Generically suppercompact cardinals as reflection principles

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Generically supercompact cardinals

- For a family $\mathcal{P}$ of p.o.s, a cardinal κ is said to be generically supercompact by $\mathcal{P}$ $:\Leftrightarrow$ for any $\lambda \geq \kappa$, there is a p.o. $\mathbb{P} \in \mathcal{P}$ with $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$, and classes j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t.

(1) $j : V \overset{\preceq}{\to} M \subseteq \mathsf{V}[\mathbb{G}]$;

(2) $crit(j) = \kappa$, $j(\kappa) > \lambda$; and

(3) $j''\lambda \in M$.

- We call j as above a λ-generically supercompact embedding for κ.

Fact 1. Suppose that κ is a (really) supercompact cardinal, $\mu < \kappa$ a regular uncountable cardinal, and $\mathbb{P}_0 = \mathrm{Col}(\mu, \kappa)$.
<u>Then</u>, for a $(\mathsf{V}, \mathbb{P}_0)$-generic $\mathbb{G}_0$,

$$\mathsf{V}[\mathbb{G}_0] \models \text{“}\mu^+ \text{ is a generically supercompact cardinal for } {<}\mu\text{-closed p.o.s”}.$$

Generically supercompact cardinals

Fact 1. Suppose that κ is a (really) supercompact cardinal, $\mu < \kappa$ a regular uncountable cardinal, and $\mathbb{P}_0 = \mathrm{Col}(\mu, \kappa)$.
Then, for a $(\mathsf{V}, \mathbb{P}_0)$-generic $\mathbb{G}_0$,

$\mathsf{V}[\mathbb{G}_0] \models$ " μ^+ is a generically supercompact cardinal for $<\mu$-closed p.o.s".

Proof. ▶ Note that $\mathsf{V}[\mathbb{G}_0] \models$ " $\mu^+ = \kappa$".

- For $\lambda \geq \kappa$, let $j : V \overset{\preceq}{\to} M$ be a λ-supercompact embedding for κ. Then we have

$$j(\mathbb{P}_0) \underbrace{=}_{\text{by elementarity}} \mathrm{Col}(\underbrace{\mu,}_{=j(\mu)} \; j(\kappa))^M \overbrace{=}^{\text{by closedness of } M} \mathrm{Col}(\mu, j(\kappa))^{\mathsf{V}}.$$

- For a $(V[\mathbb{G}_0], \mathrm{Col}(\mu, j(\kappa) \setminus \kappa))$-generic filter $\mathbb{G}$, the lifting $\tilde{j} : V[\mathbb{G}_0] \overset{\preceq}{\to} \underbrace{M[\mathbb{G}_0][\mathbb{G}]}_{\subseteq \mathsf{V}[\mathbb{G}_0][\mathbb{G}]}$; $\underset{\sim}{a}^{\mathbb{G}_0} \mapsto j(\underset{\sim}{a})^{\mathbb{G}_0 * \mathbb{G}}$ witnesses the generic λ-supercompactness of $\underbrace{\kappa}_{=(\mu^+)^{V[\mathbb{G}_0]}}$ by μ-closed p.o.s in $\mathsf{V}[\mathbb{G}_0]$.

□ (Fact 1.)

Generic supercompactness by $<\mu$-closed p.o.s

- The generic supercompactness by $<\mu$-closed p.o.s is first-order formalizable:

Theorem 2. For regular uncountable κ and μ,

κ is generically supercompact by $<\mu$-closed p.o.s
$\Leftrightarrow$ for any $\lambda \geq \kappa$, there is a $<\mu$-closed p.o. $\mathbb{P}$ s.t.
$\Vdash_{\mathbb{P}}$ "there is a V-normal ultrafilter on $\mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}})$".

to the proof of Theorem 7

- The proof of Theorem 2 is done by imitating the proof of Solovay-Reinhardt characterization of supercompactness in terms of existence of normal filters.

Generic supercompactness by $<\mu$-closed p.o.s (2/4)

Theorem 2. For regular uncountable κ and μ,

κ is generically supercompact by $<\mu$-closed p.o.s
$\Leftrightarrow$ for any $\lambda \geq \kappa$, there is a $<\mu$-closed p.o. $\mathbb{P}$ s.t.
$\Vdash_{\mathbb{P}}$ "there is a V-normal ultrafilter on $\mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}})$".

