

On reflection, heredity, and absoluteness of topological properties

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2022年6月7日 (16:00 ~ JST), 至 **Set-Theoretic and Geometric Topology,
and their applications to related fields**

The following slides are typeset by $\text{up}\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$ with beamer class, and
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The subject of the talk is related to the research supported by
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- ▶ Reflection of topological properties
- ▶ Reflection number of a topological property
- ▶ Reflection number as degree of heredity and compactness
- ▶ Reflection number of non-metrizability
- ▶ Fodor-type Reflection Principle
- ▶ Generic large cardinals
- ▶ Reflection and non-reflection down to the continuum
- ▶ Potential metrizability

* We assume in the following that topological spaces are Hausdorff (to remain on the safe side).

** Some of the results mentioned at the end of the talk appear in a future joint paper with Hiroshi Sakai.

Reflection of topological properties

refl. hered. abs. (3/21)

- ▶ For a topological space X satisfying some property P , it is very often the case that there is a subspace/ many subspaces Y of X of smaller size (in terms of cardinality) which also satisfy the property P . This is what I shall call here:

reflection of the topological property P down to Y (or down to $|Y|$).

- ▷ The following facts are such examples:

Proposition 1. Suppose that $X = \langle X, \tau \rangle$ is non-separable. Then there is a subspace Y of X of cardinality \aleph_1 which is also non-separable.

Proof.

Proposition 2. ([Hajnal-Juhász 1976]) For a topological space $X = \langle X, \tau \rangle$ and $p \in X$, if $\chi(p, X) = \kappa$ for a regular uncountable κ , then there is a subspace Y of X of cardinality $\leq \kappa$ s.t. $p \in Y$ and $\chi(p, Y) = \kappa$. The condition “of cardinality $\leq \kappa$ ” above is optimal.

Proof.

[back to Examples](#)

Reflection number of a topological property

- ▶ Suppose that P and Q are properties of topological spaces. In the following, we mainly treat cases where P is a “bad property” for topological spaces satisfying Q in the sense that if X is a topological space with the property Q and $Y \subseteq X$ is a subspace with the property P , then all intermediate spaces Z with $Y \subseteq Z \subseteq X$ satisfy the property P .
- ▷ For P and Q as above, we define the **reflection number** of the property P under the spaces with the property Q as

$\text{rn}(P, Q) := \min\{\kappa \in \text{Card} : \text{for any topological space } X \text{ with the property } Q, \text{ if } X \text{ satisfies the property } P, \text{ then there is a subspace } Y \text{ of } X \text{ with the property } P \text{ s.t. } |Y| < \kappa\}$

- ▶ if the class $\{\kappa \in \text{Card} : \dots\}$ in the definition of $\text{rn}(P, Q)$ is empty, we define $\text{rn}(P, Q)$ to be ∞ .

Reflection number of a topological property (2/3)

$\text{rn}(P, Q) := \min\{\kappa \in \text{Card} : \text{for any topological space } X \text{ with the property } Q, \text{ if } X \text{ satisfies the property } P, \text{ then there is a subspace } Y \text{ of } X \text{ with the property } P \text{ s.t. } |Y| < \kappa\}$

- ▶ if the class $\{\kappa \in \text{Card} : \dots\}$ in the definition of $\text{rn}(P, Q)$ is empty, we define $\text{rn}(P, Q)$ to be ∞ .

- ▶ We shall write $\text{rn}(P, \emptyset)$ if the property Q imposes no restrictions.

Example 3. For $P =$ “non-separable”, **Proposition 1** can be reformulated as: $\text{rn}(P, \emptyset) = \aleph_2$.

Example 4. For a regular uncountable cardinal κ and $P =$ “there is a point with χ -character κ ”, **Proposition 2** can be reformulated as: $\text{rn}(P, \emptyset) = \kappa^+$.

Reflection number of a topological property (3/3)

$\text{rn}(P, Q) := \min\{\kappa \in \text{Card} : \text{for any topological space } X \text{ with the property } Q, \text{ if } X \text{ satisfies the property } P, \text{ then there is a subspace } Y \text{ of } X \text{ with the property } P \text{ s.t. } |Y| < \kappa\}$

- ▶ if the class $\{\kappa \in \text{Card} : \dots\}$ in the definition of $\text{rn}(P, Q)$ is empty, we define $\text{rn}(P, Q)$ to be ∞ .

