Generic Absoluteness Revisited

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References

[II] S.F., A. Ottenbreit Maschio Rodrigues, and H. Sakai, Strong downward Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum, Archive for Mathematical Logic, Vol.60, 3-4, (2021), 495–523. https://fuchino.ddo.jp/papers/SDLS-II-x.pdf

[S.F.1] S.F., Maximality Principles and Resurrection Axioms under a Laver generic large cardinal, (note for "Maximality Principles and Resurrection Axioms in light of a Laver generic large cardinal", in preparation) https://fuchino.ddo.jp/papers/RIMS2022-RA-MP-x.pdf

[S.F. & Usuba] S.F., and T. Usuba, On Recurrence Axioms, preprint. https://fuchino.ddo.jp/papers/recurrence-axioms-x.pdf

[S.F.2] S.F., Reflection and Recurrence, to appear in the Festschrift on the occasion of the 75. birthday of Professor Janos Makowsky, Birkhäuser, (2024). https://fuchino.ddo.jp/papers/reflection_and_recurrence-Janos-Festschrift-x.pdf

[S.F. & Gappo & Parente] S.F., T. Gappo, and F. Parente, Generic Absoluteness revisited, preprint. https://fuchino.ddo.jp/papers/generic-absoluteness-revisited-x.pdf

Outline

Viale's Absoluteness Theorem

- We discuss "generalizations" of the following theorem (see Theorem 11 15).

Theorem 1. (M.Viale, Theorem 1.4 in [1]) Assume that MM^{++} holds, and there are class many Woodin cardinals. Then, for any stationary preserving p.o. $\mathbb{P}$ with $\Vdash_{\mathbb{P}}$"BMM", we have

$$\mathcal{H}(\aleph_2)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}(\aleph_2)^{\mathsf{V}[\mathbb{G}]} \quad \text{for } (\mathsf{V}, \mathbb{P})\text{-generic } \mathbb{G}.$$

- MM^{++} is the double plus version of Martin's Maximum.

⟦ For any stationary preserving $\mathbb{P}$, any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \aleph_2$, and set $\mathcal{S}$ of $\mathbb{P}$-names of stationary subsets of ω_1 with $|\mathcal{S}| < \aleph_2$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ over $\mathbb{P}$ s.t. $\underset{\sim}{S}[\mathbb{G}] \subseteq \omega_1$ is stationary for all $\underset{\sim}{S} \in \mathcal{S}$. ⟧

- BMM stands for Bounded Martin's Maximum.

⟦ For any stationary preserving $\mathbb{P}$, family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \aleph_2$ s.t. each $D \in \mathcal{D}$ is generated by $D' \subseteq D$ with $|D'| < \aleph_2$, there is a $\mathcal{D}$-generic filter $\mathbb{G}$ over $\mathbb{P}$. ⟧

[1] Matteo Viale, Martin's maximum revisited, Archive of Mathematical Logic, Vol.55, (2016), 295–316.

Bagaria's Absoluteness Theorem

Notation: For an ordinal α, let $\alpha^{(+)} := \sup(\{|\beta|^+ : \beta < \alpha\})$.

Note that $\alpha^{(+)} = \alpha$ if α is a cardinal. Otherwise, we have $\alpha^{(+)} = |\alpha|^+$.

- Viale's Theorem 1. is based on Bagaria's Absoluteness Theorem.

Theorem 2. (Bagaria's Absoluteness Theorem, Theorem 5 in [2])
For an uncountable cardinal κ and a class $\mathcal{P}$ of p.o.s closed under forcing equivalence, and restriction, the following are equivalent:

(a) $\mathsf{BFA}_{<\kappa}(\mathcal{P})$.

(b) For any $\mathbb{P} \in \mathcal{P}$, Σ_1-formula φ in $\mathcal{L}_\varepsilon$ and $a \in \mathcal{H}(\kappa)$, $\Vdash_{\mathbb{P}}$ "$\varphi(a)$" $\Leftrightarrow \varphi(a)$.

(c) For any $\mathbb{P} \in \mathcal{P}$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, $\mathcal{H}(\kappa)^{\mathsf{V}} \prec_{\Sigma_1} \mathcal{H}((\kappa^{(+)})^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]}$.

- $\mathsf{BFA}_{<\kappa}(\mathcal{P})$ is the Bounded Forcing Axiom for $\mathcal{P}$.

$[\![$ For any $\mathbb{P} \in \mathcal{P}$ and any family of $\mathcal{D}$ dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \kappa$, and s.t. each $D \in \mathcal{D}$ is generated by some $D' \subseteq D$ with $|D'| < \kappa$, ... $]\!]$

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[2] Joan Bagaria, Bounded forcing axioms as principles of generic absoluteness, Archive of Mathematical Logic, Vol.39, (2000), 393-401.

Recurrence Axioms

- Recurrence Axiom for a class $\mathcal{P}$ of p.o.s and a set A ([S.F. & Usuba]) is the axiom scheme expressing:

$(\mathcal{P}, A)$-RcA : For any $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(\overline{x})$ and $\overline{a} \in A$,
if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$" for a $\mathbb{P} \in \mathcal{P}$, then
there is a ground W of the universe V s.t. $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$.

* An inner model W of V is called a ground if there is a p.o. $\mathbb{P} \in \mathsf{W}$ and $(\mathsf{W}, \mathbb{P})$-generic $\mathbb{G}$ s.t. $\mathsf{V} = \mathsf{W}[\mathbb{G}]$.

Recurrence Axiom (2/2)

- The following is a natural strengthening of the Recurrence Axiom ([S.F. & Usuba]):

$(\mathcal{P}, A)$-RcA$^+$: For any $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(\overline{x})$ and any $\overline{a} \in A$,
if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$" for a $\mathbb{P} \in \mathcal{P}$, then
there is a $\mathcal{P}$-ground W of the universe V s.t. $\overline{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\overline{a})$.

* An inner model W of V is called a $\mathcal{P}$-ground if there is a p.o. $\mathbb{P} \in \mathsf{W}$ with $\mathsf{W} \models$ "$\mathbb{P} \in \mathcal{P}$", and $(\mathsf{W}, \mathbb{P})$-generic $\mathbb{G}$ s.t. $\mathsf{V} = \mathsf{W}[\mathbb{G}]$.

Recurrence Axiom$^+$ = Maximality Principle

- A non-empty class $\mathcal{P}$ of p.o.s is iterable if it satisfies: ⓪ $\{\mathbb{1}\} \in \mathcal{P}$, ① $\mathcal{P}$ is closed w.r.t. forcing equivalence (i.e. if $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P} \sim \mathbb{P}'$ then $\mathbb{P}' \in \mathcal{P}$), ② closed w.r.t. restriction, and ③ for any $\mathbb{P} \in \mathcal{P}$ and $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$, $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" implies $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \mathcal{P}$.

* For an iterable $\mathcal{P}$, an $\mathcal{L}_\varepsilon$-formula $\varphi(\overline{a})$ with parameters $\overline{a}$ ($\in$ V) is said to be a $\mathcal{P}$-button if there is $\mathbb{P} \in \mathcal{P}$ s.t. for any $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of p.o. with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", we have $\Vdash_{\mathbb{P}*\underset{\sim}{\mathbb{Q}}}$ "$\varphi(\overline{a})$".

* If $\varphi(\overline{a})$ is a $\mathcal{P}$-button then we call $\mathbb{P}$ as above a push of the button $\varphi(\overline{a})$.

- The Maximality Principle MP($\mathcal{P}, A$) for an iterable $\mathcal{P}$ is the assertion expressed as an axiom scheme in $\mathcal{L}_\varepsilon$ (Hamkins [3]):

MP($\mathcal{P}, A$): For any $\mathcal{L}_\varepsilon$-formula $\varphi(\overline{x})$ and $\overline{a} \in A$, if $\varphi(\overline{a})$ is a $\mathcal{P}$-button then $\varphi(\overline{a})$ holds.

[3] Joel Hamkins, A simple maximality principle, The Journal of Symbolic Logic, Vol.68, no.7, (2003), 527–550.

Recurrence Axiom$^+$ = Maximality Principle (2/2)

Proposition 3. Suppose that $\mathcal{P}$ is an iterable class of p.o.s and A a set (of parameters). $(\mathcal{P}, A)$-RcA$^+$ is equivalent to MP$(\mathcal{P}, A)$.

Proof. Identity crisis

Inner Model Hypothesis (IMH) (Sy-D. Friedman) If a property φ holds in an inner model of an outer model, then there is an inner model of the universe which also satisfies the property φ.

Proposition 4. For a class $\mathcal{P}$ of p.o.s with $\{\mathbb{1}\} \in \mathcal{P}$ and a set A (of parameters), $(\mathcal{P}, A)$-RcA$^+$ is equivalent to the ZFC version of IMH :

For any $\mathcal{L}_\varepsilon$-formula $\varphi = \varphi(\overline{x})$ and any $\overline{a} \in A$, if a $\mathbb{P} \in \mathcal{P}$ forces "there is a ground M with $\overline{a} \in M$ satisfying $\varphi(\overline{a})$", then there is a $\mathcal{P}$-ground W of V s.t. $\overline{a} \in$ W and W $\models \varphi(\overline{a})$.

- These equivalences in Propositions 3, 4 are also mentioned in [4] .

[4] Neil Barton, Andrés Eduardo Caicedo, Gunter Fuchs, Joel David Hamkins, Jonas Reitz, and Ralf Schindler, Inner-Model Reflection Principles, Studia Logica, Vol.108, (2020),573–595.