Proof. $(\Rightarrow)$:

- Let $\lambda \geq \kappa$ and let $\mathbb{P}$ be a $<\mu$-closed p.o. with $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ and classes j, $M \subseteq \mathsf{V}[\mathbb{G}]$ s.t. $j : V \xrightarrow{\preceq} M$ is a λ-generically supercompact embedding for κ.
- ▷ In particular, $j''\lambda \in M$.
- In $\mathsf{V}[\mathbb{G}]$, let

$$U_j := \{A \in V \,:\, A \subseteq \mathcal{P}_\kappa(\lambda)^{\mathsf{V}},\, j''\lambda \in j(A)\}.$$

- ▷ U_j is a V-normal ultrafilter on $\mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}})$.

Generic supercompactness by $<\mu$-closed p.o.s (3/4)

Theorem 2. For regular uncountable κ and μ,
κ is generically supercompact by $<\mu$-closed p.o.s
$\Leftrightarrow$ for any $\lambda \geq \kappa$, there is a $<\mu$-closed p.o. $\mathbb{P}$ s.t.
$\Vdash_{\mathbb{P}}$ "there is a V-normal ultrafilter on $\mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}})$".

Proof. ($\Leftarrow$):

- Let $\lambda \geq \kappa$ and let $\mathbb{P}$ be a $<\mu$-closed p.o. with $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ and V-normal ultrafilter $U \in \mathsf{V}[\mathbb{G}]$ on $\mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}})$.
- $\mathcal{W} := \{f \in V \,:\, f : \mathcal{P}_\kappa(\lambda)^{\mathsf{V}} \to \mathsf{V}\}$
- For $f, g \in \mathcal{W}$, $f \sim_U g :\Leftrightarrow \{x \in \mathcal{P}_\kappa(\lambda)^{\mathsf{V}} \,:\, f(x) = g(x)\} \in U$;
  $f \in_U g :\Leftrightarrow \{x \in \mathcal{P}_\kappa(\lambda)^{\mathsf{V}} \,:\, f(x) \in g(x)\} \in U$.
- $\sim_U$ is a congruence relation to $\in_U$.
  We write $f/\sim_U \in_U g/\sim_U :\Leftrightarrow f \in_U g$.

closedness of $\mathbb{P}$ is needed here!

Claim. $\in_U$ is an extensional, well-founded and set-like rel. on $\mathcal{W}/\sim_U$.

- Let M be a Mostowski-collapse of $\langle \mathcal{W}/\sim_U, \in_U \rangle$. Let j be the mapping which corresponds to the mapping $: \mathsf{V} \to \mathcal{W}/\sim_U$; $a \mapsto const_a/\sim_U$. Then $j : \mathsf{V} \xrightarrow{\preccurlyeq} M$ is a λ-generically supercompact embedding for κ. $\square$ (Theorem 2)

Some more details of the proof:

- Let $\lambda \geq \kappa$ and let $\mathbb{P}$ be a $<\mu$-closed p.o. with $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ and V-normal ultrafilter $U \in \mathsf{V}[\mathbb{G}]$ on $\mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}})$.

- $\mathcal{W} := \{f \in V \ :\ f : \mathcal{P}_\kappa(\lambda)^{\mathsf{V}} \to \mathsf{V}\}$
- For f, $g \in \mathcal{W}$, $f \sim_U g :\Leftrightarrow \{x \in \mathcal{P}_\kappa(\lambda)^{\mathsf{V}} \ :\ f(x) = g(x)\} \in U$;
  $f \in_U g :\Leftrightarrow \{x \in \mathcal{P}_\kappa(\lambda)^{\mathsf{V}} \ :\ f(x) \in g(x)\} \in U$.
- $\sim_U$ is a congruence relation to $\in_U$.
  We write $f/\sim_U \in_U g/\sim_U :\Leftrightarrow f \in_U g$.

Claim. $\in_U$ is an extensional, well-founded and set-like rel. on $\mathcal{W}/\sim_U$.

$\vdash$ To show the well-foundedness, suppose for contradiction that there is a sequence $\langle f_n \ :\ n \in \omega\rangle$ in $\mathcal{W}$, s.t. $f_{n+1} \in_U f_n$ for all $n \in \omega$.

