is inaccessible. Thus it is enough to show that κ satisfies ② of the definition of the weak extendibility.

- Suppose that $\theta > \kappa$ and $M \prec V_\theta$ is s.t. $\kappa + 1 \subseteq M$ and $|M| = \kappa$. Let φ^* be an $\mathcal{L}^{\mathrm{II}}$-sentence in the signature $\{\underset{\sqcup}{\in}\}$ s.t.

  $\langle |\mathfrak{A}|, \underset{\sqcup}{\in}^{\mathfrak{A}}\rangle \models \varphi^* \Leftrightarrow \underset{\sqcup}{\in}^{\mathfrak{A}}$ is well-founded and extensional binary relation, and the Mostowski collapse of $(\langle |\mathfrak{A}|, \underset{\sqcup}{\in}^{\mathfrak{A}}\rangle) = \langle V_\gamma, \in\rangle$ for some γ

- Let

$$T := \{\varphi^*\} \cup \{\varphi(\underset{\sqcup}{c}_{a_0}, \ldots) : \varphi \text{ is a first-order formula in the signature } \{\underset{\sqcup}{\in}\}, a_0, \ldots \in M \text{ and } M \models \varphi(a_0, \ldots)\}$$
$$\cup \{\forall_x (x \underset{\sqcup}{\in} \underset{\sqcup}{c}_\alpha \leftrightarrow \textstyle\bigvee\!\!\bigvee_{\beta<\alpha} x \equiv \underset{\sqcup}{c}_\beta) : \alpha < \kappa\}$$
$$\cup \{\underset{\sqcup}{c}_\alpha \underset{\sqcup}{\in} \underset{\sqcup}{d} : \alpha < \kappa\}$$
$$\cup \{\underset{\sqcup}{d} \underset{\sqcup}{\in} \underset{\sqcup}{c}_\kappa\}.$$

Proof of Theorem 4. (3/3)

- The signature of the $\mathcal{L}^{\mathrm{II}}_{\kappa,\omega}$-theory T is $\{\underset{\sqcup}{\in}, \underset{\sqcup}{d}\} \cup \{\underset{\sqcup}{c}_a : a \in M\}$ and it is of cardinality κ.

Claim. T is $<\kappa$-satisfiable.

$\vdash$ Suppose that $T_0 \in [T]^{<\kappa}$. Then M can be expanded to a model of T_0 $\dashv$

- By the assumption on κ, it follows that T is satisfiable. Let $\mathfrak{B}$ be a model of T. By $\mathfrak{B} \models_{\mathcal{L}^{\mathrm{II}}} \varphi^*$, we can take the Mostowski collapse $\mathfrak{B}^*$ of $\mathfrak{B}$, and $|\mathfrak{B}^*| = V_{\overline{\theta}}$ for some ordinal $\overline{\theta}$. Note that we have $\underset{\sqcup}{\in}^{\mathfrak{B}^*} = \in$. By the definition $j : M \to V_{\overline{\theta}}\,;\ a \mapsto [\underset{\sqcup}{c}_a]^{\mathfrak{B}^*}$, we obtain $j : M \preccurlyeq_\kappa V_{\overline{\theta}}$.

Theorem 4. ([∞]) (2) If $\mathfrak{wcn}(\mathcal{L}^{\mathrm{II}}) < \infty$, then $\mathfrak{wcn}(\mathcal{L}^{\mathrm{II}})$ is the least weakly extendible cardinal.

The proof is similar to that of (1) with an additional trick used in the proof of the corresponding theorem in ([Magidor 1971]).

□ (Theorem 4)