Solution(s) of Continuum Problem under Recurrence Axiom

- For a family Γ of formulas (in $\mathcal{L}_\varepsilon$), we consider the following restricted version of Recurrence Axiom:

$(\mathcal{P}, A)_\Gamma$-RcA$^+$: For any Γ-formula $\varphi = \varphi(\overline{x})$ and $\overline{a} \in A$, if $\Vdash_{\mathbb{P}}$ "$\varphi(\overline{a})$" for a $\mathbb{P} \in \mathcal{P}$, then there is a $\mathcal{P}$-ground W of the universe V s.t. $\overline{a} \in$ W and W $\models \varphi(\overline{a})$.

- Let $\kappa_{\mathfrak{refl}} := \max\{\aleph_2, 2^{\aleph_0}\}$.

  * $\kappa_{\mathfrak{refl}}$ is a cardinal which appears as the reflection point (cardinal κ s.t. reflection down to $< \kappa$ holds) in many natural reflection principles. Also we have $\kappa_{\mathfrak{refl}}$ = the tightly $\mathcal{P}$-Laver-gen. large cardinal for many natural settings of $\mathcal{P}$ and "large cardinal" if the generic large cardinal exists

Proposition 5. ([S.F. & Usuba]) If $\mathcal{P}$ contains a p.o. which adds a real, as well as a p.o. which (preserves $\aleph_1{}^{\mathsf{V}}$ but) collapses $\aleph_2{}^{\mathsf{V}}$ (e.g. $\mathcal{P}$ = proper p.o.s), then $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA implies $2^{\aleph_0} = \aleph_2$.

Proposition 6. ([S.F. & Usuba]) If $\mathcal{P}$ contains a p.o. which preserves $\aleph_1{}^{\mathsf{V}}$ but collapses $\aleph_2$, and also a p.o. which collapses $\aleph_1{}^{\mathsf{V}}$ (e.g. $\mathcal{P}$ = all p.o.s), then $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA implies $2^{\aleph_0} = \aleph_1$.

Proposition 5. ([S.F. & Usuba]) If $\mathcal{P}$ contains a p.o. which adds a real, as well as a p.o. which (preserves $\aleph_1{}^V$ but) collapses $\aleph_2{}^V$ (e.g. $\mathcal{P}$ = proper p.o.s), then $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA implies $2^{\aleph_0} = \aleph_2$.

Proposition 6. ([S.F. & Usuba]) If $\mathcal{P}$ contains a p.o. which preserves $\aleph_1{}^V$ but collapses $\aleph_2$, and also a p.o. which collapses $\aleph_1{}^V$ (e.g. $\mathcal{P}$ = all p.o.s), then $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA implies $2^{\aleph_0} = \aleph_1$.

- In Proposition 5, I put "preserves $\aleph_1{}^V$ but" in parentheses because of the following Lemma 7, (1):

Lemma 7. ([S.F. & Usuba]) (1) Suppose that $(\mathcal{P}, \mathcal{H}(\aleph_2))_{\Sigma_1}$-RcA holds. Then all elements of $\mathcal{P}$ are stat. preserving.
(2) Assume $(\mathcal{P}, A)_{\Sigma_1}$-RcA. If $\mathcal{P}$ contains a p.o. adding a real, then $\mathcal{P}(\omega) \notin A$. If $\mathcal{P}$ contains a p.o. collapsing $\kappa > \omega$ then $\kappa \notin A$.

- Lemma 7, (2) shows that $\mathcal{H}(\kappa_{\mathfrak{refl}})$ and $\mathcal{H}(2^{\aleph_0})$ in Recurrence Axioms in Lemmas 5,6 are maximal possible.

Proof of Propositions 5,6 & Lemma 7.

Proposition 8. Suppose that all $\mathbb{P} \in \mathcal{P}$ preserve cardinals, and $\mathcal{P}$ contains p.o.s adding at least κ many reals for each $\kappa \in$ Card (This is the case e.g. if $\mathcal{P} =$ ccc p.o.s). Then

(a) $(\mathcal{P}, \emptyset)_{\Sigma_2}$-RcA$^+$ implies that $2^{\aleph_0}$ is very large.

(b) $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_2}$-RcA$^+$ implies that $2^{\aleph_0}$ is a limit cardinal.

Thus, if $2^{\aleph_0}$ is regular in addition, then $2^{\aleph_0}$ is weakly inaccessible.

(c) If there is a weakly inaccessible cardinal above $2^{\aleph_0}$, then $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_2}$-RcA$^+$ implies that $2^{\aleph_0}$ is a limit of inaccessible cardinals.

Proof. (a): To prove e.g. that $2^{\aleph_0} > \aleph_\omega$, let $\mathbb{P} \in \mathcal{P}$ be s.t. $\Vdash_{\mathbb{P}}$ "$2^{\aleph_0} > \aleph_\omega$". Then by $(\mathcal{P}, \emptyset)_{\Sigma_2}$-RcA$^+$, there is a $\mathcal{P}$-ground W of V s.t. W $\models 2^{\aleph_0} > \aleph_\omega$. Since V is $\mathcal{P}$-gen. extension of W and $\mathcal{P}$ preserves cardinals, it follows that V $\models 2^{\aleph_0} > \aleph_\omega$.

(b): Suppose $\mu < 2^{\aleph_0}$. Then $\mu \in \mathcal{H}(2^{\aleph_0})$. There is $\mathbb{P} \in \mathcal{P}$ s.t. $\Vdash_{\mathbb{P}}$ "$2^{\aleph_0} > \mu^+$". By $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_2}$-RcA$^+$, it follows that there is a $\mathcal{P}$-ground W of V which satisfies this statement. Since $\mathcal{P}$ preserves cardinals it follows that V $\models 2^{\aleph_0} > \mu^+$. (c): ... □ (Proposition 8)

- Maximality Principles and hence also Recurrence Axioms have relatively low consistency strength.

Theorem 9. (Hamkins [3], Asperó [5]) The following theories are equiconsistent to each other and they are also equiconsistent with ZFC + there are stationarily many inaccessibles:

ZFC + MP(all p.o.s, $\mathcal{H}(2^{\aleph_0})$), ZFC + MP(c.c.c p.o.s, $\mathcal{H}(2^{\aleph_0})$),
ZFC + MP(proper p.o.s, $\mathcal{H}(2^{\aleph_0})$),
ZFC + MP(semi-proper p.o.s, $\mathcal{H}(2^{\aleph_0})$).

- **Caution!!** The exact consistency strength of ZFC + MP(stationary preserving p.o.s, $\mathcal{H}(2^{\aleph_0})$) is not known and its lower bound is much higher than the consistency strength in Theorem 9.

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[3] Joel Hamkins, A simple maximality principle, The Journal of Symbolic Logic Vol.68, no.7, (2003), 527–550.
[5] David Asperó, A Maximal Bounded Forcing, The Journal of Symbolic Logic, Vol.67, No.1 (2002), 130–142.

- The following Ikegami-Trang Absoluteness Theorem extends Bagaria's Absoluteness Th.

Theorem 10. (Ikegami, and Trang [6]) For an iterable class $\mathcal{P}$ of p.o.s, and a cardinal κ the following are equivalent:

(a) $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_1}$-$\mathsf{RcA}^+$. (b) $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_1}$-$\mathsf{RcA}$. (c) $\mathsf{BFA}_{<\kappa}(\mathcal{P})$. □

- Theorem 10 together with Proposition 5 implies $\mathsf{BFA}_{<\kappa_{\mathfrak{refl}}}(\text{proper p.o.s}) \rightarrow 2^{\aleph_0} = \aleph_2$.

Theorem 11. ([S.F. & Gappo & Parente]) Suppose that $\mathcal{P}$ is an iterable Σ_n-definable class of p.o.s for $n \geq 2$ and $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_n \cup \Gamma}$-$\mathsf{RcA}^+$ holds for an uncountable cardinal κ where Γ is a set of formulas which are conjunction of a Σ_2-formula and a Π_2-formula.

- Then, for any $\mathbb{P} \in \mathcal{P}$ s.t. $\Vdash_{\mathbb{P}}$ "$\mathsf{BFA}_{<\kappa}(\mathcal{P})$", we have $\mathcal{H}(\mu^+)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}(\mu^+)^{\mathsf{V}[\mathbb{G}]}$ for all $\mu < \kappa$ and for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$.
- Thus, we have $\mathcal{H}(\kappa)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}((\kappa^{(+)})^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]}$. cf.: Viale's Theorem

Proof of Theorem 11

[6] Daisuke Ikegami and Nam Trang, On a class of maximality principles, Archive for Mathematical Logic, Vol. 57, (2018), 713–725.

Tightly $\mathcal{P}$-Laver-gen. ultrahuge cardinal

- For an iterable class $\mathcal{P}$ of p.o.s, a cardinal κ is said to be (tightly) $\mathcal{P}$-Laver-generically ultrahuge, if

  for any $\lambda > \kappa$ and $\mathbb{P} \in \mathcal{P}$ there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", s.t. for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are $j, M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa} M$, $j(\kappa) > \lambda$, $\mathbb{P}, \mathbb{H}, (V_{j(\lambda)})^{\mathsf{V}[\mathbb{H}]} \in M$ and $|\mathbb{P} * \underset{\sim}{\mathbb{Q}}| \leq j(\kappa)$ (more precisely: $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a p.o. of size $\leq j(\kappa)$).

Theorem 12. ([S.F. & Gappo & Parente]) If κ is tightly $\mathcal{P}$-Laver-gen. ultrahuge for an iterable class $\mathcal{P}$. Then $(\mathcal{P}, \mathcal{H}(\kappa))_{\Gamma}$-RcA^{+} holds.

* $\Gamma =$ conjunctions of Σ_2 and Π_2 formulas.

Proof

- On the other hand:

Theorem 13. ([S.F.1]) Tightly $\mathcal{P}$-Laver-gen. ultrahugeness does not imply MP$(\mathcal{P}, \emptyset)$ (under the assumption of a large cardinal slightly more than the ultrahuge).

