Maximality Principles and generic Large Cardinal Axioms

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Maximality Principles and Recurrence Axioms

- Maximality Principle for a (two step) iterable class $\mathcal{P}$ of p.o.s and a set A of parameters (notation: $\mathsf{MP}(\mathcal{P}, A)$) is the assertion:

  if a statement $\varphi(\bar{a})$ for $\bar{a} \in A$ is forced by a p.o. $\mathbb{P} \in \mathcal{P}$, and by any further $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$ for $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", then $\varphi(\bar{a})$ holds. [1], [2]

- Maximality Principles are strengthenings of the principle we called **Recurrence Aioms** [3], [4]:

- Recurrence Axiom for $\mathcal{P}$ and A (notation: $(\mathcal{P}, A)$-RcA) is the assertion:

  if some $\mathbb{P} \in \mathcal{P}$ forces $\varphi(\bar{a})$ for $\bar{a} \in A$, then there is a ground M of the universe V s.t. $\bar{a} \in M$ and $M \models \varphi(\bar{a})$.

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[1] J.D. Hamkins, A simple maximality principle, JSL, Vol.68 (2003), 527–550.
[2] J. Stavi, and J. Väänänen, Reflection principles for the continuum, Logic and algebra, Contemporary Mathematics 302, AMS (2002), 9–84.
[3] N. Barton, A.E. Caicedo, G. Fuchs, J.D. Hamkins, J. Reitz, and R. Schindler, Inner-Model Reflection Principles, Studia Log., 108 (2020), 573–595.
[4] S.F., and T. Usuba, On recurrence axioms, APAL, Vol.176, (10), (2025).

Maximality Principles and Recurrence Axioms (2/2)

Theorem 1. (S.F., and Usuba [4], see also Barton et al.B[3]) For an iterable $\mathcal{P}$, the Maximality Principle $\mathsf{MP}(\mathcal{P}, A)$ is equivalent to the following:

$((\mathcal{P}, A)\text{-RcA}^+)$: *if* $\mathbb{P} \in \mathcal{P}$ *forces* $\varphi(\bar{a})$ *for* $\bar{a} \in A$, *then there is a* *$\mathcal{P}$-ground* M *of the universe* V *s.t.* $\bar{a} \in M$ *and* $M \models \varphi(\bar{a})$.

- ▷ An inner model of ZFC in the universe V is a ground in V if there is a p.o. $\mathbb{P} \in M$ and $(M, \mathbb{P})$-generic $\mathbb{G}$ s.t. $\mathsf{V} = M[\mathbb{G}]$.
- ▷ A ground M in V is a $\mathcal{P}$-ground in V if $\mathbb{P}$ as above satisfies $M \models \mathbb{P} \in \mathcal{P}$.

- For *all* := all p.o.s, the Maximality Principle $(\mathit{all}, A)\text{-RcA}^+$ is equivalent to the Recurrence Axiom $(\mathit{all}, A)\text{-RcA}$.
- $(\mathit{all}, \emptyset)\text{-RcA}$ ($+$ LC) implies that any mathematical statement proved to be consistent with ZFC ($+$ LC) by forcing is true in some ground of V.
- Recurrence Axioms satisfy the monotonicity: $\mathcal{P} \subseteq \mathcal{P}'$ and $A \subseteq A'$ imply $(\mathcal{P}', A')\text{-RcA} \Rightarrow (\mathcal{P}, A)\text{-RcA}$.