- ▶ We shall write $\text{rn}(P, \emptyset)$ if the property Q imposes no restrictions.

Example 5. For $P =$ “there is a point with χ -character $\geq \kappa$ ”, we have $\text{rn}(P, \emptyset) = \infty$.

Proof. Let λ be an uncountable regular cardinal and $X := \lambda + 1$ be with the topology generated from

$$\tau := \{\{\alpha\} : \alpha < \lambda\} \cup \{A \cup \{\lambda\} : A \subseteq \lambda, |\lambda \setminus A| < \lambda\}.$$

- ▶ $\chi(\alpha, X) = 1$ for all $\alpha \in \lambda$ and $\chi(\lambda, X) = \lambda$.
- ▶ $\chi(\alpha, Y) = 1$ for all $\alpha \in Y$ for any $Y \in [X]^{<\lambda}$.

□ (Example 5)

Reflection number as degree of heredity and compactness refl. hered. abs. (7/21)

- ▶ The reflection number $\text{rn}(P, Q)$ for a “bad” property P (in the sense of **the previous slide**) may be regarded as a **degree of heredity of P** : we can find always a small subspace $Y \subseteq X$ (of cardinality $< \text{rn}(P, Q)$) for the space X with $X \models P \wedge Q$ s.t. all subspace of X above Y satisfy P .
- ▶ $\text{rn}(P, Q)$ can also be seen as the **compactness** (in model-theoretic sense) **of the property $\neg P$** : the contraposition of the property in the definition of $\text{rn}(P, Q)$ implies the following.
- ▷ For any topological space $X \models Q$, if $Y \models \neg P$ for all subspace Y of X of cardinality $< \text{rn}(P, Q)$, then $X \models \neg P$.

Reflection number of non-metrizability

refl. hered. abs. (8/21)

- ▶ ([Hajnal-Juhász 1976]) Example 5 actually shows that $\text{rn}(P, \emptyset) = \infty$ for $P = \text{"non-metrizable"}$.
- ▶ Note that every metrizable space is first countable. The equality $\text{rn}(P, Q) = \aleph_1$? for $P = \text{"non-metrizable"}$ and $Q = \text{"first countable"}$ is known as **Hamburger's Problem** and (its consistency) is still unsolved.


Proposition 6. ([Hajnal-Juhász 1976]) $\text{rn}(P, Q) = \infty$ for $P = \text{"non-metrizable"}$ and $Q = \text{"first countable"}$ is consistent.

- Proof.** Let κ be a regular cardinal $\geq \aleph_2$ and $S \subseteq E_\omega^\kappa (= \{\alpha < \kappa : \text{cf}(\alpha) = \omega\})$ be a **non-reflecting stationary** set. Then S as a subspace of κ with the order topology is first countable.
- ▶ S is not perfectly normal (nor meta-Lindelöf, and hence not metrizable).
 - ▷ We can show by induction on $\alpha < \kappa$ that $S \cap \alpha$ is metrizable.
 - ▶ Note that S as above exists for $\kappa = \lambda^+$ for λ s.t. \square_λ holds.
 - ▶ Thus, e.g. $V = L$ implies $\text{rn}(P, Q) = \infty$. □ (Proposition 6.)

Reflection number of non-metrizability (2/4)


refl. hered. abs. (9/21)

- ▶ The proof of Theorem 6 shows that we need the consistency strength of quite large large cardinals to get $\text{rn}(P, Q) < \infty$ for P, Q as above: $\neg \square_\lambda$ for an end-segment of **Card** implies the consistency of a **Woodin Cardinal**.

Theorem 7. ([Bagaria-Magidor 2014]) Let $P =$ “non-metrizable” and $Q =$ “first countable”. If κ is ω_1 -strongly compact, then $\text{rn}(P, Q) \leq \kappa$. 

Corollary 8. For $P =$ “non-metrizable” and $Q =$ “first countable”, $\text{rn}(P, Q) < \infty$ is independent (modulo a certain large cardinal).

- ▶ We shall discuss later more results related to Theorem 7.

Theorem 9. ([Dow 1988]) $\text{rn}(P, Q) = \aleph_2$ holds for $P =$ “non-metrizable” and $Q =$ “countably compact”. 