- $A_n := \{x \in \mathcal{P}_\kappa(\lambda)^{\mathsf{V}} \ :\ f_{n+1}(x) \in f(n)\}$.
- Since $\mathbb{P}$ does not add any new ω-sequence, $\langle f_n \ :\ n \in \omega\rangle \in \mathsf{V}$. Thus $\bigcap_{n\in\omega} A_n \in U$ (Lemma A1). For $x \in \bigcap_{n\in\omega} A_n \in U$, we have $f_1(x) \ni f_2(x) \ni f_3(s) \ni \cdots$. $\lightning$ ... $\dashv$

Generic supercompactness by other $\mathcal{P}$s

Problem. Can generic supercompactness by a class $\mathcal{P}$ adding new ω-sequences first-order definable?

Is there any "nice" first-order definable property which can replace the generic supercompactness by $\mathcal{P}$?

The assertion

> "V is a generic extension of an inner model by adding supercompact many Cohen reals"

for example, is first-order formalizable and implies the generic supercompactness by c.c.c. p.o.s. However, this statement is too artificial to be considered as a "nice" set-theoretic principle.

Some Cardinal arithmetic

Lemma 3. Suppose that κ is a gen. supercompact cardinal by $<\mu$-closed forcing. Then we have $2^{<\mu} < \kappa$.

In particular, if $\kappa = \mu^+$ and κ is gen. supercompact by $<\mu$-closed forcing, then we have $2^{<\mu} = \mu$.

Proof. Suppose otherwise and let $\lambda = 2^{<\mu} \geq \kappa$.

- Let $\mathbb{P}$ be a $<\mu$-closed p.o. with a $(\mathsf{V}, \mathbb{P})$-generic $\mathbb{G}$ and j, $M \in \mathsf{V}[\mathbb{G}]$ s.t. $\mathsf{V}[\mathbb{G}] \models j : \mathsf{V} \overset{\preceq}{\to} M$, $crit(j) = \kappa$, $j(\lambda) \geq j(\kappa) > \lambda$, and (*) $j''\lambda \in M$.
- We have $\mathcal{P}_\mu(\mu)^{\mathsf{V}} \subseteq \mathcal{P}_\mu(\mu)^M \subseteq \mathcal{P}_\mu(\mu)^{\mathsf{V}[\mathbb{G}]}$.
- Since $\mathbb{P}$ is μ-closed, $\mathcal{P}_\mu(\mu)^{\mathsf{V}} = \mathcal{P}_\mu(\mu)^{\mathsf{V}[\mathbb{G}]}$. Thus, $\mathcal{P}_\mu(\mu)^{\mathsf{V}} = \mathcal{P}_\mu(\mu)^M$ and

$$M \models |\lambda| \underbrace{=}_{\text{the bijection showing this is in } M \text{ because of (*)}} |\mathcal{P}_\mu(\mu)^{\mathsf{V}}| = |\mathcal{P}_\mu(\mu)^M| = |\mathcal{P}_{j(\mu)}(j(\mu))^M| \underbrace{=}_{\text{by elementarity}} j(\lambda).$$

↯ □ (Lemma 3.)

to the proof of Theorem 7

Game Reflection Principle

- For a set A and $\mathcal{A} \subseteq {}^{\mu>}A$, we consider the following game $\mathcal{G}^{\mu>A}(\mathcal{A})$ for players I and II:

$$\begin{array}{c|cccccccccc} \mathrm{I} & a_0 & & a_1 & & a_2 & & \cdots & a_\xi & & \cdots \\ \hline \mathrm{II} & & b_0 & & b_1 & & b_2 & \cdots & & b_\xi & \cdots \end{array} \quad (\xi < \mu)$$

  where $a_\xi, b_\xi \in A$ for $\xi < \mu$.

  ▷ II wins this match if

  $\langle a_\xi, b_\xi : \xi < \eta \rangle \in \mathcal{A}$ and $\langle a_\xi, b_\xi : \xi < \eta \rangle ^\frown \langle a_\eta \rangle \notin \mathcal{A}$ for some $\eta < \mu$; or $\langle a_\xi, b_\xi : \xi < \mu \rangle \in [\mathcal{A}]$

  where $[\mathcal{A}] := \{ f \in {}^{\mu}A : f \restriction \xi \in \mathcal{A}$ for all $\xi < \mu \}$.