Laver-generic Large Cardinal Axioms imply fragments of MP

- For a class $\mathcal{P}$ of p.o.s and a notion LC of large cardinal, the Laver-generic Large Cardinal Axiom for $\mathcal{P}$ and LC (notation: $\mathcal{P}$-LgLCA for LC) is the assertion for $\kappa_{\mathfrak{refl}} := \max\{\aleph_2, 2^{\aleph_0}\}$:

  for any $\lambda > \kappa_{\mathfrak{refl}}$ and $\mathbb{P} \in \mathcal{P}$, there is a $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" s.t. for $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$-generic $\mathbb{H}$, there are j, $M \subseteq \mathsf{V}[\mathbb{H}]$ with

  (a) $j : \mathsf{V} \xrightarrow{\preceq}_{\kappa_{\mathfrak{refl}}} M$,

  (b) $j(\kappa_{\mathfrak{refl}}) > \lambda$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$,

  (c) *(tightness)* $|\,\mathsf{RO}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})\,| \leq j(\kappa_{\mathfrak{refl}})$, *and*

  (d) *M satisfies the closedness property corresponding to* LC.

  for "extendible" the closedness property is "$V_{j(\lambda)}{}^{\mathsf{V}[\mathbb{H}]} \in M$" ↓

- Let Γ be the set of formulas which are conjunctions of Σ_2 formula and Π_2 formula.

Theorem 2. (S.F., and Usuba[4], S.F., Gappo, and Parente[5], S.F.[6]) $\mathcal{P}$-LgLCA for extendible implies the following restricted form of MP:

$(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_\Gamma$-RcA$^+$: *if $\mathbb{P} \in \mathcal{P}$ forces $\varphi(\overline{a})$ for $\underline{\varphi \in \Gamma}$ and $\overline{a} \in \mathcal{H}(\kappa_{\mathfrak{refl}})$, then there is a $\underline{\mathcal{P}\text{-ground}}$ M of* V *s.t. $\overline{a} \in M$ and $M \models \varphi(\overline{a})$.* ◻

Laver-generic Large Cardinal Axioms imply fragments of MP (2/2)

- The fragment $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_\Gamma$-RcA$^+$ of MP is strong enough for many applications of MP.

Lemma 3. (S.F.,and Usuba[4], Theorem 3.3) For most of (natural settings of) $\mathcal{P}$, $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_{\Sigma_2}$- RcA$^+$ decides the size of the continuum to be either $\aleph_1$, or $\aleph_2$, or "very large".

Theorem 4. (S.F,Gappo, and Parente[5], Theorem 4.1) For Σ_2 definable $\mathcal{P}$, $(\mathcal{P}, \mathcal{H}(\kappa_{\mathfrak{refl}}))_\Gamma$-RcA$^+$ implies a generalization of Viale's Absoluteness Theorem.

- $\mathcal{P}$-LgLCA for hyperhuge is introduced as in the definition on a previous slide with the closure property "$j''j(\lambda) \in M$".
- ▷ $\mathcal{P}$-LgLCA for hyperhuge implies $\mathcal{P}$-LgLCA for extendible by definition.
  - ▷ In the following, we show that even $\mathcal{P}$-LgLCA for hyperhuge does not imply the full MP (Main Theorem 9 below).

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[5] S.F.,T. Gappo, and F. Parente, Generic Absoluteness revisited, to appear in JSL.

[6] S.F., Extendible cardinals and Laver-generic large cardinal axioms for extendibility, preprint: https://arxiv.org/abs/2506.03572

Outline

Preliminaries

- We usually assume that the class $\mathcal{P}$ of p.o.s satisfies some "nice" properties (see S.F.,and Usuba[4]).

- A class $\mathcal{P}$ of p.o.s is called iterable if
  - $\{\mathbb{1}\} \in \mathcal{P}$,
  - $\mathcal{P}$ is closed w.r.t. forcing equivalence, and restriction, and
  - For any $\mathbb{P} \in \mathcal{P}$, and any $\mathbb{P}$-name $\mathbb{Q}$ of a p.o. with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$", we have $\mathbb{P} * \underset{\sim}{\mathbb{Q}} \in \tilde{\mathcal{P}}$.

- An iterable class $\mathcal{P}$ of p.o.s is said to be transfinitely iterable if it permits transfinite iteration with some suitable support s.t. the limit in an iteration at an inaccessible cardinal for the support is the direct limit, and Iteration Lemmas hold for the iteration of $\mathcal{P}$ p.o.s.

- A transfinitely iterable class $\mathcal{P}$ of p.o.s is nice if it is ω_1 preserving and either $\mathcal{P}$ provably contains p.o.s collapsing $\aleph_2$ or it provably contains p.o.s adding reals.