Reflection number of non-metrizability (3/4)

ref. hered. abs. (10/21)

Corollary 10. Suppose that X is locally countably compact and all subspaces Y of X of cardinality $\leq \aleph_1$ are metrizable, then X is first countable.

Proof. Let $p \in X$. Then there is $p \in O \subseteq X$ s.t. \overline{O} is countably compact. By Dow's theorem (Theorem 7) \overline{O} is metrizable. Thus $\chi(p, \overline{O}) \leq \aleph_0$. But then $\aleph_0 \geq \chi(p, O) = \chi(p, X)$. \square (Corollary 8)

Proposition 11. It is consistent that $\text{rn}(P, Q) = \infty$ for $P = \text{"non-metrizable"}$ and $Q = \text{"locally countably compact"}$.

(Fan?) Question For a countably compact space X , if all subspaces of cardinality $\leq \aleph_1$ are first countable, does it follow that X is first countable?

Proposition 11. It is consistent that $\text{rn}(P, Q) = \infty$
for $P = \text{“non-metrizable”}$ and $Q = \text{“locally countably compact”}$.

Proof. Suppose that κ is a regular cardinal $\geq \aleph_2$ and $S \subseteq E_\omega^\kappa$ is non-reflecting stationary subset of E_ω^κ .

- ▶ For each $\xi \in S$ let $l_\xi \subseteq \xi$ be a set of successor ordinals of order-type ω cofinal in ξ .
- ▷ Let $X := \bigcup \{l_\xi : \xi \in S\} \cup S$ with the topology generated from $\tau := \{\{\alpha\} : \alpha \in \bigcup \{l_\xi : \xi \in S\}\} \cup \{l_\xi \setminus \beta \cup \{\xi\} : \xi \in S, \beta < \xi\}$.
- ▶ X is first countable and locally compact.
- ▷ $X \cap \beta$ is metrizable for all $\beta < \kappa$ (by induction).
- ▷ X is not meta-Lindelöf (use Fodor's Lemma).
- ▶ Thus, similarly to the proof of Proposition 6, $V = L$ implies $\text{rn}(P, Q) = \infty$. □ (Proposition 11)

- **Fodor-type Reflection Principle (FRP for short)** is the following assertion:

FRP: For all regular $\kappa \geq \aleph_1$, any stationary $S \subseteq E_\omega^\kappa$ and mapping $g : S \rightarrow [\kappa]^{<\aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

- (1) $\text{cf}(I) = \omega_1$;
- (2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;
- (3) for any regressive $f : S \cap I \rightarrow \kappa$ s.t. $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}''\{\xi^*\}$ is stationary in $\text{sup}(I)$.

- FRP follows from Martin's Maximum but in contrast to it, **FRP is preserved by ccc generic extension**. Hence FRP is compatible with any size of the continuum (it can be also forced under CH).

Theorem 12. ([Fuchino-Juhász-Soukup-Szentmichlószy-Usuba 2010], [Fuchino-Sakai-Soukup-Usuba ∞]) $\text{rn}(P, Q) = \aleph_2$ for $P =$ "non-metrizable" and $Q =$ "locally countably compact" is equivalent to FRP.

- For a class \mathcal{P} of p.o.s, a cardinal κ is said to be **generically supercompact by \mathcal{P}** (**\mathcal{P} -gen. supercompact** for short) if, for any $\lambda \geq \kappa$, there are σ -closed p.o. \mathbb{P} , (V, \mathbb{P}) -generic \mathbb{G} , j , $M \subseteq V[\mathbb{G}]$ s.t. $V[\mathbb{G}] \models j : V \xrightarrow{\checkmark}_{\kappa} M$ (M is transitive, κ is the critical point of j), $j(\kappa) > \lambda$ and $j''\lambda \in M$.

Theorem 13. ([Dow-Tall-Weiss 1990]) Suppose that X is a non-metrizable space, $\delta \in \text{Card}$ and $\mathbb{P} = \text{Fn}(\delta, 2)$, the p.o. with finite conditions adding δ many Cohen reals. Then we have

$$\Vdash_{\mathbb{P}} \text{“}\check{X} \text{ is non-metrizable”}.$$



Proposition 14. (see [Fuchino-O.M.Rodrigues-Sakai 202?])

