Resurrection and Recurrence

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References

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Outline

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Eternal Recurrence (永遠の回帰): RcA

ewige Wiederkunft

- For a class $\mathcal{P}$ of p.o.s and a set A (of parameters) the Recurrence Axiom for $\mathcal{P}$ and A ($(\mathcal{P}, A)$-RcA, for short) is the following assertion formulated as an axiom scheme in $\mathcal{L}_\varepsilon$:

$(\mathcal{P}, A)$-RcA : For any $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(\overline{x})$ and $\overline{a} \in A$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$" for a $\mathbb{P} \in \mathcal{P}$, then there is a ground W of the universe V s.t. $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$.

* An inner model W of V is called a ground if there is a p.o. $\mathbb{P} \in \mathsf{W}$ and $(\mathsf{W}, \mathbb{P})$-generic $\mathbb{G} \in \mathsf{V}$ s.t. $\mathsf{V} = \mathsf{W}[\mathbb{G}]$.

- Recurrence Axiom does refer to the Eternal Recurrence in the set theoretic-multiverse
(in terms of the time line expressed by set-generic extension):

$(\mathcal{P}, A)$-RcA **claims:** "if something (formulated with parameters in A) happens in one of the (near) future universes, then it is already happened in a (not so distant) past universe."

* We think that the nearness of a future universe can be measured in inverse proportion to the extent of $\mathcal{P}$.

Eternal Recurrence (永遠の回帰 ewige Wiederkunft): (2/2)

- A natural strengthening of the Recurrence Axiom:

$(\mathcal{P}, A)$-RcA$^+$: For any $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(\overline{x})$ and $\overline{a} \in A$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$" for a $\mathbb{P} \in \mathcal{P}$, then there is a $\mathcal{P}$-ground W of the universe V s.t. $\overline{a} \in$ W and W $\models \varphi(\overline{a})$.

* An inner model W of V is called a $\mathcal{P}$-ground if there is a p.o. $\mathbb{P} \in$ W with $\mathbb{P} \in \mathcal{P}$ and $(\mathsf{W}, \mathbb{P})$-generic $\mathbb{G}$ s.t. W $\models \mathbb{P} \in \mathcal{P}$ and V $=$ W$[\mathbb{G}]$.

$(\mathcal{P}, A)$-RcA$^+$ **can be interpreted as it is saying**:

"if something (formulated with parameters from A) happens in one of the near future universes, then it already happened in a near past universe."

RcA is a variation of known principles

- A non-empty class $\mathcal{P}$ of p.o.s is iterable if it satisfies: ⓪ $\{\mathbb{1}\} \in \mathcal{P}$, ① $\mathcal{P}$ is closed w.r.t. forcing equivalence (i.e. if $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P} \sim \mathbb{P}'$ then $\mathbb{P}' \in \mathcal{P}$), ② closed w.r.t. restriction, and ③ for any $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$, $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" implies $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathcal{P}$.
- For an iterable $\mathcal{P}$, an $\mathcal{L}_\varepsilon$-formula $\varphi(\bar{a})$ with parameters $\bar{a}$ $(\in \mathsf{V})$ is said to be a $\mathcal{P}$-button if there is $\mathbb{P} \in \mathcal{P}$ s.t. for any $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of p.o. with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", we have $\Vdash_{\mathbb{P} * \underset{\sim}{\mathbb{Q}}}$ "$\varphi(\bar{a}^{\checkmark})$".
- ▷ If $\varphi(\bar{a})$ is a $\mathcal{P}$-button then we call $\mathbb{P}$ as above a push of the button $\varphi(\bar{a})$.
- The Maximality Principle $\mathsf{MP}(\mathcal{P}, A)$ introduced in [Hamkins] is the following assertion expressed as an axiom scheme in $\mathcal{L}_\varepsilon$:

$\mathsf{MP}(\mathcal{P}, A)$: For any $\mathcal{L}_\varepsilon$-formula $\varphi(\bar{x})$ and $\bar{a} \in A$, if $\varphi(\bar{a})$ is a $\mathcal{P}$-button then $\varphi(\bar{a})$ holds.

[Hamkins] Joel Hamkins, A simple maximality principle, The Journal of Symbolic Logic Vol.68, no.7, (2003), 527–550.

Proposition 1. Suppose that $\mathcal{P}$ is an iterable class of p.o.s and A a set (of parameters). $(\mathcal{P}, A)$-RcA$^+$ is equivalent to MP$(\mathcal{P}, A)$.

Proof. Suppose that $(\mathcal{P}, A)$-RcA$^+$ holds. We show that MP$(\mathcal{P}, A)$ holds.

- Suppose that $\mathbb{P} \in \mathcal{P}$ is a push of the $\mathcal{P}$-button $\varphi(\overline{a})$. Let $\varphi'(\overline{x})$ be the formula saying (*) for any $\mathbb{Q} \in \mathcal{P}$, $\Vdash_{\mathbb{Q}}$ "$\varphi(\overline{x}^{\checkmark})$" holds.
  - Then we have $\Vdash_{\mathbb{P}}$ "$\varphi'(\overline{a}^{\checkmark})$". By $(\mathcal{P}, A)$-RcA$^+$, there is a $\mathcal{P}$-ground W of V s.t. $\overline{a} \in$ W and W $\models \varphi'(\overline{a})$ holds.
  - By the definition (*) of φ', it follows that V $\models \varphi(\overline{a})$ holds.
- Now suppose that MP$(\mathcal{P}, A)$ holds, and $\mathbb{P} \in \mathcal{P}$ is s.t. $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a}^{\checkmark})$" for $\overline{a} \in A$. Let φ'' be a formula saying:
  (**) there is a $\mathcal{P}$-ground N s.t. $\overline{x} \in N$ and $N \models \varphi(\overline{x})$.
  Then $\varphi''(\overline{a})$ is a $\mathcal{P}$-button and $\mathbb{P}$ is its push.
  By MP$(\mathcal{P}, A)$, $\varphi''(\overline{a})$ holds in V and hence there is a $\mathcal{P}$-ground W of V s.t. $\overline{a} \in$ W and W $\models \varphi(\overline{a})$. This shows that $(\mathcal{P}, A)$-RcA$^+$holds.