Preliminaries (2/4)

Theorem 5.2

Proposition 5. (see S.F.[6]) For a nice transfinitely iterable Σ_2 $\mathcal{P}$, the consistency of "ZFC +there is a hyperhuge (extendible resp.)κ" implies the consistency of ZFC + $\mathcal{P}$-LgLCA for huperhuge(extendible resp.).

Proof. The iteration of p.o.s of $\mathcal{P}$ of length κ with the appropriate support along with the Laver function for the respective large cardinal produces a model of $\mathcal{P}$-LgLCA for hyperhuge/extendible. □

Theorem 6. (S.F.,and Usuba[4], A special case of Theorem 5.2 + 5.3) For a class $\mathcal{P}$ of p.o.s, if $\mathcal{P}$-LgLCA for hyperhuge holds, then there is the bedrock which is a $\leq \kappa_{\mathfrak{refl}}$-ground $\overline{M}$ of V, and $(\kappa_{\mathfrak{refl}})^V$ is a hyperhuge cardinal in $\overline{M}$. □

Corollary 7. For nice transfinitely iterable Σ_2 $\mathcal{P}$, $\mathcal{P}$-LgLCA for hyperhuge is equiconsistent with the assertion "a hyperhuge cardinal exists". □

Corollary 8. If $\mathcal{P}$-LgLCA for hyperhuge holds then every ground of V is a $\leq \kappa_{\mathfrak{refl}}$-ground of V. □

Preliminaries (3/4)

- Laver generic Large Cardinal Axiom has the following important variation obtained by setting $\mathcal{P} = all$, and replacing $\kappa_{\mathfrak{refl}}$ by $2^{\aleph_0}$:
- For a notion LC of large cardinal the Laver-generic Large Cardinal Axiom for all posets for LC (LgLCAA for LC for short) is the assertion:

> *for any* $\lambda > 2^{\aleph_0}$ *and p.o.* $\mathbb{P}$, *there is a* $\mathbb{P}$*-name* $\underset{\sim}{\mathbb{Q}}$ *of a p.o. s.t. for* $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$*-generic* $\mathbb{H}$, *there are* j, $M \subseteq \mathsf{V}[\mathbb{H}]$ *with*
>
> (a) $j : \mathsf{V} \overset{\preceq}{\rightarrow}_{2^{\aleph_0}} M$,
>
> (b) $j(2^{\aleph_0}) > \lambda$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$,
>
> (c) *(tightness)* $|\mathsf{RO}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})| \leq j(\kappa)$, *and*
>
> (d) M *satisfies the closedness property corresponding to* LC.

- LgLCAA for LC implies CH (see S.F.[6], Lemma 8.1).
- Note that $2^{\aleph_0}$ in the definition above cannot be replaced by $\kappa_{\mathfrak{refl}}$ since $\mathcal{P}$-LgLCA implies that $\mathcal{P}$ is stationary preserving (see S.F.[6], Theorem 4.5, (1)).

Preliminaries (4/4)

Theorem 5.2

Proposition 5'. (see S.F.[6]) The consistency of "ZFC + there is a hyperhuge (extendible resp.) κ" implies the consistency of ZFC + LgLCAA for huperhuge (extendible resp.).

Proof. The iteration of p.o.s of length κ with the appropriate support along with the Laver function for the respective large cardinal produces a model of LgLCAA for hyperhuge/extendible. □

Theorem 6'. (S.F.,and Usuba[4], A special case of Theorem 5.2 + 5.3) If LgLCAA for hyperhuge holds, then there is the bedrock which is a $\leq 2^{\aleph_0}$-ground $\overline{M}$ of V, and $(2^{\aleph_0})^{\mathsf{V}}$ is a hyperhuge cardinal in $\overline{M}$. □

Corollary 7'. $\mathcal{P}$-LgLCAA for hyperhuge is equiconsistent with the assertion "a (genuin) hyperhuge cardinal exists". □

Corollary 8'. If LgLCAA for hyperhuge holds then every ground of V is a $\leq 2^{\aleph_0}$-ground of V. □

Main Theorem 9

Main Theorem 9. (S.F.[7]) (1) Suppose that $\mathcal{P}$ is a nice transfinitely iterable class of p.o.s. Then the $\mathcal{P}$-LgLCA for hyperhuge does not imply $(\mathcal{P}, \emptyset)$-RcA.