If κ is Cohen-gen. supercompact, then $\text{rn}(P, Q) \leq \kappa$ for $P = \text{“non-metrizable”}$ and $Q = \text{“first countable”}$.

Proof

Corollary 15. ([Dow-Tall-Weiss 1990]) $\text{rn}(P, Q) \leq 2^{\aleph_0}$

is consistent modulo large cardinals for P, Q as above.

Proof

- ▶ Let $P :=$ “non-metrizable”, $Q :=$ “first countable” and $Q_0 :=$ locally countably compact.
- ▶ The statement of Corollary 15 can be still improved by starting from two supercompact cardinals.

Theorem 16. (see [Fuchino-O.M.Rodrigues-Sakai 202?]) $\text{rn}(P, Q_0) = \aleph_2 + \text{rn}(P, Q) \leq 2^{\aleph_0}$ is consistent modulo certain large cardinals.

Proof. Start from two supercompact cardinals κ_0, κ_1 with $\kappa_0 < \kappa_1$.

- ▶ Use κ_0 to force FRP. The forcing can be chosen to be small enough so that the supercompactness of κ_1 survives the extension.
- ▶ In the generic extension, force with $\mathbb{P} = \text{Fn}(\kappa_1, 2)$. Since \mathbb{P} is ccc, FRP survives in the second generic extension. Thus, by [Theorem 12](#), $\text{rn}(P, Q_0) = \aleph_2$ holds in the second generic extension.
- ▶ In the second generic extension, we have $\kappa_1 = 2^{\aleph_0}$ and it is Cohen-gen. supercompact. Thus by [Proposition 14](#) we have $\text{rn}(P, Q) \leq 2^{\aleph_0}$. □ (Theorem 16)

- ▶ Let $P :=$ “non-metrizable”, $Q :=$ “first countable” and $Q_0 :=$ locally countably compact.

Theorem 17. (van Douwen, see [Fuchino-O.M.Rodrigues-Sakai 202?])
 $\mathfrak{b} < \mathfrak{rn}(P, Q)$.

Theorem 18. (see [Fuchino-O.M.Rodrigues-Sakai 202?])
 $MA + \mathfrak{rn}(P, Q_0) = \aleph_2 + \mathfrak{rn}(P, Q) \not\leq 2^{\aleph_0}$ is consistent modulo certain large cardinals.

Proof. The proof is similar to that of Theorem 16. Start again from two supercompact cardinals $\kappa_0 < \kappa_1$.

- ▶ Use κ_0 to force FRP and then force $MA + 2^{\aleph_0} = \kappa_1$ by a ccc p.o..
- ▷ FRP is preserved by ccc of the second extension.
- ▶ In the second generic extension, we forced $\mathfrak{b} = 2^{\aleph_0}$. Hence by Theorem 17, we have $\mathfrak{rn}(P, Q) \not\leq 2^{\aleph_0}$. □ (Theorem 18)

- ▶ Let $P :=$ “non-metrizable”, $Q :=$ “first countable” and $Q_0 :=$ locally countably compact.
- ▶ In the proof of Theorem 18, κ_1 is ccc-gen. supercompact.
- ▶ The construction in the proofs of Theorem 16 and Theorem 18 can be further refined by using certain new type of mixed support iteration to make 2^{\aleph_0} a strongform of gen. large cardinal (what we called **Laver-generic large cardinal**) to obtain strong stationary reflection type properties or Rado Conjecture type reflection with reflection number around the continuum together with either $\text{rn}(P, Q) \leq 2^{\aleph_0}$ or $\text{rn}(P, Q) \not\leq 2^{\aleph_0}$.
(see [Fuchino-O.M.Rodrigues-Sakai 202?])
- ▷ These results suggest that the reflection of non-metrizability should be regarded as a reflection of the type quite different from the other more standard reflection properties.