- The proof of Theorem 13 can be modified to show the non-implication of $(\mathcal{P}, \emptyset)_{\Pi_3}$-RcA from a generic large cardinal for many instances of $\mathcal{P}$. "Γ" in Theorem 12 for such $\mathcal{P}$ is almost optimal.

super-$C^{(\infty)}$ ...

Generic absoluteness under $\mathcal{P}$-Laver-gen. large cardinals

- The following is a corollary of Theorem 11 (and Theorem 12 for (2)) :

Corollary 14. (1) Suppose that $(\mathcal{P}, \mathcal{H}(\kappa))$-RcA$^+$ holds for an iterable $\mathcal{P}$. Then, for for any $\mathbb{P} \in \mathcal{P}$ s.t. $\Vdash_{\mathbb{P}}$" $\mathsf{BFA}_{<\kappa}(\mathcal{P})$", we have $\mathcal{H}(\mu^+)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}(\mu^+)^{\mathsf{V}[\mathbb{G}]}$ for all $\mu < \kappa$ and for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$. Thus, $\mathcal{H}(\kappa)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}((\kappa^{(+))^{\mathsf{V}[\mathbb{G}]}})^{\mathsf{V}[\mathbb{G}]}$.

(2) Suppose that κ is tightly $\mathcal{P}$-Laver-gen. ultrahuge for an iterable and Σ_2-definable $\mathcal{P}$. Then, for for any $\mathbb{P} \in \mathcal{P}$ s.t. $\Vdash_{\mathbb{P}}$" $\mathsf{BFA}_{<\kappa}(\mathcal{P})$", we have $\mathcal{H}(\mu^+)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}(\mu^+)^{\mathsf{V}[\mathbb{G}]}$ for all $\mu < \kappa$ and for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$. Thus, $\mathcal{H}(\kappa)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}((\kappa^{(+))^{\mathsf{V}[\mathbb{G}]}})^{\mathsf{V}[\mathbb{G}]}$. ⧉

- By a direct proof, we can improve (2) of the Corollary 14:

Theorem 15.([S.F. & Gappo & Parente]) For an iterable class $\mathcal{P}$ of p.o.s, suppose that $\mathsf{BFA}_{<\kappa}(\mathcal{P})$ holds, and κ is tightly $\mathcal{P}$-Laver-gen. huge. Then, for any $\mathbb{P} \in \mathcal{P}$ s.t. $\Vdash_{\mathbb{P}}$" $\mathsf{BFA}_{<\kappa}(\mathcal{P})$", we have $\mathcal{H}(\mu^+)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}(\mu^+)^{\mathsf{V}[\mathbb{G}]}$ for all $\mu < \kappa$ and for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$. Thus, $\mathcal{H}(\kappa)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}((\kappa^{(+)})^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]}$.

Theorem 15.([S.F. & Gappo & Parente]) For an iterable class $\mathcal{P}$ of p.o.s, suppose that $\mathsf{BFA}_{<\kappa}(\mathcal{P})$ holds, and κ is tightly $\mathcal{P}$-Laver-gen. huge. Then, for any $\mathbb{P} \in \mathcal{P}$ s.t. $\Vdash_{\mathbb{P}}$ "$\mathsf{BFA}_{<\kappa}(\mathcal{P})$", we have $\mathcal{H}(\mu^+)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}(\mu^+)^{\mathsf{V}[\mathbb{G}]}$ for all $\mu < \kappa$ and for $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$. Thus, $\mathcal{H}(\kappa)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}((\kappa^{(+)})^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]}$.

Proof

- $\mathsf{BFA}_{<\kappa}(\mathcal{P})$ in the assumption of Theorem 15 is absorbed in the Laver-genericity part of the assumption if we assume the Lever-genericity for a slightly (?) stronger notion of large cardinal:

Theorem 16. ([7], see also [S.F. & Gappo & Parente]) (1) Suppose that κ is $\mathcal{P}$-Laver-gen. supercompact. Then $\mathsf{FA}_{<\kappa}(\mathcal{P})$ holds.

(2) If all elements of the class $\mathcal{P}$ of p.o.s are stationary preserving and κ is $\mathcal{P}$-Laver-gen. supercompact, then $\mathsf{FA}^{+<\kappa}_{<\kappa}(\mathcal{P})$ holds.

[7] S.F., A. Ottenbreit Maschio Rodrigues, and H. Sakai, Strong Löwenheim-Skolem theorems for stationary logics, II — reflection down to the continuum, Archive for Mathematical Logic, Volume 60, issue 3-4, (2021), 495–523.

Ground Axiom and generic absoluteness

- The Ground Axiom (GA) asserts that there is no proper ground of the universe V.

Theorem 17. MM^{++} + "there are class many supercompact cardinals" is consistent with GA.

Proof. MM^{++} is preserved by $<\omega_2$-directed closed forcing (Larson, Cox [8], Theorem 4.7). Starting from a model with cofinally many supercompact cardinals, use the first supercompact to force MM^{++}. Then the class forcing just like that in the proof of Laver's indestructibility theorem will produce a desired model. □ (Theorem 17)

Corollary 18. (cf. [S.F. & Gappo & Parente]) The conclusion of Viale's Theorem :

$$\mathcal{H}(\aleph_2)^{\mathsf{V}} \prec_{\Sigma_2} \mathcal{H}(\aleph_2)^{\mathsf{V}[\mathbb{G}]} \quad \text{for all stationary preserving } \mathbb{P} \text{ and } (\mathsf{V},\mathbb{P})\text{-generic } \mathbb{G}$$

is consistent with GA.

Proof. By Viale's Absoluteness Theorem and Theorem 17. □ (Corollary 18)

[8] Sean D. Cox, Forcing axioms, approachability, and stationary set reflection, The Journal of Symbolic Logic Volume 86, Number 2, June 2021, 499–530.

Ground Axiom and generic absoluteness (2/2)

Theorem 17. MM^{++} + "there are class many supercompact cardinals" is consistent with GA.

Lemma 19. $\mathsf{GA} + \mathfrak{b} > \aleph_1$ implies $\neg\,(ccc, \emptyset)_{\Sigma_2}$-RcA and $\neg\,(ccc, \emptyset)_{\Pi_2}$-RcA.

Proof. Assume that $\mathsf{GA} + \mathsf{MA} + \neg\mathsf{CH}$ holds. Let $\mathbb{P}$ be a p.o. adding $\aleph_1$ Cohen reals then we have $\Vdash_{\mathbb{P}}$" $\mathfrak{b} = \aleph_1$ ". If $(ccc, \emptyset)_{\Sigma_2}$-RcA$^+$ holds then, since $\mathfrak{b} = \aleph_1$ is Σ_2, there is a ground satisfying this equation. The ground must be different from V since $\mathsf{V} \models \mathfrak{b} > \aleph_1$. This is a contradiction.

- For $\neg\,(ccc, \emptyset)_{\Pi_1}$-RcA$^+$, argue similarly e.g. using the fact that $\mathfrak{b} < \mathfrak{d}$ is Π_2.

□ (Lemma 19)

Corollary 20.([S.F. & Gappo & Parente]) MM^{++} + "there are class many supercompact cardinals" does not imply the existence of a tightly $\mathcal{P}$-Laver gen. ultrahuge cardinal for any class $\mathcal{P}$ of p.o.s containing p.o. for adding $\aleph_1$ many Cohen reals.

Proof. Work in $\mathsf{ZFC} + \mathsf{MM}^{++}$ + "there are class many supercompact cardinals" + GA (Theorem 17). By Lemma 19 and Theorem 12 , this theory proves that there is no tightly $\mathcal{P}$-Laver-gen. ultrahuge cardinal. □ (Corollary 20)

Some (presumably relatively easiy) open problems

- Is the conclusion of Theorems 11 and 15 consistent with GA for $\mathcal{P}$ other than "stationary preserving" and with the continuum other than $\aleph_2$?

- Does (tightly) $\mathcal{P}$-Laver-gen. supercompactness already imply $\neg$GA ?

Thank you for your attention!

ご清聴ありがとうございました.

관심을 가져 주셔서 감사합니다

Dziękuję za uwagę.

Ich danke Ihnen für Ihre Aufmerksamkeit.

Tightly super-$C^{(\infty)}$ $\mathcal{P}$-Laver-gen. ultrahuge cardinal

- The following strengthening of tightly $\mathcal{P}$-Laver-gen. ultrahugeness of κ (which is formulated in an axiom scheme) implies $\mathsf{MP}(\mathcal{P}, \mathcal{H}(\kappa))$.
- For a natural number n, we call a cardinal κ super-$C^{(n)}$-hyperhuge if for any $\lambda_0 > \kappa$ there are $\lambda \geq \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, and j, $M \subseteq \mathsf{V}$ s.t. $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$, $j(\kappa) > \lambda$, ${}^{j(\lambda)}M \subseteq M$ and $V_{j(\lambda)} \prec_{\Sigma_n} \mathsf{V}$.
- κ is super-$C^{(n)}$-ultrahuge if the condition above holds with "${}^{j(\lambda)}M \subseteq M$" replaced by "${}^{j(\kappa)}M \subseteq M$ and $V_{j(\lambda)} \subseteq M$".
- ▷ If κ is super-$C^{(n)}$-hyperhuge then it is super-$C^{(n)}$-ultrahuge.
- We shall also say that κ is super-$C^{(\infty)}$-hyperhuge (super-$C^{(\infty)}$-ultrahuge, resp.) if it is super $C^{(n)}$-hyperhuge (super-$C^{(n)}$-ultrahuge, resp.) for all natural number n.
- A similar kind of strengthening of the notions of large cardinals which we call here "super-$C^{(n)}$" appears also in Boney [Boney]. It is called "$C^{(n)+}$", and is considered there in connection with extendibility.

