Reflection Theorems on non-existence of orthonormal bases

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(Historical) Background

- An inner product space (over $K = \mathbb{R}$ or $\mathbb{C}$) is also called pre-Hilbert space.

- ▷ The inner product of $\mathbf{a}, \mathbf{b} \in X$ is denoted here by $(\mathbf{a}, \mathbf{b})$.

- The completion of a pre-Hilbert space X is a Hilbert space. Thus it can be assumed without loss of generality that X is a subspace of the Hilbert space $\ell_2(\kappa)$ where κ is the density $d(X)$ of the space X (and the inner product of X is the restriction of the inner product of $\ell_2(\kappa)$).

Fact 1

If X is a separable pre-Hilbert space then X has an orthonormal basis.

Proof. By Gram-Schmidt orthogonalization process. □ (Fact 1)

Fact 2 (Halmos (197?), see [Gudder 1974])

There are pre-Hilbert spaces without any orthonormal bases.

Sketch of a proof.

- Let B be a linear basis (Hamel basis) of the linear space $\ell_2(\omega)$ extending $\{\mathbf{e}^{\omega}_n : n \in \omega\}$. Note that $|B| = 2^{\aleph_0}$.
- Let $\omega_1 \leq \lambda \leq 2^{\aleph_0}$ and $f : B \to \{\mathbf{e}^{\lambda}_{\alpha} : \alpha < \lambda\} \cup \{\mathbf{0}_{\ell_2(\lambda)}\}$ be a surjection s.t. $f(\mathbf{e}^{\omega}_n) = \mathbf{0}_{\ell_2(\lambda)}$ for all $n \in \omega$. Note that f generates a linear mapping from the linear space $\ell_2(\omega)$ to a dense subspace of $\ell_2(\lambda)$.
- Let $U = \{\langle \mathbf{b}, f(\mathbf{b})\rangle : \mathbf{b} \in B\}$ and $X = [U]_{\ell_2(\omega)\oplus\ell_2(\lambda)}$. Then this X is as desired since $\{\langle \mathbf{e}^{\omega}_n, \mathbf{0}\rangle : n \in \omega\}$ is a maximal orthonormal system in X and hence $\dim(X) = \aleph_0$ while we have $cls_{\ell_2(\omega)\oplus\ell_2(\lambda)}(X) = \ell_2(\omega) \oplus \ell_2(\lambda)$ and hence $d(X) = \lambda$.

$\square$ (Fact 2)

Fact 3 ([Buhagiara et al.])

For any pre-Hilbert space X, *we have* $d(X) \leq |X| \leq (\dim(X))^{\aleph_0}$.

Some natural questions

- Are there pre-Hilbert spaces with $\dim(X) = d(X)$ which have no orthonormal bases ?
- For which cardinals κ and λ are there pre-Hilbert spaces X without orthonormal basis s.t. $\dim(X) = \kappa$ and $d(X) = \lambda$?
- How can we characterize pre-Hilbert spaces X without any orthonormal bases ?
- Are there pre-Hilbert spaces whose non-existence of orthonormal bases is not absolute (in terms of generic extension of the universe of the set theory e.g. preserving cardinals and not adding reals) ?
- For a pre-Hilbert space X without any orthonormal bases, can we always find a sub space Y of X of density $\aleph_1$ which also has no orthonormal bases ?

Some natural questions

- Are there pre-Hilbert spaces with $\dim(X) = d(X)$ which have no orthonormal bases ? — Yes. E.g. "Halmos' example" $\oplus\, \ell_2(\lambda)$
- For which cardinals κ and λ are there pre-Hilbert spaces X without orthonormal basis s.t. $\dim(X) = \kappa$ and $d(X) = \lambda$?
  — All κ, λ s.t. $\kappa \leq \lambda$ and $\lambda \leq \kappa^{\aleph_0}$ (by the same trick as above).
- How can we characterize pre-Hilbert spaces X without any orthonormal bases ? — See the next slides.
- Are there pre-Hilbert spaces whose non-existence of orthonormal bases is not absolute (in terms of generic extension of the universe of the set theory e.g. preserving cardinals and not adding reals) ?
  — Yes. See the next slides.
- For a pre-Hilbert space X without any orthonormal bases, can we always find a sub space Y of X of density $\aleph_1$ which also has no orthonormal bases ? — The answer is independent from ZFC. See the next slides.

The characterization

Theorem 4 ([F.])