Potential metrizability

ref. hered. abs. (17/21)

- ▶ The consistency of $\text{rn}(P, Q) = \aleph_2$ for $P = \text{“non-metrizable”}$, and $Q = \text{“first countable”}$. (Hamburger’s Problem) is very difficult to establish since non-metrizability of a topological space in general is neither preserved by σ -closed p.o. nor by arbitrary ccc p.o. (cf. the proof of Proposition 14) — **non-absoluteness of non-metrizability for σ -closed or ccc generic extensions.**
- ▶ We can often solve a problem by changing the question itself:
 - ▷ For a class \mathcal{P} of p.o.s, let

$\text{rn}^{\mathcal{P}}(P, Q) := \min \{ \kappa : \text{for any topological space } X \text{ with } X \models Q, \text{ if all subspace of } X \text{ of size } < \kappa \text{ satisfy } P, \text{ then there is } \mathbb{P} \in \mathcal{P} \text{ s.t. } \Vdash_{\mathbb{P}} \text{“} X \text{ satisfies } P \text{”} \}$

▷ Cf.

$\text{rn}(P, Q) = \min \{ \kappa : \text{for any topological space } X \text{ with } X \models Q, \text{ if all subspace of } X \text{ of size } < \kappa \text{ satisfy } P, \text{ then } X \text{ satisfies } P \}$

▷ For a class \mathcal{P} of p.o.s, let

$\text{rn}^{\mathcal{P}}(P, Q) := \min \{ \kappa : \text{for any topological space } X \text{ with } X \models Q, \\ \text{if all subspace of } X \text{ of size } < \kappa \text{ satisfy } P, \\ \text{then there is } \mathbb{P} \in \mathcal{P} \text{ s.t. } \Vdash_{\mathbb{P}} \text{“} X \text{ satisfies } P \text{”} \}$

▷ Cf.

$\text{rn}(P, Q) = \min \{ \kappa : \text{for any topological space } X \text{ with } X \models Q, \\ \text{if all subspace of } X \text{ of size } < \kappa \text{ satisfy } P, \\ \text{then } X \text{ satisfies } P \}$

$\text{rn}^{\mathcal{P}^-}(P, Q) := \min \{ \kappa : \text{for any topological space } X \text{ with } X \models Q, \\ \text{if all subspace of } X \text{ of size } < \kappa \text{ satisfy } P, \\ \text{then there is } \mathbb{P} \in \mathcal{P} \text{ s.t.} \\ \Vdash_{\mathbb{P}} \text{“in an inner model, } X \text{ satisfies } P \text{”} \}$

▶ If \mathcal{P} contains the trivial p.o., $\text{rn}^{\mathcal{P}^-}(P, Q) \leq \text{rn}^{\mathcal{P}}(P, Q) \leq \text{rn}(P, Q)$.

Theorem 19. For regular cardinals κ , μ with $\mu < \kappa$ the following are equivalent:

- (a) κ is μ^+ -cc gen. supercompact.
- (b) For any $\lambda > \kappa$, there are μ -cc \mathbb{P} , (\mathbb{V}, \mathbb{P}) -generic \mathbb{G} and j , $M \subseteq V[\mathbb{G}]$ s.t. $V[\mathbb{G}] \models j : V \xrightarrow{\sim}_{\kappa} M$, $j(\kappa) > \lambda$ and $([M]^\lambda)^{V[\mathbb{G}]} \subseteq M$.

► Let, again, $P :=$ “non-metrizable”, and $Q :=$ “first countable”.

► An argument similar to that of the proof of Proposition 14 combined with Theorem 19 proves the following:

Theorem 20. If κ is ccc gen. supercompact, then $\text{rn}^{\text{ccc}}(P, Q) \leq \kappa$. \square

► Theorem 19 is not available for σ -closed p.o.s. Though we still have

Theorem 21. If κ is σ -closed gen. supercompact, then $\text{rn}^{\sigma\text{-closed}}(P, Q) \leq \kappa$. \square

Potential metrizable (3/3)

- ▶ Let, again, $P :=$ “non-metrizable”,
 $Q :=$ “first countable” and $Q_0 :=$ “locally compact”.
- ▶ Theorem 20 on the last slide combined with the idea of Theorem 18 implies the following:

Corollary 22. $MA + \mathfrak{rn}(P_0, Q) = \aleph_2$, $\mathfrak{rn}^{\text{ccc}}(P, Q) \leq 2^{\aleph_0} + \mathfrak{rn}(P, Q) \not\leq 2^{\aleph_0}$ is consistent modulo large cardinals.



