# Shelah's Singular Compactness Theorem
## and its variants

Sakaé Fuchino (渕野 昌)

Graduate School of System Informatics
Kobe University
(神戸大学大学院 システム情報学研究科)

http://fuchino.ddo.jp/index-j.html

**BARCELONA SET THEORY SEMINAR**

(2016 年 12 月 29 日 (05:39 CEST) version)

**2016 年 10 月 21 日 (於 University of Barcelona)**

This presentation is typeset by pLaTeX with beamer class.
These slides are downloadable as
http://fuchino.ddo.jp/slides/UB-seminar2016-10-21.pdf

# Shelah's Singular Compactness Theorem



- **Shelah's Singular Compactness Theorem (SCT)** states, roughly speaking, that for a class of structures $\mathcal{F}$, if $A \in \mathcal{F}$ of singular cardinal is "almost free" then $A$ is free.

- **Some instances of the SCT:** The following statements are theorems in $\mathrm{ZFC}$:

(1) For an abelian group $A$ of singular cardinality, if all subgroups of $A$ of cardinality strictly less than $|A|$ are free, then $A$ is free.

(2) For a graph $E$ of singular cardinality, if all subgraphs of $E$ of cardinality strictly less than $|E|$ are of countable coloring number, then $E$ also has the countable coloring number.

## Shelah's Singular Compactness Theorem (2/2) 

- **SCT for non-hereditary notions of "freeness":** The following statements are theorems in $\mathrm{ZFC}$:

(3) For a Boolean algebra $B$ of singular cardinality $\lambda$, if there are cofinally many $\kappa < \lambda$ s.t.

$$\{C \leq B : |C| = \kappa^+, C \text{ is free}\}$$

contains a club $\subseteq [B]^{\kappa^+}$, then $B$ is free.

(4) For a pre-Hilbert space $X$ (over $\mathbb{R}$ or $\mathbb{C}$) which is a dense subspace of $\ell_2(\lambda)$ for a singular $\lambda$, if there are cofinally many $\kappa < \lambda$ s.t.

$$\{u \in [\lambda]^{\kappa^+} : X \downarrow u \text{ is a non-pathological pre-Hilbert space}\}$$

contains a club in $[\lambda]^{\kappa^+}$, then $X$ is non-pathological.

# Applications



- $(2)$ and $(4)$ are used as a part of the proof of the following equivalence theorem:

**Theorem.** The following are equivalent:

(a) FRP;

(b) A graph $E$ is of countable coloring number if and only if all subgraph of $E$ of cardinality $\leq \aleph_1$ are of countable coloring number.

(c) For any pre-Hilbert space $X$ which is a dense subspace of $\ell_2(\kappa)$ for some cardinal $\kappa > \omega$, $X$ is pathological if and only if $\{u \in [\kappa]^{\aleph_1} : X \downarrow u \text{ is pathological}\}$ is stationary in $[\kappa]^{\aleph_1}$.

# Reflection of maximal chromatic number



**Theorem.** (Ottenbreit-Sakai/Foreman-Laver) The combination of the following assertions $(\alpha) + (\beta)$ is is consistent (modulo a huge cardinal):

$(\alpha)$ For any graph $E$ of cardinality $\aleph_3$, if $E$ is of maximal chromatic number (i.e. if $chr(E) = |\,E\,|$) then there is a subgraph $E'$ of $E$ of cardinality $\aleph_1$ with maximal chromatic number.

$(\beta)$ There is a graph $F$ of cardinality $\aleph_2$ of maximal chromatic number s.t. all subgraphs of $F$ of cardinality $\aleph_1$ are countable chromatic.

**Problem.** Does SCT holds for maximal chormatic number?

# Trivia: Reflection of non-maximal chromatic number



**Fact.** The following holds in $\mathrm{ZFC}$:

For any graph $E$ of cardinality $\aleph_2$ with non-maximal chromatic number (i.e. $chr(E) < |\, E\,|$) there is a subgraph $E'$ of $E$ of cardinality $\aleph_1$ with non-maximal chromatic number.

In the following "stationarily many" means in the sense of Woodin:

**Proposition.** (H. Sakai) The following are equialent:

(a) For any graph $E$ of cardinality $\aleph_2$ with non-maximal chromatic number there are stationarily many subgraphs $E'$ of $E$ of cardinality $\aleph_1$ with non-maximal chromatic number;

(b) Chang's Conjecture.

**Problem.** Does SCT holds for non-maximal chormatic number?


Thank you for your attention.

## Coloring number of a graph

- A graph $E = \langle E, K \rangle$ has **coloring number** $\leq \kappa \in \mathrm{Card}$ if there is a well-ordering $\sqsubseteq$ on $E$ s.t. for all $p \in E$ the set

$$\{q \in E \,:\, q \sqsubseteq p \text{ and } q\,K\,p\}$$

has cardinality $< \kappa$.

- The coloring number $col(E)$ of a graph $E$ is the minimal cardinal among such $\kappa$ as above.

## Pathological pre-Hilbert spaces

- A pre-Hilbert space (inner-product space) $X$ over $K$ $(= \mathbb{R}$ or $\mathbb{C})$is said to be **pathological** if there is no orthonormal basis of of $X$ over $K$.
- For an infinite set $S$,

  $\ell_2(S) = \{\mathbf{u} \in {}^S K \,:\, \sum_{x\in S}(\mathbf{u}(x))^2 < \infty\}$,

  where $\sum_{x\in S}(\mathbf{u}(x))^2$ is defined as $\sup\{\sum_{x\in A}(\mathbf{u}(x))^2 \,:\, A \in [S]^{<\aleph_0}\}$.
  $\ell_2(S)$ with with coordinatewise addition and scalar multiplication, as well as the inner product defined by

  $(\mathbf{u}, \mathbf{v}) = \sum_{x\in S} \mathbf{u}(x)\overline{\mathbf{v}(x)}$ for $\mathbf{u}$, $\mathbf{v} \in \ell_2(S)$.

  is a/the Hilbert space of density $|\,S\,|$.
- For $X \subseteq \ell_2(S)$ and $u \subseteq S$, $X \downarrow u = \{\mathbf{u} \in X \,:\, \mathrm{supp}(\mathbf{u}) \subseteq u\}$.

# FRP

(FRP) For any regular $\kappa > \omega_1$, any stationary $S \subseteq E^{\omega}_{\kappa}$ and any mapping $g : S \to [\kappa]^{\aleph_0}$, there is $\alpha^* \in E^{\omega_1}_{\kappa}$ s.t.

(*) $\alpha^*$ is closed w.r.t. $g$ (that is, $g(\alpha) \subseteq \alpha^*$ for all $\alpha \in S \cap \alpha^*$) and, for any $I \in [\alpha^*]^{\aleph_1}$ closed w.r.t. $g$, closed in $\alpha^*$ w.r.t. the order topology and with $\sup(I) = \alpha^*$, if $\langle I_\alpha : \alpha < \omega_1 \rangle$ is a filtration of $I$ then $\sup(I_\alpha) \in S$ and $g(\sup(I_\alpha)) \cap \sup(I_\alpha) \subseteq I_\alpha$ hold for stationarily many $\alpha < \omega_1$