(2) LgLCAA for hyperhuge does not imply $(\mathcal{P}, \emptyset)$-RcA for any $\mathcal{P}$ containing enough collapsing or adding reals.

▷ The following Lemma is used in the proof:

Lemma 10. Suppose that λ is an inaccessible cardinal and $n \geq 2$. Then, for any p.o. $\mathbb{P}$ with $|\mathbb{P}| < \lambda$, λ is $C^{(n)}$ if and only if $\Vdash_{\mathbb{P}}$ "λ is $C^{(n)}$".

Proof.

[7] S.F., Resurrection and Maximality in light of Laver-generic large cardinal, in preparation (pre-preprint: https://fuchino.ddo.jp/papers/RIMS2022-RA-MP-x.pdf)

Main Theorem 9. (S.F.[7]) (1) Suppose that $\mathcal{P}$ is a nice transfinitely iterable class of p.o.s. Then the $\mathcal{P}$-LgLCA for hyperhuge does not imply $(\mathcal{P}, \emptyset)$-RcA.

(2) ...

Proof. We only prove Main Theorem 9., (1). (2) can be proved similarly.

- Let $m \geq 2$ be large enough (for the following argument). In particular, we assume that the statement "$\underline{x}$ is hyperhuge" is Σ_m and V_μ for $C^{(m)}$ cardinal μ is absolute for the properties of $\mathcal{P}$ used in the following.
- We work in the theory: ZFC + "κ is the least hyperhuge cardinal" + "$\lambda_0 < \lambda_1$ are the first two inaccessible cardinals above κ which are $C^{(m)}$".
- We have $V_{\lambda_1} \models$ "κ is a hyperhuge cardinal". By Proposition 5., there is $\mathbb{P} \in \mathcal{P} \cap V_{\lambda_1}$ with cardinality κ s.t., for $(V_{\lambda_1}, \mathbb{P})$-generic $\mathbb{G}$, $V_{\lambda_1}[\mathbb{G}] \models \mathcal{P}$-LgLCA for hyperhuge.

▷ Thus the following claim proves the theorem:

Proof of Main Theorem 9 (2/2)

Claim. $V_{\lambda_1}[\mathbb{G}] \models \neg(\mathcal{P}, \emptyset)$-RcA.

$\vdash$ Assume, towards a contradiction, that

(*) $V_{\lambda_1}[\mathbb{G}] \models (\mathcal{P}, \emptyset)$-RcA.

▷ We have

$V_{\lambda_1} \models$ " λ_0 is the unique inaccessible which is also $C^{(m)}$ above κ."

by the choice of κ, λ_0, λ_1.

▷ Thus, by Lemma 10. ,

(**) $V_{\lambda_1}[\mathbb{G}] \models$ " λ_0 is the unique inaccessible which is also $C^{(m)}$ above κ."

▷ By the assumption on $\mathcal{P}$ there is $\mathbb{Q} \in \mathcal{P}^{V_{\lambda_1}[\mathbb{G}]}$ s.t.
$V_{\lambda_1}[\mathbb{G}] \models$ " $\Vdash_{\mathbb{Q}}$ " there is no inaccessible cardinal which is $C^{(m)}$ "".

▷ By (*) it follows that there is a ground M of $V_{\lambda_1}[\mathbb{G}]$ s.t.
$M \models$ " there is no inaccessible cardinal which is $C^{(m)}$".

By Corollary 8. , M is a $\leq (\kappa_{\mathfrak{refl}})^{V_{\lambda_1}[\mathbb{G}]}$-ground of $V_{\lambda_1}[\mathbb{G}]$. This is a contradiction to (**). $\dashv$ ◫ (Main Theorem 9.)