Suppose that X is a pre-Hilbert space. Then X has an orthonormal basis if and only if there are separable subspaces X_α, $\alpha < \delta$ of X orthogonal to each other (for $\delta = d(X)$) s.t. $\oplus_{\alpha<\delta} X_\alpha$ is a dense subspace of X.

- The non trivial direction of Theorem 4 is proved by an iterated application of the following Lemma:

Lemma 5

Suppose that X is a pre-Hilbert space and X is a dense subspace of $\ell_2(S)$. If $\mathcal{B} \subseteq X$ is an orthonormal basis, then, for any $S_0 \subseteq S$, there is an $A \subseteq S$ s.t. $S_0 \subseteq A$, $|A| = |S_0| + \aleph_0$, $X \downarrow A$ is a dense subspace of $\ell_2(S) \downarrow A$, $\mathcal{B}_A = \{\mathbf{b} \in \mathcal{B} : \mathrm{supp}(\mathbf{b}) \subseteq A\}$ is an orthonormal basis of $X \downarrow A$ and $\mathcal{B}_A^- = \mathcal{B} \setminus \mathcal{B}_A$ is an orthonormal basis of $X \downarrow (S \setminus A)$.
In particular, we have $X = (X \downarrow A) \oplus (X \downarrow (S \setminus A))$.

Sketch of the proof of Lemma 5

Lemma 5

Suppose that X is a pre-Hilbert space and X is a dense subspace of $\ell_2(S)$. If $\mathcal{B} \subseteq X$ is an orthonormal basis, then, for any $S_0 \subseteq S$, there is an $A \subseteq S$ s.t. $S_0 \subseteq A$, $|A| = |S_0| + \aleph_0$, $X \downarrow A$ is a dense subspace of $\ell_2(S) \downarrow A$, $\mathcal{B}_A = \{\mathbf{b} \in \mathcal{B} : \mathrm{supp}(\mathbf{b}) \subseteq A\}$ is an orthonormal basis of $X \downarrow A$ and $\mathcal{B}_A^- = \mathcal{B} \setminus \mathcal{B}_A$ is an orthonormal basis of $X \downarrow (S \setminus A)$.
In particular, we have $X = (X \downarrow A) \oplus (X \downarrow (S \setminus A))$.

Proof.

- Let θ be a sufficiently large regular cardinal. Let $M \prec \mathcal{H}(\theta)$ be s.t. X, $S \in M$, $S_0 \subseteq M$ and $|M| = |S_0| + \aleph_0$.
- By elementarity of M, $A = S \cap M$ is then as desired. $\square$ (**Lemma 5**)

Some applications of Theorem 4

- The next Lemma follows easily from Theorem 4:

Lemma 6

If at least one of X and Y does not have any orthonormal bases then $X \oplus Y$ does not have any orthonormal bases.

Corollary 7 (A strengthening of the converse of [Buhagiara et al.])

For any infinite cardinals κ and λ s.t. either $\aleph_0 < \kappa = \lambda$, or, $\kappa < \lambda$ and $\lambda \leq \kappa^{\aleph_0}$, there is a pre-Hilbert space X with $\dim(X) = \kappa$ and $d(X) = \lambda$ without any orthonormal bases.

Proof.

- For $\kappa < \lambda$ with $\lambda \leq \kappa^{\aleph_0}$ a modification of Halmos' example will do.
- For $\aleph_0 < \kappa = \lambda$, "Halmos' example for $\aleph_1$" $\oplus \ell_2(\lambda)$ will do by **Lemma 6**. □ (**Corollary 7**)

An example of a maximal orthonormal system

- (Hiroshi Fujita) Let $\mathbf{b} \in \ell_2(\omega + 1)$ be defined by
  (1) $\mathbf{b}(\omega) = 1$;
  (2) $\mathbf{b}(n) = \frac{1}{n+1}$, for $n \in \omega$.

  Let $X = [\{\mathbf{e}_n^{\omega+1} : n \in \omega\} \cup \{\mathbf{b}\}]_{\ell_2(\omega+1)}$
  (i.e. the subspace of $\ell_2(\omega + 1)$ spanned by $\{\mathbf{e}_n^{\omega+1} : n \in \omega\} \cup \{\mathbf{b}\}$).

- ▷ $B = \{\mathbf{e}_n^{\omega+1} : n \in \omega\}$ is a maximal orthonormal system in X but B is not an orthonormal basis of X.

- ▷ X has an orthonormal basis since it is separative.