- For regular cardinals μ, κ with $\omega < \mu < \kappa$,
  The Game Reflection Principle for $< \mu$ and $< \kappa$ is the assertion:

GRP$^{<\mu}(<\kappa)$: For any set A of regular cardinality $\geq \kappa$ and μ-club $\mathcal{C} \subseteq [A]^{<\kappa}$, if the player II has no winning strategy in $\mathcal{G}^{\mu>A}(\mathcal{A})$ for some $\mathcal{A} \subseteq {}^{\mu>}A$, there is $B \in \mathcal{C}$ s.t. the player II has no winning strategy in $\mathcal{G}^{\mu>B}(\mathcal{A} \cap {}^{\mu>}B)$.

Game Reflection Principle (2/4)

$\mathrm{GRP}^{<\mu}(<\kappa)$: For any set A of regular cardinality $\geq \kappa$ and μ-club $\mathcal{C} \subseteq [A]^{<\kappa}$, if the player II has no winning strategy in $\mathcal{G}^{\mu>A}(\mathcal{A})$ for some $\mathcal{A} \subseteq {}^{\mu>}A$, there is $B \in \mathcal{C}$ s.t. the player II has no winning strategy in $\mathcal{G}^{\mu>B}(\mathcal{A} \cap {}^{\mu>}B)$.

Lemma 4. For any uncountable regular cardinals μ_0 μ, κ with $\mu_0 \leq \mu < \kappa$, $\mathrm{GRP}^{<\mu}(<\kappa)$ implies $\mathrm{GRP}^{<\mu_0}(<\kappa)$.

- The "Strong Game Reflection Principle" Bernhard König introduced in his 2004 paper [König] is $\mathrm{GRP}^{<\omega_1}(<\aleph_2)$ in our terminology.

Game Reflection Principle (3/4)

Proposition 5. (Lemma 4.11 in [1]) For a regular uncountable μ and $\kappa = \mu^+$, if κ is gen. supercompact by $<\mu$-closed forcing, then $\mathsf{GRP}^{<\mu}(<\kappa)$ holds.

Proof. Suppose that $\lambda \geq \kappa$, $\mathcal{A} \subseteq {}^{\mu>}\lambda$, and the set $\{S \in \mathcal{P}_\mu(\lambda) : \text{II has a w.s. in } \mathcal{G}^{\mu>S}(\mathcal{A} \cap {}^{\mu>}S)\}$ contains a μ-club $\mathcal{C}$.

- We want to show that II has a w.s. in $\mathcal{G}^{\mu>\lambda}(\mathcal{A})$.
- Let $\mathbb{P}$ be a $<\mu$-closed p.o. with $(\mathsf{V}, \mathbb{P})$-gen. $\mathbb{G}$ s.t. there are j, $M \subseteq \mathsf{V}[\mathbb{G}]$ with $j : \mathsf{V} \overset{\preceq}{\to} M$, $crit(j) = \kappa$, $j(\kappa) > \lambda$, and (*) $j''\lambda \in M$.
- In M, we have $j''\lambda \in j(\mathcal{C})$. Thus, the player II has a w.s. in $\mathcal{G}^{\mu>j''\lambda}(j(\mathcal{A}) \cap {}^{\mu>}j''\lambda)$.
- By the closedness (*) of M, M also thinks that II has a w.s. in $\mathcal{G}^{\mu>\lambda}(\mathcal{A}) \cong \mathcal{G}^{\mu>j''\lambda}(j(\mathcal{A}) \cap {}^{\mu>}j''\lambda)$.
- Again by the closedness (*) II has a w.s. in $\mathcal{G}^{\mu>\lambda}(\mathcal{A})$ in $\mathsf{V}[\mathbb{G}]$.
- Since $\mathbb{P}$ is $<\mu$-closed, it follows that II has a w.s. in $\mathcal{G}^{\mu>\lambda}(\mathcal{A})$ in V.

to the proof of Theorem 7

□ (Proposition 5)

Game Reflection Principle (4/4)

Theorem 7. ([König], [I]) For a regular uncountable cardinal μ and $\kappa = \mu^+$,

κ is gen. supercompact by $<\mu$-closed p.o.s. $\Leftrightarrow$

$2^{<\mu} = \mu$ and $\mathsf{GRP}^{<\mu}(<\kappa)$.