[Boney] Will Boney, Model Theoretic Characterizations of Large Cardinals, Israel Journal of Mathematics, 236, (2020), 133–181.

Tightly super-$C^{(\infty)}$ $\mathcal{P}$-Laver-gen. ultrahuge cardinal (2/6)

- For a natural number n and an iterable class $\mathcal{P}$ of p.o.s, a cardinal κ is super-$C^{(n)}$ $\mathcal{P}$-Laver-generically ultrahuge (super-$C^{(n)}$ $\mathcal{P}$-Laver-gen. ultrahuge, for short) if, for any $\lambda_0 > \kappa$ and for any $\mathbb{P} \in \mathcal{P}$, there are a $\lambda \geq \lambda_0$ with $V_\lambda \prec_{\Sigma_n} \mathsf{V}$, a $\mathcal{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", and j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$, $j(\kappa) > \lambda$, $\mathbb{P}$, $\mathbb{H}$, ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$ and ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$.
  - A super-$C^{(n)}$ $\mathcal{P}$-Laver-generically ultrahuge cardinal κ is tightly super-$C^{(n)}$ $\mathcal{P}$-Laver-generically ultrahuge (tightly super-$C^{(n)}$ $\mathcal{P}$-Laver-gen. ultrahuge, for short), if $|\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,| \leq j(\kappa)$.
- Super-$C^{(\infty)}$ $\mathcal{P}$-Laver-gen. ultrahugeness and tightly super-$C^{(\infty)}$ $\mathcal{P}$-Laver gen. ultrahugeness are defined similarly to super-$C^{(\infty)}$ ultrahugeness.
- Note that, in general, super-$C^{(\infty)}$ hyperhugeness and super-$C^{(\infty)}$ ultrahugeness are notions unformalizable in the language of ZFC without introducing a new constant symbol for κ since we need infinitely many $\mathcal{L}_\varepsilon$-formulas to formulate them.
  - Exceptions are ...

Tightly super-$C^{(\infty)}$ $\mathcal{P}$-Laver-gen. ultrahuge cardinal (3/6)

- Exceptions are when we are talking about a cardinal in a set model being with one of these properties, or when we are talking about a cardinal definable in V having these properties in an inner model. In the latter case, the situation is formalizable with infinitely may $\mathcal{L}_\varepsilon$-sentences.

Note that if κ is the $\mathcal{P}$–Laver gen. supercompact cardinal for a stationary preserving and iterable $\mathcal{P}$, then $MA^{++}(\mathcal{P}<\kappa)$ holds ([II]).

- In contrast, the super-$C^{(\infty)}$ $\mathcal{P}$-Laver gen. ultrahugeness of κ is expressible in infinitely many $\mathcal{L}_\varepsilon$-sentences. This is because a $\mathcal{P}$-Laver gen. large cardinal κ for relevant classes $\mathcal{P}$ of p.o.s is uniquely determined as $\kappa_{\mathfrak{refl}}$ or $2^{\aleph_0}$ (see e.g. [II] or [S.F.]).

Theorem 21. ([S.F. & Usuba]) Suppose that $\mathcal{P}$ is an iterable class of p.o.s and κ is tightly super-$C^{(\infty)}$ $\mathcal{P}$-Laver-gen. ultrahuge. Then $(\mathcal{P}, \mathcal{H}(\kappa))$-RcA$^+$ (i.e. $\mathrm{MP}(\mathcal{P}, \mathcal{H}(\kappa))$) holds.

Proof. Similarly to Theorem 12.

Corollary 21a. "there is a tightly super-C^∞ (stationary preserving p.o.s) -Laver-gen. hyperhuge cardinal" is strictly stronger than MM^{++}.

Tightly super-$C^{(\infty)}$ $\mathcal{P}$-Laver-gen. ultrahuge cardinal (4/6)

- Consistency of tightly super-$C^{(\infty)}$ $\mathcal{P}$-Laver-gen. ultrahuge cardinal for reasonable $\mathcal{P}$ follows from 2-huge.

Lemma 22. ([S.F. & Usuba]) Suppose that κ is 2-huge with the 2-huge elementary embedding j, that is, $j : \mathsf{V} \xrightarrow{\prec}_{\kappa} M \subseteq \mathsf{V}$, for some $M \subseteq \mathsf{V}$ and ${}^{j^2(\kappa)}M \subseteq M$. Then
$V_{j(\kappa)} \models$" κ is super-$C^{(\infty)}$-hyperhuge cardinal", and for each $n \in \omega$,
$V_{j(\kappa)} \models$" there are stationarily many super-$C^{(n)}$-hyperhuge cardinals". □

Theorem 23. ([S.F. & Usuba]) Suppose that μ is an inaccessible cardinal and κ is super-$C^{(\infty)}$-hyperhuge in V_μ. Then there is a Laver function $f : \kappa \to V_\kappa$ for super-$C^{(\infty)}$-hyperhugeness in V_μ. □

Tightly super-$C^{(\infty)}$ $\mathcal{P}$-Laver-gen. ultrahuge cardinal (5/6)

Theorem 24. ([S.F. & Usuba]) (1) Suppose that μ is inaccessible and $\kappa < \mu$ is super-$C^{(\infty)}$-ultrahuge in V_μ. Let $\mathbb{P} = \mathrm{Col}(\aleph_1, \kappa)$. Then, in $V_\mu[\mathbb{G}]$, for any $V_\mu, \mathbb{P}$-generic $\mathbb{G}$, $\aleph_2^{V_\mu[\mathbb{G}]}$ $(= \kappa)$ is tightly super-$C^{(\infty)}$ σ-closed-Laver-gen. ultrahuge and CH holds.

(2) Suppose that μ is inaccessible and $\kappa < \mu$ is super-$C^{(\infty)}$-ultrahuge with a Laver function $f : \kappa \to V_\kappa$ for super-$C^{(\infty)}$-ultrahugeness in V_μ. If $\mathbb{P}$ is the CS-iteration of length κ for forcing PFA along with f, then, in $V_\mu[\mathbb{G}]$ for any $(V_\mu, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{V_\mu[\mathbb{G}]}$ $(= \kappa)$ is tightly super-$C^{(\infty)}$ proper-Laver-gen. ultrahuge and $2^{\aleph_0} = \aleph_2$ holds.

(2′) Suppose that μ is inaccessible and $\kappa < \mu$ is super-$C^{(\infty)}$-ultrahuge with a Laver function $f : \kappa \to V_\kappa$ for super-$C^{(\infty)}$-ultrahugeness in V_μ. If $\mathbb{P}$ is the RCS-iteration of length κ for forcing MM along with f, then, in $V_\mu[\mathbb{G}]$ for any $(V_\mu, \mathbb{P})$-generic $\mathbb{G}$, $\aleph_2^{V_\mu[\mathbb{G}]}$ $(= \kappa)$ is tightly super-$C^{(\infty)}$ semi-proper-Laver-gen. ultrahuge and $2^{\aleph_0} = \aleph_2$ holds.

(3) Suppose that μ is inaccessible and κ is super-$C^{(\infty)}$-ultrahuge with a Laver function $f : \kappa \to V_\kappa$ for super-$C^{(\infty)}$-ultrahugeness in V_μ. If $\mathbb{P}$ is a FS-iteration of length κ for forcing MA along with f, then, in $V_\mu[\mathbb{G}]$ for any $(V_\mu, \mathbb{P})$-generic $\mathbb{G}$, $2^{\aleph_0}$ $(= \kappa)$ is tightly super-$C^{(\infty)}$ c.c.c.-Laver-gen. ultrahuge, and κ is very large in $V_\mu[\mathbb{G}]$.

(4) Suppose that μ is inaccessible and κ is super-$C^{(\infty)}$-ultrahuge with a Laver function $f : \kappa \to V_\kappa$ for super-$C^{(\infty)}$-ultrahugeness in V_μ. If $\mathbb{P}$ is a FS-iteration of length κ along with f enumerating "all" p.o.s, then, in $V_\mu[\mathbb{G}]$ for any $(V_\mu, \mathbb{P})$-generic $\mathbb{G}$, $2^{\aleph_0}$ $(= \aleph_1)$ is tightly super-$C^{(\infty)}$ all p.o.s-Laver-gen. ultrahuge, and CH holds.

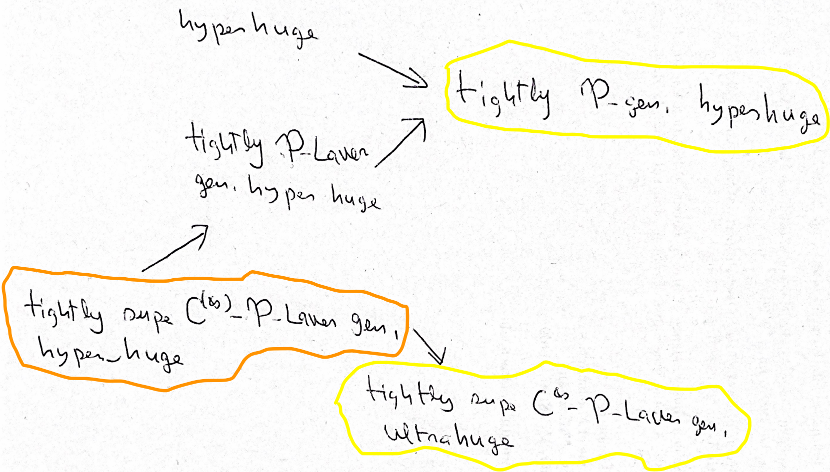
□

Bedrock of tightly $\mathcal{P}$-gen. hyperhuge cardinal

- Recall that a cardinal κ is hyperhuge, if for every $\lambda > \kappa$, there is $j : \mathsf{V} \overset{\prec}{\to}_\kappa M \subseteq \mathsf{V}$ s.t. $\lambda < j(\kappa)$ and ${}^{j(\lambda)}M \subseteq M$. A hyperhuge cardinal κ can be characterized in terms of existence of κ-complete normal ultrafilters with certain additional properties (e.g. see [S.F. & Usuba]).
- For a class $\mathcal{P}$ of p.o.s, a cardinal κ is tightly $\mathcal{P}$-generic hyperhuge (tightly $\mathcal{P}$-gen. hyperhuge, for short) if for any $\lambda > \kappa$, there is $\mathbb{Q} \in \mathcal{P}$ s.t. for a $(\mathsf{V}, \mathbb{Q})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$, $\lambda < j(\kappa)$, $|\,\mathbb{Q}\,| \leq j(\kappa)$, and $j''j(\lambda), \mathbb{H} \in M$.
- For a class $\mathcal{P}$ of p.o.s, a cardinal κ is tightly $\mathcal{P}$-Laver-generically hyperhuge (tightly $\mathcal{P}$-Laver-gen. hyperhuge, for short) if for any $\lambda > \kappa$, and $\mathbb{P} \in \mathcal{P}$ there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t. for a $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \overset{\prec}{\to}_\kappa M$, $\lambda < j(\kappa)$, $|\,\mathbb{P} * \underset{\sim}{\mathbb{Q}}\,| \leq j(\kappa)$, and $j''j(\lambda), \mathbb{H} \in M$.