- Suppose that κ is a regular cardinal and $S \subseteq E^{\omega}_{\kappa} = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega\}$ is stationary in κ.
- For $\alpha < \kappa$, let $\mathbf{b}_\alpha \in \ell_2(\kappa)$ be defined by

$$\mathbf{b}_\alpha = \begin{cases} \mathbf{e}^{\kappa}_{\alpha}, & \text{if } \alpha \notin S; \\ \text{an element of } \ell_2(\alpha+1) \text{ corresponding} & \\ \text{to } \mathbf{b} \text{ in the previous example}, & \text{if } \alpha \in S. \end{cases}$$

▷ Then $[\{\mathbf{b}_\alpha : \alpha < \kappa\}]_{\ell_2(\kappa)}$ does not have any orthonormal basis.

Modifications

▷ If $S \subseteq \kappa$ is non-reflecting stationary then all subspaces of X of density $< \kappa$ have a orthonormal basis while X does not.

▷ If $S \subseteq E^{\omega}_{\omega_1}$ is stationary and co-stationary then shooting a club into the complement of S adds an orthonormal basis to X.

Theorem 8 ([F.])

Each of the following assertions is equivalent to FRP :

(a) *For any regular $\kappa > \omega_1$ and any dense subspace X of $\ell_2(\kappa)$ (with the inner product induced from the inner product of $\ell_2(\kappa)$), if X does not have any orthonormal basis then*

$$S_X = \{\alpha < \kappa : X \downarrow \alpha \text{ d.n. have any orthonormal basis}\}$$

is stationary in κ.

(b) *For any regular $\kappa > \omega_1$ and any dense subspace X of $\ell_2(\kappa)$, if X does not have any orthonormal basis then*

$$S_X^{\aleph_1} = \{U \in [\kappa]^{\aleph_1} : X \downarrow U \text{ d.n. have any orthonormal basis}\}$$

is stationary in $[\kappa]^{\aleph_1}$.

Singular Compactness Theorem

- The proof of the forward direction of Theorem 8 is done by induction on the density κ of the pre-Hilbert spaces X.

- For singular κ we use the following theorem (in ZFC) which can be proved similarly to the proof of Shelah's Singular Compactness Theorem given in [Hodges 1981]:

Theorem 9 (Singular Compactness Theorem, [F.])

For any singular λ and any pre-Hilbert space X which is a dense subspace of $\ell_2(\lambda)$, if

(**) $\mathcal{N}_\kappa^X = \{u \in [\lambda]^{\kappa^+} : X \downarrow u \text{ has an orthonormal basis}\}$ *contains a club in* $[\lambda]^{\kappa^+}$ *for cofinally many* $\kappa < \lambda$,

then X also has an orthonormal basis.

Almost disjoint sequences

- The bacward direction of Theorem 8 is proved by the following characterization of FRP and the construction of pre-Hilbert space similar to the one in the previous slides using an ADS sequcence in place of a stationary set.

Theorem 10 ([F., Sakai, Soukup and Usuba])

TFAE over ZFC*:*

(a) FRP*;*

(b) $\mathrm{ADS}^-(\kappa)$ *does not hold for all regular uncountable* $\kappa > \omega_1$*.*

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Wszystkiego najlepszego, Olek!

Dziękuję za uwagę.

Fodor-type Reflection Principle (FRP)

(FRP) For any regular $\kappa > \omega_1$, any stationary $E \subseteq E^{\omega}_{\kappa}$ and any mapping $g : E \to [\kappa]^{\aleph_0}$, there is $\alpha^* \in E^{\omega_1}_{\kappa}$ s.t.

(*) α^* is closed w.r.t. g (that is, $g(\alpha) \subseteq \alpha^*$ for all $\alpha \in E \cap \alpha^*$) and, for any $I \in [\alpha^*]^{\aleph_1}$ closed w.r.t. g, closed in α^* w.r.t. the order topology and with $\sup(I) = \alpha^*$, if $\langle I_\alpha : \alpha < \omega_1 \rangle$ is a filtration of I then $\sup(I_\alpha) \in E$ and $g(\sup(I_\alpha)) \cap \sup(I_\alpha) \subseteq I_\alpha$ hold for stationarily many $\alpha < \omega_1$

(see [F., Sakai, Soukup and Usuba]).

- $\mathcal{F} = \langle I_\alpha : \alpha < \lambda \rangle$ is a filtration of I if $\mathcal{F}$ is a continuously increasing $\subseteq$-sequence of subsets of I of cardinality $< |I|$ s.t. $I = \bigcup_{\alpha<\lambda} I_\alpha$.

- FRP follows from Martin's Maximum or Rado's Conjecture. FRP is a large cardinal property: it implies the total failure of the square principle. FRP is known to be equivalent to many mathematical reflection statements like: A locally compact space is non-metrizable if and only if it has a non-metrizable subspace of cardinality $\aleph_1$.