$\square$ (Proposition 1)

RcA is a variation of known principles (3/3)

Inner Model Hypothesis (IMH) (Sy D. Friedman) If a property φ holds in an inner model of an outer model, then there is an inner model of the universe which also satisfies the property φ.

- $(\mathcal{P}, A)$-RcA is also equivalent to a set-generic version of S. Friedman's Inner Model Hypothesis with the same parameters $\mathcal{P}$ and A.
- The following Proposition can be proved similarly to Proposition 1:

Proposition 2. For a class $\mathcal{P}$ of p.o.s with $\{\mathbb{1}\} \in \mathcal{P}$ and a set A (of parameters), $(\mathcal{P}, A)$-RcA is equivalent to the following assertion:

For any $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(\overline{x})$ and $\overline{a} \in A$, if a $\mathbb{P} \in \mathcal{P}$ forces "there is a ground M with $\overline{a} \in M$ satisfying $\varphi(\overline{a})$", then there is a ground W of V s.t. $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$.

- These facts in Propositions 1, 2 are also mentioned in [Barton, et al.] as characterizations of variations of the Maximality Principle.

[Barton, et al.] Neil Barton, Andr s Eduardo Caicedo, Gunter Fuchs, Joel David Hamkins, Jonas Reitz, and Ralf Schindler, Inner-Model Reflection Principles, Studia Logica, Vol.108, (2020),573–595.

Solution(s) of Continuum Problem

- For a family Γ of formulas (in $\mathcal{L}_\varepsilon$) let us consider the following weakening of Recurrence Axiom:

$(\mathcal{P}, A)_\Gamma$-RcA : For any Γ-formula $\varphi = \varphi(\overline{x})$ and $\overline{a} \in A$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$" for a $\mathbb{P} \in \mathcal{P}$, then there is a ground W of the universe V s.t. $\overline{a} \in \mathrm{W}$ and $\mathrm{W} \models \varphi(\overline{a})$.

- Let $\kappa_{\mathfrak{refl}} := \max\{\aleph_2, 2^{\aleph_0}\}$.

  $\kappa_{\mathfrak{refl}}$ is a cardinal which appears as the reflection cardinal (cardinal κ s.t. reflection down to $< \kappa$ holds) of many natural reflection principles.

Proposition 3. If $\mathcal{P}$ contains a p.o. which adds a real, as well as a p.o. which (preserves ${\aleph_1}^{\mathrm{V}}$ but) collapses ${\aleph_2}^{\mathrm{V}}$ (e.g. $\mathcal{P}$ = proper p.o.s) then $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA implies $2^{\aleph_0} = \aleph_2$.

Proposition 4. If $\mathcal{P}$ contains a p.o. which preserves ${\aleph_1}^{\mathrm{V}}$ but collapses $\aleph_2$, and also a p.o. which collapses ${\aleph_1}^{\mathrm{V}}$ (e.g. $\mathcal{P}$ = all p.o.s) then $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA implies $2^{\aleph_0} = \aleph_1$.

Solution(s) of Continuum Problem (2/3)

Proposition 3. If $\mathcal{P}$ contains a p.o. which adds a real, as well as a p.o. which (preserves ${\aleph_1}^{\mathsf{V}}$ but) collapses ${\aleph_2}^{\mathsf{V}}$ (e.g. $\mathcal{P}$ = proper p.o.s) then $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA implies $2^{\aleph_0} = \aleph_2$.

Proposition 4. If $\mathcal{P}$ contains a p.o. which preserves ${\aleph_1}^{\mathsf{V}}$ but collapses $\aleph_2$, and also a p.o. which collapses ${\aleph_1}^{\mathsf{V}}$ (e.g. $\mathcal{P}$ = all p.o.s) then $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA implies $2^{\aleph_0} = \aleph_1$.

- In Proposition 3, I put "preserves ${\aleph_1}^{\mathsf{V}}$ but" in parentheses because of the following Lemma 5, (1):

Lemma 5. (1) Suppose that $(\mathcal{P}, \mathcal{H}(\aleph_2))_{\Sigma_1}$-RcA holds. Then all elements of $\mathcal{P}$ are $\aleph_1$-preserving and stationary preserving.
(2) Assume $(\mathcal{P}, A)_{\Sigma_1}$-RcA. If $\mathcal{P}$ contains a p.o. adding a real, then $\mathcal{P}(\omega) \notin A$. If $\mathcal{P}$ contains a p.o. collapsing $\kappa > \omega$ then $\kappa \notin A$.

- Lemma 5 also shows that $\mathcal{H}(\kappa_{\mathfrak{refl}})$ and $\mathcal{H}(2^{\aleph_0})$ in Lemmas 3,4 are maximal possible.

Proof of Propositions 3,4 and Lemma 5.

Solution(s) of Continuum Problem (3/3)

- Recurrence Axioms seem to be quite reasonable requirements.
- If we demand that a maximal (but of course consistent) instance of "Recurrence" should hold then we arrive at either

Recurrence Axioms are monotonic in parameters

(1) (all p.o.s, $\mathcal{H}(2^{\aleph_0})$)-RcA, or

(2) (semi-proper p.o.s, $\mathcal{H}(\kappa_{\mathfrak{refl}})$)-RcA.

(see Lemma 5). (1) and (2) are incompatible: By Proposition 3 , (1) implies CH while (2) implies $2^{\aleph_0} = \aleph_2$ by Proposition 4 .

- The conflict between (1) and (2) above can be (almost) resolved by considering:

(1)' (all p.o.s, H)-RcA for a reasonable $H \subseteq \mathcal{H}(2^{\aleph_0})$, and

(2) (semi-proper p.o.s, $\mathcal{H}(\kappa_{\mathfrak{refl}})$)-RcA.

- This combination is consistent (e.g. modulo 2-huge) and follows from an axiom (introduced later) which also implies almost all known "preferable" axioms like MM^{++} and a strong form of Resurrection Axiom.
- Note that (1)' + (2) implies $2^{\aleph_0} = \aleph_2$.

Consistency strength of RcA

- Maximality Principles and hence also Recurrence Axioms have relatively low consistency strength.