The condition $2^{<\mu} = \mu$ follows from $\mathsf{GRP}^{<\mu}(<\kappa)$ if $\mu = \omega_1$:

Theorem 8. ([König], [I]) $\mathsf{GRP}^{<\omega_1}(<\kappa)$ implies $2^{\aleph_0} < \kappa$.

Proof of Theorem 7: "$\Rightarrow$" follows from Lemma 3 and Proposition 5 .

The proof for "$\Leftarrow$" is too involved to be presented here.

- **A very rough idea of "$\Leftarrow$":**

Game Reflection Principle (4/4)

Theorem 7. ([König], [I]) For a regular uncountable cardinal μ and $\kappa = \mu^+$,

κ is gen. supercompact by $<\mu$-closed p.o.s. $\Leftrightarrow$

$2^{<\mu} = \mu$ and $\mathrm{GRP}^{<\mu}(<\kappa)$.

Proof. **A very rough idea of "$\Leftarrow$":**
By Theorem 2, it is enough to show that for each $\lambda \geq \kappa$ there is a $<\mu$-closed p.o. $\mathbb{P}$ s.t. $\mathbb{P}$ forces a V-normal ultrafilter.

- ▷ We design a game in which the player II tries to obtain the set $\{b_\xi : \xi < \mu\}$ which encodes a filter basis while the player I challenges by presenting a regressive function a_ξ and demands that player II should choose the move b_ξ which should witness the V-normality for this regressive function.
- ▷ We prove that the player II has a w.s. in the game under $\mathrm{GRP}^{<\mu}(<\kappa)$ ($2^{<\mu} = \mu$ is necessary for this proof), and that in the generic extension with $<\mu$-closed forcing collapsing enough cardinals, the player I can enumerate all the regressive functions and a wined game for II creates a V-normal filter. □ (Theorem 7)

Game Reflection Principle is a very strong reflection statement

Theorem 8. ([König], [I]) For a regular cardinal $\kappa > \aleph_1$, $\mathsf{GRP}^{<\omega_1}(<\kappa)$ implies the Rado Conjecture $\mathsf{RC}(<\kappa)$ with reflection point $<\kappa$.

Theorem 9. ([I]) Suppose that κ is a regular uncountable cardinal s.t. $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$ holds. Then $\mathsf{GRP}^{<\omega_1}(<\kappa)$ implies the Downward Löwenheim-Skolem Theorem $\mathsf{SDLS}_+(\mathcal{L}^{\aleph_0,II}_{stat}, <\kappa)$ for stationary logic with reflection point $<\kappa$.

Reflection down to $< \aleph_2$

Game Reflection Principle ($\mathsf{GRP}^{<\omega_1}(<\aleph_2)$)
$\Leftrightarrow$ ω_2 is generically supercompact
by σ-closed forcing

$\mathsf{MM}^{+\omega_1}$

$\mathsf{PFA}^{+\omega_1}$

[I] Theorem 4.7,
Lemma 4.1 and Lemma 4.2

$\mathsf{SDLS}(\mathcal{L}^{\aleph_0,II}_{stat}, <\aleph_2)$
$\Leftrightarrow \mathsf{DRP}(\mathsf{IC}_{\aleph_0}) + \mathsf{CH}$

$\mathsf{MA}^{+\omega_1}(\sigma\text{-closed})$

MM

B.König

[I] Lemma 2.1,(1) and
Lemma 3.5,(2)

[I] Lemma 2.1,(2)
and Lemma 3.5,(1)

$\mathsf{MA}^{+}(\sigma\text{-closed})$

$\mathsf{SDLS}^{-}(\mathcal{L}^{\aleph_0}_{stat}, <\aleph_2) \Leftrightarrow \mathsf{DRP}(\mathsf{IC}_{\aleph_0})$

Rado Conjecture ($\mathsf{RC}(<\aleph_2)$)