Bedrock of tightly $\mathcal{P}$-gen. hyperhuge cardinal (2/6)

For an iterable $\mathcal{P}$:



Bedrock of tightly $\mathcal{P}$-gen. hyperhuge cardinal (3/6)

- For a cardinal κ, a ground W of the universe V is called a $\leq\kappa$-ground if there is a p.o. $\mathbb{P}\in\mathsf{W}$ of cardinality $\leq\kappa$ (in the sense of V) and $(\mathsf{W},\mathbb{P})$-generic filter $\mathbb{G}$ s.t. $\mathsf{V}=\mathsf{W}[\mathbb{G}]$.
- Let

$$\overline{\mathsf{W}}:=\bigcap\{\mathsf{W}\ :\ \mathsf{W}\text{ is a }\leq\kappa\text{-ground}\}.$$

Since there are only set many $\leq\kappa$-grounds, $\overline{W}$ contains a ground by Theorem 1.3 in [Usuba]. We shall call $\overline{\mathsf{W}}$ defined above the $\leq\kappa$-mantle of V.

- The following theorem generalizes Theorem 1.6 in [Usuba].

Theorem 25. ([S.F. & Usuba]) Suppose that $\mathcal{P}$ is any class of p.o.s. If κ is a tightly $\mathcal{P}$-gen. hyperhuge cardinal, then the $\leq\kappa$-mantle is the smallest ground of V (i.e. it is the bedrock of V) and it is also a $\leq\kappa$-ground.

[Usuba] Toshimichi Usuba, The downward directed grounds hypothesis and very large cardinals, Journal of Mathematical Logic, Vol. 17(2) (2017), 1–24.

Bedrock of tightly $\mathcal{P}$-gen. hyperhuge cardinal (4/6)

Theorem 25. ([S.F. & Usuba]) Suppose that $\mathcal{P}$ is any class of p.o.s. If κ is a tightly $\mathcal{P}$-gen. hyperhuge cardinal, then the $\leq\kappa$-mantle is the smallest ground of V (i.e. it is the bedrock of V) and it is also a $\leq\kappa$-ground.

A very rough sketch of the Proof.

- Analyzing the proof of Theorem 25, we also obtain:

Theorem 26. ([S.F. & Usuba]) Suppose that $\mathcal{P}$ is any class of p.o.s. If κ is a tightly $\mathcal{P}$-gen. hyperhuge cardinal, then κ is a hyperhuge cardinal in the bedrock $\overline{W}$ of V. □

Theorem 27. ([S.F. & Usuba]) Suppose that $\mathcal{P}$ is any class of p.o.s. If κ is a tightly super-$C^{(n)}$ $\mathcal{P}$-gen. hyperhuge cardinal, then κ is a super-C^n-hyperhuge cardinal in the bedrock $\overline{W}$ of V. □

- These Theorems have many strong consequences. Some of them are ...

Corollary 28.([S.F. & Usuba]) Suppose that $\mathcal{P}$ is the class of all p.o.s. Then the following theories are equiconsistent:

(a) ZFC + "there is a hyperhuge cardinal".

(b) ZFC + "there is a tightly $\mathcal{P}$-Laver gen. hyperhuge cardinal".

(c) ZFC + "there is a tightly $\mathcal{P}$-gen. hyperhuge cardinal".

(d) ZFC + "bedrock $\overline{\mathsf{W}}$ exists and ω_1 is a hyperhuge cardinal in $\overline{\mathsf{W}}$". □

Corollary 29.([S.F. & Usuba]) Suppose that $\mathcal{P}$ is one of the following classes of p.o.s: all semi-proper p.o.s; all proper p.o.s; all ccc p.o.s; all σ-closed p.o.s. Then the following theories are equiconsistent:

(a) ZFC + "there is a hyperhuge cardinal".

(b) ZFC + "there is a tightly $\mathcal{P}$-Laver gen. hyperhuge cardinal".

(c) ZFC + "there is a tightly $\mathcal{P}$-gen. hyperhuge cardinal".

(d) ZFC + "bedrock $\overline{\mathsf{W}}$ exists and $\kappa_{\mathfrak{refl}}$ is a hyperhuge cardinal in $\overline{\mathsf{W}}$". □

Cf.: Theorem 24, and Theorem 27.

Equiconsistency as the Eternal Recurrence (2/2)

Corollary 30.([S.F. & Usuba]) Suppose that $\mathcal{P}$ is the class of all p.o.s. Then the following theories are equiconsistent:

(a) ZFC + "there is a tightly super-$C^{(\infty)}$ $\mathcal{P}$-Laver gen. hyperhuge cardinal".

(b) ZFC + "bedrock $\overline{\mathsf{W}}$ exists and ω_1^{V} is a super-$C^{(\infty)}$-hyperhuge cardinal in $\overline{\mathsf{W}}$".

Corollary 31.([S.F. & Usuba]) Suppose that $\mathcal{P}$ is one of the following classes of p.o.s: all semi-proper p.o.s; all proper p.o.s; all ccc p.o.s; all σ-closed p.o.s. Then the following theories are equiconsistent:

(a) ZFC + "there is a tightly super-$C^{(\infty)}$ $\mathcal{P}$-Laver gen. hyperhuge cardinal".

(b) ZFC + "bedrock $\overline{\mathsf{W}}$ exists and ${\kappa_{\mathfrak{refl}}}^{\mathsf{V}}$ is a super-$C^{(\infty)}$-hyperhuge cardinal in $\overline{\mathsf{W}}$".

Toward the Laver-generic Maximum

- The existence of tightly super-$C^{(\infty)}$ $\mathcal{P}$-Laver gen. superhuge cardinal for the class $\mathcal{P}$ of all semi-proper p.o.s is one of the strongest principle we considered so far. It implies the tightly super-$C^{(\infty)}$ $\mathcal{P}$-Laver gen. superhuge cardinal is $2^{\aleph_0} = \aleph_2$ and MM^{++} holds (see [II] or [S.F.1]), the existence of the bedrock (Theorem 25), and $(\mathcal{P}, \mathcal{H}(\aleph_2))$-$\mathsf{RcA}^+$ (Theorem 21).
  - MM^{++} implies many preferable set-theoretic axioms/principles including Woodin's (*) ([Aspero-Schindler]).

[Aspero-Schindler] David Asperó, and Ralf Schindler, Martin's Maximum++ implies Woodin's axiom (*). Annals of Mathematics, 193(3), (2021), 793-835.

  - $(\mathcal{P}, \mathcal{H}(\aleph_2))$-$\mathsf{RcA}^+$ claims that any property (even with any subset of ω_1 as parameter) forcable by a semi-proper p.o., is a theorem in some semi-proper ground. E.g. Cichón's Maximum is what happens in a semi-proper ground.

- Strong forms of Resurrection Axiom are also consequences of the existence of the super-$C^{(\infty)}$ (semi-proper)-Laver gen. large cardinal:

Toward the Laver-generic Maximum (2/4)

- Suppose that $\mathcal{P}$ is a class of p.o.s and $\mu^\bullet$ is a definition of a cardinal (e.g. "$\aleph_1$", "$\aleph_2$", "$2^{\aleph_0}$")
  - The following boldface version of the Resurrection Axioms is considered in [Hamkins-Johnstone]:

$\mathbb{RA}^{\mathcal{P}}_{\mathcal{H}(\mu^\bullet)}$: For any $A \subseteq \mathcal{H}(\mu^\bullet)$ and any $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ of p.o. s.t. $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" and, for any $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there is $A^* \subseteq \mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}$ s.t. $(\mathcal{H}(\mu^\bullet)^{\mathsf{V}}, A, \in) \prec (\mathcal{H}(\mu^\bullet)^{\mathsf{V}[\mathbb{H}]}, A^*, \in)$.

Theorem 32. [S.F.1] For an iterable class of p.o.s $\mathcal{P}$, if $\kappa_{\mathfrak{refl}}$ is tightly $\mathcal{P}$-Laver-gen. superhuge, then $\mathbb{RA}^{\mathcal{P}}_{\mathcal{H}(\kappa_{\mathfrak{refl}})}$ holds. □

[Hamkins-Johnstone] Joel David Hamkins, and Thomas A. Johnstone, Strongly uplifting cardinals and the boldface resurrection axioms, Archive for Mathematical Logic Vol.56, (2017), 1115–1133.