Theorem 6. ([Hamkins]) The following theories are equiconsistent to each other and they are also equiconsistent with ZFC + there are stationarily many inaccessibles:

ZFC + MP(all p.o.s, $\mathcal{H}(\omega_1)$),

ZFC + MP(c.c.c p.o.s, $\mathcal{H}(2^{\aleph_0})$),

ZFC + MP(proper p.o.s, $\mathcal{H}(2^{\aleph_0})$).

[Hamkins] Joel Hamkins, A simple maximality principle, The Journal of Symbolic Logic Vol.68, no.7, (2003), 527–550.

- Maximality Principles increase the consistency strength of large cardinals.

Example 7. Suppose that $(\mathcal{P}, \emptyset)$-RcA holds for a class $\mathcal{P}$ of p.o.s s.t. $\mathcal{P}$ contains enough collapsing p.o.s. If there is an inaccessible cardinal then there are class many inaccessible cardinal. $\Box$

Tightly $\mathcal{P}$-Laver-gen. ultrahuge cardinal

- For an iterable class $\mathcal{P}$ of p.o.s, a cardinal κ is said to be (tightly) $\mathcal{P}$-Laver-generically ultrahuge ((tightly) $\mathcal{P}$-Laver-gen. ultrahuge, for short), if, for any $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", s.t. for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are $j, M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{P}, \mathbb{H}, (V_{j(\lambda)})^{\mathsf{V}[\mathbb{H}]} \in M$ and $|\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,| \leq j(\kappa)$ (more precisely: $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a p.o. of size $\leq j(\kappa)$).

Theorem 8. ([S.F. & Usuba]) Suppose that κ is tightly $\mathcal{P}$-Laver-gen. ultrahuge for an iterable class $\mathcal{P}$. Then $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_2}$-RcA$^+$ holds.

Proof of Theorem 8.

Theorem 9. ([S.F.]) Tightly $\mathcal{P}$-Laver-gen. ultrahugeness does not imply $\mathsf{MP}(\mathcal{P}, \emptyset)$ (under the assumption of a large cardinal slightly more than the ultrahugeness).

- The proof of Theorem 9 can be modified to prove the non-implication of $(\mathcal{P}, \emptyset)_{\Pi_3}$-RcA from a generic large cardinal. In particular "Σ_2" in Theorem 8 is optimal.

Tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver-gen. ultrahuge cardinal

- The following strengthening of tightly $\mathcal{P}$-Laver-gen. ultrahugeness of κ (which is formulated in an axiom scheme) implies $\mathrm{MP}(\mathcal{P}, \mathcal{H}(\kappa))$.
- For a natural number n, we call a cardinal κ super $C^{(n)}$-hyperhuge if for any $\lambda_0 > \kappa$ there are $\lambda \geq \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, and j, $M \subseteq \mathsf{V}$ s.t. $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$, $j(\kappa) > \lambda$, ${}^{j(\lambda)}M \subseteq M$ and $V_{j(\lambda)} \prec_{\Sigma_n} \mathsf{V}$.
- κ is super $C^{(n)}$-ultrahuge if the condition above holds with "${}^{j(\lambda)}M \subseteq M$" replaced by "${}^{j(\kappa)}M \subseteq M$ and $V_{j(\lambda)} \subseteq M$".
  - ▷ If κ is super $C^{(n)}$-hyperhuge then it is super $C^{(n)}$-ultrahuge.
- We shall also say that κ is super $C^{(\infty)}$-hyperhuge (super $C^{(\infty)}$-ultrahuge, resp.) if it is super $C^{(n)}$-hyperhuge (super $C^{(n)}$-ultrahuge, resp.) for all natural number n.
- A similar kind of strengthening of the notions of large cardinals which we call here "super $C^{(n)}$" appears also in Boney [Boney]. It is called "$C^{(n)+}$", and is considered there in connection with extendibility.

[Boney] Will Boney, Model Theoretic Characterizations of Large Cardinals, Israel Journal of Mathematics, 236, (2020), 133–181.

Tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver-gen. ultrahuge cardinal (2/6)

- For a natural number n and an iterable class $\mathcal{P}$ of p.o.s, a cardinal κ is super $C^{(n)}$- $\mathcal{P}$-Laver-generically ultrahuge (super $C^{(n)}$- $\mathcal{P}$-Laver-gen. ultrahuge, for short) if, for any $\lambda_0 > \kappa$ and for any $\mathbb{P} \in \mathcal{P}$, there are a $\lambda \geq \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, a $\mathcal{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", and j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$, $j(\kappa) > \lambda$, $\mathbb{P}$, $\mathbb{H}$, ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$ and ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$.
- ▷ A super $C^{(n)}$- $\mathcal{P}$-Laver-generically ultrahuge cardinal κ is tightly super $C^{(n)}$- $\mathcal{P}$-Laver-generically ultrahuge (tightly super $C^{(n)}$- $\mathcal{P}$-Laver-gen. ultrahuge, for short), if $|\mathbb{P} * \underset{\sim}{\mathbb{Q}}| \leq j(\kappa)$.
- Super $C^{(\infty)}$- $\mathcal{P}$-Laver-gen. ultrahugeness and tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver gen. ultrahugeness are defined similarly to super $C^{(\infty)}$-ultrahugeness.
- Note that, in general, super $C^{(\infty)}$-hyperhugeness and super $C^{(\infty)}$-ultrahugeness are notions unformalizable in the language of ZFC without introducing a new constant symbol for κ since we need infinitely many $\mathcal{L}_\varepsilon$-formulas to formulate them.
- ▷ Exceptions are ...

Tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver-gen. ultrahuge cardinal (3/6)

- Exceptions are when we are talking about a cardinal in a set model being with one of these properties, or when we are talking about a cardinal definable in V having these properties in an inner model. In the latter case, the situation is formalizable with infinitely may $\mathcal{L}_\varepsilon$-sentences.
- In contrast, the super $C^{(\infty)}$-$\mathcal{P}$-Laver gen. ultrahugeness of κ is expressible in infinitely many $\mathcal{L}_\varepsilon$-sentences. This is because a $\mathcal{P}$-Laver gen. large cardinal κ for relevant classes $\mathcal{P}$ of p.o.s is uniquely determined as $\kappa_{\mathfrak{refl}}$ or $2^{\aleph_0}$ (see e.g. [II] or [S.F.]).