$\mathsf{RP}_{\mathsf{IC}_{\aleph_0}}$

Ph.Doebler

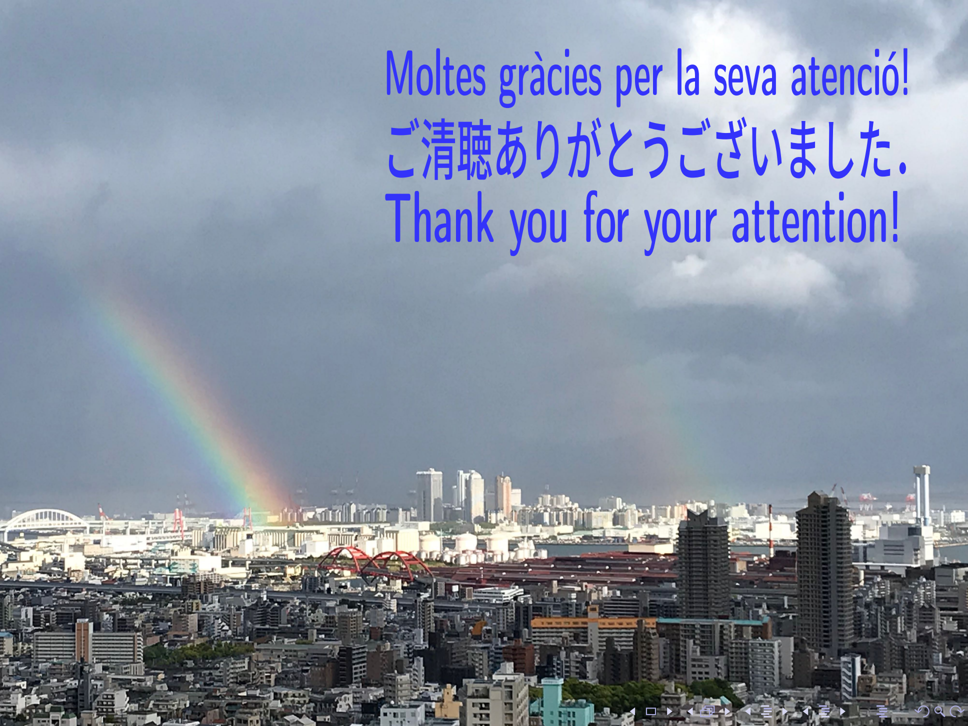
Axiom R $\Leftrightarrow \mathsf{RP}_{\mathsf{IU}_{\aleph_0}}$

Semi-stationary Reflection (SSR)

Fodor-type Reflection Principle (FRP)

Doebler-Scindler

A strong version of Changs Conjecture



Moltes gràcies per la seva atenció!

ご清聴ありがとうございました.

Thank you for your attention!

Downward Löwenheim-Skolem Theorem for stationary Logic (1/2)

- The logic $\mathcal{L}^{\aleph_0,II}$ is the monadic second-order logic with second-order variables X, Y, Z etc. which are interpreted as countable sets of the underlying set of the structure. second order quantifiers $\exists$ (and its dual $\forall$) are allowed.
- The logic has a built-in relation symbol ε which connects first and second order variables as "$x \, \varepsilon \, X$ with the obvious interpretation.
- $\mathcal{L}^{\aleph_0,II}_{stat}$ is an extension of $\mathcal{L}^{\aleph_0,II}$ in which a new second order quantifier "*stat*" is also allowed with the interpretation

$$\mathfrak{A} \models stat\, X \varphi(a_0, \ldots, a_{m-1}, B_0, \ldots, B_{n-1}, X) \;\Leftrightarrow$$

$$\{B \in [\,|\mathfrak{A}|\,]^{\aleph_0} \;:\; \mathfrak{A} \models \varphi(a_0, \ldots, a_{m-1}, B_0, \ldots, B_{n-1}, B)\}$$

is stationary.

$\mathrm{SDLS}_+(\mathcal{L}^{\aleph_0,II}_{stat}, <\kappa)$: For any structure $\mathfrak{A}$ (with a countable signature), there are stationarily may $M \in [\,|\mathfrak{A}|\,]^{<\kappa}$ s.t. $\mathfrak{A} \restriction M \prec_{\mathcal{L}^{\aleph_0,II}_{stat}} \mathfrak{A}$.

Downward Löwenheim-Skolem Theorem for stationary Logic (2/2)

Proposition A6. (M. Magidor) $\mathrm{SDLS}_+(\mathcal{L}^{\aleph_0,II}_{stat}, <\aleph_2)$ implies Fodor-Type Reflection Principle.

Proposition A7. ([I]) $\mathrm{SDLS}_+(\mathcal{L}^{\aleph_0,II}_{stat}, <\kappa)$ implies $2^{\aleph_0} < \kappa$.