Toward the Laver-generic Maximum (3/4)

- With a Lever-genericity corresponding to a larger large cardinal, we obtain the "tight" version of Unbounded Resurrection Principle in [Tsaprounis]:

TUR($\mathcal{P}$) : For any $\lambda > \kappa_{\mathfrak{refl}}$, and $\mathbb{P} \in \mathcal{P}$, there exists a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$"$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t., for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-gen. $\mathbb{H}$, there are $\lambda^* \in \mathsf{On}$, and $j_0 \in \mathsf{V}[\mathbb{H}]$ s.t. $j_0 : \mathcal{H}(\lambda)^{\mathsf{V}} \overset{\prec}{\to}_{\kappa_{\mathfrak{refl}}} \mathcal{H}(\lambda^*)^{\mathsf{V}[\mathbb{H}]}$, $j_0(\kappa_{\mathfrak{refl}}) > \lambda$, and $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is forcing equivalent to a p.o. of size $j_0(\kappa_{\mathfrak{refl}})$.

Theorem 33. [S.F.1] For an iterable class $\mathcal{P}$, if $\kappa_{\mathfrak{refl}}$ is tightly $\mathcal{P}$-Laver gen. ultrahuge, then TUR($\mathcal{P}$) holds.

[Tsaprounis] Tsaprounis, On resurrection axioms, The Journal of Symbolic Logic, Vol.80, No.2, (2015), 587–608.

Toward the Laver-generic Maximum (4/4)

- We can even establish the consistency of:
- $2^{\aleph_0}$ is tightly super-$C^{(\infty)}$ (semi-proper)-Laver gen. superhuge $+$ (all p.o.s, $\mathcal{H}(\aleph_1)^{\overline{W}}$)-RcA

A construction of a model: Work in a model V_λ where κ is super-$C^{(\infty)}$-hyperhuge. Then $V_\kappa \prec V_\lambda$. Take an inaccessible $\delta < \kappa$ with $V_\delta \prec V_\lambda$. Use this to force (all p.o.s, $\mathcal{H}(\aleph_1)$)-RcA. κ is still super-$C^{(\infty)}$-hyperhuge in the generic extension, so we can use it to force $2^{\aleph_0}$ to be tightly super-$C^{(\infty)}$ (semi-proper)-Laver gen. superhuge. (all p.o.s, $\mathcal{H}(\aleph_1)^{\overline{W}}$)-RcA survives this forcing. □

- **Open Problems:**
- Is there any natural axiom which would imply the combination of the principles above?
- A (possibly) related question: Is there anything similar to HOD dichotomy for the bedrock under a (tightly generic/tightly Laver-generic) very large cardinal?

\huge 終り

Recurrence Axioms are monotonic in parameters

- For classes of p.o.s $\mathcal{P}$, $\mathcal{P}'$ and sets A, A' of parameters, if $\mathcal{P} \subseteq \mathcal{P}'$ and $A \subseteq A'$, then we have

$$(\mathcal{P}', A')\text{-RcA} \;\Rightarrow\; (\mathcal{P}, A)\text{-RcA}.$$

- Note that, in general, we do not have similar implication between $\mathrm{MP}(\mathcal{P}, A)$ and $\mathrm{MP}(\mathcal{P}', A')$.

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Proof of Propositions 5,6 and Lemma 7.

Proposition 5. If $\mathcal{P}$ contains a p.o. which adds a real, as well as a p.o. which (preserves ${\aleph_1}^{\mathsf{V}}$ but) collapses ${\aleph_2}^{\mathsf{V}}$ (e.g. $\mathcal{P}$ = proper p.o.s) then $(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA implies $2^{\aleph_0}=\aleph_2$.

Proof. Suppose that $\mathcal{P}$ is as above and $(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA holds.

- $2^{\aleph_0}\geq\aleph_2$: Otherwise CH holds. Then $\mathcal{P}(\omega)^{\mathsf{V}}\in\mathcal{H}(\kappa_{\mathfrak{refl}})$. Hence "$\exists x\,(x\subseteq\omega\wedge x\notin\mathcal{P}(\omega)^{\mathsf{V}})$" is a Σ_1-formula with parameters from $\mathcal{H}(\kappa_{\mathfrak{refl}})$ and $\mathbb{P}\in\mathcal{P}$ adding a real forces (the formula in forcing language corresponding to) this formula.
- ▷ By $(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA, the formula must hold in a ground. This is a contradiction.

- $2^{\aleph_0}\leq\aleph_2$: If $2^{\aleph_0}>\aleph_2$ then ${\aleph_1}^{\mathsf{V}},{\aleph_2}^{\mathsf{V}}\in\mathcal{H}(2^{\aleph_0})\subseteq\mathcal{H}(\kappa_{\mathfrak{refl}})$. Let $\mathbb{P}\in\mathcal{P}$ be a p.o. which preserves $\aleph_1$ but collapses $\aleph_2$.
- ▷ Letting $\psi(x,y)$ a Σ_1-formula saying "$\exists f\,(f$ is a surjection from x to $y)$", we have $\Vdash_{\mathbb{P}}$"$\psi(({\aleph_1}^{\mathsf{V}})^{\checkmark},({\aleph_2}^{\mathsf{V}})^{\checkmark})$".
- ▷ By $(\mathcal{P},\mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA, the formula $\psi({\aleph_1}^{\mathsf{V}},{\aleph_2}^{\mathsf{V}})$ must hold in a ground. This is a contradiction. □

back

Proof of Propositions 5,6 and Lemma 7. (2/3)

Proposition 6. If $\mathcal{P}$ contains a p.o. which preserves ${\aleph_1}^{V}$ but collapses $\aleph_2$, and also a p.o. which collapses ${\aleph_1}^{V}$ (e.g. $\mathcal{P}$ = all p.o.s) then $(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))_{\Sigma_1}$-RcA implies $2^{\aleph_0} = \aleph_1$.

Proof. We have $2^{\aleph_0} \leq \aleph_2$, by the second half of the proof of Proposition 5.

- If $2^{\aleph_0} = \aleph_2$, then ${\aleph_1}^{V} \in \mathcal{H}(2^{\aleph_0})$.
- Let $\mathbb{P} \in \mathcal{P}$ be a p.o. collapsing ${\aleph_1}^{V}$. I.e. $\Vdash_{\mathbb{P}}$ " ${\aleph_1}^{V}$ is countable". Since "$\cdots$ is countable" is Σ_1, there is a ground M s.t. $M \models$ " ${\aleph_1}^{V}$ is countable". This is a contradiction. □ (Proposition 6)

back

Proof of Propositions 5,6 and Lemma 7. (3/3)

Lemma 7. (1) Suppose that $(\mathcal{P}, \mathcal{H}(\aleph_2))_{\Sigma_1}$-RcA holds. Then all elements of $\mathcal{P}$ are $\aleph_1$-preserving and stationary preserving.

(2) Assume $(\mathcal{P}, A)_{\Sigma_1}$-RcA. If $\mathcal{P}$ contains a p.o. adding a real, then $\mathcal{P}(\omega) \notin A$. If $\mathcal{P}$ contains a p.o. collapsing $\kappa > \omega$ then $\kappa \notin A$.

Proof. (1): Suppose otherwise and $\mathbb{P} \in \mathcal{P}$ is s.t. $\Vdash_{\mathbb{P}}$"${\aleph_1}^{\mathsf{V}}$ is countable". Note that $\omega, \aleph_1 \in \mathcal{H}(\kappa_{\mathfrak{refl}})$.

- By $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA, it follows that there is a ground W of V s.t. W $\models$"${\aleph_1}^{\mathsf{V}}$ is countable". This is a contradiction.
- Suppose that $\mathbb{P} \in \mathcal{P}$ destroy the stationarity of $S \subseteq \omega_1$. Note that ω_1, $S \in \mathcal{H}(\aleph_2)$. Let $\varphi = \varphi(y, z)$ be the Σ_1-formula

  $$\exists x\,(y \text{ is a club subset of the ordinal } y \text{ and } z \cap x = \emptyset).$$

  Then we have $\Vdash_{\mathbb{P}}$"$\varphi(\omega_1, S)$". By $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_1}$-RcA, it follows that there is a ground W $\subseteq$ V s.t. $S \in$ W and W $\models \varphi(\omega_1, S)$. This is a contradiction.

(2): By the first part of the proof of Proposition 5, and the proof of Proposition 6. □ (Lemma 7)

Proof of Theorem 12.

Theorem 12. ([S.F. & Gappo & Parente]) If κ is tightly $\mathcal{P}$-Laver-gen. ultrahuge for an iterable class $\mathcal{P}$. Then $(\mathcal{P}, \mathcal{H}(\kappa))_\Gamma$-RcA$^+$ holds.

Proof. We prove the case $\Gamma = \Sigma_2$. p-Lg-RcA-0 in ...-revisited.pdf

Lemma 12a. If α is a limit ordinal and V_α satisfies a large enough fragment of ZFC, then for any $\mathbb{P} \in V_\alpha$ and $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, we have $V_\alpha[\mathbb{G}] = {V_\alpha}^{\mathsf{V}[\mathbb{G}]}$.

- Assume that κ is tightly $\mathcal{P}$-Laver gen. ultrahuge for an iterable class $\mathcal{P}$ of p.o.s. ▷ Suppose that $\varphi = \varphi(x)$ is Σ_2-formula (in $\mathcal{L}_\varepsilon$), $a \in \mathcal{H}(\kappa)$, and $\mathbb{P} \in \mathcal{P}$ is s.t.

* The general case of a Γ-formula is proved similarly.

(a) $\mathsf{V} \models \Vdash_{\mathbb{P}}$ "$\varphi(a)$".

- Let $\lambda > \kappa$ be s.t. $\mathbb{P} \in \mathsf{V}_\lambda$ and

(0) $V_\lambda \prec_{\Sigma_n} \mathsf{V}$ for a sufficiently large n.