- ▶ For a supercompact κ , if all cardinals below κ are collapsed to cardinality \aleph_1 by σ -closed forcing, $\kappa = \aleph_2$ in the generic extension is σ -closed gen. supercompact. This together with Theorem 21 implies:

Corollary 23. $\mathfrak{rn}^{\sigma\text{-closed}}(P, Q) = \aleph_2$ is consistent modulo large cardinals.



Thank you for your attention!
ご清聴ありがとうございました。

관심을 가져 주셔서 감사합니다

Gracias por su atención.

Dziękuję za uwagę.

Grazie per l'attenzione.

Dank u voor uw aandacht.

Ich danke Ihnen für Ihre Aufmerksamkeit.

Corollary 15. (see [Fuchino-O.M.Rodrigues-Sakai 202?]) $\text{rn}(P, Q) \leq 2^{\aleph_0}$ is consistent modulo large cardinals for P, Q as above.

Proof. Suppose that κ is a supercompact cardinal. Let $\mathbb{P} = \text{Fn}(\kappa, 2)$ and let \mathbb{G} be a (V, \mathbb{P}) -generic filter.

► Then $V[\mathbb{G}] \models \kappa = 2^{\aleph_0}$ and κ is a Cohen-gen. supercompact in $V[\mathbb{G}]$.

▷ Thus, by Proposition 14, we have $V[\mathbb{G}] \models \text{rn}(P, Q) \leq 2^{\aleph_0}$.

□ (Corollary 15)

back

Proposition 14. (see [Fuchino-O.M.Rodrigues-Sakai 202?])

If κ is Cohen-gen. supercompact, then $\text{rn}(P, Q) \leq \kappa$
for $P = \text{"non-metrizable"}$ and $Q = \text{"first countable"}$.

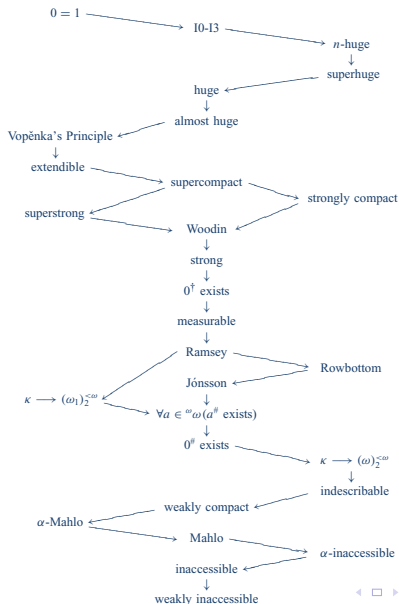
Proof. Suppose that X is a non-metrizable space with

(0) $\chi(p, X) \leq \aleph_0$ for all $p \in X$.

- ▶ W.l.o.g., $X = \langle \theta, \tau \rangle$ for some ordinal θ and an open base τ on θ .
Let $\lambda \geq \theta$ be sufficiently large and let $\mathbb{P} = \text{Fn}(\mu, 2)$ for some cardinal μ s.t., for a (V, \mathbb{P}) -generic filter \mathbb{G} , there are classes $j, M \subseteq V[\mathbb{G}]$ s.t. (1) $V[\mathbb{G}] \models j : V \overset{\lambda}{\rightarrow} \kappa M$, (2) $j(\kappa) > \lambda$ and (3) $j''\lambda \in M$.
- ▶ Let $\tau'' = \{j(O) \cap j''\theta : O \in \tau\}$. Then we have $\langle j''\theta, \tau'' \rangle$, $\langle \theta, \tau \rangle \in M$, and $M \models \langle \theta, \tau \rangle \cong \langle j''\theta, \tau'' \rangle$ by (3).
- ▶ By Dow-Tall-Weiss theorem (Theorem 13), $V[\mathbb{G}] \models \langle j''\theta, \tau'' \rangle \text{ is non-metrizable}$.
- ▷ By (0), $M \models \langle j''\theta, \tau'' \rangle \text{ is a sub-space of } \langle j(\theta), j(\tau) \rangle$.
- ▶ Thus, $M \models \text{"there is a non-metrizable subspace } Y \text{ of } j(X) \text{ of cardinality } < j(\kappa)\text{"}$. By elementarity, it follows that $V \models \text{"there is a non-metrizable subspace } Y \text{ of } X \text{ of cardinality } < \kappa\text{"}$. \square

Chart of Large cardinals in [Kanamori 2003]

The arrows indicates direct implications or relative consistency implications, often both.



back

Proposition 2. (Hajnal and Juhász) For a topological space $X = \langle X, \tau \rangle$ and $p \in X$, if $\chi(p, X) = \kappa$ for a regular uncountable κ , then there is a subspace Y of X of cardinality $\leq \kappa$ s.t. $p \in Y$ and $\chi(p, Y) = \kappa$. The condition “of cardinality $\leq \kappa$ ” above is optimal.