Theorem A8. ([I]) $\mathrm{SDLS}_+(\mathcal{L}^{\aleph_0,II}_{stat}, <\kappa)$ is equivalent to $2^{\aleph_0} < \kappa$ + Diagonal Reflection Principle of S.Cox for internally club sets down to $<\kappa$.

Back

Rado Conjecture (1/2)

- A tree $T = \langle T, \leq_T \rangle$ is special if T is a countable union of pairwise incomparable sets (anti-chains) $T = \bigcup_{n\in\omega} A_n$.
- For a cardinal κ, Rado Conjecture with reflection point $< \kappa$ is the principle:

RC$(< \kappa)$: For any non-special tree T there is a subtree $T' \subseteq T$ of size $< \kappa$ s.t. T' is non-special.

▷ The classical Rado Conjecture RC is the principle RC$(\leq \aleph_2)$.

Rado Conjecture (2/2)

▷ The classical Rado Conjecture RC is the principle RC($\leq \aleph_2$).

Theorem A3. (Ph. Doebler) RC implies Semi-Stationary Reflection (which implies in turn a strong version of Chang's Conjecture).

Theorem A4. (S.F., H.Sakai, V.Torres-Perez, T.Usuba) RC implies Fodor-type Reflection Principle (and this principle is known to be equivalent to may "mathematical" reflection statements).

Back

μ-club family of $[A]^{<\kappa}$

- For a regular cardinals $\mu < \kappa$ and a set A,
  $\mathcal{C} \subseteq [A]^{<\kappa}$ is μ-club $:\Leftrightarrow$

  $\mathcal{C}$ is cofinal in $[A]^{<\kappa}$ w.r.t. $\subseteq$, and we have $\bigcup_{\alpha<\nu} c_\alpha \in \mathcal{C}$ for any $\subseteq$-increasing sequence $\langle c_\alpha \in \mathcal{C} : \alpha < \nu\rangle$ in $\mathcal{C}$ with $\mu \leq \mathrm{cf}(\nu) < \kappa$.

Lemma A2. For regular μ_0, μ with $\mu_0 < \mu$, if $\mathcal{C} \subseteq [A]^{<\kappa}$ is μ_0-club, then $\mathcal{C}$ is μ-club.

Back

V-normal ultrafilter

- Suppose that we are living in a universe W and V is an inner model.
  - In W, $U \subseteq \mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}})$ is a V-normal ultrafilter

:$\Leftrightarrow$

① $\emptyset \notin U$; For any A, $A' \in U$, $A \cap A' \in U$; If $A \in U$, $A \subseteq A' \subseteq \mathcal{P}_\kappa(\lambda)^{\mathsf{V}}$, then $A' \in U$; for any $A \in \mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}})$, either $A \in U$ or $\mathcal{P}_\kappa(\lambda)^{\mathsf{V}} \setminus A \in U$; and

② For any $x_0 \in \mathcal{P}_\kappa(\lambda)^{\mathsf{V}}$, $\{x \in \mathcal{P}_\kappa(\lambda)^{\mathsf{V}} : x_0 \subseteq x\} \in U$;

③ For any $\langle A_\xi : \xi \in \lambda \rangle \in V$, if $\{A_\xi : \xi < \lambda\} \subseteq U$, then $\triangle_{\xi\in\lambda} A_\xi := \{x \in \mathcal{P}_\kappa(\lambda)^{\mathsf{V}} : x \in A_\xi \text{ for all } \xi \in x\} \in U$. Back

Lemma A1. For V-normal U and $\langle A_n : n \in \omega \rangle \in \mathsf{V}$ with $A_n \in U$ for all $n \in \omega$, we have $\bigcap_{n\in\omega} A_n \in U$

Proof. Let $A_\xi := \mathcal{P}_\kappa(\lambda)^{\mathsf{V}}$ for all $\xi \in \lambda \setminus \omega$. Then $U \ni \triangle_{\xi\in\lambda} A_\xi \cap \{x \in \mathcal{P}^{\mathsf{V}}(\mathcal{P}_\kappa(\lambda)^{\mathsf{V}}) : \omega \subseteq x\} \subseteq \bigcap_{n\in\omega} A_n$.

Back to the proof of Claim

$\square$ (Lemma A1.)