Super-$C^{(\infty)}$-Laver-generic Large Cardinal Axioms imply the full MP

- For a class $\mathcal{P}$ of p.o.s, the Super-$C^{(\infty)}$-$\mathcal{P}$-Laver-generic Large Cardinal Axiom for LC (Super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for LC, for short) holds, if

  for any $n \in \mathbb{N}$, $\lambda_0 > \kappa_{\mathfrak{refl}}$ *and* $\mathbb{P} \in \mathcal{P}$, *there are* $\lambda \geq \lambda_0$ *with* $V_\lambda \prec_{\Sigma_n} \mathsf{V}$ *and a* $\mathbb{P}$*-name* $\underset{\sim}{\mathbb{Q}}$ *with* $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\mathbb{Q}} \in \mathcal{P}$" *s.t. for* $(\mathsf{V}, \mathbb{P} * \underset{\sim}{\mathbb{Q}})$*-generic* $\mathbb{H}$, *there are* j, $M \subseteq \mathsf{V}[\mathbb{H}]$ *with*

  (a) $j : \mathsf{V} \overset{\prec}{\rightarrow}_{\kappa_{\mathfrak{refl}}} M$,
  (b) $j(\kappa_{\mathfrak{refl}}) > \lambda$, $\mathbb{P}$, $\mathbb{P} * \underset{\sim}{\mathbb{Q}}$, $\mathbb{H} \in M$,
  (c) *(tightness)* $|\mathsf{RO}(\mathbb{P} * \underset{\sim}{\mathbb{Q}})| \leq j(\kappa_{\mathfrak{refl}})$,
  (d) M *satisfies the closedness property corresponding to* LC, *and*
  (e) ${V_{j(\lambda)}}^{\mathsf{V}[\mathbb{H}]} \prec_{\Sigma_n} \mathsf{V}[\mathbb{H}]$.

- Note that "$V_\lambda \prec_{\Sigma_n} \mathsf{V}$" is formalizable in ZFC but not the assertion "$V_\lambda \prec \mathsf{V}$.
- Note that Super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for LC is an axiom scheme.
- The Super-$C^{(\infty)}$-LgLCAA for LC is defined similarly.

Theorem 11. (S.F.,and Usuba[4], Theorem 4.10) (1) The super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for extendible implies: $\mathsf{MP}(\mathcal{P},\mathcal{H}(\kappa_{\mathfrak{refl}}))$.

(2) The super-$C^{(\infty)}$-LgLCAA for extendible implies: $\mathsf{MP}(\mathit{all},\mathcal{H}(2^{\aleph_0}))$.

Theorem 6". (S.F.,and Usuba[4], Theorem 5.8) (1) For a class $\mathcal{P}$ of p.o.s, if the super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for hyperhuge holds, then there is the bedrock $\overline{M}$ which is a $\leq\kappa_{\mathfrak{refl}}$-ground of V, and $(\kappa_{\mathfrak{refl}})^{\mathsf{V}}$ is a super-C^{∞} hyperhuge cardinal in $\overline{M}$.

(2) If the super-$C^{(\infty)}$-LgLCAA for hyperhuge holds, then there is the bedrock $\overline{M}$ which is a $\leq 2^{\aleph_0}$-ground of V, and $(2^{\aleph_0})^{\mathsf{V}}$ is a super-C^{∞} hyperhuge cardinal in $\overline{M}$.

Super-$C^{(\infty)}$-Laver-generic Large Cardinal Axioms imply the full MP (3/3)

- The consistency of each of the super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for hyperhuge for nice transfinitely iterable $\mathcal{P}$ and the super-$C^{(\infty)}$-LgLCAA for hyperhuge follows from the existence of a 2-huge cardinal. (S.F.,and Usuba[4])
- The consistency of each of the super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA for extendible for nice transfinitely iterable $\mathcal{P}$ and the super-$C^{(\infty)}$-LgLCAA for extendible follows from the existence of an almost huge cardinal (S.F.[6]).