Proof. Let θ be sufficiently large regular cardinals and let $M \prec \mathcal{H}(\theta)$ be s.t. $p, X, \tau, \kappa \in M$, $\kappa \subseteq M$ and $|M| = \kappa$.

► Let $Y := X \cap M$. ► Let $\{B_\alpha : \alpha < \kappa\}$ be a neighborhood basis of p in X of size κ s.t. $\langle B_\alpha : \alpha < \kappa \rangle \in M$ (there is such a sequence by elementarity). ▷ The following Claims says implies Y is as desired.

Claim. $\{B_\alpha \cap Y : \alpha < \kappa\}$ is a neighborhood basis of p in Y and no $\{B_\alpha \cap Y : \alpha \in I\}$ for $I \in [\kappa]^{< \kappa}$ is a neighborhood basis of p in Y .

⊢ For the second half of the claim, suppose $I \in [\kappa]^{< \kappa}$. By regularity of κ , there is $\beta < \kappa$ s.t. $I \subseteq \beta$. $\mathcal{H}(\theta)$ knows that $\{B_\alpha : \alpha < \beta\}$ is not a neighborhood basis of p . Thus, M also knows it. It follows that $\{B_\alpha \cap Y : \alpha < \beta\}$ is not a neighborhood basis of p in Y . Hence neither $\{B_\alpha : \alpha \in I\}$.

⊢

(2/2)

Proposition 2. (Hajnal and Juhász) For a topological space $P = \langle P, \tau \rangle$ and $x \in P$, if $\chi(x, X) = \kappa$ for a regular uncountable κ , then there is a subspace Y of X of cardinality $\leq \kappa$ s.t. $x \in Y$ and $\chi(x, Y) = \kappa$. The condition “of cardinality $\leq \kappa$ ” above is optimal.

Proof.

► To see that the condition “of cardinality $\leq \kappa$ ” above is optimal (in particular, that it cannot be replaced by “of cardinality $< \kappa$ ”), consider e.g. $X = \kappa + 1$ with order topology of the canonical ordering of $\kappa + 1$:

▷ $\chi(\kappa, X) = \kappa$.

▷ For any $Y \in [\kappa + 1]^{<\kappa}$ with $\kappa \in Y$, $\chi(\kappa, Y) \leq |Y| < \kappa$.

□ (Proposition 2)

back

Proposition 1. Suppose that $X = \langle X, \tau \rangle$ is non-separable. Then there is a subspace Y of X of cardinality \aleph_1 which is also non-separable.

Proof.

- ▶ Note that $|X| \geq \aleph_1$.
- ▷ Let θ be a sufficiently large regular cardinal with $\langle X, \tau \rangle \in \mathcal{H}(\theta)$. Let $M \prec \mathcal{H}(\theta)$ be internally cofinal (w.r.t. countable sets: M is said to be **internally cofinal** if $[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$ w.r.t. \subseteq) s.t. $\langle X, \tau \rangle \in M$ and $|M| = \aleph_1$. Let $Y := X \cap M$.
- ▷ Then $|Y| = \aleph_1$.
- ▷ We show that Y (as a subspace of X) is non-separable: Suppose that $c \subseteq Y$ is an arbitrary countable set. Since $c \subseteq M$, there is a countable set $a \in M$ with $c \subseteq a \subseteq Y$. By elementarity, $M \models$ “ a is not a dense subset of X ”. Thus, there is $O \in \tau \cap M$ and $y \in X \cap M (= Y)$ s.t. $y \in O$ and $M \models a \cap O = \emptyset$. It follows that $a \cap O = \emptyset$ by elementarity. This shows that a is not a dense subset of Y and hence $c \subseteq a$ is neither a dense subset of Y . \square (Proposition 1)