Theorem 10. ([S.F. & Usuba]) Suppose that $\mathcal{P}$ is an iterable class of p.o.s and κ is super $C^{(\infty)}$-$\mathcal{P}$-Laver-gen. ultrahuge. Then $(\mathcal{P}, \mathcal{H}(\kappa))$-$\mathsf{RcA}^+$ holds.

Proof. Similarly to Theorem 8. □

- Consistency of tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver-gen. ultrahuge cardinal for reasonable $\mathcal{P}$ follows from 2-huge.

Lemma 11. ([S.F. & Usuba]) Suppose that κ is 2-huge with the 2-huge elementary embedding j, that is, $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M \subseteq \mathsf{V}$, for some $M \subseteq \mathsf{V}$ and ${}^{j^2(\kappa)}M \subseteq M$. Then $V_{j(\kappa)} \models$ "κ is super $C^{(\infty)}$-hyperhuge cardinal", and for each $n \in \omega$, $V_{j(\kappa)} \models$ "there are stationarily many super $C^{(n)}$-hyperhuge cardinals".

Theorem 12. ([S.F. & Usuba]) Suppose that μ is an inaccessible cardinal and κ is super $C^{(\infty)}$-hyperhuge in V_μ. Then there is a Laver function $f : \kappa \to V_\kappa$ for super $C^{(\infty)}$-hyperhugeness in V_μ.

Theorem 13. ([S.F. & Usuba]) (1) Suppose that μ is inaccessible and $\kappa < \mu$ is super $C^{(\infty)}$-ultrahuge in V_μ. Let $\mathbb{P} = \mathrm{Col}(\aleph_1, \kappa)$. Then, in $V_\mu[\mathbb{G}]$, for any $V_\mu, \mathbb{P}$-generic $\mathbb{G}$, $\aleph_2^{V_\mu[\mathbb{G}]}$ $(= \kappa)$ is tightly super $C^{(\infty)}$-σ-closed-Laver-gen. ultrahuge and CH holds.

(2) Suppose that μ is inaccessible and $\kappa < \mu$ is super $C^{(\infty)}$-ultrahuge with a Laver function $f : \kappa \to V_\kappa$ for super $C^{(\infty)}$-ultrahugeness in V_μ. If $\mathbb{P}$ is the CS-iteration of length κ for forcing PFA along with f, then, in $V_\mu[\mathbb{G}]$ for any $(V_\mu, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{V_\mu[\mathbb{G}]}$ $(= \kappa)$ is tightly super $C^{(\infty)}$-proper-Laver-gen. ultrahuge and $2^{\aleph_0} = \aleph_2$ holds.

(2′) Suppose that μ is inaccessible and $\kappa < \mu$ is super $C^{(\infty)}$-ultrahuge with a Laver function $f : \kappa \to V_\kappa$ for super $C^{(\infty)}$-ultrahugeness in V_μ. If $\mathbb{P}$ is the RCS-iteration of length κ for forcing MM along with f, then, in $V_\mu[\mathbb{G}]$ for any $(V_\mu, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{V_\mu[\mathbb{G}]}$ $(= \kappa)$ is tightly super $C^{(\infty)}$-semi-proper-Laver-gen. ultrahuge and $2^{\aleph_0} = \aleph_2$ holds.

(3) Suppose that μ is inaccessible and κ is super $C^{(\infty)}$-ultrahuge with a Laver function $f : \kappa \to V_\kappa$ for super $C^{(\infty)}$-ultrahugeness in V_μ. If $\mathbb{P}$ is a FS-iteration of length κ for forcing MA along with f, then, in $V_\mu[\mathbb{G}]$ for any $(V_\mu, \mathbb{P})$-generic $\mathbb{G}$, $2^{\aleph_0}$ $(= \kappa)$ is tightly super $C^{(\infty)}$-c.c.c.-Laver-gen. ultrahuge, and κ is very large in $V_\mu[\mathbb{G}]$.

(4) Suppose that μ is inaccessible and κ is super $C^{(\infty)}$-ultrahuge with a Laver function $f : \kappa \to V_\kappa$ for super $C^{(\infty)}$-ultrahugeness in V_μ. If $\mathbb{P}$ is a FS-iteration of length κ along with f enumerating "all" p.o.s, then, in $V_\mu[\mathbb{G}]$ for any $(V_\mu, \mathbb{P})$-generic $\mathbb{G}$, $2^{\aleph_0}$ $(= \aleph_1)$ is tightly super $C^{(\infty)}$-all p.o.s-Laver-gen. ultrahuge, and CH holds.

$\square$

Bedrock of tightly $\mathcal{P}$-gen. hyperhuge cardinal

- Recall that a cardinal κ is hyperhuge, if for every $\lambda > \kappa$, there is $j : \mathsf{V} \overset{\prec}{\to}_{\kappa} M \subseteq \mathsf{V}$ s.t. $\lambda < j(\kappa)$ and ${}^{j(\lambda)}M \subseteq M$. A hyperhuge cardinal κ can be characterized in terms of existence of κ-complete normal ultrafilters with certain additional properties (e.g. see [S.F. & Usuba]).
- For a class $\mathcal{P}$ of p.o.s, a cardinal κ is tightly $\mathcal{P}$-generic hyperhuge (tightly $\mathcal{P}$-gen. hyperhuge, for short) if for any $\lambda > \kappa$, there is $\mathbb{Q} \in \mathcal{P}$ s.t. for a $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \overset{\prec}{\to}_{\kappa} M$, $\lambda < j(\kappa)$, $|\,\mathbb{Q}\,| \leq j(\kappa)$, and $j''j(\lambda), \mathbb{H} \in M$.
- For a class $\mathcal{P}$ of p.o.s, a cardinal κ is tightly $\mathcal{P}$-Laver-generically hyperhuge (tightly $\mathcal{P}$-Laver-gen. hyperhuge, for short) if for any $\lambda > \kappa$, and $\mathbb{P} \in \mathcal{P}$ there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t. for a $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \overset{\prec}{\to}_{\kappa} M$, $\lambda < j(\kappa)$, $|\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,| \leq j(\kappa)$, and $j''j(\lambda), \mathbb{H} \in M$.