Open Problem: Does the super-$C^{(\infty)}$-$\mathcal{P}$-LgLCA or the super-$C^{(\infty)}$-LgLCAA for extendible also imply the existence of bedrock?

- ▷ C.f.: For genuine large cardinals the assumption of a hyperhuge cardinal can be reduced to the existence of an extendible cardinal in connection with the existence of the bedrock (see [8], [9]).

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[8] T. Usuba, The downward directed grounds hypothesis and very large cardinals, JML, Vol. 17(2) (2017), 1–24.

[9] T. Usuba, Extendible cardinals and the mantle, AML Vol.58, (2019), 71-75.

Jörg さん

還暦おめでとうございます.

Herzlichen Glückwunsch zum 60. Geburtstag.

Congratulations on your 60th birthday.

환갑을 축하합니다.

Felicidades por tu 60 cumpleaños.

Lemma 10. Suppose that λ is an inaccessible cardinal and $n \geq 2$. Then, for any p.o. $\mathbb{P}$ with $|\mathbb{P}| < \lambda$, λ is $C^{(n)}$ if and only if $\Vdash_{\mathbb{P}}$ "λ is $C^{(n)}$".

Proof. W.l.o.g., we may assume that $\mathbb{P} \in V_\lambda$.

- Assume first that $\lambda \in C^{(n)}$ (in V). Let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic set, and φ a Σ_n-formula. For arbitrary $\overline{a} \in {V_\lambda}^{\mathsf{V}[\mathbb{G}]}$ $(= V_\lambda[\mathbb{G}])$, there is $\mathbb{P}$-name $\underset{\sim}{\overline{a}} \in V_\lambda$ s.t. $\overline{a} = \underset{\sim}{\overline{a}}[\mathbb{G}]$.
  - ▷ Note that the formula expressing "$\underline{\mathbb{p}} \Vdash_{\underline{\mathbb{P}}}$ "$\varphi(\underline{\overline{a}})$"" with free variables $\underline{\mathbb{p}}$, $\underline{\mathbb{P}}$, $\underline{\overline{a}}$ is Σ_n. Thus $\mathsf{V}[\mathbb{G}] \models \varphi(\overline{a}) \Leftrightarrow$ there is $\mathbb{p} \in \mathbb{G}$ s.t. $\mathsf{V} \models \mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{\overline{a}})$" $\Leftrightarrow$ there is $\mathbb{p} \in \mathbb{G}$ s.t. $V_\lambda \models \mathbb{p} \Vdash_{\mathbb{P}}$ "$\varphi(\underset{\sim}{\overline{a}})$" $\Leftrightarrow$ ${V_\lambda}^{\mathsf{V}[\mathbb{G}]} = V_\lambda[\mathbb{G}] \models \varphi(\overline{a})$.
  - ▷ This shows the direction "$\Rightarrow$" of the equivalence.
- Suppose now that $\mathsf{V} \models \; \Vdash_{\mathbb{P}}$ "$\lambda \in C^{(n)}$". Let $\mathbb{G}$ be a $(\mathsf{V}, \mathbb{P})$-generic filter, $\varphi = \varphi(\overline{x})$ a Σ_n formula, and $\overline{a} \in V_\lambda$. We have $V_\lambda \models \varphi(\overline{a}) \Leftrightarrow {V_\lambda}^{\mathsf{V}[\mathbb{G}]} \models$ "$\underset{\smile}{\mathsf{V}} \models \varphi(\overline{a})$". The latter condition can be formulated as a Σ_n-formula with an additional parameter from ${V_\lambda}^{\mathsf{V}[\mathbb{G}]}$. Thus this last condition is equivalent to $\mathsf{V}[\mathbb{G}] \models$ "$\underset{\smile}{\mathsf{V}} \models \varphi(\overline{a})$" $\Leftrightarrow \mathsf{V} \models \varphi(\overline{a})$. This shows that $\mathsf{V} \models \lambda \in C^{(n)}$. ◻ (Lemma 10.)

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