In particular, we may assume that we have chosen the n above so that a sufficiently large fragment of ZFC holds in V_λ in the sense of Lemma 12a.

Proof of Theorem 12. (2/3)

Let $\underset{\sim}{\mathbb{Q}}$ be a $\mathbb{P}$-name s.t. $\Vdash_{\mathbb{P}}$"$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", and for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with

(1) $j : \mathsf{V} \overset{\prec}{\to}_{\kappa} M$,

(2) $j(\kappa) > \lambda$,

(3) $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{P}$, $\mathbb{H}$, ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \in M$, and

(4) $|\mathbb{P} * \underset{\sim}{\mathbb{Q}}| \leq j(\kappa)$.

By (4), we may assume that the underlying set of $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ is $j(\kappa)$ and $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in {V_{j(\lambda)}}^{\mathsf{V}}$.

Let $\mathbb{G} := \mathbb{H} \cap \mathbb{P}$. Note that $\mathbb{G} \in M$ by (3) and we have

Since ${V_{j(\lambda)}}^{M}$ $(= V_{j(\lambda)}^{\mathsf{V}[\mathbb{H}]})$ satisfies a sufficiently large fragment of ZFC by elementarity of j, and hence the equality follows by Lemma 12a

(5) $${V_{j(\lambda)}}^{M} \underbrace{=}_{\text{by (3)}} {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \overbrace{=} {V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{H}].$$

Thus, by (3), choice (0) of λ, and by the definability of grounds, we have ${V_{j(\lambda)}}^{\mathsf{V}} \in M$ and ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] \in M$.

Proof of Theorem 12. (3/3)

Claim 12b. ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] \models \varphi(a)$.

$\vdash$ By Lemma 12a, ${V_\lambda}^{\mathsf{V}}[\mathbb{G}] = {V_\lambda}^{\mathsf{V}[\mathbb{G}]}$, and ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] = {V_{j(\lambda)}}^{\mathsf{V}[\mathbb{G}]}$ by (5). By (0), both ${V_\lambda}^{\mathsf{V}}[\mathbb{G}]$ and $V^{\mathsf{V}}_{j(\lambda)}[\mathbb{G}]$ satisfy large enough fragment of ZFC. Thus

(6) ${V_\lambda}^{\mathsf{V}}[\mathbb{G}] \prec_{\Sigma_1} {V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}]$.

By (a) and (0) we have ${V_\lambda}^{\mathsf{V}}[\mathbb{G}] \models \varphi(a)$. By (6) and since φ is Σ_2, it follows that ${V_{j(\lambda)}}^{\mathsf{V}}[\mathbb{G}] \models \varphi(a)$. $\dashv$ (Claim 12b.)

Thus we have

(7) $M \models$"there is a $\mathcal{P}$-ground N of $V_{j(\lambda)}$ s.t. $N \models \varphi(a)$".

By the elementarity (1), it follows that

(6) $\mathsf{V} \models$"there is a $\mathcal{P}$-ground N of V_λ s.t. $N \models \varphi(a)$".

Now by (0), it follows that there is a $\mathcal{P}$-ground W of V s.t. $\mathsf{W} \models \varphi(a)$. $\square$ (Theorem 12) $\square$

A very rough sketch of the Proof of Theorem 14.

Theorem 14. ([S.F. & Usuba]) Suppose that $\mathcal{P}$ is any class of p.o.s. If κ is a tightly $\mathcal{P}$-gen. hyperhuge cardinal, then the $\leq\kappa$-mantle is the smallest ground of V (i.e. it is the bedrock of V) and it is also a $\leq\kappa$-ground.

A rough sketch of the Proof.

- Suppose that κ is tightly $\mathcal{P}$-gen. hyperhuge and let $\overline{\mathsf{W}}$ be the $\leq\kappa$-mantle.
- By Theorem 1.3 in [Usuba], it is enough to show that, for any ground $\mathsf{W}\subseteq\overline{\mathsf{W}}$ is actually a $\leq\kappa$-ground and hence $\mathsf{W}=\overline{\mathsf{W}}$ holds.
- Let $\mathsf{W}\subseteq\overline{\mathsf{W}}$ be a ground. Let μ be the cardinality (in the sense of V) of a p.o. $\mathbb{S}\in\mathsf{W}$ s.t. there is a $(\mathsf{W},\mathbb{S})$-generic $\mathbb{F}$ s.t. $\mathsf{V}=\mathsf{W}[\mathbb{F}]$. W.l.o.g., $\mu\geq\kappa$.
- By Laver-Woodin Theorem, there is $r\in\mathsf{V}$ s.t. $\mathsf{W}=\Phi(\cdot,r)^{\mathsf{V}}$ for an $\mathcal{L}_\varepsilon$-formula Φ.
- Let $\theta\geq\mu$ be s.t. $r\in V_\theta$, and for a sufficiently large natural number n, we have ${V_\theta}^{\mathsf{V}}\prec_{\Sigma_n}\mathsf{V}$. By the choice of θ, $\Phi(\cdot,r)^{{V_\theta}^{\mathsf{V}}}=\Phi(\cdot,r)^{\mathsf{V}}\cap{V_\theta}^{\mathsf{V}}=\mathsf{W}\cap{V_\theta}^{\mathsf{V}}$ $={V_\theta}^{\mathsf{W}}$. Let $\mathbb{Q}\in\mathcal{P}$ s.t. for $(\mathsf{V},\mathbb{Q})$-generic $\mathbb{H}$, there are j, $M\subseteq\mathsf{V}[\mathbb{H}]$ with $j:\mathsf{V}\overset{\prec}{\rightarrow}_\kappa M$, $\theta<j(\kappa)$, $|\mathbb{Q}|\leq j(\kappa)$, ${V_{j(\theta)}}^{\mathsf{V}[\mathbb{H}]}\subseteq M$, and $\mathbb{H}$, $j''j(\theta)\in M$.

  ... (back and forth with j) ... Thus ${V_\theta}^{\overline{\mathsf{W}}}\subseteq{V_\theta}^{\mathsf{W}}$. Since θ can be arbitrary large, It follows that $\overline{\mathsf{W}}\subseteq\mathsf{W}$.

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Proof of Theorem 11.

- Suppose that $\mathbb{P} \in \mathcal{P}$ is s.t. $\Vdash_{\mathbb{P}}$ "$\mathsf{BFA}_{<\kappa}(\mathcal{P})$" and $\mathbb{G}$ is a $(\mathsf{V}, \mathbb{P})$-generic set. Let $\varphi = \varphi(x)$ be a Σ_2-formula in $\mathcal{L}_\varepsilon$, and $\varphi(x) = \exists y\, \psi(x, y)$ for a Π_1-formula ψ in $\mathcal{L}_\varepsilon$. Let $\mu < \kappa$ and $a \in \mathcal{H}(\mu^+)$ ($\subseteq \mathcal{H}(\kappa)$). We have to show that $\mathcal{H}(\mu^+)^{\mathsf{V}} \models \varphi(a) \Leftrightarrow \mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \varphi(a)$.
- Suppose first that $\mathcal{H}(\mu^+)^{\mathsf{V}} \models \varphi(a)$. Let $b \in \mathcal{H}(\mu^+)^{\mathsf{V}}$ be s.t. $\mathcal{H}((\mu^+)^{\mathsf{V}})^{\mathsf{V}} \models \psi(a, b)$. Since we have $\mathsf{V} \models \mathsf{BFA}_{<\kappa}(\mathcal{P})$ by Ikegami-Trang Theorem 10 , it follows that $\mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \psi(a, b)$ by Bagaria's Absoluteness Theorem 2 , and thus $\mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \varphi(a)$.
  Suppose now $\mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \varphi(a)$. By $(\mathcal{P}, \mathcal{H}(\kappa))_{\Sigma_n \cup \Gamma}$-$\mathsf{RcA}^+$, there is a $\mathcal{P}$-ground W of V s.t.
  - * $\mathsf{W} \models$ "$\mathsf{BFA}_{<\mu^+}(\mathcal{P}) \wedge \mathcal{H}(\mu^+) \models \varphi(a)$".

  Note that the formula in (*) is Σ_n if $n \geq 3$ and Γ if $n = 2$.

Proof of Theorem 11. (2/2)

Let $b \in \mathcal{H}((\mu^+)^{\mathsf{W}})^{\mathsf{W}}$ be s.t. $\mathsf{W} \models$"$\mathcal{H}(\mu^+) \models \psi(a, b)$". By Bagaria's Absoluteness Theorem 2, and since V is a $\mathcal{P}$-generic extension of W, it follows that $\mathsf{V} \models$"$\mathcal{H}(\mu^+) \models \psi(a, b)$" and hence $\mathcal{H}(\mu^+)^{\mathsf{V}} \models \varphi(a)$.

- For the last statement of the present theorem, let φ be a Σ_2-formula, and $a \in \mathcal{H}(\kappa)$. If $\mathcal{H}(\kappa) \models \varphi(a)$, then, by Lemma A1 below, there is $\mu < \kappa$ s.t. $\mathcal{H}(\mu^+) \models \varphi(a)$. By the first part of the theorem, it follows that $\mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \varphi(a)$. Thus $\mathcal{H}((\kappa^{(+)})^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \varphi(a)$ by Lemma A1.
  If $\mathcal{H}((\kappa^{(+)})^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \varphi(a)$, then there is $\mu < \kappa$ s.t. $\mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \varphi(a)$ (this is also shown using Lemma A1). Hence $\mathcal{H}((\mu^+)^{\mathsf{V}}) \models \varphi(a)$ by the first part of the theorem.