Bedrock of tightly $\mathcal{P}$-gen. hyperhuge cardinal (2/6)

For an iterable $\mathcal{P}$:

hyperhuge $\rightarrow$ tightly $\mathcal{P}$-gen. hyperhuge

tightly $\mathcal{P}$-Laver gen. hyperhuge $\rightarrow$ tightly $\mathcal{P}$-gen. hyperhuge

tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver gen. hyperhuge $\rightarrow$ tightly $\mathcal{P}$-Laver gen. hyperhuge

tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver gen. hyperhuge $\rightarrow$ tightly super C^{∞}-$\mathcal{P}$-Laver gen. ultrahuge

Bedrock of tightly $\mathcal{P}$-gen. hyperhuge cardinal (3/6)

- For a cardinal κ, a ground W of the universe V is called a $\leq\kappa$-ground if there is a p.o. $\mathbb{P} \in$ W of cardinality $\leq\kappa$ (in the sense of V) and (W, $\mathbb{P}$)-generic filter $\mathbb{G}$ s.t. V = W[$\mathbb{G}$].
- Let

$$\overline{\mathsf{W}} := \bigcap\{\mathsf{W} : \mathsf{W} \text{ is a } \leq\kappa\text{-ground}\}.$$

Since there are only set many $\leq\kappa$-grounds, $\overline{W}$ contains a ground by Theorem 1.3 in [Usuba]. We shall call $\overline{\mathsf{W}}$ defined above the $\leq\kappa$-mantle of V.

- The following theorem generalizes Theorem 1.6 in [Usuba].

Theorem 14. ([S.F. & Usuba]) Suppose that $\mathcal{P}$ is any class of p.o.s. If κ is a tightly $\mathcal{P}$-gen. hyperhuge cardinal, then the $\leq\kappa$-mantle is the smallest ground of V (i.e. it is the bedrock of V) and it is also a $\leq\kappa$-ground.

[Usuba] Toshimichi Usuba, The downward directed grounds hypothesis and very large cardinals, Journal of Mathematical Logic, Vol. 17(2) (2017), 1–24.

Bedrock of tightly $\mathcal{P}$-gen. hyperhuge cardinal (4/6)

Theorem 14. ([S.F. & Usuba]) Suppose that $\mathcal{P}$ is any class of p.o.s. If κ is a tightly $\mathcal{P}$-gen. hyperhuge cardinal, then the $\leq\kappa$-mantle is the smallest ground of V (i.e. it is the bedrock of V) and it is also a $\leq\kappa$-ground.

A very rough sketch of the Proof.

- Analyzing the proof of Theorem 14, we also obtain:

Theorem 15. ([S.F. & Usuba]) Suppose that $\mathcal{P}$ is any class of p.o.s. If κ is a tightly $\mathcal{P}$-gen. hyperhuge cardinal, then κ is a hyperhuge cardinal in the bedrock $\overline{W}$ of V.

Theorem 16. ([S.F. & Usuba]) Suppose that $\mathcal{P}$ is any class of p.o.s. If κ is a tightly super $C^{(n)}$-$\mathcal{P}$-gen. hyperhuge cardinal, then κ is a super C^n hyperhuge cardinal in the bedrock $\overline{W}$ of V.

- These Theorems have many strong consequences. Some of them are ...

Corollary 17.([S.F. & Usuba]) Suppose that $\mathcal{P}$ is the class of all p.o.s. Then the following theories are equiconsistent:

(a)ZFC + “there is a hyperhuge cardinal”.

(b)ZFC + “there is a tightly $\mathcal{P}$-Laver gen. hyperhuge cardinal”.

(c)ZFC + “there is a tightly $\mathcal{P}$-gen. hyperhuge cardinal”.

(d)ZFC + “bedrock $\overline{\mathsf{W}}$ exists and ω_1 is a hyperhuge cardinal in $\overline{\mathsf{W}}$”. □

Corollary 18.([S.F. & Usuba]) Suppose that $\mathcal{P}$ is one of the following classes of p.o.s: all semi-proper p.o.s; all proper p.o.s; all ccc p.o.s; all σ-closed p.o.s. Then the following theories are equiconsistent:

(a)ZFC + “there is a hyperhuge cardinal”.

(b)ZFC + “there is a tightly $\mathcal{P}$-Laver gen. hyperhuge cardinal”.

(c)ZFC + “there is a tightly $\mathcal{P}$-gen. hyperhuge cardinal”.

(d)ZFC + “bedrock $\overline{\mathsf{W}}$ exists and $\kappa_{\mathfrak{refl}}$ is a hyperhuge cardinal in $\overline{\mathsf{W}}$”.

□

Cf.: Theorem 13 , and Theorem 16 .

Equiconsistency as the Eternal Recurrence (2/2)

Corollary 19.([S.F. & Usuba]) Suppose that $\mathcal{P}$ is the class of all p.o.s. Then the following theories are equiconsistent:

(a) ZFC + "there is a tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver gen. hyperhuge cardinal".

(b) ZFC + "bedrock $\overline{\mathsf{W}}$ exists and ω_1^{V} is a super $C^{(\infty)}$-hyperhuge cardinal in $\overline{\mathsf{W}}$".

Corollary 20.([S.F. & Usuba]) Suppose that $\mathcal{P}$ is one of the following classes of p.o.s: all semi-proper p.o.s; all proper p.o.s; all ccc p.o.s; all σ-closed p.o.s. Then the following theories are equiconsistent:

(a) ZFC + "there is a tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver gen. hyperhuge cardinal".

(b) ZFC + "bedrock $\overline{\mathsf{W}}$ exists and ${\kappa_{\mathfrak{refl}}}^{\mathsf{V}}$ is a super $C^{(\infty)}$-hyperhuge cardinal in $\overline{\mathsf{W}}$".