Maximal orthonormal systems and Orthonormal bases

- For a pre-Hilbert space X over K $(= \mathbb{R}$ or $\mathbb{C})$, $B \subseteq X$ is an orthonormal system if $(x, x) = 1$ for all $x \in B$ and $(x, y) = 0$ for all distinct x, $y \in B$.

- $B \subseteq X$ is an orthonormal basis of X if it is an orthonormal system and B spans a dense subspace of X.

- (By Axiom of Choice) any pre-Hilbert space X has a maximal orthonormal system A. If X is a Hilbert space then such A is always an orthonormal basis.

- In case of (not necessarily complete) pre-Hilbert space X a maximal orthonormal system A of X need not to be an orthonormal basis even if X does have an orthonormal basis!

- For $x \in S$ the standard unit vector $\mathbf{e}_x^S \in \ell_2(S)$ is defined by $\mathbf{e}_x^S(y) = \delta_{x,y}$ for any $y \in S$.
  $B = \{\mathbf{e}_x^S : x \in S\}$ is an orthonormal basis of $\ell_2(S)$.

The Hilbert spaces $\ell_2(S)$

- For $K = \mathbb{R}$ or $\mathbb{C}$ and a set S, let

$$\ell_2(S) = \{\mathbf{u} \in {}^S K \;:\; \sum_{x \in S} (\mathbf{u}(x))^2 < \infty\},$$

  where $\sum_{x \in S} (\mathbf{u}(x))^2 = \sup\{\sum_{x \in A} (\mathbf{u}(x))^2 \;:\; A \in [S]^{<\aleph_0}\}$.

- $\ell_2(S)$ is endowed with a natural structure of inner product space with coordinatewise addition and scalar multiplication, as well as the inner product defined by

$$(\mathbf{u}, \mathbf{v}) = \sum_{x \in S} \mathbf{u}(x)\overline{\mathbf{v}(x)} \quad \text{for } \mathbf{u},\, \mathbf{v} \in \ell_2(S).$$

▷ It is easily seen that $\ell_2(S)$ is a/the Hilbert space of density $|\,S\,|$.

- Thus any pre-Hilbert space X of density λ can be embedded densely into $\ell_2(\lambda)$ as a subspace.

Wstecz

Dimension of a pre-Hilbert space

- For a pre-Hilbert space X the cardinality of any maximal orthonormal system $A \subseteq X$ is the same (this is proved easily by elementary cardinal arithmetic and Bessel's inequality). This cardinality is called the dimension of X and denoted by $\dim(X)$.

- Note that if $\dim(X) < d(X)$ then X cannot have any orthonormal bases.

Wstecz

$\mathrm{supp}(\mathbf{a})$, $X \downarrow A$

- Let X be a pre-Hilbert space s.t. X is a dense subspace of $\ell_2(S)$ where $|S| \geq d(X)$ and the inner product of X coincides with that of $\ell_2(S)$ restricted to X.

- For X as above and for $\mathbf{a} \in X$, the support of $\mathbf{a}$ is defined as

$$\mathrm{supp}(\mathbf{a}) = \{\alpha \in S : \mathbf{a}(\alpha) \neq 0\}.$$

- For $A \subseteq S$,

$$X \downarrow A = \{\mathbf{a} \in X : \mathrm{supp}(\mathbf{a}) \subseteq A\}.$$

Wstecz

$\mathsf{ADS}^-(\kappa)$

- For a regular cardinal κ, $\mathsf{ADS}^-(\kappa)$ is the assertion that there is a stationary set $E \subseteq E^\omega_\kappa$ and a sequence $\langle A_\alpha : \alpha \in E \rangle$ s.t.
  (1) $A_\alpha \subseteq \alpha$ and $\mathrm{ot}(A_\alpha) = \omega$ for all $\alpha \in E$;
  (2) for any $\beta < \kappa$, there is a mapping $f : E \cap \beta \to \beta$ s.t. $f(\alpha) < \sup(A_\alpha)$ for all $\alpha \in E \cap \beta$ and $A_\alpha \setminus f(\alpha)$, $\alpha \in E \cap \beta$ are pairwise disjoint.

- We shall call $\langle A_\alpha : \alpha \in E \rangle$ as above an $\mathsf{ADS}^-(\kappa)$**-sequence**.

- Note that it follows from (1) and (2) that A_α, $\alpha \in E$ are pairwise almost disjoint.

Wstecz