  □ (Theorem 11)

Lemma A1. (Levy) $\mathcal{H}(\kappa) \prec_{\Sigma_1} \mathsf{V}$ for any cardinal $\kappa > \aleph_0$. □

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Proof of Proposition 3

Proposition 3. Suppose that $\mathcal{P}$ is an iterable class of p.o.s and A a set (of parameters). $(\mathcal{P}, A)$-RcA^+ is equivalent to $\mathsf{MP}(\mathcal{P}, A)$.

Proof. ▶ Suppose that $(\mathcal{P}, A)$-RcA^+ holds. We show that $\mathsf{MP}(\mathcal{P}, A)$ holds. Let $\mathbb{P} \in \mathcal{P}$ be a push of the $\mathcal{P}$-button $\varphi(\bar{a})$.

▷ Let $\varphi'(\bar{x})$ be the formula saying (*) $\forall \underline{\mathbb{Q}} (\underline{\mathbb{Q}} \in \mathcal{P} \rightarrow \Vdash_{\underline{\mathbb{Q}}} “\varphi(\bar{x})”$.

▷ Then we have $\Vdash_{\mathbb{P}} “\varphi'(\bar{a})”$. By $(\mathcal{P}, A)$-RcA^+, there is a $\mathcal{P}$-ground W of V s.t. $\bar{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi'(\bar{a})$ holds.

▷ By the definition (*) of φ', it follows that $\mathsf{V} \models \varphi(\bar{a})$ holds.

▶ Now suppose that $\mathsf{MP}(\mathcal{P}, A)$ holds, and $\mathbb{P} \in \mathcal{P}$ is s.t. $\Vdash_{\mathbb{P}} “\varphi(\bar{a})”$ for $\bar{a} \in A$.

▷ Let φ'' be a formula saying:

(**) “there is a $\mathcal{P}$-ground N s.t. $\bar{x} \in N$ and $N \models \varphi(\bar{x})$”. [9]

Then $\varphi''(\bar{a})$ is a $\mathcal{P}$-button and $\mathbb{P}$ is its push.

By $\mathsf{MP}(\mathcal{P}, A)$, $\varphi''(\bar{a})$ holds in V and hence there is a $\mathcal{P}$-ground W of V s.t. $\bar{a} \in \mathsf{W}$ and $\mathsf{W} \models \varphi(\bar{a})$. This shows $(\mathcal{P}, A)$-RcA^+. □ (Proposition 3)

[9] This is formalizable in the language of ZFC by Laver-Woodin Theorem. See:

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[9a] Jonas Reitz, The Ground Axiom, JSL, Vol.72, No.4 (2007), 1299–1317.

[9b] Joan Bagaria, Joel David Hamkins, Konstantinos Tsaprounis, Toshimichi Usuba, Superstrong and other large cardinals are never Laver indestructible, AML, Vol.55 (2016), 19–35.

Proof of Theorem 15.

Proof. Suppose that $\underline{\Vdash_{\mathbb{P}} \text{“} \mathcal{H}(\mu^+) \models \varphi(\bar{a}) \text{”}}$ for $\mathbb{P} \in \mathcal{P}$ with $\Vdash_{\mathbb{P}} \text{“} \mathsf{BFA}_{<\kappa}(\mathcal{P}) \text{”}$, $\mu < \kappa$, Σ_2-formula φ and for $\bar{a} \in \mathcal{H}(\mu^+)$.

- Let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic set. Then we have

  (1) $\mathsf{V}[\mathbb{G}] \models \text{“} \mathsf{BFA}_{<\kappa}(\mathcal{P}) \;\wedge\; \mathcal{H}(\mu^+) \models \varphi(\bar{a}) \text{”}$.

- Let $\varphi = \exists y \psi(\bar{x}, y)$ where ψ is a Π_1-formula in $\mathcal{L}_\varepsilon$.
  Let $b \in \mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]}$. be s.t. $\mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{G}]})^{\mathsf{V}[\mathbb{G}]} \models \psi(\bar{a}, b)$.

- Since κ is tightly $\mathcal{P}$-Laver-gen. huge, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}} \text{“} \underset{\sim}{\mathbb{Q}} \in \mathcal{P} \text{”}$ s.t., for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$ with

  (2) $\mathbb{G} \subseteq \mathbb{H}$ (under the identification $\mathbb{P} \lessdot \mathbb{P} * \underset{\sim}{\mathbb{Q}}$),

  there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ s.t. $j : \mathsf{V} \overset{\prec}{\rightarrow}_\kappa M$,

  (3) $|\, \mathbb{P} * \underset{\sim}{\mathbb{Q}} \,| \leq j(\kappa)$ (by tightness),
  (4) $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$ and
  (5) $j''j(\kappa) \in M$.

  By (1), (2) and Bagaria's Absoluteness Theorem 2 (applied to $V[\mathbb{G}]$), we have $\mathsf{V}[\mathbb{H}] \models \text{“} \psi(\bar{a}, b) \text{”}$ and hence $\mathsf{V}[\mathbb{H}] \models \text{“} \mathcal{H}(\mu^+) \models \psi(\bar{a}, b) \text{”}$.

Proof of Theorem 15. (2/2)

- By (3), (4) and (5), there is a $\mathbb{P}$-name of b in M. By (4), it follows that $b \in M$. By similar argument, we have $\mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{H}]})^{\mathsf{V}[\mathbb{H}]} \subseteq M$ and hence $\mathcal{H}((\mu^+)^{\mathsf{V}[\mathbb{H}]})^{\mathsf{V}[\mathbb{H}]} = \mathcal{H}((\mu^+)^M)^M \in M$. Thus we have $M \models$ "$\mathcal{H}(\mu^+) \models \psi(\bar{a}, b)$".
- By elementarity, it follows that $\mathsf{V} \models$ "$\mathcal{H}(\mu^+) \models \exists y \psi(\bar{a}, y)$", and hence $\mathsf{V} \models$ "$\mathcal{H}(\mu^+) \models \varphi(\bar{a})$" as desired.

▷ Suppose now that $\mathbb{P}$, μ, φ, $\bar{a}$ are as above and assume that $\mathsf{V} \models$ "$\mathcal{H}(\mu^+) \models \varphi(\bar{a})$" holds. For Π_1-formula ψ as above let $b \in \mathcal{H}(\mu^+)^{\mathsf{V}}$ be s.t. $V \models$ "$\mathcal{H}(\mu^+) \models \psi(\bar{a}, b)$".
Since $\mathsf{V} \models \mathsf{BFA}_{<\kappa}(\mathcal{P})$ by assumption, it follows that $\mathsf{V}[\mathbb{G}] \models \psi(\bar{a}, b)$ by Bagaria's Absoluteness Theorem 2 , and hence $\mathsf{V}[\mathbb{G}] \models \varphi(\bar{a})$.
The last assertion of the theorem follows by the same argument as that given at the end of the proof of Theorem 11. □ (Theorem 15.)

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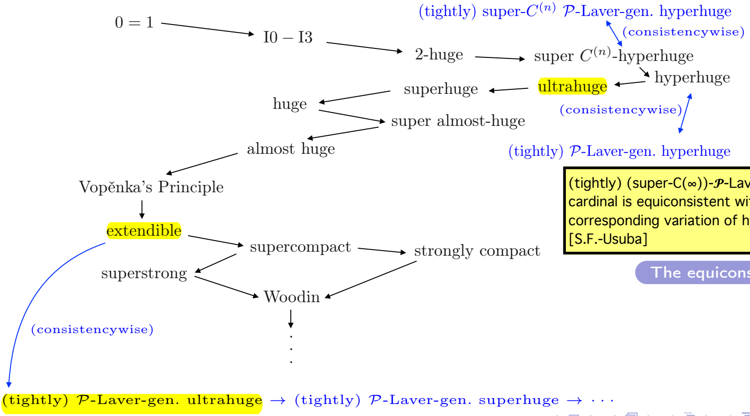
Additional slide 2: Identity crisis (or a resolution thereof)

- I am working on the following conjecture (suggested by G. Goldberg):

Proposition. A model with a/the tightly $\mathcal{P}$-Laver generically ultrahuge cardinal can be obtained starting from a model with an extendible cardinal.

Conjecture. A model with a/the tightly super-$C^{(\infty)}$ $\mathcal{P}$-Laver generically ultrahuge cardinal can be obtained starting from a model with a super-$C^{(\infty)}$ extendible cardinal, and this cardinal has relatively low consistency strength.



(tightly) (super-C(∞))-$\mathcal{P}$-Laver-gen. hyperhuge cardinal is equiconsistent with the corresponding variation of hyperhuge cardinal [S.F.-Usuba]

The equiconsistency

Additional slide 1: Identity crisis (or a resolution thereof)

- For many combination of $\mathcal{P}$, A, and Γ the exact consistency strength of $\mathsf{MP}(\mathcal{P}, A)_\Gamma$ is known: they are usually quite low and compatible with $V = L$.
  - For example for $\mathcal{P} =$ ccc p.o.s, proper p.o.s, or semi-proper p.o.s, $\mathsf{MP}(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))$ is known to be compatible with $V = L$.
  - An exception is when $\mathcal{P} =$ stationary preserving p.o.s. The known lower bound of $\mathsf{MP}(\mathcal{P}, \mathcal{H}(2^{\aleph_0}))$ implies e.g. much nore than $0^\#$ exists.
- On the other hand,

Theroem 34. MM^{++} (or even MM^{++} with class many, stationarily many etc. supercompact cardinals) does not imply any of $\mathsf{MP}(\mathcal{P}, \emptyset)$ for any non-trivial $\mathcal{P}$.

Proof. MM^{++} (with class many supercompact cardinals) is compatible with GA (Ground Axiom) while $\mathsf{MP}(\mathcal{P}, \emptyset)$ for any non-trivial $\mathcal{P}$ implies $\neg$ GA. □

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