Toward the Laver-generic Maximum

- The existence of tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver gen. superhuge cardinal for the class $\mathcal{P}$ of all semi-proper p.o.s is one of the strongest principle we considered so far. It implies the tightly super $C^{(\infty)}$-$\mathcal{P}$-Laver gen. superhuge cardinal is $2^{\aleph_0} = \aleph_2$ and MM^{++} holds (see [II] or [S.F.]), the existence of the bedrock (Theorem 14), and $(\mathcal{P}, \mathcal{H}(\aleph_2))$-RcA$^+$ (Theorem 10).
  - ▷ MM^{++} implies many preferable set-theoretic axioms/principles including Woodin's (*) ([Aspero-Schindler]).

[Aspero-Schindler] David Asperó, and Ralf Schindler, Martin's Maximum++ implies Woodin's axiom (*). Annals of Mathematics, 193(3), (2021), 793-835.

  - ▷ $(\mathcal{P}, \mathcal{H}(\aleph_2))$-RcA$^+$ claims that any property (even with any subset of ω_1 as parameter) forcable by a semi-proper p.o., is a theorem in some semi-proper ground. E.g. Cichón's Maximum is what happens in a semi-proper ground.

- Strong forms of Resurrection Axiom are also consequences of the existence of the super $C^{(\infty)}$-(semi-proper)-Laver gen. large cardinal:

Toward the Laver-generic Maximum (2/4)

- Suppose that $\mathcal{P}$ is a class of p.o.s and $\mu^\bullet$ is a definition of a cardinal (e.g. "$\aleph_1$", "$\aleph_2$", "$2^{\aleph_0}$")
  - The following boldface version of the Resurrection Axioms is considered in [Hamkins-Johnstone]:

$\mathbb{RA}^{\mathcal{P}}_{\mathcal{H}(\mu^\bullet)}$: For any $A \subseteq \mathcal{H}(\mu^\bullet)$ and any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of p.o. s.t. $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there is $A^* \subseteq \mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}$ s.t. $(\mathcal{H}(\mu^\bullet)^{\mathsf{V}}, A, \in) \prec (\mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}, A^*, \in)$.

Theorem 21. [S.F.] For an iterable class of p.o.s $\mathcal{P}$, if $\kappa_{\mathfrak{refl}}$ is tightly $\mathcal{P}$-Laver-gen. superhuge, then $\mathbb{RA}^{\mathcal{P}}_{\mathcal{H}(\kappa_{\mathfrak{refl}})}$ holds. $\square$

[Hamkins-Johnstone] Joel David Hamkins, and Thomas A. Johnstone, Strongly uplifting cardinals and the boldface resurrection axioms, Archive for Mathematical Logic Vol.56, (2017), 1115–1133.

Toward the Laver-generic Maximum (3/4)

- With a Lever-genricity corresponding to a larger large cardinal, we obtain the "tight" version of Unbounded Resurrection Principle in [Tsaprounis]:

TUR($\mathcal{P}$) : For any $\lambda > \kappa_{\mathfrak{refl}}$, and $\mathbb{P} \in \mathcal{P}$, there exists a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$"$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t., for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-gen. $\mathbb{H}$, there are $\lambda^* \in \mathsf{On}$, and $j_0 \in \mathsf{V}[\mathbb{H}]$ s.t. $j_0 : \mathcal{H}(\lambda)^{\mathsf{V}} \overset{\prec}{\rightarrow}_{\kappa_{\mathfrak{refl}}} \mathcal{H}(\lambda^*)^{\mathsf{V}[\mathbb{H}]}$, $j_0(\kappa_{\mathfrak{refl}}) > \lambda$, and $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a p.o. of size $j_0(\kappa_{\mathfrak{refl}})$.

Theorem 22. [S.F.] For an iterable class $\mathcal{P}$, if $\kappa_{\mathfrak{refl}}$ is tightly $\mathcal{P}$-Laver gen. ultrahuge, then TUR($\mathcal{P}$) holds.

[Tsaprounis] Tsaprounis, On resurrection axioms, The Journal of Symbolic Logic, Vol.80, No.2, (2015), 587–608.

Toward the Laver-generic Maximum (4/4)

- We can even establish the consistency of:
  - $2^{\aleph_0}$ is tightly super $C^{(\infty)}$-(semi-proper)-Laver gen. superhuge $+$ (all p.o.s, $\mathcal{H}(\aleph_1)^{\overline{W}}$)-RcA

A construction of a model: Work in a model V_λ where κ is super $C^{(\infty)}$ hyperhuge. Then $V_\kappa \prec V_\lambda$. Take an inaccessible $\delta < \kappa$ with $V_\delta \prec V_\lambda$. Use this to force (all p.o.s, $\mathcal{H}(\aleph_1)$)-RcA. κ is still super $C^{(\infty)}$ hyperhuge in the generic extension, so we can use it to force $2^{\aleph_0}$ to be tightly super $C^{(\infty)}$-(semi-proper)-Laver gen. superhuge. (all p.o.s, $\mathcal{H}(\aleph_1)^{\overline{W}}$)-RcA survives this forcing.

- **Open Problems:**
  - Is there any natural axiom which would imply the combination of the principles above?
  - A (possibly) related question: Is there anything similar to HOD dichotomy for the bedrock under a (tightly generic/tightly Laver-generic) very large cardinal?

Thank you for your attention!

ご清聴ありがとうございました.

관심을 가져 주셔서 감사합니다

Σας ευχαριστώ για την προσοχή σας.

Dziękuję za uwagę.

Ich danke Ihnen für Ihre Aufmerksamkeit.

Recurrence Axioms are monotonic in parameters

- For classes of p.o.s $\mathcal{P}$, $\mathcal{P}'$ and sets A, A' of parameters, if $\mathcal{P} \subseteq \mathcal{P}'$ and $A \subseteq A'$, then we have

  $$(\mathcal{P}', A')\text{-RcA} \;\Rightarrow\; (\mathcal{P}, A)\text{-RcA}.$$

- Note that, in general, we do not have similar implication between $\mathrm{MP}(\mathcal{P}, A)$ and $\mathrm{MP}(\mathcal{P}', A')$.

back

Proof of Propositions 3,4 and Lemma 5.

Proposition 3. If $\mathcal{P}$ contains a p.o. which adds a real, as well as a p.o. which (preserves ${\aleph_1}^{\mathsf{V}}$ but) collapses ${\aleph_2}^{\mathsf{V}}$ (e.g. $\mathcal{P}$ = proper p.o.s) then $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA implies $2^{\aleph_0} = \aleph_2$.

Proof. Suppose that $\mathcal{P}$ is as above and $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA holds.

- $2^{\aleph_0} \geq \aleph_2$: Otherwise CH holds. Then $\mathcal{P}(\omega)^{\mathsf{V}} \in \mathcal{H}(\kappa_{\mathfrak{refl}})$. Hence "$\exists x\,(x \subseteq \omega \wedge x \notin \mathcal{P}(\omega)^{\mathsf{V}})$" is a Σ_1-formula with parameters from $\mathcal{H}(\kappa_{\mathfrak{refl}})$ and $\mathbb{P} \in \mathcal{P}$ adding a real forces (the formula in forcing language corresponding to) this formula.
- By $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA, the formula must hold in a ground. This is a contradiction.

- $2^{\aleph_0} \leq \aleph_2$: If $2^{\aleph_0} > \aleph_2$ then ${\aleph_1}^{\mathsf{V}}, {\aleph_2}^{\mathsf{V}} \in \mathcal{H}(2^{\aleph_0}) \subseteq \mathcal{H}(\kappa_{\mathfrak{refl}})$. Let $\mathbb{P} \in \mathcal{P}$ be a p.o. which preserves $\aleph_1$ but collapses $\aleph_2$.
- Letting $\psi(x, y)$ a Σ_1-formula saying "$\exists f\,(f$ is a surjection from x to $y)$", we have $\Vdash_{\mathbb{P}}$"$\psi(({\aleph_1}^{\mathsf{V}})^{\checkmark}, ({\aleph_2}^{\mathsf{V}})^{\checkmark})$".
- By $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA, the formula $\psi({\aleph_1}^{\mathsf{V}}, {\aleph_2}^{\mathsf{V}})$ must hold in a ground. This is a contradiction. $\square$

back

Proof of Propositions 3,4 and Lemma 5. (2/3)

Proposition 4. If $\mathcal{P}$ contains a p.o. which preserves ${\aleph_1}^{\mathsf{V}}$ but collapses $\aleph_2$, and also a p.o. which collapses ${\aleph_1}^{\mathsf{V}}$ (e.g. $\mathcal{P} =$ all p.o.s) <u>then</u> $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA implies $2^{\aleph_0} = \aleph_1$.

Proof. We have $2^{\aleph_0} \le \aleph_2$, by the second half of the proof of Proposition 3.

- If $2^{\aleph_0} = \aleph_2$, then ${\aleph_1}^{\mathsf{V}} \in \mathcal{H}(2^{\aleph_0})$.
- Let $\mathbb{P} \in \mathcal{P}$ be a p.o. collapsing ${\aleph_1}^{\mathsf{V}}$. I.e. $\Vdash_{\mathbb{P}}$"${\aleph_1}^{\mathsf{V}}$ is countable". Since "$\cdots$ is countable" is Σ_1, there is a ground M s.t. $M \models$"${\aleph_1}^{\mathsf{V}}$ is countable". This is a contradiction. $\Box$ (Proposition 4)

back

Proof of Propositions 3,4 and Lemma 5. (3/3)

Lemma 5. (1) Suppose that $(\mathcal{P}, \mathcal{H}(\aleph_2))_{\Sigma_1}$-RcA holds. Then all elements of $\mathcal{P}$ are $\aleph_1$-preserving and stationary preserving.

(2) Assume $(\mathcal{P}, A)_{\Sigma_1}$-RcA. If $\mathcal{P}$ contains a p.o. adding a real, then $\mathcal{P}(\omega) \notin A$. If $\mathcal{P}$ contains a p.o. collapsing $\kappa > \omega$ then $\kappa \notin A$.

Proof. (1): Suppose otherwise and $\mathbb{P} \in \mathcal{P}$ is s.t. $\Vdash_{\mathbb{P}}$“ ${\aleph_1}^{\mathsf{V}}$ is countable”. Note that $\omega, \aleph_1 \in \mathcal{H}(\kappa_{\mathfrak{refl}})$.

- By $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA, it follows that there is a ground W of V s.t. W $\models$“ ${\aleph_1}^{\mathsf{V}}$ is countable”. This is a contradiction.
- Suppose that $\mathbb{P} \in \mathcal{P}$ destroy the stationarity of $S \subseteq \omega_1$. Note that ω_1, $S \in \mathcal{H}(\aleph_2)$. Let $\varphi = \varphi(y, z)$ be the Σ_1-formula
  $$\exists x\,(y \text{ is a club subset of the ordinal } y \text{ and } z \cap x = \emptyset).$$
  Then we have $\Vdash_{\mathbb{P}}$“ $\varphi(\omega_1, S)$ ”. By $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA, it follows that there is a ground W $\subseteq$ V s.t. $S \in$ W and W $\models \varphi(\omega_1, S)$. This is a contradiction.

(2): By the first part of the proof of Proposition 3 , and the proof of Proposition 4 . □ (Lemma 5)

Proof of Theorem 8.

Theorem 8. ([S.F. & Usuba]) Suppose that κ is tightly $\mathcal{P}$-Laver-gen. ultrahuge for an iterable class $\mathcal{P}$. Then $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_2}$-RcA$^+$ holds.

Proof. We use the following

Lemma 8a. If α is a limit ordinal and V_α satisfies a large enough fragment of ZFC, then for any $\mathbb{P} \in V_\alpha$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, we have $V_\alpha[\mathbb{G}] = {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$.

- Assume that κ is tightly $\mathcal{P}$-Laver gen. ultrahuge for an iterable class $\mathcal{P}$ of p.o.s. ▷ Suppose that $\varphi = \varphi(x)$ is Σ_2 formula (in $\mathcal{L}_\varepsilon$), $a \in \mathcal{H}(\kappa)$, and $\mathbb{P} \in \mathcal{P}$ is s.t.

  (a) $\mathsf{V} \models \ \Vdash_{\mathbb{P}}$ "$\varphi(\check{a})$".

- Let $\lambda > \kappa$ be s.t. $\mathbb{P} \in \mathsf{V}_\lambda$ and

  (0) $V_\lambda \prec_{\Sigma_n} \mathsf{V}$ for a sufficiently large n.

  In particular, we may assume that we have chosen the n above so that a sufficiently large fragment of ZFC holds in V_λ in the sense of Lemma 8a.

Proof of Theorem 8. (2/3)

Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name s.t. $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", and for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with

(1) $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa} M$,

(2) $j(\kappa) > \lambda$,

(3) $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{P}$, $\mathbb{H}$, ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$, and

(4) $|\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,| \leq j(\kappa)$.

By (4), we may assume that the underlying set of $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is $j(\kappa)$ and $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in {V_{j(\lambda)}}^{\mathsf{V}}$.

Let $\mathbb{G} := \mathbb{H} \cap \mathbb{P}$. Note that $\mathbb{G} \in M$ by (3) and we have

(5) $${V_{j(\lambda)}}^{M} \underbrace{=}_{\text{by (3)}} {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \overbrace{=}^{\substack{\text{Since } {V_{j(\lambda)}}^{M} \ (= V_{j(\lambda)}^{\mathsf{V}[\mathbb{H}]}) \text{ satisfies a sufficiently large fragment of ZFC} \\ \text{by elementarity of } j \text{, and hence the equality follows by Lemma 8a}}} {V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{H}].$$

Thus, by (3) and by the definability of grounds, we have ${V_{j(\lambda)}}^{\mathsf{V}} \in M$ and ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] \in M$.

Proof of Theorem 8. (3/3)

Claim 8b. ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] \models \varphi(a)$.

$\vdash$ By Lemma 8a , ${V_\lambda}^{\mathsf{V}}[\mathbb{G}] = {V_\lambda}^{\mathsf{V}[\mathbb{G}]}$, and ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] = {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{G}]}$ by (5) .
By (0) , both ${V_\lambda}^{\mathsf{V}}[\mathbb{G}]$ and $V^{\mathsf{V}}_{j(\lambda)}[\mathbb{G}]$ satisfy large enough fragment of ZFC. Thus

(6) ${V_\lambda}^{\mathsf{V}}[\mathbb{G}] \prec_{\Sigma_1} {V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}]$.

By (a) and (0) we have ${V_\lambda}^{\mathsf{V}}[\mathbb{G}] \models \varphi(a)$. By (6) and since φ is Σ_2, it follows that ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] \models \varphi(a)$. $\dashv$ (Claim 8b.)

Thus we have

(7) $M \models$“ there is a $\mathcal{P}$-ground N of $V_{j(\lambda)}$ s.t. $N \models \varphi(a)$”.

By the elementarity (1), it follows that

(6) $\mathsf{V} \models$“ there is a $\mathcal{P}$-ground N of V_λ s.t. $N \models \varphi(a)$”.

Now by (0) , it follows that there is a $\mathcal{P}$-ground W of V s.t. $\mathsf{W} \models \varphi(a)$. ⧉ (Theorem 8)⧉

A very rough sketch of the Proof of Theorem 14.

Theorem 14. ([S.F. & Usuba]) Suppose that $\mathcal{P}$ is any class of p.o.s. If κ is a tightly $\mathcal{P}$-gen. hyperhuge cardinal, then the $\leq\kappa$-mantle is the smallest ground of V (i.e. it is the bedrock of V) and it is also a $\leq\kappa$-ground.

A rough sketch of the Proof.

- Suppose that κ is tightly $\mathcal{P}$-gen. hyperhuge and let $\overline{\mathsf{W}}$ be the $\leq\kappa$-mantle.
- By Theorem 1.3 in [Usuba], it is enough to show that, for any ground $\mathsf{W}\subseteq\overline{\mathsf{W}}$ is actually a $\leq\kappa$-ground and hence $\mathsf{W}=\overline{\mathsf{W}}$ holds.
- Let $\mathsf{W}\subseteq\overline{\mathsf{W}}$ be a ground. Let μ be the cardinality (in the sense of V) of a p.o. $\mathbb{S}\in\mathsf{W}$ s.t. there is a $(\mathsf{W},\mathbb{S})$-generic $\mathbb{F}$ s.t. $\mathsf{V}=\mathsf{W}[\mathbb{F}]$. W.l.o.g., $\mu\geq\kappa$.
- By Laver-Woodin Theorem, there is $r\in\mathsf{V}$ s.t. $\mathsf{W}=\Phi(\cdot,r)^{\mathsf{V}}$ for an $\mathcal{L}_\varepsilon$-formula Φ.
- Let $\theta\geq\mu$ be s.t. $r\in V_\theta$, and for a sufficiently large natural number n, we have ${V_\theta}^{\mathsf{V}}\prec_{\Sigma_n}\mathsf{V}$. By the choice of θ, $\Phi(\cdot,r)^{{V_\theta}^{\mathsf{V}}}=\Phi(\cdot,r)^{\mathsf{V}}\cap{V_\theta}^{\mathsf{V}}=\mathsf{W}\cap{V_\theta}^{\mathsf{V}}$ $={V_\theta}^{\mathsf{W}}$. Let $\mathbb{Q}\in\mathcal{P}$ s.t. for $(\mathsf{V},\mathbb{Q})$-generic $\mathbb{H}$, there are j, $M\subseteq\mathsf{V}[\mathbb{H}]$ with $j:\mathsf{V}\overset{\prec}{\to}_\kappa M$, $\theta<j(\kappa)$, $|\mathbb{Q}|\leq j(\kappa)$, ${V_{j(\theta)}}^{\mathsf{V}[\mathbb{H}]}\subseteq M$, and $\mathbb{H}$, $j''j(\theta)\in M$.
  ... (back and forth with j) ... Thus ${V_\theta}^{\overline{\mathsf{W}}}\subseteq{V_\theta}^{\mathsf{W}}$. Since θ can be arbitrary large, It follows that $\overline{\mathsf{W}}\subseteq\mathsf{W}$.

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